Calcolo (2010) 47: 129–147
DOI 10.1007/s10092-009-0017-6

Analysis of finite element methods for the Brinkman problem

Mika Juntunen · Rolf Stenberg

Received: 1 September 2009 / Accepted: 3 November 2009 / Published online: 18 November 2009

Abstract The parameter dependent Brinkman problem, covering a field of problems from the Darcy equations to the Stokes problem, is studied. A mathematical framework is introduced for analyzing the problem. Using this uniform a priori and a posteriori estimates for two families of finite element methods are proved. Nitsche's method for imposing boundary conditions is discussed.

Keywords Brinkman equation · Stokes equation · Darcy equation · Nitsche's method · Mixed finite element methods · Stabilized methods

Mathematics Subject Classification (2000) 65N30

1 Introduction

The purpose of this paper is to analyze finite element methods for the Brinkman equations modeling porous media flow. The model is usually derived by homogenization assuming a high porosity, cf. [1–3, 12, 16]. The equations are, in fact, a whole range of equations with Darcy's equations and Stokes equations as limits. As a consequence, it is not trivial to design efficient finite element methods. If they are efficient for the Darcy problem that is not necessarily the case for Stokes, and vice versa. Tied to this are the norms used in the analysis for the velocity and pressure, respectively. Roughly speaking, they change place when going from one extreme to the other.

M. Juntunen · R. Stenberg (✉)
Institute of Mathematics and Systems Analysis, Helsinki University of Technology (TKK), P.O. Box 1100, 02015 Espoo, Finland
e-mail: rolf.stenberg@tkk.fi

M. Juntunen
e-mail: mika.juntunen@tkk.fi

The plan of the paper is as follows. In the next section we introduce a framework using two scales of norms for analyzing the problem. We do not use the approach of [13] since that does not include the Stokes limit. The recent paper [5] contains an a priori analysis, but the choice of norms in different from ours. In Sect. 3 we consider a family of classical mixed finite element methods. We prove the stability (in the chosen norms) and derive both a priori and a posteriori error estimates. Next, we perform the same analysis for a family of stabilized finite element methods. In Sect. 5 we follow [10] and discuss the enforcement of Dirichlet boundary conditions by Nitsche's method.

In a forthcoming paper [9] we present the results of numerical tests with the finite element methods.

2 The Brinkman problem

Let $\Omega \subset \mathbb{R}^N$ be a domain with polygonal or polyhedral boundary. The Brinkman problem is the parameter dependent equations

$$-t^2 \boldsymbol{A}\boldsymbol{u} + \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1}$$

$$\operatorname{div} \boldsymbol{u} = g \quad \text{in } \Omega, \tag{2}$$

where the parameter $0 \le t \le C$. Above we denote $\boldsymbol{A}\boldsymbol{u} = \operatorname{div} \boldsymbol{\varepsilon}(\boldsymbol{u})$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)/2$. For $t > 0$ the equations are formally a Stokes problem for which we assume homogeneous essential boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{3}$$

In the limit $t = 0$, we obtain the Darcy problem with the natural boundary conditions

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{4}$$

Since the boundary conditions are homogenous, the compatibility condition $g \in L_0^2(\Omega)$ is required for the load in both cases. The same condition; $p \in L_0^2(\Omega)$, is imposed in order to have a unique pressure.

The natural energy norm for the velocity is

$$t^2 \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0^2 + \|\boldsymbol{v}\|_0^2, \tag{5}$$

and the natural solution space is $\boldsymbol{V}$; the completion of $[C_0^\infty(\Omega)]^N$ with respect to this norm. By Korn's inequality the energy norm is equivalent to

$$\|\boldsymbol{v}\|_t^2 = t^2 \|\nabla \boldsymbol{v}\|_0^2 + \|\boldsymbol{v}\|_0^2, \tag{6}$$

which is the norm to be used in the sequel. For $t > 0$ we hence have

$$\boldsymbol{V} = [H_0^1(\Omega)]^N, \tag{7}$$

but the equivalence is not uniform, for $0 < t \le C$ it holds

$$C_1 t \|\boldsymbol{v}\|_1 \le \|\boldsymbol{v}\|_t \le C_2 \|\boldsymbol{v}\|_1. \tag{8}$$

(Here and in the sequel all constants C and C_i are assumed independent of t and the mesh parameter h.) For $t = 0$ the space is

$$\boldsymbol{V} = [L^2(\Omega)]^N. \tag{9}$$

Hence, when $t > 0$ is "small", the equations are best considered as a singular perturbation of the Darcy equations. Note that the essential boundary conditions disappear from the energy space in the limit $t = 0$.

The space for the pressure is defined through the norm

$$|||q|||_t = \sup_{\boldsymbol{v} \in \boldsymbol{V}} \frac{\langle \boldsymbol{v}, \nabla q \rangle}{\|\boldsymbol{v}\|_t}, \tag{10}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing in $\boldsymbol{V} \times \boldsymbol{V}^*$. In other words, the distributional gradient of the pressure is required to lie in the dual $\boldsymbol{V}^*$. The space is denoted by Q:

$$Q = \{q \in L_0^2(\Omega) \mid |||q|||_t < \infty\}. \tag{11}$$

Note that for $(\boldsymbol{v}, q) \in \boldsymbol{V} \times Q$ it holds

$$\langle \boldsymbol{v}, \nabla q \rangle = \begin{cases} -(\operatorname{div} \boldsymbol{v}, q) & \text{for } t > 0, \\ (\boldsymbol{v}, \nabla q) & \text{for } t = 0, \end{cases} \tag{12}$$

where $(\cdot, \cdot)$ denotes the L^2-inner products. For $t > 0$ the Babuška-Brezzi condition

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_1} \ge C \|q\|_0 \quad \forall q \in L_0^2(\Omega) \tag{13}$$

implies that $Q = L_0^2(\Omega)$, but again the equivalence is not uniformly valid. For $0 < t < C$ we have

$$C_1 \|q\|_0 \le |||q|||_t \le C_2 t^{-1} \|q\|_0. \tag{14}$$

For $t = 0$ we have

$$|||q|||_t \equiv \|\nabla q\|_0 \tag{15}$$

and $Q = H^1(\Omega) \cap L_0^2(\Omega)$.

