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Lagrange–Hermite interpolation on the real semiaxis

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Abstract In order to approximate continuous functions on $[0, +\infty)$, we consider a Lagrange–Hermite polynomial, interpolating a finite section of the function at the zeros of some orthogonal polynomials and, with its first $(r-1)$ derivatives, at the point 0. We give necessary and sufficient conditions on the weights for the uniform boundedness of the related operator. Moreover, we prove optimal estimates for the error of this process in the weighted L^p and uniform metric.

Keywords Hermite–Lagrange interpolation · Approximation by algebraic polynomials · Orthogonal polynomials · Generalized Laguerre weights · Real semiaxis

Mathematics Subject Classification 41A05 · 41A10

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1 Introduction

In this paper we discuss the weighted polynomial interpolation of continuous functions on $[0, +\infty)$, which are $(r-1)$-times differentiable at 0 and can increase with order $\mathcal{O}\left(e^{x^\beta/2}\right)$, $\beta > 1/2$, for $x \to \infty$.

The case $\beta = 1$ has been treated in [1,5,15,20,22,24–26], where the Authors considered Lagrange polynomials based on Laguerre zeros (see also [8]). Here we choose as a tool the orthonormal system $\{p_m(w)\}_m$ in $(0, +\infty)$ related to a generalized Laguerre weight of the form $w(x) = x^\alpha e^{-x^\beta}$. From the numerical point of view, we observe that the weight w is nonclassical in general and, for the construction of the orthonormal system in the case $\beta \neq 1$, we can use the procedure introduced in [2] (see also [16]).

The presence of the derivatives of the function in 0 leads in a natural way to the construction of Lagrange–Hermite polynomial $\mathcal{L}_{m,r}(w, f)$ based at the zeros of the polynomial $p_m(w)$, 0 as a multiple node and another additional node. Applying the operator $\mathcal{L}_{m,r}(w)$ to a suitable finite section of the function f, we obtain a new interpolation process, that we will denote by $\mathcal{L}^*_{m,r}(w)$. This new operator is not a projector into the set of all polynomials of degree at most $m+r$, $\mathbb{P}_{m+r}$, but on a special subset $\mathcal{P}^*_{m,r} \subset \mathbb{P}_{m+r}$ that can replace the space $\mathbb{P}_{m+r}$, in the sense we are going to show. Thus the projector $\mathcal{L}^*_{m,r}(w)$ can be profitably used in quadrature rules and in the numerical treatment of Boundary Value Problems on $(0, \infty)$ (see, e.g., [6,27]).

We estimate the error of the process in weighted L^p and uniform metric and compare it with the order of convergence of the best weighted polynomial approximation (that can be found in [21]). The error estimates are sharp for the considered classes of functions. All the results of this paper are new and cover the ones in the literature.

The paper is structured as follows. In Sect. 2 we recall some basic facts and give some preliminary results, in Sect. 3 we will state our main results and in Sect. 4 we will prove them.

2 Preliminary results

In the sequel $\mathcal{C}$ will stand for a positive constant that can assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ when $\mathcal{C}$ is independent of $a, b, \ldots$. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leq \mathcal{C}$. Finally, we denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m.

In order to introduce some interpolation processes we consider the weight

$$w(x) = x^\alpha e^{-x^\beta}, \quad x \in (0, +\infty), \tag{2.1}$$

with $\alpha > -1$, $\beta > \frac{1}{2}$, and the corresponding sequence of orthonormal polynomials $\{p_m(w)\}_m$, with positive leading coefficient γ_m. The zeros $x_k = x_{m,k}(w)$ of $p_m(w)$, $m \geq 1$, are located as follows

$$\left(\frac{\sqrt{a_m}}{m}\right)^2 < x_1 < x_2 < \cdots < x_m < a_m\left(1-\frac{c}{m^{2/3}}\right),$$

where

$$a_m = a_m(\sqrt{w}) = \left[\frac{2^{2\beta-2}\Gamma(\beta)^2}{\Gamma(2\beta)}\right]^{1/\beta}(4m+2\alpha+1)^{1/\beta} \sim m^{1/\beta}$$

is the Mhaskar–Rakhmanov–Saff number related to the weight $\sqrt{w}$ (see also [18, 19]). If $\beta = 1$ the nodes x_k are the Laguerre zeros. Nevertheless, in general w is a nonstandard weight and, for the computation of the zeros and the Christoffel numbers, we may use the Mathematica package "*OrthogonalPolynomials*" introduced in [2].

For any continuous function in $[0,+\infty)$, $f \in C^0([0,+\infty))$, and $(r-1)$-times differentiable at 0, briefly $f \in C_r^0, r \geq 1$, we define the Lagrange–Hermite polynomial $\mathcal{L}_{m,r}(w, f)$ as follows

$$\mathcal{L}_{m,r}(w, f, x_k) = f(x_k), \quad k = 1, \ldots, m+1,$$

with $x_{m+1} := a_m$, and

$$\mathcal{L}_{m,r}(w, f)^{(j)}(0) = f^{(j)}(0), \quad j = 0, 1, \ldots, r-1,$$

where $f^{(0)} \equiv f$. Here we used also an idea introduced by Szabados for Lagrange and Hermite–Fejér interpolation, adding the additional node a_m (see [28]).

The polynomial $\mathcal{L}_{m,r}(w, f)$ can be written as

$$\mathcal{L}_{m,r}(w, f, x) = \sum_{k=1}^{m+1} \frac{x^r}{x_k^r} \ell_k(w, x) f(x_k) + (a_m - x) p_m(w, x) \sum_{i=0}^{r-1} \frac{x^i}{i!} \left(\frac{f}{(a_m - \cdot) p_m(w)}\right)^{(i)}(0), \quad (2.2)$$

where

$$\ell_k(w, x) = \frac{p_m(w, x)}{(x - x_k) p_m'(w, x_k)} \frac{(a_m - x)}{(a_m - x_k)}, \quad k = 1, 2, \ldots, m,$$

and

$$\ell_{m+1}(w, x) = \frac{p_m(w, x)}{p_m(w, a_m)}$$

are the fundamental Lagrange polynomials. It is easily seen that

$$\mathcal{L}_{m,r}(w, P) = P, \quad P \in \mathbb{P}_{m+r},$$

and $\mathcal{L}_{m,r}(w)$ is a projector from C^0_r into $\mathbb{P}_{m+r}$.

We are now going to introduce another Lagrange–Hermite type operator $\mathcal{L}^*_{m,r}(w)$, modifying the operator $\mathcal{L}_{m,r}(w)$. To this end, for a fixed $\theta \in (0, 1)$ and for a sufficiently large m we define an index $j = j(m, \theta)$ such that

$$x_j = \min_k \{x_k : x_k \ge \theta a_m\}. \tag{2.3}$$

Of course if m is small we let $j = m$. Hence we define the new Lagrange–Hermite type polynomial as follows

$$\begin{aligned}\mathcal{L}^*_{m,r}(w, f, x) &= \sum_{k=1}^{j} \frac{x^r}{x_k^r} \ell_k(w, x) f(x_k)\\ &\quad + (a_m - x) p_m(w, x) \sum_{i=0}^{r-1} \frac{x^i}{i!} \left(\frac{f}{(a_m - \cdot) p_m(w)} \right)^{(i)} (0).\end{aligned} \tag{2.4}$$

By this definition,

$$\begin{aligned}\mathcal{L}^*_{m,r}(w, f, x_k) &= f(x_k), \quad k = 1, \ldots, j,\\ \mathcal{L}^*_{m,r}(w, f, x_k) &= 0, \quad k = j+1, \ldots, m+1,\end{aligned}$$

and

$$\mathcal{L}^*_{m,r}(w, f)^{(i)}(0) = f^{(i)}(0), \quad i = 0, 1, \ldots, r-1.$$

Moreover, we observe that $\mathcal{L}^*_{m,r}(w)$ does not preserve all the polynomials of degree at most $m + r$, for example $\mathcal{L}^*_{m,r}(w, 1) \neq 1$.

Nevertheless, if we introduce the following set of polynomials

$$\mathcal{P}^*_{m,r} = \{Q \in \mathbb{P}_{m+r} : Q(x_i) = 0,\ i > j\}.$$

then it is easy to show that for any $f \in C^0_r$, $\mathcal{L}^*_{m,r}(w, f) \in \mathcal{P}^*_{m,r}$ and for any $Q \in \mathcal{P}^*_{m,r}$, $\mathcal{L}^*_{m,r}(w, Q) = Q$, therefore $\mathcal{L}^*_{m,r}(w, f)$ is a projector from C^0_r into $\mathcal{P}^*_{m,r}$.

In order to show some approximation properties of the projectors defined above, we are going to define some function spaces.

2.1 Function spaces

Let us consider a weight of the form

$$u(x) = x^\gamma \mathrm{e}^{-x^\beta/2}, \quad x \in (0, +\infty), \tag{2.5}$$

with $\beta > \frac{1}{2}$ and the following weighted function spaces associated to u.

