Calcolo (2014) 51:363–366
DOI 10.1007/s10092-013-0089-1

A note on the growth factor in Gaussian elimination for accretive-dissipative matrices

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Received: 28 December 2012 / Accepted: 7 May 2013 / Published online: 15 May 2013

Abstract This short note proves that if A is accretive-dissipative, then the growth factor for such A in Gaussian elimination is less than 4. If A is a Higham matrix, i.e., the accretive-dissipative matrix A is complex symmetric, then the growth factor is less than $2\sqrt{2}$. The result obtained improves those of George et al. in [Numer. Linear Algebra Appl. **9**, 107–114 (2002)] and is one step closer to the final solution of Higham's conjecture.

Keywords Accretive-dissipative matrix · Higham matrix · Growth factor · Gaussian elimination.

Mathematics Subject Classification (2010) 65F05 · 15A45

1 Introduction

Let $\mathbb{M}_n(\mathbf{C})$ be the set of $n \times n$ complex matrices. For any $A = (a_{ij}) \in \mathbb{M}_n(\mathbf{C})$, A^* stands for the conjugate transpose of A. Similarly, x^* means the conjugate transpose of x for any $x \in \mathbf{C}^n$. $A \in \mathbb{M}_n(\mathbf{C})$ is accretive-dissipative if it can be written as $A = B + iC$, where $B = \frac{A+A^*}{2}$ and $C = \frac{A-A^*}{2i}$ are both (Hermitian) positive definite. If B, C are real symmetric positive definite, then A is called a Higham matrix.[1]

Consider the linear system

$$Ax = b \tag{1}$$

[1] In [2], accretive-dissipative matrix is called generalized Higham matrix.

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and let $A^{(k)} = (a_{ij}^{(k)})$ be the matrix resulted from applying the first k $(1 \leq k \leq n-1)$ steps of Gaussian elimination to A. The quantity

$$\rho_n(A) \equiv \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}|}$$

is called the growth factor (in Gaussian elimination) of A. For more information on the numerical significance of investigating growth factor and its connection with the stability of Gaussian elimination, we refer to [3] and references threrein. It is proved in [3] that if A in (1) is a Higham matrix, then no pivoting is needed in Gaussian elimination. Higham [3] also conjectured that $\rho_n(A) \leq 2$ for such a matrix.

George et al. [2] made some progress concerning this conjecture. They obtained the following result:

Theorem 1 *Let $A \in \mathbb{M}_n(\mathbf{C})$ be accretive-dissipative. Then $\rho_n(A) < 3\sqrt{2}$. If A is a Higham matrix, then $\rho_n(A) < 3$.*

They proved Theorem 1 via a stronger result, viz,

Theorem 2 *Let $A \in \mathbb{M}_n(\mathbf{C})$ be accretive-dissipative. Then*

$$\frac{|a_{jj}^{(k)}|}{|a_{jj}|} < 3, \quad j = 1, \ldots, n; \quad k = 1, \ldots, n-1. \tag{2}$$

2 The Main Theorem

In this article, we show a tighter bound than (2). As a result, Theorem 1 is improved. Our result can be read as follows:

Theorem 3 *Let $A \in \mathbb{M}_n(\mathbf{C})$ be accretive-dissipative. Then*

$$\frac{|a_{jj}^{(k)}|}{|a_{jj}|} < 2\sqrt{2}, \quad j = 1, \ldots, n; \quad k = 1, \ldots, n-1. \tag{3}$$

Consequently, $\rho_n(A) < 4$. If A is a Higham matrix, then $\rho_n(A) < 2\sqrt{2}$.

Proof Readers are assumed to have read [2]. The first few steps are the same as the proof in [2], so we skip them. We start from $a_{jj} = b_{jj} + ic_{jj}$ and the fact that $b_{jj}, c_{jj} > 0$.

Setting

$$a_{jj}^{(k)} = \beta + i\gamma, \quad \beta, \gamma \in \mathbf{R},$$

then we have

$$\beta = b_{jj} - b^* X_k b + c^* X_k c - 2\text{Re}(b^* Y_k c)$$

and

$$\gamma = c_{jj} + b^* Y_k b - c^* Y_k c - 2\mathrm{Re}(b^* X_k c),$$

where

$$X_k = (B_k + C_k B_k^{-1} C_k)^{-1} \tag{4}$$
$$Y_k = (C_k + B_k C_k^{-1} B_k)^{-1} \tag{5}$$

with

$$\begin{bmatrix} B_k & b \\ b^* & b_{jj} \end{bmatrix} \text{ and } \begin{bmatrix} C_k & c \\ c^* & c_{jj} \end{bmatrix} \tag{6}$$

positive definite. It is known that $\beta, \gamma > 0$.