Define the bilinear forms

$$a(\boldsymbol{u}, \boldsymbol{v}) = t^2 (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})) + (\boldsymbol{u}, \boldsymbol{v}), \tag{16}$$

$$b(\boldsymbol{v}, p) = \langle \boldsymbol{v}, \nabla p \rangle \tag{17}$$

and

$$\mathcal{B}(\boldsymbol{u}, p; \boldsymbol{v}, q) = a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b(\boldsymbol{u}, q). \tag{18}$$

The weak formulation of the problem is then: Find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ such that

$$\mathcal{B}(\boldsymbol{u}, p; \boldsymbol{v}, q) = \mathcal{L}(\boldsymbol{v}, q) \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{V} \times Q, \tag{19}$$

where

$$\mathcal{L}(\boldsymbol{v}, q) = (\boldsymbol{f}, \boldsymbol{v}) - (g, q). \tag{20}$$

By definition of the norms and Korn's inequality, Brezzi's conditions for a saddle point problem are satisfied, namely

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq C \|\boldsymbol{v}\|_t^2 \quad \forall \boldsymbol{v} \in \boldsymbol{V} \quad \text{and} \quad \sup_{\boldsymbol{v} \in \boldsymbol{V}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_t} \geq |\!|\!| q |\!|\!|_t \quad \forall q \in Q. \tag{21}$$

These two imply the stability condition

$$\sup_{(\boldsymbol{v}, q) \in \boldsymbol{V} \times Q} \frac{\mathcal{B}(\boldsymbol{w}, r; \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_t + |\!|\!| q |\!|\!|_t} \geq C\big(\|\boldsymbol{w}\|_t + |\!|\!| r |\!|\!|_t\big) \quad \forall (\boldsymbol{w}, r) \in \boldsymbol{V} \times Q \tag{22}$$

by which the solution is unique.

3 Mixed finite element methods

We assume a partitioning $\mathcal{C}_h$ of the domain Ω into simplices. With $K \in \mathcal{C}_h$ we denote an element of the partitioning, and the maximum size of $K \in \mathcal{C}_h$ is denoted by h. With Γ_h we denote the internal edges/faces of the partitioning.

The finite element spaces are a generalization of the classical MINI element [4] and they are defined as

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{V} \cap [C(\Omega)]^N \mid \boldsymbol{v}|_K \in [P_k(K) \cup B_{k+N}(K)]^N \ \forall K \in \mathcal{C}_h\}, \tag{23}$$

$$Q_h = \{q \in L_0^2(\Omega) \cap C(\Omega) \mid q|_K \in P_k(K) \ \forall K \in \mathcal{C}_h\}, \tag{24}$$

where $P_k(K)$ denotes the polynomials of degree k and

$$B_{k+N}(K) = P_{k+N}(K) \cap H_0^1(K)$$

are the bubbles of degree $k + N$. In the analysis will also use the subspace $\overline{\boldsymbol{V}}_h \subset \boldsymbol{V}_h$ where the "bubbles" are left out:

$$\overline{\boldsymbol{V}}_h = \{\boldsymbol{v} \in \boldsymbol{V} \cap [C(\Omega)]^N \mid \boldsymbol{v}|_K \in [P_k(K)]^N \ \forall K \in \mathcal{C}_h\}. \tag{25}$$

The finite element formulations is: find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that

$$\mathcal{B}(\boldsymbol{u}_h, p_h; \boldsymbol{v}, q) = \mathcal{L}(\boldsymbol{v}, q) \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{V}_h \times Q_h. \tag{26}$$

3.1 Stability

To prove the stability of our formulation we have to verify the two conditions, the ellipticity and the inf-sup condition. For this we will utilize the following discrete counterpart of the norm (10)

$$|||q|||_{t,h}^2 = \sum_{K\in\mathcal{C}_h} \frac{h_K^2}{t^2+h_K^2} \|\nabla q\|_{0,K}^2. \tag{27}$$

This norm is also important in practice, since it can be readily computed.

First, we prove the inf-sup condition with this norm.

Lemma 1 *There is a constant $C > 0$ such that*

$$\sup_{v\in \boldsymbol{V}_h} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_t} \geq C|||q|||_{t,h} \quad \forall q \in Q_h. \tag{28}$$

Proof For $q \in Q_h$ given, it holds $\nabla q|_K \in [P_{k-1}(K)]^N$, and we can define $\boldsymbol{v} \in \boldsymbol{V}_h$ through

$$\boldsymbol{v}|_K = \left(\frac{h_K^2}{t^2+h_K^2}\right) b_K \nabla q|_K, \tag{29}$$

where b_K is the cubic/quartic bubble on K. For $\boldsymbol{v}$ it holds

$$b(\boldsymbol{v},q) = (\boldsymbol{v},\nabla q) \geq C \sum_{K\in\mathcal{C}_h} \frac{h_K^2}{t^2+h_K^2}\|\nabla q\|_{0,K}^2 = C|||q|||_{t,h}^2 \tag{30}$$

and

$$\begin{aligned}\|\boldsymbol{v}\|_t^2 = t^2\|\nabla \boldsymbol{v}\|_0^2 + \|\boldsymbol{v}\|_0^2 &\leq C \sum_{K\in\mathcal{C}_h} (t^2 h_K^{-2}+1)\|\boldsymbol{v}\|_{0,K}^2 \\ &\leq C \sum_{K\in\mathcal{C}_h} (t^2 h_K^{-2}+1)\left(\frac{h_K^2}{t^2+h_K^2}\right)^2 \|\nabla q\|_{0,K}^2 = C|||q|||_{t,h}^2. \end{aligned} \tag{31}$$

Combining (30) and (31) completes the proof. □

Next, we use the 'Pitkäranta-Verfürth'-trick (see [15, 18]) to prove the stability in the continuous norm.