For $1 \le p < \infty$ and $\gamma > -1/p$, by L_u^p we denote the set of all measurable functions f such that

$$\|f\|_{L_u^p} := \|fu\|_p = \left(\int_0^{+\infty} |fu|^p(x)\,\mathrm{d}x\right)^{1/p} < \infty.$$

For $p = \infty$ and $\gamma \ge 0$, by a slight abuse of notation, we set

$$L_u^\infty := C_u = \left\{ f \in C^0(0,+\infty) : \lim_{\substack{x\to 0^+ \\ x\to+\infty}} f(x)u(x) = 0 \right\},$$

and we equip this space with the norm

$$\|f\|_{L_u^\infty} := \|fu\|_\infty = \sup_{x\in(0,+\infty)} |f(x)u(x)|.$$

Note that the Weierstrass theorem implies the limit conditions in the definition of C_u.

Subspaces of L_u^p, $1 \le p \le \infty$, are the Sobolev spaces, given by

$$W_s^p(u) = \left\{ f \in L_u^p : \ f^{(s-1)} \in AC(0,+\infty), \ \|f^{(s)}\varphi^s u\|_p < \infty \right\}, \quad 1 \le s \in \mathbb{Z},$$

where $AC(0,+\infty)$ denotes the set of all functions which are absolutely continuous on every closed subset of $(0,+\infty)$ and $\varphi(x) = \sqrt{x}$. We equip these spaces with the norm

$$\|f\|_{W_s^p(u)} = \|fu\|_p + \|f^{(s)}\varphi^s u\|_p.$$

In order to define some further function spaces, we consider the s-th modulus of smoothness of $f \in L_u^p$, $1 \le p \le \infty$, $s \ge 1$,

$$\Omega_\varphi^s(f,t)_{u,p} = \sup_{0<h\le t} \left\| \Delta_{h\varphi}^s(f)\,u \right\|_{L^p(I_{rh})},$$

where

$$\Delta_{h\varphi}^s f(x) = \sum_{i=0}^s (-1)^i \binom{s}{i} f\left(x + (s-i)h\varphi(x)\right)$$

is the forward finite difference of order s with variable step $h\varphi(x)$, $\varphi(x) = \sqrt{x}$ and $I_{sh} = [8s^2h^2, \mathcal{C}h^{-\frac{1}{(\beta-1/2)}}]$, $h > 0$.

Let $f \in L_u^p$, $1 \le p \le \infty$. Then the following estimate

$$\Omega_\varphi^s(f,t)_{u,p} \le \mathcal{C} \sup_{0<h\le t} h^s \|f^{(s)}\varphi^s u\|_{L^p(I_{sh})}, \tag{2.6}$$

holds with $\mathcal{C} \neq \mathcal{C}(f,t)$, provided the norm on the right-hand side is finite (see [21]).

Using these moduli of smoothness we can define the Zygmund spaces as

$$Z_\lambda^p(u) = \left\{ f \in L_u^p : \sup_{t>0} \frac{\Omega_\varphi^s(f,t)_{u,p}}{t^\lambda} < \infty,\ s > \lambda \right\}, \quad 0 < \lambda \in \mathbb{R},$$

with the norm

$$\|f\|_{Z_\lambda^p(u)} = \|fu\|_p + \sup_{t>0} \frac{\Omega_\varphi^s(f,t)_{u,p}}{t^\lambda}, \quad s > \lambda.$$

Let us denote by

$$E_m(f)_{u,p} = \inf_{P_m \in \mathbb{P}_m} \|(f - P_m)\, u\|_p$$

the error of best weighted polynomial approximation in L_u^p, $1 \leq p \leq \infty$. Estimates for the error of best weighted approximation have been proved in [21]. In particular it has been shown that

$$\lim_{m\to\infty} E_m(f)_{u,p} = 0 \quad \forall f \in L_u^p. \tag{2.7}$$

For our aims it is sufficient to recall the following weak Jackson inequality

$$E_m(f)_{u,p} \leq \mathcal{C} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^s(f,t)_{u,p}}{t}\, \mathrm{d}t \tag{2.8}$$

that holds for any function $f \in L_u^p$, $1 \leq p \leq \infty$.

In particular, if $f \in Z_\lambda^p$, by (2.8) we deduce

$$E_m(f)_{u,p} \leq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^\lambda \sup_{t>0} \frac{\Omega_\varphi^s(f,t)_{u,p}}{t^\lambda}, \tag{2.9}$$

and by (2.6) we have

$$E_m(f)_{u,p} \leq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^s \|f^{(s)} \varphi^s u\|_p, \tag{2.10}$$

for any $f \in W_s^p$, $s \geq 1$ and $1 \leq p \leq \infty$.

We are now able to show an important property of the polynomials in $\mathcal{P}_{m,r}^*$. To this end we need some further notation. We say that P_M is a polynomial of quasi best approximation for $f \in L_u^p$, $1 \leq p \leq \infty$, if, for some $\mathcal{C} \geq 1$,

$$\|(f - P_M)\, u\|_p \leq \mathcal{C} E_M(f)_{u,p}.$$

Moreover, we denote by

$$E^*_{m,r}(f)_{u,p} = \inf_{Q \in \mathcal{P}^*_{m,r}} \|(f - Q)u\|_p,$$

the error of best approximation by means of polynomials of $\mathcal{P}^*_{m,r}$. The relation between $E_m(f)_{u,p}$ and $E^*_m(f)_{u,p}$ is established by the following Lemma.

Lemma 1 *Let w, u be the weights defined in (2.1) and (2.5) with $\alpha > -1$, $\beta > \frac{1}{2}$ and $u \in L^p$, $1 \le p \le \infty$. Let $f \in L^p_u$ and $P_M \in \mathbb{P}_M$ one of its polynomial of quasi best approximation, with*

$$M = \left\lfloor \left(\frac{\theta}{\theta + 1}\right) m \right\rfloor \tag{2.11}$$

for a fixed $\theta \in (0, 1)$. Then we have

$$E^*_{m,r}(f)_{u,p} \le \|[f - \mathcal{L}^*_{m,r}(w, P_M)]u\|_p \le \mathcal{C}\{E_M(f)_{u,p} + \mathrm{e}^{-cm}\|fu\|_p\}, \tag{2.12}$$

where $\mathcal{C}, c$ are independent of m, r, f. Moreover, for $s \ge 1$, we get

$$\left(\frac{\sqrt{a_m}}{m}\right)^s \|\mathcal{L}^*_{m,r}(w, P_M)^{(s)}\varphi^s u\|_p \le \mathcal{C}\left\{\mathrm{e}^{-cm}\|fu\|_p + \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^s_\varphi(f,t)_{u,p}}{t}\,\mathrm{d}t\right\}. \tag{2.13}$$

Using Lemma 1 and (2.7) it follows that the order of convergence of $E^*_{m,r}(f)_{u,p}$ is the same as that of $E_{m,r}(f)_{u,p}$. Therefore, $\bigcup_m \mathcal{P}^*_{m,r}$ is dense in L^p_u. In the next Section we will show the behaviour of the operators $\mathcal{L}_{m,r}(w)$ and $\mathcal{L}^*_{m,r}(w)$ in different function spaces.

3 Main results

The following results hold true in weighted uniform norm. In order to state it we need a further definition. We say that a function is quasi increasing on I if there exists a constant $\mathcal{C}$ such that $f(x) \le \mathcal{C} f(y)$ for $x < y$, $x, y \in I$.

Theorem 2 *Let w and u be the previously defined weights, with $\gamma \ge 0$ and $\alpha > -1$. For every $f \in C^0_r \cap C_u$, $r \ge 1$, we have*

$$\left\|\mathcal{L}_{m,r}(w, f)\, u\right\|_\infty \le \mathcal{C}\left\{(\log m)\|fu\|_\infty + \left(\frac{\sqrt{a_m}}{m}\right)^{2\gamma} \sum_{i=0}^{r-1} \frac{|f^{(i)}(0)|}{i!}\left(\frac{\sqrt{a_m}}{m}\right)^{2i}\right\}, \tag{3.1}$$

where $\mathcal{C} \ne \mathcal{C}(m, f)$, if and only if

$$\frac{\alpha}{2} + \frac{1}{4} \le \gamma + r \le \frac{\alpha}{2} + \frac{5}{4}. \tag{3.2}$$

Moreover, under the assumptions (3.2)*, if* $\left|f^{(i)}\right|$ *is quasi increasing on* $[0, \alpha_m]$*, with* $\alpha_m = a_m/m^2$ *and* $i = 0, 1, \ldots, r-1$*, we get*

$$\left\|\left[f - \mathcal{L}_{m,r}(w, f)\right] u\right\|_\infty \le \mathcal{C}(\log m) E_{m+r}(f)_{u,\infty} + \mathcal{O}\left(\left(\frac{\sqrt{a_m}}{m}\right)^{r-1}\right) \tag{3.3}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$ *and the constant in "*$\mathcal{O}$*" is independent of* m*.*

Therefore, in weighted spaces of continuous functions, the behaviour of the operators $\left\{\mathcal{L}_{m,r}(w)\right\}_m$ is comparable with similar interpolation processes based on Jacobi zeros on bounded intervals (see, e.g., [15,17]).

We emphasize that if r and γ are given, then, in order to approximate a function f, we can always choose some α such that the Lagrange–Hermite polynomial converges with the order of the best polynomial approximation times the extra factor $\log m$. In fact, we can rewrite (3.2) as follows

$$2r + 2\gamma - \frac{5}{2} \le \alpha \le 2r + 2\gamma - \frac{1}{2}.$$

Moreover, since the Lebesgue constants related to the processes $\{\mathcal{L}^*_{m,r}(w)\}_m$ are bounded by the ones related to $\{\mathcal{L}_{m,r}(w)\}_m$, by Theorem 2 and Lemma 1, we deduce the following error estimate for the "truncated" process.