By the Cauchy-Schwarz inequality and the arithmetic mean-geometric mean inequality, we have

$$\pm 2\mathrm{Re}(b^* Y_k c) \leq 2\sqrt{(b^* Y_k b)(c^* Y_k c)} \leq b^* Y_k b + c^* Y_k c; \tag{7}$$
$$\pm 2\mathrm{Re}(b^* X_k c) \leq 2\sqrt{(b^* X_k b)(c^* X_k c)} \leq b^* X_k b + c^* X_k c. \tag{8}$$

From (4) and (5) we have [2, Lemma 2.3]

$$X_k \leq \frac{1}{2} C_k^{-1} \quad \text{and} \quad Y_k \leq \frac{1}{2} B_k^{-1}, \tag{9}$$

where the inequality is in the sense of Loewner partial order. Also from (6), we know

$$b_{jj} > b^* B_k^{-1} b \quad \text{and} \quad c_{jj} > c^* C_k^{-1} c \tag{10}$$

Compute

$$\begin{aligned}
|a_{jj}^{(k)}| &= |\beta + i\gamma| \\
&\leq \beta + \gamma \\
&= b_{jj} - b^* X_k b + c^* X_k c - 2\mathrm{Re}(b^* Y_k c) \\
&\quad + c_{jj} + b^* Y_k b - c^* Y_k c - 2\mathrm{Re}(b^* X_k c) \\
&\leq b_{jj} - b^* X_k b + c^* X_k c + (b^* Y_k b + c^* Y_k c) \quad \text{(by (7))} \\
&\quad + c_{jj} + b^* Y_k b - c^* Y_k c + (b^* X_k b + c^* X_k c) \quad \text{(by (8))} \\
&= b_{jj} + 2b^* Y_k b + c_{jj} + 2c^* X_k c \\
&\leq b_{jj} + b^* B_k^{-1} b + c_{jj} + c^* C_k^{-1} c \quad \text{(by (9))} \\
&< 2(b_{jj} + c_{jj}) \quad \text{(by (10))} \\
&\leq 2\sqrt{2} |b_{jj} + i c_{jj}| \\
&= 2\sqrt{2} |a_{jj}|
\end{aligned}$$

This completes the proof of (3). To show the remaining claims, we need the following facts:

Fact 1. [2, Corollary 2.3] *The property of being an accretive-dissipative matrix is hereditary under Gaussian elimination.*

Fact 2. [2, Lemma 2.1, 2.2] If $A = (a_{lj}) \in \mathbb{M}_n(\mathbf{C})$ is accretive-dissipative, then $\sqrt{2}\max_{l} |a_{ll}| \geq \max_{l\neq j} |a_{lj}|$. If A is a Higham matrix, then $\max_{l} |a_{ll}| \geq \max_{l,j} |a_{lj}|$.

Suppose $\max_{j,k} |a_{jj}^{(k)}| = |a_{j_0 j_0}^{(k_0)}|$ for some j_0, k_0, then

$$\rho_n(A) = \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\max_{i,j} |a_{ij}|} \leq \frac{\sqrt{2}\max_{j,k} |a_{jj}^{(k)}|}{\max_{i,j} |a_{ij}|} \leq \frac{\sqrt{2}|a_{j_0 j_0}^{(k_0)}|}{|a_{j_0 j_0}|} < 4.$$

Similarly, we can show that if A is a Higham matrix, then $\rho_n(A) < 2\sqrt{2}$. The proof is thus complete. □

We remark that Fact 2 in the preceding proof has been extended to norm inequalities for accretive-dissipative operator matrices; see [4].

3 Conclusion

Compared with Theorems 1 and 2, it might look minor to improve the upper bound from 3 to $2\sqrt{2}$, but it is one step closer to the final solution of Higham's conjecture. Moreover, the approach in the previous proof may also apply to other related results; see e.g. [1]. In [5], we have used a similar idea to improve a result on Fischer type determinantal inequalities for accretive-dissipative matrices.

Acknowledgments The author thanks Prof. Henry Wolkowicz for valuable discussion on a draft of this paper, and both referees for their comments on the presentation of the submitted version.

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