Lemma 2 *There is a constant $C > 0$ such that*

$$\sup_{\boldsymbol{v}\in \boldsymbol{V}_h} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_t} \geq C|||q|||_t \quad \forall q \in Q_h. \tag{32}$$

Proof Due to the continuous inf-sup condition (21), there exist $\boldsymbol{w} \in \boldsymbol{V}$ such that

$$b(\boldsymbol{w}, q) \geq |||q|||_t^2 \quad \text{and} \quad \|\boldsymbol{w}\|_t \leq |||q|||_t \quad \forall q \in Q_h. \tag{33}$$

With $\tilde{\boldsymbol{w}} \in \bar{\boldsymbol{V}}_h$ we denote the Clément-Scott-Zhang interpolant [6, 7] of $\boldsymbol{w}$. For this it holds

$$\sum_{K \in K_h} h_K^{-2} \|\boldsymbol{w} - \tilde{\boldsymbol{w}}\|_{0,K}^2 \leq C \|\nabla \boldsymbol{w}\|_0^2, \tag{34}$$

$$\|\tilde{\boldsymbol{w}}\|_0 \leq C \|\boldsymbol{w}\|_0 \quad \text{and} \quad \|\nabla \tilde{\boldsymbol{w}}\|_0 \leq C \|\nabla \boldsymbol{w}\|_0. \tag{35}$$

This gives

$$\begin{aligned}\sum_{K \in \mathcal{C}_h} \left(\frac{t + h_K}{h_K}\right)^2 \|\tilde{\boldsymbol{w}} - \boldsymbol{w}\|_{0,K}^2 &\leq 2 \sum_{K \in \mathcal{C}_h} \left(\left(\frac{t}{h_K}\right)^2 + 1\right) \|\tilde{\boldsymbol{w}} - \boldsymbol{w}\|_{0,K}^2 \\ &\leq C \|\boldsymbol{w}\|_t^2. \end{aligned} \tag{36}$$

Using the estimates above, we obtain

$$\begin{aligned} b(\tilde{\boldsymbol{w}}, q) &= (\tilde{\boldsymbol{w}}, \nabla q) \\ &= (\boldsymbol{w}, \nabla q) + (\tilde{\boldsymbol{w}} - \boldsymbol{w}, \nabla q) \\ &\geq |||q|||_t^2 - \sum_{K \in \mathcal{C}_h} \frac{h_K}{t + h_K} \|\nabla q\|_{0,K} \frac{t + h_K}{h_K} \|\tilde{\boldsymbol{w}} - \boldsymbol{w}\|_{0,K} \\ &\geq |||q|||_t^2 - |||q|||_{t,h} \left(\sum_{K \in \mathcal{C}_h} \left(\frac{t + h_K}{h_K}\right)^2 \|\tilde{\boldsymbol{w}} - \boldsymbol{w}\|_{0,K}^2 \right)^{1/2} \\ &\geq |||q|||_t^2 - C |||q|||_{t,h} \|\boldsymbol{w}\|_t \\ &\geq \left(|||q|||_t - C |||q|||_{t,h}\right) \|\boldsymbol{w}\|_t. \end{aligned} \tag{37}$$

Now, if $|||q|||_t - C|||q|||_{t,h} < 0$, then the assertion follows from Lemma 1. Otherwise, (37) and (35) give

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_t} \geq C_1 |||q|||_t - C_2 |||q|||_{t,h}. \tag{38}$$

Combining this estimate and Lemma 1, with $0 < \alpha < 1$, we get

$$\begin{aligned} \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_t} &= \alpha \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_t} + (1 - \alpha) \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_t} \\ &\geq \alpha C_1 |||q|||_t - \alpha C_2 |||q|||_{t,h} + (1 - \alpha) C |||q|||_{t,h} \\ &= \alpha C_1 |||q|||_t + (C - \alpha(C + C_2)) |||q|||_{t,h}. \end{aligned} \tag{39}$$

Choosing α such that $0 < \alpha < C/(C + C_2)$ proves the assertion. □

Lemmas 1 and 2 give the two stability results.

Theorem 1 *There is a constant $C > 0$ such that*

$$\sup_{(\boldsymbol{v},q)\in \boldsymbol{V}_h\times Q_h} \frac{\mathcal{B}(\boldsymbol{w},r;\boldsymbol{v},q)}{\|\boldsymbol{v}\|_t + |||q|||_t} \geq C\big(\|\boldsymbol{w}\|_t + |||r|||_t\big)$$
$$\forall(\boldsymbol{w},r)\in \boldsymbol{V}_h\times Q_h. \tag{40}$$

Theorem 2 *There is a constant $C > 0$ such that*

$$\sup_{(\boldsymbol{v},q)\in \boldsymbol{V}_h\times Q_h} \frac{\mathcal{B}(\boldsymbol{w},r;\boldsymbol{v},q)}{\|\boldsymbol{v}\|_t + |||q|||_{t,h}} \geq C\big(\|\boldsymbol{w}\|_t + |||r|||_{t,h}\big)$$
$$\forall(\boldsymbol{w},r)\in \boldsymbol{V}_h\times Q_h. \tag{41}$$

3.2 A priori estimate

The stability estimate of Theorem 1 and the consistency gives the following quasi-optimality result.

Theorem 3 *There exists a constant $C > 0$ such that*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_t + |||p-p_h|||_t \leq C\Big\{\inf_{\boldsymbol{v}\in\boldsymbol{V}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_t + \inf_{q\in Q_h}|||p-q|||_t\Big\}. \tag{42}$$

Standard interpolation estimates then give.

Theorem 4 *Assume that the problem has a smooth solution. Then it holds*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_t + |||p-p_h|||_t = \mathcal{O}(h^k). \tag{43}$$

When measuring the error in the computable mesh dependent norm for the pressure we get the following theorem.