Corollary 3 *Let* $\theta \in (0, 1)$*. Under the assumptions of Theorem* 2*, for every* $f \in C^0_r$*,* $r \ge 1$*, for* m *sufficiently large (say* $m > m_0$*) we have*

$$\left\|\left[f - \mathcal{L}^*_{m,r}(w, f)\right] u\right\|_\infty \le \mathcal{C}(\log m) E_M(f)_{u,\infty} + \mathcal{O}\left(\left(\frac{\sqrt{a_m}}{m}\right)^{r-1}\right),$$

where M *given by* (2.11)*,* $\mathcal{C} \neq \mathcal{C}(m, f)$ *and the constant in "*$\mathcal{O}$*" is independent of* m*.*

In weighted L^p-spaces the behaviour of $\left\{\mathcal{L}_{m,r}(w)\right\}_m$ is not optimal, while for the new process $\left\{\mathcal{L}^*_{m,r}(w)\right\}_m$ we can state the following

Theorem 4 *Let* $u \in L^p$*,* $1 < p < \infty$ *and* $\theta \in (0, 1)$ *be fixed. Then, for every function* $f \in C^0_r$*,* $r \ge 1$*, we have*

$$\left\|\mathcal{L}^*_{m,r}(w, f)\, u\right\|_p \sim \left(\sum_{k=1}^{j} \Delta x_k |f u|^p(x_k)\right)^{1/p} + \left(\frac{\sqrt{a_m}}{m}\right)^{2(\gamma+1/p)} \sum_{i=0}^{r-1} \frac{|f^{(i)}(0)|}{i!} \left(\frac{\sqrt{a_m}}{m}\right)^{2i} \tag{3.4}$$

if and only if

$$\frac{\alpha}{2} + \frac{1}{4} - \frac{1}{p} < \gamma + r < \frac{\alpha}{2} + \frac{5}{4} - \frac{1}{p}, \tag{3.5}$$

where the constants in "$\sim$" depend on θ and are independent of m and f.

If the function $f \in L^p_u$ fulfills the additional assumption

$$\frac{\Omega^s_\varphi(f,t)_{u,p}}{t^{1+1/p}} \in \mathrm{L}^1[0,1] \tag{3.6}$$

for some $s \geq 1$, then f is continuous in $(0,+\infty)$ (see [21]). Moreover, using the same arguments as in [22], we obtain

$$\left(\sum_{k=1}^{j} \Delta x_k |fu|^p(x_k)\right)^{1/p} \leq \mathcal{C}\left[\|fu\|_p + \left(\frac{\sqrt{a_m}}{m}\right)^{1/p} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^s_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t\right].$$

Hence, if $f \in L^p_u$ fulfills (3.6) and is $(r-1)$-times differentiable at 0, the bound (3.4) becomes

$$\begin{aligned}\left\|\mathcal{L}^*_{m,r}(w,f)\,u\right\|_p \leq \mathcal{C}\Bigg[&\|fu\|_p + \left(\frac{\sqrt{a_m}}{m}\right)^{1/p} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^s_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t \\ &+ \left(\frac{\sqrt{a_m}}{m}\right)^{2(\gamma+1/p)} \sum_{i=0}^{r-1} \frac{|f^{(i)}(0)|}{i!}\left(\frac{\sqrt{a_m}}{m}\right)^{2i}\Bigg],\end{aligned} \tag{3.7}$$

where $\mathcal{C}$ depends on $\theta \in (0,1)$ (fixed) and is independent of m and f. Note that if $f \in \mathcal{P}^*_{m,r}$, (3.7) becomes an equivalence.

Theorem 5 *Under the assumptions (3.5), let $f \in L^p_u \cap C^0_r$ satisfy (3.6) for some $s \geq r$ and $\left|f^{(i)}\right|$ be quasi increasing on $[0,\alpha_m]$, with $\alpha_m = a_m/m^2$ and $i = 0,1,\ldots,r-1$. Then, for $m > m_0$, we have*

$$\|[f - \mathcal{L}^*_{m,r}(w,f)]u\|_p \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{1/p} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^s_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t + \mathcal{O}\left(\left(\frac{\sqrt{a_m}}{m}\right)^{r-1}\right) \tag{3.8}$$

where $\mathcal{C}$ depends on θ and is independent of m and f, the constant in "$\mathcal{O}$" is independent of m.

In particular, under the assumptions of Theorem 5, if $f \in Z^p_\lambda(u)$, $1/p < \lambda \leq r-1$, the error estimate (3.8) becomes

$$\left\|\left[f - \mathcal{L}^*_{m,r}(w,f)\right]u\right\|_p = \mathcal{O}\left(\left(\frac{\sqrt{a_m}}{m}\right)^{\lambda}\right),$$

where the constant in $\mathcal{O}$ is independent of m and f, that is the order of the best approximation in $Z^p_\lambda(u)$ [see (2.9)].

Analogously, if $f \in W_s^p(u)$, $s \le r-1$, from Theorem 5 we get

$$\|[f-\mathcal{L}^*_{m,r}(w,f)]u\|_p = \mathcal{O}\left(\left(\frac{\sqrt{a_m}}{m}\right)^s\right),$$

where the constant in $\mathcal{O}$ is independent of m and f [see (2.10)]. While, for $f \in W_r^p(u)$, the following corollary holds.

Corollary 6 *Under the assumptions* (3.5)*, let* $f \in W_r^p(u) \cap C_r^0$, $1 < p < \infty$*, such that* $\left|f^{(i)}\right|$ *is quasi increasing on* $[0, \alpha_m]$, $\alpha_m = a_m/m^2$*, for* $i = 0, 1, \ldots, r-1$ *and* m *sufficiently large (say* $m > m_0$*). Then we have*

$$\left\|\mathcal{L}^*_{m,r}(w,f)\,u\right\|_p \le \mathcal{C}\left\{\|fu\|_p + \left(\frac{\sqrt{a_m}}{m}\right)^r \|f^{(r)}\varphi^r u\|_p\right\} \tag{3.9}$$

and

$$\left\|\left[f-\mathcal{L}^*_{m,r}(w,f)\right]u\right\|_p \le \mathcal{C}\left\{\left(\frac{\sqrt{a_m}}{m}\right)^r \|f^{(r)}\varphi^r u\|_p + \mathrm{e}^{-cm}\|fu\|_p\right\} \tag{3.10}$$

where $\mathcal{C}$, c *depend on* θ *and are independent of* m *and* f.

Finally, the following remark is of some interest.

Remark 7 The behaviours of the two interpolation processes $\{\mathcal{L}_{m,r}(w)\}_m$ and $\{\mathcal{L}^*_{m,r}(w)\}_m$ are essentially equivalent in weighted uniform norm. Nevertheless, the "truncated" process $\{\mathcal{L}^*_{m,r}(w)\}_m$ has the advantage of dropping $c\,m$ terms, $c < 1$. This turns out to be useful in the numerical treatment of functional equations, since the dimension of the matrices obtained from the discretization of operators are strongly reduced with an evident computational saving.

On the other hand, the two operators behave in a completely different way in weighted L^p-norm. In fact, the equivalence (3.4) of Theorem 4 is not true for the "nontruncated" operator $\{\mathcal{L}_{m,r}(w)\}_m$.

Moreover, all the constants in Corollary 3, Theorems 4 and 5 are independent of m and f, but depend on θ. For instance, the constant in the upper bound of (3.4) is $\mathcal{O}\left((1-\theta)^{-3/4}\right)$ for $\theta \to 1$ and so θ cannot assume value 1.

We refer to Sect. 4 for more details.

4 Proofs

First of all we recall some weighted polynomial inequalities with the weight u defined in (2.5) which will be used in the sequel (see [21]).

Let $1 \le p \le \infty$ and $a_m = a_m(u) \sim m^{1/\beta}$. For any $P_m \in \mathbb{P}_m$, the restricted range inequalities

$$\|P_m u\|_p \le \mathcal{C}\|P_m u\|_{L^p(I_m)}, \quad I_m = \left[c\,\frac{a_m}{m^2}, a_m\right], \tag{4.1}$$

and

$$\|P_m u\|_{L^p[a_m(1+\delta),+\infty)} \le \mathcal{C}e^{-cm}\|P_m u\|_p, \quad \delta > 0, \tag{4.2}$$

hold with $\mathcal{C} \neq \mathcal{C}(m, P_m)$ and $c \neq c(m, P_m)$. Moreover, we recall the Bernstein and Markov inequalities

$$\|P_m' \varphi u\|_p \le \mathcal{C}\frac{m}{\sqrt{a_m}}\|P_m u\|_p, \quad \varphi(x) = \sqrt{x}, \tag{4.3}$$

and

$$\|P_m' u\|_p \le \mathcal{C}\frac{m^2}{a_m}\|P_m u\|_p, \tag{4.4}$$

where $\mathcal{C} \neq \mathcal{C}(m, P_m)$ in both cases. Finally, for $1 \le p < \infty$, we will need the following Nikolskii inequality

$$\|P_m u\|_\infty \le \mathcal{C}\left(\frac{m^2}{a_m}\right)^{\frac{1}{p}}\|P_m u\|_p, \quad \mathcal{C} \neq \mathcal{C}(m, P_m). \tag{4.5}$$

We also need some estimates for the polynomials of the orthonormal system $\{p_m(w)\}_{m\in\mathbb{N}}$, where w is the weight defined in (2.1). The generalized Laguerre weight w is a nonstandard weight and the estimates for the related orthogonal polynomials can be deduced by [7,11,12,21,22].