Theorem 5 *There exists $C > 0$ such that*

$$\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{u}_h\|_t + |||p-p_h|||_{t,h}\\
&\quad\leq C\Bigg(\inf_{\boldsymbol{v}\in\boldsymbol{V}_h}\Bigg\{\|\boldsymbol{u}-\boldsymbol{v}\|_t + t\Bigg(\sum_{K\in\mathcal{C}_h} h_K^{-2}\|\boldsymbol{u}-\boldsymbol{v}\|_{0,K}^2\Bigg)^{1/2}\Bigg\}\\
&\qquad+\inf_{q\in Q_h}\big\{|||p-q|||_{t,h} + |||p-q|||_t\big\}\Bigg).
\end{aligned} \tag{44}$$

Proof By the triangle inequality

$$\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{u}_h\|_t + |||p-p_h|||_{t,h}\\
&\quad\leq \|\boldsymbol{u}-\boldsymbol{v}\|_t + |||p-q|||_{t,h} + \|\boldsymbol{u}_h-\boldsymbol{v}\|_t + |||p_h-q|||_{t,h}.
\end{aligned} \tag{45}$$

Hence, we have to bound

$$\|\boldsymbol{u}_h - \boldsymbol{v}\|_t + |||p_h - q|||_{t,h}.$$

Using the stability estimate of Theorem 2 we know there exists $(\boldsymbol{w}, r) \in \boldsymbol{V}_h \times Q_h$, with

$$\|\boldsymbol{w}\|_t + |||r|||_{t,h} \le C \tag{46}$$

such that

$$\|\boldsymbol{u}_h - \boldsymbol{v}\|_t + |||p_h - q|||_{t,h} \le \mathcal{B}(\boldsymbol{u}_h - \boldsymbol{v}, p_h - q; \boldsymbol{w}, r). \tag{47}$$

By the consistency we have

$$\mathcal{B}(\boldsymbol{u}_h - \boldsymbol{v}, p_h - q; \boldsymbol{w}, r) = \mathcal{B}(\boldsymbol{u} - \boldsymbol{v}, p - q; \boldsymbol{w}, r). \tag{48}$$

Using Schwartz inequality we then get

$$\begin{aligned}
&\mathcal{B}(\boldsymbol{u} - \boldsymbol{v}, p - q; \boldsymbol{w}, r)\\
&\quad = t^2(\nabla(\boldsymbol{u} - \boldsymbol{v}), \nabla \boldsymbol{w} + (\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{w}) + \langle \boldsymbol{w}, p - q\rangle + (\boldsymbol{u} - \boldsymbol{v}, \nabla r)\\
&\quad \le t\|\nabla(\boldsymbol{u} - \boldsymbol{v})\|_0 t\|\nabla \boldsymbol{w}\|_0 + \|\boldsymbol{u} - \boldsymbol{v}\|_0\|\boldsymbol{w}\|_0 + \|\boldsymbol{w}\|_t |||p - q|||_t\\
&\qquad + \left(\sum_{K\in\mathcal{C}_h}\left(\frac{t + h_K}{h_K}\right)^2\|\boldsymbol{u} - \boldsymbol{v}\|_{0,K}^2\right)^{1/2}\left(\sum_{K\in\mathcal{C}_h}\left(\frac{h_K}{t + h_K}\right)^2\|\nabla r\|_{0,K}^2\right)^{1/2}\\
&\quad \le C\left(\|\boldsymbol{u} - \boldsymbol{v}\|_t + |||p - q|||_t + t\left(\sum_{K\in\mathcal{C}_h} h_K^{-2}\|\boldsymbol{u} - \boldsymbol{v}\|_{0,K}^2\right)^{1/2}\right).
\end{aligned} \tag{49}$$

Combining (45)–(49) proves (44). □

3.3 A posteriori estimate

In this section we will introduce and analyze a residual based a posteriori estimator. In an earlier paper we have done this for the related scalar reaction-diffusion [11]. The element wise estimator is defined by

$$\begin{aligned}
&E_K(\boldsymbol{u}_h, p_h)^2\\
&\quad = \frac{h_K^2}{t^2 + h_K^2}\|t^2 \boldsymbol{A}\boldsymbol{u}_h - \boldsymbol{u}_h - \nabla p_h + \boldsymbol{f}\|_{0,K}^2\\
&\qquad + (t^2 + h_K^2)\|\operatorname{div} \boldsymbol{u}_h - g\|_{0,K}^2\\
&\qquad + \frac{h_K}{t^2 + h_K^2}\|[\![t^2 \boldsymbol{\varepsilon}_n(\boldsymbol{u}_h)]\!]\|_{0,\partial K\setminus\partial\Omega}^2 + \frac{t^2 + h_K^2}{h_K}\|\boldsymbol{u}_h \cdot \boldsymbol{n}\|_{0,\partial K\cap\partial\Omega}^2
\end{aligned} \tag{50}$$

and the global estimator is

$$\eta = \left(\sum_{K\in\mathcal{C}_h} E_K(\boldsymbol{u}_h, p_h)^2\right)^{1/2}. \tag{51}$$

Here $\boldsymbol{\varepsilon}_n(\cdot)$ denotes the normal derivative and $[\![\cdot]\!]$ is the jump. Note, that the last term in (50) vanishes when $t > 0$.

In the limit $t = 0$ (or as $t < h$) the a posteriori estimator becomes

$$E_K(\boldsymbol{u}_h, p_h)^2 \approx \|\boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}\|_{0,K}^2 + h_K^2 \|\operatorname{div} \boldsymbol{u}_h - g\|_{0,K}^2$$
$$+ h_E \|\boldsymbol{u}_h \cdot \boldsymbol{n}\|_{0,\partial K \cap \partial \Omega}^2,$$

which is the estimator for the Darcy problem. On the other hand, if $t \geq C > 0$, the estimator can be expressed as (since $\boldsymbol{u}_h|_{\partial\Omega} = \boldsymbol{0}$)

$$E_K(\boldsymbol{u}_h, p_h)^2 \approx h_K^2 \|t^2 \boldsymbol{A}\boldsymbol{u}_h - \boldsymbol{u}_h - \nabla p_h + \boldsymbol{f}\|_{0,K}^2 + \|\operatorname{div} \boldsymbol{u}_h - g\|_{0,K}^2$$
$$+ h_E \|[\![t^2 \boldsymbol{\varepsilon}_n(\boldsymbol{u}_h)]\!]\|_{0,\partial K \setminus \partial\Omega}^2,$$

which is the standard Stokes estimator.

For our analysis we will need a saturation assumption. The partitioning $\mathcal{C}_h$ is refined into $\mathcal{C}_{h/2}$ by dividing each triangle/tetrahedron K into four/eight elements with mesh size less or equal to $h_K/2$. By $(\boldsymbol{u}_{h/2}, p_{h/2}) \in \boldsymbol{V}_{h/2} \times Q_{h/2}$ we denote the finite element solution on the refined mesh.

Assumption 6 There exists a positive constant $\beta < 1$ such that

$$\|\boldsymbol{u} - \boldsymbol{u}_{h/2}\|_t + |\!|\!| p - p_{h/2} |\!|\!|_{t,h} \leq \beta\big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_t + |\!|\!| p - p_h |\!|\!|_{t,h}\big). \tag{52}$$

The main result is the following theorem.

Theorem 7 *Let Assumption 6 hold. Then there exists* $C > 0$ *such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_t + |\!|\!| p - p_h |\!|\!|_{t,h} \leq C\eta. \tag{53}$$

Proof By the triangle inequality the saturation assumption gives

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_t + |\!|\!| p - p_h |\!|\!|_{t,h} \leq \frac{1}{1-\beta}\big(\|\boldsymbol{u}_{h/2} - \boldsymbol{u}_h\|_t + |\!|\!| p_{h/2} - p_h |\!|\!|\big). \tag{54}$$

From the stability, Theorem 2, there exists $(\boldsymbol{v}, q) \in \boldsymbol{V}_{h/2} \times Q_{h/2}$, with

$$\|\boldsymbol{v}\|_t + |\!|\!| q |\!|\!|_{t,h} \leq C, \tag{55}$$

such that

$$\|\boldsymbol{u}_{h/2} - \boldsymbol{u}_h\|_t + |\!|\!| p_{h/2} - p_h |\!|\!|_{t,h} \leq \mathcal{B}(\boldsymbol{u}_{h/2} - \boldsymbol{u}_h, p_{h/2} - p_h; \boldsymbol{v}, q). \tag{56}$$

Let now $(\tilde{\boldsymbol{v}}, \tilde{q}) \in \bar{\boldsymbol{V}}_h \times Q_h$ be the normal Lagrange interpolants to $(\boldsymbol{v}, q)$. Since $\bar{\boldsymbol{V}}_h \subset \boldsymbol{V}_h$ and $\bar{\boldsymbol{V}}_h \subset \boldsymbol{V}_{h/2}$ (and $Q_h \subset Q_{h/2}$) it holds

$$\mathcal{B}(\boldsymbol{u}_{h/2} - \boldsymbol{u}_h, p_{h/2} - p_h; \tilde{\boldsymbol{v}}, \tilde{q}) = 0.$$

Hence we have

$$\mathcal{B}(\boldsymbol{u}_{h/2}-\boldsymbol{u}_h,p_{h/2}-p_h;\boldsymbol{v},q)=\mathcal{B}(\boldsymbol{u}_{h/2}-\boldsymbol{u}_h,p_{h/2}-p_h;\boldsymbol{v}-\tilde{\boldsymbol{v}},q-\tilde{q}). \tag{57}$$

Writing out the right hand side, using the fact that $(\boldsymbol{u}_{h/2},p_{h/2})$ satisfies

$$\mathcal{B}(\boldsymbol{u}_{h/2},p_{h/2};\boldsymbol{v}-\tilde{\boldsymbol{v}},q-\tilde{q})=(\boldsymbol{f},\boldsymbol{v}-\tilde{\boldsymbol{v}})-(g,q-\tilde{q}) \tag{58}$$

and integrating by parts, we have

$$\begin{aligned}
&\mathcal{B}(\boldsymbol{u}_{h/2}-\boldsymbol{u}_h,p_{h/2}-p_h;\boldsymbol{v}-\tilde{\boldsymbol{v}},q-\tilde{q})\\
&\quad=(\boldsymbol{f},\boldsymbol{v}-\tilde{\boldsymbol{v}})-(g,q-\tilde{q})-t^2(\boldsymbol{\varepsilon}(\boldsymbol{u}_h),\boldsymbol{\varepsilon}(\boldsymbol{v}-\tilde{\boldsymbol{v}}))-(\boldsymbol{u}_h,\boldsymbol{v}-\tilde{\boldsymbol{v}})\\
&\qquad-(\boldsymbol{v}-\tilde{\boldsymbol{v}},\nabla p_h)-(\boldsymbol{u}_h,\nabla(q-\tilde{q}))\\
&\quad=\sum_{K\in\mathcal{C}_h}\big\{(t^2\boldsymbol{A}\boldsymbol{u}_h-\boldsymbol{u}_h-\nabla p_h+\boldsymbol{f},\boldsymbol{v}-\tilde{\boldsymbol{v}})_K+t^2\langle\boldsymbol{\varepsilon}_n(\boldsymbol{u}_h),\boldsymbol{v}-\tilde{\boldsymbol{v}}\rangle_{\partial K}\\
&\qquad+(\operatorname{div}\boldsymbol{u}_h-g,q-\tilde{q})_K-\langle\boldsymbol{u}_h\cdot\boldsymbol{n},q-\tilde{q}\rangle_{\partial K\cap\partial\Omega}\big\}.
\end{aligned} \tag{59}$$

Since, $\boldsymbol{v}$, $\tilde{\boldsymbol{v}}$, q and $\tilde{q}$, all are in finite element subspaces, scaling arguments give

$$\begin{aligned}
&\Big(\sum_{K\in\mathcal{C}_h}\Big(\frac{t+h_K}{h_K}\Big)^2\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{0,K}^2\Big)^{1/2}\\
&\quad\le C\Big(\sum_{K\in\mathcal{C}_h}\big(t^2\|\nabla\boldsymbol{v}\|_{0,K}^2+\|\boldsymbol{v}\|_{0,K}^2\big)\Big)^{1/2}\\
&\quad\le C\|\boldsymbol{v}\|_t\le C,
\end{aligned} \tag{60}$$