The estimate (which can be deduced from [9], see also [19,21])

$$|p_m(w,x)|\left(x + \frac{a_m}{m^2}\right)^{\frac{\alpha}{2}+\frac{1}{4}} e^{-\frac{x^\beta}{2}} \sqrt[4]{|a_m - x| + \frac{a_m}{m^{2/3}}} \sim \frac{|x - x_d|}{\Delta x_{d\pm1}} \tag{4.6}$$

holds with $x \in [0, +\infty)$ and x_d a zero closest to x. From (4.6) it follows that

$$|p_m(w,x)|\sqrt{w(x)\sqrt{x(a_m - x)}} \le \mathcal{C}, \quad x \in I_m. \tag{4.7}$$

Moreover, the following proposition will be useful.

Proposition 8 *Let w be the weight defined by (2.1) and $\{p_m(w)\}_m$ its related orthonormal system. Then we have*

$$|p_m(w,0)| \sim \left(\frac{m}{\sqrt{a_m}}\right)^{\alpha+\frac{1}{2}}\frac{1}{\sqrt[4]{a_m}}, \tag{4.8}$$

$$|p_m^{(k)}(w,0)| \le \mathcal{C}\left(\frac{m}{\sqrt{a_m}}\right)^{2k}|p_m(w,0)|, \tag{4.9}$$

and

$$\left|\left(\frac{1}{p_m(w)}\right)^{(k)}(0)\right| \le \mathcal{C}\left(\frac{m}{\sqrt{a_m}}\right)^{2k}\frac{1}{|p_m(w,0)|}, \tag{4.10}$$

where $\mathcal{C}$ and the constant in "$\sim$" are independent of m.

Proof Equivalence (4.8) easily follows from (4.6) for $x = 0$.

In order to prove (4.9) we set

$$g(x) = \left(x + \frac{a_m}{m^2}\right)^{\frac{\alpha}{2}+\frac{1}{4}} \mathrm{e}^{-\frac{x^\beta}{2}} \sqrt[4]{|a_m - x| + \frac{a_m}{m^{2/3}}}.$$

Then, using (4.4) with u replaced by g, by (4.6), we obtain

$$|p_m^{(k)}(w, x)| \le \mathcal{C} \left(\frac{m}{\sqrt{a_m}}\right)^{2k} \frac{\|p_m(w)g\|_\infty}{|g(x)|} \le \mathcal{C} \left(\frac{m}{\sqrt{a_m}}\right)^{2k} \frac{1}{|g(x)|}.$$

Hence, for $x = 0$, by (4.8), we get

$$|p_m^{(k)}(w, 0)| \le \mathcal{C} \left(\frac{m}{\sqrt{a_m}}\right)^{2k} \frac{1}{\left(\frac{\sqrt{a_m}}{m}\right)^{\alpha+\frac{1}{2}} \sqrt[4]{a_m}} \le \mathcal{C} \left(\frac{m}{\sqrt{a_m}}\right)^{2k} |p_m(w, 0)|.$$

Finally, let us show (4.10) by induction. For $k = 1$, by (4.9) we have

$$\left|\left(\frac{1}{p_m(w)}\right)'(0)\right| = \frac{|p_m'(w, 0)|}{|p_m(w, 0)|^2} \le \mathcal{C} \left(\frac{m}{\sqrt{a_m}}\right)^2 \frac{1}{|p_m(w, 0)|}.$$

For $k > 1$, from the identity

$$\left(\frac{1}{p_m(w)} p_m(w)\right)^{(k)}(0) = \sum_{i=0}^{k} \binom{k}{i} \left(\frac{1}{p_m(w)}\right)^{(i)}(0) p_m^{(k-i)}(w, 0) = 0$$

it follows that

$$-\left(\frac{1}{p_m(w)}\right)^{(k)}(0) p_m(w, 0) = \sum_{i=0}^{k-1} \binom{k}{i} \left(\frac{1}{p_m(w)}\right)^{(i)}(0) p_m^{(k-i)}(w, 0)$$

Then, using the induction hypothesis and (4.9), we obtain

$$\begin{aligned} |p_m(w, 0)| \left|\left(\frac{1}{p_m(w)}\right)^{(k)}(0)\right| &= \left|\sum_{i=0}^{k-1} \binom{k}{i} \left(\frac{1}{p_m(w)}\right)^{(i)}(0) p_m^{(k-i)}(w, 0)\right| \\ &\le \mathcal{C} \left(\frac{m}{\sqrt{a_m}}\right)^{2k}, \end{aligned}$$

which completes the proof. □

Furthermore, the following relation will be useful (see [10,21])

$$\frac{1}{|p_m'(w, x_k)|\sqrt{w(x_k)}} \sim \Delta x_k \sqrt[4]{x_k} \sqrt[4]{|a_m - x_k| + \frac{a_m}{m^{2/3}}}. \tag{4.11}$$

By (4.6) and (4.11) if $x \in \left[\left(\frac{\sqrt{a_m}}{m}\right)^2, a_m\right]$ we have

$$u(x)\frac{|\ell_k(w,x)|}{u(x_k)} \le \mathcal{C}\left(\frac{x}{x_k}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}\left(\frac{a_m-x}{a_m-x_k}\right)^{3/4}\frac{\Delta x_k}{|x-x_k|}, \quad (x \ne x_k). \tag{4.12}$$

Moreover, if x_d is a node closest to x, we get (see [4])

$$u(x)\frac{|\ell_d(w,x)|}{u(x_d)} \sim 1. \tag{4.13}$$

Proof of Lemma 1 We are going to prove (2.12) only for $p < \infty$, since the case $p = \infty$ is simpler.

Let $P_M \in \mathbb{P}_M$, with $M = \left\lfloor \frac{\theta m}{\theta+1} \right\rfloor$, be a polynomial of quasi best approximation for $f \in L^p_u$. So we can write

$$E^*_{m,r}(f)_{u,p} \le \left\| \left[f - \mathcal{L}^*_{m,r}(w, P_M)\right] u \right\|_p .$$

Since

$$\mathcal{L}^*_{m,r}(w, P_M, x) = PM(x) - \sum_{k=j+1}^{m} \frac{\ell_k(w,x)x^r}{x_k^r} P_M(x_k) \tag{4.14}$$

then, letting $v^r(x) = x^r$, we get

$$E^*_{m,r}(f)_{u,p} \le \mathcal{C}E_M(f)_{u,p} + \left\| \sum_{k=j+1}^{m} \frac{\ell_k(w)v^r u}{v^r(x_k)u(x_k)}(P_M u)(x_k) \right\|_p .$$

By (4.1) the second summand on the right-hand side is dominated by

$$\mathcal{C}a_m^{1/p} \|P_M u\|_{L^\infty[\theta a_m,+\infty)} \sup_{x\in I_m} \sum_{k=j+1}^{m} \frac{|\ell_k(w,x)|v^r(x)u(x)}{v^r(x_k)u(x_k)},$$

and, using (4.12) and (4.13), we have

$$\sup_{x\in I_m} \sum_{k=j+1}^{m} \frac{|\ell_k(w,x)|v^r(x)u(x)}{v^r(x_k)u(x_k)} \le \mathcal{C}m^\tau, \tag{4.15}$$

for some $\tau > 0$. Moreover, by (4.2) and (4.5), we obtain

$$\begin{aligned}\left\| \sum_{k=j+1}^{m} \frac{\ell_k(w)v^r u}{v^r(x_k)u(x_k)}(P_M u)(x_k) \right\|_p &\le \mathcal{C} m^\tau a_m^{1/p} \|P_M u\|_{L^\infty[\theta a_m,+\infty)} \\ &\le \mathcal{C} m^\tau a_m^{1/p} \mathrm{e}^{-cm} \|P_M u\|_\infty \\ &\le \mathcal{C} m^{\tau+\frac{2}{p}} \mathrm{e}^{-cm} \|P_M u\|_p \\ &\le \mathcal{C} \mathrm{e}^{-cm} \|f u\|_p. \end{aligned} \tag{4.16}$$

Let us now prove (2.13). By (4.14) we have

$$\begin{aligned}&\left(\frac{\sqrt{a_m}}{m}\right)^s \|\mathcal{L}^*_{m,r}(w, P_M)^{(s)} \varphi^s u\|_p \\ &\le \left(\frac{\sqrt{a_m}}{m}\right)^s \|P_M^{(s)} \varphi^s u\|_p + \left(\frac{\sqrt{a_m}}{m}\right)^s \left\| \left(\sum_{k=j+1}^{m} \frac{\ell_k(w)v^r}{v^r(x_k)} P_M(x_k) \right)^{(s)} \varphi^s u \right\|_p\end{aligned}$$

For the first term on the right-hand side we use the following estimate (see [21])

$$\left(\frac{\sqrt{a_m}}{m}\right)^s \|P_M^{(s)} \varphi^s u\|_p \le \mathcal{C} \int_0^{\sqrt{a_m}/m} \frac{\Omega_\varphi^s(f,t)_{u,p}}{t} \mathrm{d}t.$$