$$\begin{aligned}
&\Big(\sum_{K\in\mathcal{C}_h}\frac{t^2+h_K^2}{h_K}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{0,\partial K}^2\Big)^{1/2}\\
&\quad\le C\Big(\sum_{K\in\mathcal{C}_h}\frac{t^2+h_K^2}{h_K}h_K^{-1}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{0,K}^2\Big)^{1/2}\\
&\quad= C\Big(\sum_{K\in\mathcal{C}_h}\Big(\frac{t^2}{h_K^2}+1\Big)\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{0,K}^2\Big)^{1/2}\\
&\quad\le C\Big(\sum_{K\in\mathcal{C}_h}\big(t^2\|\nabla\boldsymbol{v}\|_{0,K}^2+\|\boldsymbol{v}\|_{0,K}^2\big)\Big)\\
&\quad= C\|\boldsymbol{v}\|_t\le C
\end{aligned} \tag{61}$$

and

$$\Big(\sum_{K\in\mathcal{C}_h}\Big(\frac{h_K}{(t+h_K)^2}\|q-\tilde{q}\|_{0,\partial K}^2+(t+h_K)^{-2}\|q-\tilde{q}\|_{0,K}^2\Big)\Big)^{1/2}$$

$$\leq C\left(\sum_{K\in\mathcal{C}_h}\left(\frac{h_K}{t+h_K}\right)^2\|\nabla q\|_{0,K}^2\right)^{1/2} \leq C|\!|\!|q|\!|\!|_{t,h} \leq C. \tag{62}$$

Using the Schwartz inequality and the properties above in (59) completes the proof. □

3.4 Efficiency of the a posteriori estimate

We show that the a posteriori upper bound is also a lower bound to the error. In this sense the estimator is sharp.

Theorem 8 *There exist $C > 0$ such that*

$$C\eta^2 \leq \|\boldsymbol{u}-\boldsymbol{u}_h\|_t^2 + |\!|\!|p-p_h|\!|\!|_{t,h}^2 + \sum_{K\in\mathcal{C}_h}\left(\frac{h_K^2}{t^2+h_K^2}\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,K}^2 + (t^2+h_K^2)\|g-g_h\|_{0,K}^2\right), \tag{63}$$

where the projections $\boldsymbol{f}_h \in \boldsymbol{V}_h$ and $g_h \in Q_h$.

We use suitable cut-off functions to prove the above theorem, we refer to [19] for more details. The first cut-off function is Ψ_K; the support of Ψ_K is element K and $0 \leq \Psi_K \leq 1$. The second cut-off function is Ψ_E; the support of Ψ_E is ω_E and $0 \leq \Psi_E \leq 1$. The domain ω_E is the elements sharing edge (in 3D face) E. For the edge (or face) E we also need an extension mapping $\chi : L^2(E) \to L^2(\omega_E)$ such that in E χ is the identity operator. The proof of the lemma below follows with scaling arguments; note that p and σ are polynomials, cf. [19].

Lemma 3 *For an arbitrary element K, having edge/face E, and for arbitrary polynomials p and σ it holds*:

$$\|\Psi_K p\|_{0,K} \leq \|p\|_{0,K} \leq C\|\Psi_K^{1/2} p\|_{0,K}, \tag{64}$$

$$\|\nabla(\Psi_K p)\|_{0,K} \leq Ch_K^{-1}\|\Psi_K p\|_{0,K}, \tag{65}$$

$$\|\sigma\|_{0,E} \leq C\|\Psi_E^{1/2}\sigma\|_{0,E}, \tag{66}$$

$$Ch_E^{1/2}\|\sigma\|_{0,E} \leq \|\Psi_E\chi\sigma\|_{0,E} \leq Ch_E^{1/2}\|\sigma\|_{0,E}, \tag{67}$$

$$\|\nabla(\Psi_E\chi\sigma)\|_{0,K} \leq Ch_K^{-1}\|\Psi_E\chi\sigma\|_{0,K}. \tag{68}$$

Proof We bound the terms of $E_K(\boldsymbol{u}_h, p_h)$ separately. We begin with the first internal residual term and introduce

$$R_K^1 = t^2\boldsymbol{A}\boldsymbol{u}_h - \boldsymbol{u}_h - \nabla p_h + \boldsymbol{f}, \qquad R_{K,\mathrm{red}}^1 = t^2\boldsymbol{A}\boldsymbol{u}_h - \boldsymbol{u}_h - \nabla p_h + \boldsymbol{f}_h,$$
$$\boldsymbol{w} = \Psi_K R_{K,\mathrm{red}}^1.$$

We have, using Lemma 3,

$$
\begin{aligned}
&\|R^1_{K,\mathrm{red}}\|^2_{0,K}\\
&\quad\le C\|\Psi_K^{1/2}R^1_{K,\mathrm{red}}\|^2_{0,K} = C(R^1_{K,\mathrm{red}},\boldsymbol{w})_K\\
&\quad= C\big((R^1_K,\boldsymbol{w})_K + (\boldsymbol{f}_h-\boldsymbol{f},\boldsymbol{w})_K\big)\\
&\quad= C\big(t^2(\nabla(\boldsymbol{u}-\boldsymbol{u}_h),\nabla\boldsymbol{w})_K + (\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{w})_K\\
&\qquad+(\nabla(p_h-p),\boldsymbol{w})_K + (\boldsymbol{f}_h-\boldsymbol{f},\boldsymbol{w})_K\big)\\
&\quad\le C\big(t^2h_K^{-1}\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{0,K} + \|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,K}\\
&\qquad+\|\nabla(p-p_h)\|_{0,K} + \|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,K}\big)\|R^1_{K,\mathrm{red}}\|_{0,K}.
\end{aligned}
\tag{69}
$$

Combining the above result with

$$
\|R^1_K\|_{0,K} \le \|R^1_{K,\mathrm{red}}\|_{0,K} + \|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,K}
$$

gives

$$
\begin{aligned}
&\frac{h_K}{t+h_K}\|t^2\boldsymbol{A}\boldsymbol{u}_h - \boldsymbol{u}_h - \nabla p_h + \boldsymbol{f}\|_{0,K}\\
&\le C\left(\|\boldsymbol{u}-\boldsymbol{u}_h\|_{t,K} + |||p-p_h|||_{t,h,K} + \frac{h_K}{t+h_K}\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,K}\right).
\end{aligned}
\tag{70}
$$

Next bound the second internal residual term and introduce

$$
\begin{aligned}
R^2_K &= \operatorname{div}\boldsymbol{u}_h - g, \qquad R^2_{K,\mathrm{red}} = \operatorname{div}\boldsymbol{u}_h - g_h,\\
\boldsymbol{w} &= \Psi_K R^2_{K,\mathrm{red}}.
\end{aligned}
$$

Using Lemma 3 we get

$$
\begin{aligned}
&\|R^2_{K,\mathrm{red}}\|^2_{0,K}\\
&\quad\le C\big\|\Psi_K^{1/2}R^2_{K,\mathrm{red}}\big\|^2_{0,K} = C\big(R^2_{K,\mathrm{red}},\boldsymbol{w}\big)_K\\
&\quad= C\big((R^2_K,\boldsymbol{w})_K + (g-g_h,\boldsymbol{w})_K\big)\\
&\quad= C\big((\operatorname{div}(\boldsymbol{u}_h-\boldsymbol{u}),\boldsymbol{w})_K + (g-g_h,\boldsymbol{w})_K\big)\\
&\quad= C\bigg(\frac{t}{t+h_K}\big(\operatorname{div}(\boldsymbol{u}_h-\boldsymbol{u}),\boldsymbol{w}\big)_K\\
&\qquad+\frac{h_K}{t+h_K}(\boldsymbol{u}_h-\boldsymbol{u},\operatorname{div}\boldsymbol{w})_K + (g-g_h,\boldsymbol{w})_K\bigg)\\
&\quad\le C\big((t+h_K)^{-1}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{t,K} + \|g-g_h\|_{0,K}\big)\|R^2_{K,\mathrm{red}}\|_{0,K}.
\end{aligned}
\tag{71}
$$

Combining the result with $\|R_K^2\|_{0,K} \le \|R_{K,\mathrm{red}}^2\|_{0,K} + \|g - g_h\|_{0,K}$ gives

$$(t + h_K)\|\operatorname{div} \boldsymbol{u}_h - g\|_{0,K} \le C\big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{t,K} + (t + h_K)\|g - g_h\|_{0,K}\big). \tag{72}$$

Next we bound the internal jumps. We introduce

$$R_E^1 = t^2[\![\boldsymbol{\varepsilon}_n(\boldsymbol{u}_h)]\!], \qquad \boldsymbol{w} = \Psi_E \chi R_E^1$$

and continue with Lemma 3

$$\begin{aligned}
&\|R_E^1\|_{0,E}^2 \\
&\quad\le C\|\Psi_E^{1/2} R_E^1\|_{0,E}^2 = C\big(R_E^1, \boldsymbol{w}\big)_E \\
&\quad= C\big((R_K^1, \boldsymbol{w})_{\omega_E} - t^2(\nabla(\boldsymbol{u} - \boldsymbol{u}_h), \nabla \boldsymbol{w})_{\omega_E} \\
&\qquad - (\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{w})_{\omega_E} - (\nabla(p - p_h), \boldsymbol{w})_{\omega_E}\big) \\
&\quad\le C\big(t^2 h_K^{-1/2}\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\omega_E} + h_K^{1/2}\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\omega_E} \\
&\qquad + h_K^{1/2}\|\nabla(p - p_h)\|_{\omega_E} + h_K^{1/2}\|\boldsymbol{f} - \boldsymbol{f}_h\|_{\omega_E}\big)\|R_E^1\|_{0,E}.
\end{aligned} \tag{73}$$

Thus, we have

$$\begin{aligned}
&\frac{h_E^{1/2}}{t + h_E}\|t^2[\![\boldsymbol{\varepsilon}_n(\boldsymbol{u}_h)]\!]\|_{0,E} \\
&\qquad \le C\left(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{t,\omega_E} + \frac{h_K}{t + h_K}\|\boldsymbol{f} - \boldsymbol{f}_h)\|_{\omega_E}\right).
\end{aligned} \tag{74}$$

Lastly we bound the boundary residual. We define

$$R_E^2 = (\boldsymbol{u} - \boldsymbol{u}_h) \cdot \boldsymbol{n}, \qquad \boldsymbol{w} = \Psi_E \chi R_E^2$$

and continue with Lemma 3

$$\begin{aligned}
\|R_E^2\|_{0,E}^2 &\le C\|\Psi_E^{1/2} R_E^2\|_{0,E}^2 = C\big(R_E^2, \boldsymbol{w}\big)_E \\
&= C\big(\big(R_K^2, \boldsymbol{w}\big)_{\omega_E} + (\boldsymbol{u}_h - \boldsymbol{u}, \nabla \boldsymbol{w})_{\omega_E}\big) \\
&\le C\left(\frac{h_K^{1/2}}{t + h_K}\|\boldsymbol{u} - \boldsymbol{u}_h\|_{t,\omega_E} + h_K^{1/2}\|g - g_h\|_{\omega_E}\right. \\
&\qquad \left. + h_K^{-1/2}\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\omega_E}\right)\|R_E^2\|_{0,E}.
\end{aligned} \tag{75}$$

Hence we get

$$\frac{t + h_K}{h_K^{1/2}}\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,E} \le C\big(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{t,\omega_E} + (t + h_K)\|g - g_h\|_{\omega_E}\big). \tag{76}$$

Now we have bounded all the terms of the a posteriori estimator and combining (70), (72), (74) and (76) completes the proof. □

4 Stabilized methods

Stabilized methods enable us to use the standard finite elements without bubble degrees of freedom. Thus, the subspaces are

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{V} \cap [C(\Omega)]^N \mid \boldsymbol{v}|_K \in [P_k(K)]^N \ \forall K \in \mathcal{C}_h\}, \tag{77}$$

$$Q_h = \{q \in L_0^2(\Omega) \cap C(\Omega) \mid q|_K \in P_k(K) \ \forall K \in \mathcal{C}_h\}. \tag{78}$$