While, for the second term on the right-hand side, using s times the Bernstein inequality (4.3) and (4.16), we obtain

$$\begin{aligned}\left(\frac{\sqrt{a_m}}{m}\right)^s \left\| \left(\sum_{k=j+1}^{m} \frac{\ell_k(w)v^r}{v^r(x_k)} P_M(x_k) \right)^{(s)} \varphi^s u \right\|_p &\le \mathcal{C} \left\| \sum_{k=j+1}^{m} \frac{\ell_k(w)v^r}{v^r(x_k)} P_M(x_k) u \right\|_p \\ &\le \mathcal{C} \mathrm{e}^{-cm} \|f u\|_p.\end{aligned}$$

Hence we get (2.13). □

We recall the following estimate, concerning the distance Δx_k between two consecutive zeros (see [10,21]):

$$\Delta x_k = x_{k+1} - x_k \sim \frac{a_m}{m} \sqrt{x_k} \frac{1}{\sqrt{a_m - x_k + a_m m^{-2/3}}}, \tag{4.17}$$

where the constants in "$\sim$" are independent of m and k. We remark that from (4.17), with j given by (2.3), it follows that

$$\Delta x_k \sim \frac{\sqrt{a_m}}{m} \sqrt{x_k} \qquad k = 1, 2, \ldots j.$$

Now, letting

$$\sigma_m(f) = \sum_{i=0}^{r-1} \frac{f^{(i)}(0)}{i!} \left(\frac{\sqrt{a_m}}{m}\right)^{2i} \tag{4.18}$$

and

$$A(x) = (a_m - x)p_m(w, x) \sum_{i=0}^{r-1} \frac{x^i}{i!} \left(\frac{f}{(a_m - \cdot)p_m(w)}\right)^{(i)} (0), \tag{4.19}$$

we can state the following proposition.

Proposition 9 *With the notation* (4.18) *and* (4.19), *we have*

$$\|Au\|_\infty \leq \mathcal{C} \left(\frac{\sqrt{a_m}}{m}\right)^{2\gamma} \sigma_m(f), \quad \mathcal{C} \neq \mathcal{C}(m, f), \tag{4.20}$$

if and only if

$$\frac{\alpha}{2} + \frac{1}{4} \leq \gamma + r \leq \frac{\alpha}{2} + \frac{5}{4}. \tag{4.21}$$

Moreover, for $p \geq 1$, *we get*

$$\|Au\|_p \leq \mathcal{C} \left(\frac{\sqrt{a_m}}{m}\right)^{2(\gamma+1/p)} \sigma_m(f), \quad \mathcal{C} \neq \mathcal{C}(m, f), \tag{4.22}$$

if and only if

$$\frac{\alpha}{2} + \frac{1}{4} - \frac{1}{p} \leq \gamma + r \leq \frac{\alpha}{2} + \frac{5}{4} - \frac{1}{p}. \tag{4.23}$$

Proof We first prove (4.20). Recalling (4.19), we may write

$$|A(x)u(x)| \leq \sum_{i=0}^{r-1} b_i \frac{|D_i|}{i!}, \tag{4.24}$$

where

$$b_i = \|(a_m - \cdot)p_m(w, \cdot)uv^i\|_\infty, \quad v^i(x) = x^i$$

and

$$\begin{aligned} |D_i| &= \left| \left(\frac{f}{(a_m - \cdot)p_m(w)}\right)^{(i)} (0) \right| \\ &= \left| \sum_{\eta=0}^{i} \binom{i}{\eta} f^{(\eta)}(0) \left(\frac{1}{(a_m - \cdot)p_m(w, \cdot)}\right)^{(i-\eta)} (0) \right|. \end{aligned}$$

By (4.10) and (4.11) we get

$$\begin{aligned}
\left|\left(\frac{1}{(a_m-\cdot)p_m(w,\cdot)}\right)^{(i-\eta)}(0)\right| &= \left|\sum_{j=0}^{i-\eta}\binom{i-\eta}{j}\frac{-1^j}{(a_m)^{j+1}}\left(\frac{1}{p_m(w,\cdot)}\right)^{(i-\eta-j)}(0)\right| \\
&\le \frac{\mathcal{C}}{|p_m(w,0)|}\sum_{j=0}^{i-\eta}\binom{i-\eta}{j}\frac{1}{(a_m)^{j+1}}\left(\frac{m}{\sqrt{a_m}}\right)^{2(i-\eta-j)} \\
&= \frac{\mathcal{C}}{a_m|p_m(w,0)|}\left(\frac{1}{a_m}+\frac{m^2}{a_m}\right)^{i-\eta} \\
&\le \mathcal{C}\left(\frac{m}{\sqrt{a_m}}\right)^{2i-2\eta}\frac{1}{a_m|p_m(w,0)|}.
\end{aligned}$$

Then we have

$$|D_i| \le \frac{\mathcal{C}}{a_m|p_m(w,0)|}\sum_{\eta=0}^{i}\binom{i}{\eta}|f^{(\eta)}(0)|\left(\frac{m}{\sqrt{a_m}}\right)^{2i-2\eta}. \tag{4.25}$$

On the other hand, by (4.1), (3.2) and (4.8), we get

$$\begin{aligned}
b_i &\le \mathcal{C}\max_{x\in I_m}\left|\sqrt{w(x)}p_m(w,x)(a_m-x)x^{i+\gamma-\frac{\alpha}{2}}\right| \\
&\le \mathcal{C}\max_{x\in I_m}\left|(a_m-x)^{\frac{3}{4}}x^{i+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\right| \\
&\le \mathcal{C}(a_m)^{\frac{3}{4}}\left(\frac{\sqrt{a_m}}{m}\right)^{2i+2\gamma-\alpha-\frac{1}{2}} \sim a_m\left(\frac{\sqrt{a_m}}{m}\right)^{2i+2\gamma}|p_m(w,0)|.
\end{aligned} \tag{4.26}$$

Combining (4.25) and (4.26) in (4.24), we obtain

$$\begin{aligned}
|A(x)u(x)| &\le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{2\gamma}\sum_{i=0}^{r-1}\frac{1}{i!}\sum_{\eta=0}^{i}\binom{i}{\eta}|f^{(\eta)}(0)|\left(\frac{\sqrt{a_m}}{m}\right)^{2\eta} \\
&= \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{2\gamma}\sum_{i=0}^{r-1}\sum_{\eta=0}^{r-1}(i-\eta+1)_+^0|f^{(\eta)}(0)|\left(\frac{\sqrt{a_m}}{m}\right)^{2\eta}\binom{i}{\eta}\frac{1}{i!} \\
&= \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{2\gamma}\sum_{\eta=0}^{r-1}\left(\frac{\sqrt{a_m}}{m}\right)^{2\eta}|f^{(\eta)}(0)|\sum_{i=0}^{r-1}(i-\eta+1)_+^0\binom{i}{\eta}\frac{1}{i!} \\
&\sim \left(\frac{\sqrt{a_m}}{m}\right)^{2\gamma}\sum_{\nu=0}^{r-1}\frac{|f^{(\nu)}(0)|}{\nu!}\left(\frac{\sqrt{a_m}}{m}\right)^{2\nu},
\end{aligned}$$

since $\sum_{i=\eta+1}^{r-1}\binom{i}{\eta}\frac{1}{i!}\sim 1$.

Let us now show (4.22). Recalling (4.24), we have

$$\begin{aligned}\|(a_m-\cdot)p_m(w)uv^i\|_p &\le \mathcal{C}\|(a_m-\cdot)p_m(w)uv^i\|_{L^p(I_m)}\\ &= \mathcal{C}\|\sqrt{w}p_m(w)(a_m-\cdot)v^{i+\gamma-\frac{\alpha}{2}}\|_{L^p(I_m)}\\ &\le \mathcal{C}\|(a_m-\cdot)^{\frac{3}{4}}v^{i+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^p(I_m)}\\ &\le \mathcal{C}(a_m)^{\frac{3}{4}}\left(\frac{\sqrt{a_m}}{m}\right)^{2i+2\gamma-\alpha-\frac{1}{2}+\frac{2}{p}}\\ &= \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{2i+2\gamma+\frac{2}{p}} a_m|p_m(w,0)|\end{aligned}$$

where we have applied (4.1), (4.6), (4.8), taking also into account the assumption (4.23). Then recalling (4.25), we deduce (4.22).

The necessity of the conditions (4.21) and (4.23) can be proved using standard arguments (see [22]). We omit the details. □

Proof of Theorem 2 Let us first prove that conditions (3.2) imply inequality (3.1). Recalling (2.2) and Proposition 9, it remains to prove that

$$\left|\sum_{k=1}^{m+1} u(x)\frac{x^r}{x_k^r}\frac{\ell_k(w,x)}{u(x_k)}f(x_k)u(x_k)\right| \le \mathcal{C}(\log m)\max_{1\le i\le m+1}|fu|(x_i).$$

Let $x\in I_m=\left[c\frac{a_m}{m^2},a_m\right]$ and x_d be a zero closest to x. By using (4.7), (4.11) and (4.13), we get

$$\begin{aligned}&\sum_{k=1}^{m+1} u(x)\frac{x^r}{x_k^r}\frac{|\ell_k(w,x)|}{u(x_k)}f(x_k)u(x_k)\\ &\quad\le \mathcal{C}\left[|p_m(w,x)|\sqrt{w(x)\sqrt{x(a_m-x)}}\sum_{k\ne d,m+1}\left(\frac{x}{x_k}\right)^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\left(\frac{a_m-x}{a_m-x_k}\right)^{\frac{3}{4}}\right.\\ &\qquad\left.\frac{\Delta x_k}{|x-x_k|}+1+\left|\frac{p_m(w,x)}{p_m(w,a_m)}\right|\frac{x^r}{a_m^r}\frac{u(x)}{u(a_m)}\right]\max_{1\le i\le m+1}|fu|(x_i).\end{aligned}$$

By (4.6) the last summand on the right-hand side is bounded by $\left(\frac{x}{a_m}\right)^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}$ and so the whole right-hand side is dominated by

$$\begin{aligned}\mathcal{C}&\left(\sum_{k\ne d,m+1}\left(\frac{x}{x_k}\right)^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\left(\frac{a_m-x}{a_m-x_k}\right)^{\frac{3}{4}}\frac{\Delta x_k}{|x-x_k|}\right.\\ &\left.+1+\left(\frac{x}{a_m}\right)^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\right)\max_{1\le i\le m+1}|fu|(x_i).\end{aligned}$$

Therefore, if the parameters r, γ and α fulfill the assumptions (3.2), the sum on the right-hand side is bounded by $\mathcal{C}(\log m)$ and the last term is bounded by 1.

Let us now prove that inequality (3.1) implies conditions (3.2). To this aim, for any $f \in C_u$, consider the piecewise linear function f_1 defined as follows

$$\begin{cases} f_1^{(i)}(0) = 0, & i = 0, 1, \dots, r-1, \\ f_1(x_k) = -|f(x_k)| \operatorname{sgn}\{p_m'(w, x_k)\} \neq 0, & \text{for } 1 \le x_k \le 2, \\ f_1(x_k) = 0, & \text{otherwise.} \end{cases} \tag{4.27}$$

By (3.1) we have

$$\mathcal{C}(\log m) \max_{1 \le x_i \le 2} |f_1 u|(x_i) \ge \|\mathcal{L}_{m,r}(w, f_1)u\|_\infty \ge \|\mathcal{L}_{m,r}(w, f_1)u\|_{L^\infty[0,1]}, \tag{4.28}$$

where, for $x \in [0, 1]$ and $x_k \in [1, 2]$,

$$\begin{aligned} \mathcal{L}_{m,r}(w, f_1, x)u(x) &= -\sum_{1 \le x_k \le 2} \frac{x^r}{x_k^r} \frac{p_m(w, x)|f_1 u|(x_k)}{|p_m'(w, x_k)|(x - x_k)} \left(\frac{a_m - x}{a_m - x_k}\right) \frac{u(x)}{u(x_k)} \\ &= \sum_{1 \le x_k \le 2} \frac{x^r}{x_k^r} \frac{p_m(w, x)|f u|(x_k)}{|p_m'(w, x_k)||x - x_k|} \left(\frac{a_m - x}{a_m - x_k}\right) \frac{u(x)}{u(x_k)}, \end{aligned}$$

since $x - x_k < 0$. Now, if $x \in [0, 1]$ and $x_k \in [1, 2]$, then $|x - x_k| \le 2$ and $\frac{a_m - x}{a_m - x_k} \sim 1$. Hence, setting

$$\widetilde{A}(x) = p_m(w, x)\sqrt{w(x)\sqrt{x\,(a_m - x)}}$$

and using (4.11), we get

$$\|\mathcal{L}_{m,r}(w, f_1)u\|_{L^\infty[0,1]} \ge \mathcal{C}\|\widetilde{A}v^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^\infty[0,1]} \sum_{1 \le x_k \le 2} \frac{\Delta x_k |f u|(x_k)}{x_k^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}}. \tag{4.29}$$

Combining (4.28) and (4.29), we obtain

$$\|\widetilde{A}v^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^\infty[0,1]} \sum_{1 \le x_k \le 2} \frac{\Delta x_k |f u|(x_k)}{x_k^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}} \le \mathcal{C}(\log m) \max_{1 \le x_i \le 2} |f u|(x_i)$$

with $\mathcal{C} \neq \mathcal{C}(m, f)$, whence letting $b_i = |f u|(x_i)$ and $\|\underline{b}\|_\infty^* = \max_{1 \le x_i \le 2} b_i$, we have

$$\|\widetilde{A}v^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^\infty[0,1]} \sum_{1 \le x_k \le 2} \frac{\Delta x_k}{x_k^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}} \frac{b_k}{\|\underline{b}\|_\infty^*} \le \mathcal{C}(\log m), \quad \mathcal{C} \neq \mathcal{C}(m, f)$$

and then

$$\|\widetilde{A}v^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^\infty[0,1]} \sup_{\|\underline{d}\|_\infty^*=1} \sum_{1\le x_k\le 2} \frac{\Delta x_k}{x_k^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}} d_k \le \mathcal{C}(\log m), \quad \mathcal{C}\ne\mathcal{C}(m,f).$$

Hence we get

$$\|\widetilde{A}v^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^\infty[0,1]} \sum_{1\le x_k\le 2} \frac{\Delta x_k}{x_k^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}} \le \mathcal{C}(\log m).$$

Obviously the sum on the left-hand side is ~ 1. Moreover, by (4.6), we have

$$\|\widetilde{A}v^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^\infty[0,1]} \ge \left|\widetilde{A}\left(\frac{a_m}{m^2}\right) v^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\left(\frac{a_m}{m^2}\right)\right| \sim \left(\frac{a_m}{m^2}\right)^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}.$$

So, from

$$\left(\frac{a_m}{m^2}\right)^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}} \le \mathcal{C}(\log m)$$

it follows that $r+\gamma \ge \frac{\alpha}{2}+\frac{1}{4}$.

In order to prove the necessity of the other inequality in (3.2), we introduce the piecewise linear function f_2 defined as

$$\begin{cases} f_2^{(i)}(0)=0, & i=0,1,\dots,r-1, \\ f_2(x_k)=|f(x_k)|\,\mathrm{sgn}\{p_m'(w,x_k)\}, & \text{for} \quad 0<x_k\le 1, \\ f_2(x_k)=0, & \text{otherwise.} \end{cases} \tag{4.30}$$

Using similar arguments we obtain

$$\|\widetilde{A}v^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^\infty[1,2]} \sum_{0<x_k\le 1} \frac{\Delta x_k}{x_k^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}} \le \mathcal{C}(\log m).$$

Now, the sum on the left-hand side is $\sim \int_{x_1}^1 x^{-r-\gamma+\frac{\alpha}{2}+\frac{1}{4}}\,\mathrm{d}x$, while the norm is bounded. So, from

$$\int_{x_1}^1 x^{-r-\gamma+\frac{\alpha}{2}+\frac{1}{4}}\,\mathrm{d}x \le \mathcal{C} \sum_{0<x_k\le 1} \frac{\Delta x_k}{x_k^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}} \le \mathcal{C}(\log m)$$

we deduce $r+\gamma \le \frac{\alpha}{2}+\frac{5}{4}$ and the first part of the theorem follows.

Let us now prove (3.3). Since $\left|f^{(i)}\right|$ is quasi increasing on $[0,\alpha_m]$, with $i \in \{0,1,\dots,r-1\}$ and $\alpha_m = a_m/m^2$, we have

$$\alpha_m^{i+\gamma}\left|f^{(i)}(0)\right| \le \mathcal{C}\alpha_m^{i+\gamma}\left|f^{(i)}\left(\frac{\alpha_m}{2}\right)\right| \le \mathcal{C}\alpha_m^{i+\gamma}\left\|f^{(i)}\right\|_{L^\infty[\frac{\alpha_m}{2},\alpha_m]}.$$

Recalling a formula in [3, p.15], it follows that

$$\alpha_m^{i+\gamma}\left|f^{(i)}(0)\right| \le \mathcal{C}\alpha_m^{\gamma}\|f\|_{L^\infty[\frac{\alpha_m}{2},\alpha_m]} + \mathcal{C}\alpha_m^{r-1+\gamma}\left\|f^{(r-1)}\right\|_{L^\infty[\frac{\alpha_m}{2},\alpha_m]}$$
$$\le \mathcal{C}\|fu\|_{L^\infty[\frac{\alpha_m}{2},\alpha_m]} + \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{r-1}\left\|f^{(r-1)}\varphi^{r-1}u\right\|_{L^\infty[\frac{\alpha_m}{2},\alpha_m]}.$$

Hence (3.1) becomes

$$\left\|\mathcal{L}_{m,r}(w,f)\,u\right\|_\infty \le \mathcal{C}(\log m)\|fu\|_\infty + \mathcal{O}\left(\left(\frac{\sqrt{a_m}}{m}\right)^{r-1}\right).$$

Letting $P_{m+r} \in \mathbb{P}_{m+r}$ be the polynomial of best approximation for $f \in C_u$, it follows that

$$\left\|\left[f - \mathcal{L}_{m,r}(w,f)\right]u\right\|_\infty = \left\|\left[f - P_{m+r} - \mathcal{L}_{m,r}(w, f - P_{m+r})\right]u\right\|_\infty$$
$$\le \mathcal{C}(\log m)E_{m+r}(f)_{u,\infty} + \mathcal{O}\left(\left(\frac{\sqrt{a_m}}{m}\right)^{r-1}\right),$$

i.e. (3.3). □

In order to prove Theorem 4 we recall some known results concerning the Hilbert transform $H(f,t)$. The Hilbert transform related to the interval $(0,a)$ is defined as follows

$$H(f,t) = \int_0^a \frac{f(x)}{x-t}\,\mathrm{d}x, \quad t \in (0,a),\ a > 0,$$

where the integral is understood in the Cauchy principal value sense. Letting $v^\rho(x) = x^\rho$ and $1 < p < \infty$, the bound

$$\|(Hf)v^\rho\|_{L^p(0,1)} \le \mathcal{C}\|fv^\rho\|_{L^p(0,1)}, \quad \mathcal{C} \ne \mathcal{C}(f), \tag{4.31}$$

holds for any f such that $fv^\rho \in L^p(0,a)$ if and only if $-\frac{1}{p} < \rho < 1 - \frac{1}{p}$ (see [23]).

The following lemma, that can be found in [21] (see also [8]), will be also useful in the proof of Theorem 4.

Lemma 10 *Let* $0 < \theta < 1$, j *be given by* (2.3), $u \in L^p$, $1 \le p < \infty$ *and* $\Delta x_k = x_{k+1} - x_k$. *Then, for an arbitrary polynomial* $P \in \mathbb{P}_{lm}$ *(where* l *is a fixed integer), there exists* $\bar\theta \in (\theta, 1)$ *such that*

$$\sum_{k=1}^{j}\Delta x_k|Pu|^p(x_k) \le \mathcal{C}\int_{x_1}^{\bar\theta a_m}|Pu|^p(x)\,\mathrm{d}x, \tag{4.32}$$

with $\mathcal{C} \ne \mathcal{C}(m,f)$.

We want to emphasize that the first Marcinkiewicz inequality (4.32) does not hold if the sum on the left-hand side is extended to all the zeros of $p_m(w)$, i.e. $k = 1, \dots, m$ (see [13,14]).

Proof of Theorem 4 Taking into account Proposition 9 it remains to estimate the L^p_u-norm of the first term in (2.4). Letting $v^r(x) = x^r$ and

$$L^*_m(w, f, x) = \sum_{k=1}^{j} \ell_k(w, x) f(x_k), \quad \ell_k(w, x) = \frac{p_m(w, x)}{(x - x_k) p'_m(w, x_k)} \frac{(a_m - x)}{(a_m - x_k)},$$

this term can be rewritten as $v^r L^*_m\left(w, \frac{f}{v^r}\right)$. So, using (4.1), we have

$$\begin{aligned}\left\| v^r L^*_m\left(w, \frac{f}{v^r}\right) u \right\|_p &\le \mathcal{C} \left\| v^r L^*_m\left(w, \frac{f}{v^r}\right) u \right\|_{L^p(I_m)} \\ &= \mathcal{C} \sup_{\|g\|_q = 1} \left| \int_{I_m} x^r L^*_m\left(w, \frac{f}{x^r}, x\right) u(x) g(x)\, dx \right| \\ &=: \sup_{\|g\|_q=1} |\Gamma(g)|, \end{aligned} \tag{4.33}$$

where $I_m = \left[c\, a_m m^{-2}, a_m\right]$ and

$$\begin{aligned}\Gamma(g) &= \int_{I_m} x^r \sum_{k=1}^{j} \frac{\ell_k(w, x)}{x_k^r} f(x_k) g(x) u(x)\, dx \\ &= \sum_{k=1}^{j} \frac{f(x_k) u(x_k)}{p'_m(w, x_k)(a_m - x_k) u(x_k) x_k^r} \int_{I_m} \frac{(a_m - x) p_m(w, x) x^r g(x) u(x)}{x - x_k}\, dx.\end{aligned}$$

By using (4.11), we have

$$\begin{aligned}|\Gamma(g)| &\le \mathcal{C} \sum_{k=1}^{j} \frac{\Delta x_k |fu|(x_k)}{(a_m - x_k)^{\frac{3}{4}} x_k^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}} \left| \int_{I_m} \frac{(a_m - x) p_m(w, x) x^r g(x) u(x)}{x - x_k}\, dx \right| \\ &\le \frac{\mathcal{C}}{a_m^{3/4}} \sum_{k=1}^{j} \frac{\Delta x_k |fu|(x_k)}{x_k^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}} \left| \int_{I_m} \frac{(a_m - x) p_m(w, x) x^r g(x) u(x)}{x - x_k}\, dx \right|.\end{aligned}$$

Denoting by $G(x_k)$ the integral on the right-hand side and using the Hölder inequality, we get

$$|\Gamma(g)| \le \frac{\mathcal{C}}{a_m^{3/4}} \left(\sum_{k=1}^{j} \Delta x_k |fu|^p(x_k) \right)^{1/p} \left(\sum_{k=1}^{j} \Delta x_k \left| \frac{G(x_k)}{x_k^{r+\gamma-\alpha/2-1/4}} \right|^q \right)^{1/q}. \tag{4.34}$$

For any polynomial $0 < Q \in \mathbb{P}_{lm}$, with l a fixed integer, we can write

$$G(x_k) = \int_{I_m} \frac{(a_m - x)p_m(w,x)Q(x) - (a_m - x_k)p_m(w,x_k)Q(x_k)}{x - x_k} \frac{x^r g(x)u(x)}{Q(x)}\,\mathrm{d}x$$

and $G(t)$ is a polynomial of degree $m + lm$. Thus, using Lemma 10, we have

$$B = \left(\sum_{k=1}^{j} \Delta x_k \left|\frac{G(x_k)}{x_k^{r+\gamma-\alpha/2-1/4}}\right|^q\right)^{1/q} \le \left(\int_{I_{\bar{\theta}m}} \left|\frac{G(t)}{t^{r+\gamma-\alpha/2-1/4}}\right|^q \mathrm{d}t\right)^{1/q},$$

where $0 < \theta < \bar{\theta} < 1$ and $I_{\bar{\theta}m} = [c\,a_m m^{-2}, \bar{\theta}a_m]$. Then we obtain

$$\begin{aligned} B \le& \left(\int_{I_{\bar{\theta}m}} \left|t^{-r-\gamma+\alpha/2+1/4} \int_{I_m} \frac{(a_m - x)p_m(w,x)x^r(gu)(x)}{x - t}\,\mathrm{d}x\right|^q \mathrm{d}t\right)^{1/q} \\ &+ \left(\int_{I_{\bar{\theta}m}} \left|t^{-r-\gamma+\alpha/2+1/4}(a_m - t)p_m(w,t)Q(t) \int_{I_m} \frac{x^r(gu)(x)}{(x-t)Q(x)}\,\mathrm{d}x\right|^q \mathrm{d}t\right)^{1/q} \\ =:& B_1 + B_2. \end{aligned} \tag{4.35}$$

Let us estimate the term B_1. By (4.7) we have

$$\left|\widetilde{A}(x)\right| := |p_m(w,x)|\sqrt{w(x)\sqrt{x(a_m - x)}} \le \mathcal{C}, \quad x \in I_m.$$

Moreover, under the assumptions (3.5), $t^{-r-\gamma+\frac{\alpha}{2}+\frac{1}{4}} \in L^q$ and $t^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}} \in L^p$, we can apply (4.31), obtaining

$$\begin{aligned} B_1 &\le \left(\int_{I_m} \left|t^{-r-\gamma+\frac{\alpha}{2}+\frac{1}{4}} \int_{I_m} \frac{(a_m - x)^{\frac{3}{4}}\widetilde{A}(x)x^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}g(x)}{x - t}\,\mathrm{d}x\right|^q \mathrm{d}t\right)^{1/q} \\ &\le \mathcal{C}\left(\int_{I_m} \left|(a_m - x)^{\frac{3}{4}}\widetilde{A}(t)g(t)\right|^q \mathrm{d}t\right)^{1/q} \\ &\le \mathcal{C}a_m^{\frac{3}{4}}\|g\|_{L^q(I_m)}. \end{aligned} \tag{4.36}$$

In order to estimate B_2 we can construct a polynomial $Q \in \mathbb{P}_{lm}$ such that $Q(x) \sim x^r u(x)$ for $x \in I_m$. So, using (4.7), we have

$$\left|t^{-r-\gamma+\alpha/2+1/4}(a_m - t)p_m(w,t)Q(t)\right| \le \mathcal{C}a_m^{3/4}.$$

Then, by (4.31), we get

$$B_2 \le \mathcal{C} a_m^{3/4} \left(\int_{I_m} \left| \int_{I_m} \frac{x^r (gu)(x)}{(x-t)Q(x)} \,\mathrm{d}x \right|^q \mathrm{d}t \right)^{1/q} \le \mathcal{C} a_m^{3/4} \|g\|_{L^q(I_m)}. \tag{4.37}$$

So, taking into account (4.36), (4.37) and (4.35), it follows that

$$\left(\int_{I_{\bar\theta m}} \left| \frac{G(t)}{t^{r+\gamma-\alpha/2-1/4}} \right|^q \mathrm{d}t \right)^{1/q} \le \mathcal{C} a_m^{3/4} \|g\|_{L^q(I_m)},$$

and combining this last inequality with (4.34) and (4.33), the first term in (2.4) is bounded by

$$\mathcal{C} \left(\sum_{k=1}^{j} \Delta x_k |fu|^p (x_k) \right)^{1/p}.$$

Therefore, recalling (4.22), we have that (3.5) implies (3.4).

In order to prove that (3.4) implies (3.5) we use arguments similar to those in the proof of Theorem 4. Therefore, we are going to show only the main steps. Considering the function f_1 as in (4.27) we can write

$$\|\mathcal{L}^*_{m,r}(w, f_1)u\|_{L^p[c\, a_m m^{-2},1]} \le \|\mathcal{L}^*_{m,r}(w, f_1)u\|_p \le \mathcal{C} \left(\sum_{1 \le x_k \le 2} \Delta x_k |f_1 u|^p (x_k) \right)^{1/p},$$

where $[c\, a_m m^{-2}, 1] = [0, 1] \cap I_m$. Recalling the expression of $\mathcal{L}^*_{m,r}(w, f_1, x)$ for $x \in [c\, a_m m^{-2}, 1]$, with

$$|\widetilde{A}(x)| = |p_m(w, x)| \sqrt{w(x)\sqrt{x(a_m - x)}},$$

we obtain

$$\|\widetilde{A} v^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^p[a_m m^{-2},1]} \sum_{1 \le x_k \le 2} \frac{\Delta x_k |f_1 u|(x_k)}{x_k^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}} \le \mathcal{C} \left(\sum_{1 \le x_k \le 2} \Delta x_k |f_1 u|^p (x_k) \right)^{1/p}. \tag{4.38}$$

Now, to simplify the notation, we set

$$a_k = \frac{(\Delta x_k)^{1/q}}{x_k^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}}, \quad b_k = (\Delta x_k)^{1/p} |f_1 u|(x_k), \quad \|\underline{b}\|^*_p = \left(\sum_{1 \le x_k \le 2} b_k^p \right)^{1/p}.$$

Hence (4.38) becomes

$$\|\widetilde{A}v^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^p[c\,a_m m^{-2},1]}\sum_{1\le x_k\le 2} a_k\frac{b_k}{\|\underline{b}\|_p^*}\le \mathcal{C},\quad \mathcal{C}\ne\mathcal{C}(m,f).$$

Taking the supremum on $\underline{b}$, we get

$$\|\widetilde{A}v^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^p[c\,a_m m^{-2},1]}\left(\sum_{1\le x_k\le 2}|a_k|^q\right)^{1/q}\le \mathcal{C},$$

which implies $r+\gamma-\frac{\alpha}{2}-\frac{1}{4}>-\frac{1}{p}$, taking into account that, by (4.7), $\left|\widetilde{A}(x)\right|\le \mathcal{C}$, $x\in[c\,a_m m^{-2},1]$.

Analogously, considering the function f_2 defined in (4.30), we get

$$\|\widetilde{A}v^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^p[1,2]}\left(\sum_{0<x_k\le 1}|a_k|^q\right)^{1/q}\le \mathcal{C},$$

i.e.

$$\|\widetilde{A}v^{r+\gamma-\frac{\alpha}{2}-\frac{1}{4}}\|_{L^p[1,2]}\left(\sum_{0<x_k\le 1}\frac{\Delta x_k}{x_k^{(r+\gamma-\alpha/2-1/4)q}}\right)^{1/q}\le \mathcal{C},$$

whence $-r-\gamma+\frac{\alpha}{2}+\frac{1}{4}>-\frac{1}{q}$ that is $r+\gamma-\frac{\alpha}{2}-\frac{1}{4}<1-\frac{1}{p}$, which completes the proof. □

Proof of Theorem 5 Since $\left|f^{(i)}\right|$ is quasi increasing on $[0,\alpha_m]$, with $i\in\{0,1,\dots,r-1\}$ and $\alpha_m=a_m/m^2$, using the Hölder inequality, we have

$$\begin{aligned}\alpha_m^{i+\gamma+1/p}\left|f^{(i)}(0)\right| &\le \mathcal{C}\alpha_m^{i+\gamma+1/p}\left|f^{(i)}\left(\frac{\alpha_m}{2}\right)\right| \le \mathcal{C}\alpha_m^{i+\gamma+1/p-1}\int_{\frac{\alpha_m}{2}}^{\alpha_m}\left|f^{(i)}(x)\right|\mathrm{d}x\\ &\le \mathcal{C}\alpha_m^{i+\gamma}\left\|f^{(i)}\right\|_{L^p[\frac{\alpha_m}{2},\alpha_m]}.\end{aligned}$$

Recalling a formula in [3, p.15], it follows that

$$\begin{aligned}\alpha_m^{i+\gamma+1/p}\left|f^{(i)}(0)\right| &\le \mathcal{C}\alpha_m^{\gamma}\|f\|_{L^p[\frac{\alpha_m}{2},\alpha_m]}+\mathcal{C}\alpha_m^{r-1+\gamma}\left\|f^{(r-1)}\right\|_{L^p[\frac{\alpha_m}{2},\alpha_m]}\\ &\le \mathcal{C}\|fu\|_{L^p[\frac{\alpha_m}{2},\alpha_m]}+\mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{r-1}\left\|f^{(r-1)}\varphi^{r-1}u\right\|_{L^p[\frac{\alpha_m}{2},\alpha_m]}.\end{aligned}$$

So inequality (3.7) becomes

$$\left\|\mathcal{L}^*_{m,r}(w,f)\,u\right\|_p \le \mathcal{C}\left[\|fu\|_p + \left(\frac{\sqrt{a_m}}{m}\right)^{1/p}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^s_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t + \left(\frac{\sqrt{a_m}}{m}\right)^{r-1}\left\|f^{(r-1)}\varphi^{r-1}u\right\|_{L^p[0,\alpha_m]}\right]. \tag{4.39}$$

Let $P_M \in \mathbb{P}_M$, with $M = \left\lfloor \frac{\theta m}{\theta+1}\right\rfloor$, be a polynomial of quasi best approximation for $f \in L^p_u$, and set $Q = \mathcal{L}^*_{m,r}(w, P_M)$. Hence, by (4.39), we get

$$\|[f-\mathcal{L}^*_{m,r}(w,f)]u\|_p \le \mathcal{C}\left[\|(f-Q)\,u\|_p + \left(\frac{\sqrt{a_m}}{m}\right)^{1/p}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^s_\varphi(f-Q,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t + \left(\frac{\sqrt{a_m}}{m}\right)^{r-1}\left\|(f-Q)^{(r-1)}\,\varphi^{r-1}u\right\|_{L^p[0,\alpha_m]}\right]$$

Now, to estimate the first summand at the right-hand side we can use (2.12) and (2.8). For the second summand we note that

$$\Omega^s_\varphi(f-Q,t)_{u,p} \le \Omega^s_\varphi(f,t)_{u,p} + t^s\left\|Q^{(s)}\varphi^s u\right\|_p.$$

So, by (2.13), taking into account that

$$\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^s_\varphi(f,t)_{u,p}}{t}\,\mathrm{d}t \le \left(\frac{\sqrt{a_m}}{m}\right)^{1/p}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^s_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t,$$

inequality (3.8) follows. □

Proof of Corollary 6 Proceeding as in the proof of Theorem 5, since $\left|f^{(i)}\right|$ is quasi increasing on $[0,\alpha_m]$, with $i \in \{0,1,\ldots,r-1\}$ and $\alpha_m = a_m/m^2$, and $f \in W^p_r(u)$, we get

$$\alpha_m^{i+\gamma+1/p}\left|f^{(i)}(0)\right| \le \mathcal{C}\alpha_m^{\gamma}\,\|f\|_{L^p[\frac{\alpha_m}{2},\alpha_m]} + \mathcal{C}\alpha_m^{r+\gamma}\left\|f^{(r)}\right\|_{L^p[\frac{\alpha_m}{2},\alpha_m]} \le \mathcal{C}\,\|fu\|_{L^p[\frac{\alpha_m}{2},\alpha_m]} + \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r\left\|f^{(r)}\varphi^r u\right\|_{L^p[\frac{\alpha_m}{2},\alpha_m]}. \tag{4.40}$$

Hence, by (3.7), (2.6) and (4.40), we obtain (3.9).

Now, in order to prove (3.10), let $Q = \mathcal{L}^*_{m,r}(w, P_M)$, where $P_M \in \mathbb{P}_M$, with $M = \left\lfloor \frac{\theta m}{\theta+1}\right\rfloor$, is a polynomial of quasi best approximation for $f \in L^p_u$. By (3.9) we have

$$\left\| \left[f - \mathcal{L}^*_{m,r}(w,f) \right] u \right\|_p \le \mathcal{C} \left\| (f-Q)u \right\|_p + \left\| \mathcal{L}^*_{m,r}(w, f-Q) u \right\|_p$$
$$\le \mathcal{C} \left\| (f-Q)u \right\|_p + \left(\frac{\sqrt{a_m}}{m} \right)^r \left\| f^{(r)} \varphi^r u \right\|_p + \left(\frac{\sqrt{a_m}}{m} \right)^r \left\| Q^{(r)} \varphi^r u \right\|_p .$$

Using Lemma 1, (2.10) and (2.6), we obtain (3.10). □

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