The stabilized method is: Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that

$$\mathcal{B}_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}, q) = \mathcal{L}_h(\boldsymbol{v}, q) \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{V}_h \times Q_h, \tag{79}$$

with

$$\begin{aligned}
&\mathcal{B}_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}, q)\\
&\quad = \mathcal{B}(\boldsymbol{u}_h, p_h; \boldsymbol{v}, q)\\
&\qquad - \alpha \sum_{K \in \mathcal{C}_h} \frac{h_K^2}{t^2 + h_K^2} (t^2 \boldsymbol{A}\boldsymbol{u}_h - \boldsymbol{u}_h - \nabla p_h, t^2 \boldsymbol{A}\boldsymbol{v} - \boldsymbol{v} - \nabla q)_K
\end{aligned} \tag{80}$$

and

$$\mathcal{L}_h(\boldsymbol{v}, q) = \mathcal{L}(\boldsymbol{v}, q) - \alpha \sum_{K \in \mathcal{C}_h} \frac{h_K^2}{t^2 + h_K^2} (\boldsymbol{f}, t^2 \boldsymbol{A}\boldsymbol{v} - \boldsymbol{v} - \nabla q)_K, \tag{81}$$

with a parameter $\alpha > 0$. For the method to be consistent we assume that

$$t^2 \boldsymbol{A}\boldsymbol{u} - \boldsymbol{u} - \nabla p = \boldsymbol{f} \in [L^2(\Omega)]^N. \tag{82}$$

Then it holds

$$\mathcal{B}_h(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h; \boldsymbol{v}, q) = 0 \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{V}_h \times Q_h. \tag{83}$$

Note, that one does not have to assume that $t^2 \boldsymbol{A}\boldsymbol{u} \in [L^2(\Omega)]^2$, and $\nabla p \in L^2(\Omega)$ (contrary to some quite widespread belief).

4.1 Stability

For the analysis it is convenient to introduce the constant C_I in the following inverse inequality

$$h_K^2 \|\boldsymbol{A}\boldsymbol{w}\|_{0,K}^2 \le C_I \|\nabla \boldsymbol{w}\|_{0,K}^2 \quad \forall \boldsymbol{w} \in [P_k(K)]^N. \tag{84}$$

The stability result is then.

Theorem 9 *Assume that* $0 < \alpha < \min\{1/(2C_I), 1/2\}$. *Then there exists a constant* $C > 0$ *such that for all* $(\boldsymbol{w}, r) \in \boldsymbol{V}_h \times Q_h$

$$\sup_{(\boldsymbol{v},q) \in \boldsymbol{V}_h \times Q_h} \frac{\mathcal{B}_h(\boldsymbol{w}, r; \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_t + |||q|||_{t,h}} \ge C\big(\|\boldsymbol{w}\|_t + |||r|||_{t,h}\big). \tag{85}$$

Proof For $(\boldsymbol{w}, r) \in \boldsymbol{V}_h \times Q_h$ arbitrary we have

$$\begin{aligned}\mathcal{B}_h(\boldsymbol{w}, r; \boldsymbol{w}, -r) \\ &= t^2 \|\nabla \boldsymbol{w}\|_0^2 + \|\boldsymbol{w}\|_0^2 \\ &\quad - \alpha \sum_{K \in \mathcal{C}_h} \frac{h_K^2}{t^2 + h_K^2} \big\{ \|t^2 \boldsymbol{A}\boldsymbol{w} - \boldsymbol{w}\|_{0,K}^2 - \|\nabla r\|_{0,K}^2 \big\}. \end{aligned} \tag{86}$$

From this we get

$$\begin{aligned}\mathcal{B}_h(\boldsymbol{w}, r; \boldsymbol{w}, -r) \\ &\geq t^2 \|\nabla \boldsymbol{w}\|_0^2 + \|\boldsymbol{w}\|_0^2 + \alpha |\!|\!| r |\!|\!|_{t,h}^2 \\ &\quad - 2\alpha \sum_{K \in \mathcal{C}_h} \frac{h_K^2}{t^2 + h_K^2} t^4 \|\boldsymbol{A}\boldsymbol{w}\|_{0,K}^2 - 2\alpha \sum_{K \in \mathcal{C}_h} \frac{h_K^2}{t^2 + h_K^2} \|\boldsymbol{w}\|_{0,K}^2. \end{aligned} \tag{87}$$

Applying the inverse inequality gives

$$\begin{aligned}\mathcal{B}_h(\boldsymbol{w}, r; \boldsymbol{w}, -r) \\ &\geq (1 - 2\alpha C_I) t^2 \|\nabla \boldsymbol{w}\|_0^2 + (1 - 2\alpha) \|\boldsymbol{w}\|_0^2 + \alpha |\!|\!| r |\!|\!|_{t,h}^2. \end{aligned} \tag{88}$$

The assumption $0 < \alpha < \min\{1/(2C_I), 1/2\}$ implies the asserted stability. $\square$

Remark 1 Using the Pitkäranta-Verfürth technique it is also possible to prove the stability with the continuous norm for the pressure:

$$\begin{aligned} &\sup_{(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times Q_h} \frac{\mathcal{B}_h(\boldsymbol{w}, r; \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_t + |\!|\!| q |\!|\!|_t} \geq C \big(\|\boldsymbol{w}\|_t + |\!|\!| r |\!|\!|_t \big) \\ &\forall (\boldsymbol{w}, r) \in \boldsymbol{V}_h \times Q_h. \end{aligned} \tag{89}$$

See [8] where this is done for the Stokes problem.

Remark 2 For linear elements it suffices to have $0 < \alpha < 1/2$ since $\boldsymbol{A}\boldsymbol{w} = \boldsymbol{0}$ for linear functions.

4.2 A priori estimate

In the spirit of stabilized methods and a posteriori estimates we will formulate the a priori estimate as a quasi-optimality result that contain a term measuring the residual.

Theorem 10 *Assume that* $0 < \alpha < \min\{1/(2C_I), 1/2\}$. *Then it holds*

$$\begin{aligned} &\|\boldsymbol{u} - \boldsymbol{u}_h\|_t + |\!|\!| p - p_h |\!|\!|_{t,h} \\ &\quad \leq C \inf_{(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times Q_h} \Bigg\{ \|\boldsymbol{u} - \boldsymbol{v}\|_t + t \Bigg(\sum_{K \in \mathcal{C}_h} h_K^{-2} \|\boldsymbol{u} - \boldsymbol{v}\|_{0,K}^2 \Bigg)^{1/2} \end{aligned}$$

$$+ |||p-q|||_{t,h} + |||p-q|||_t + \left(\sum_{K\in\mathcal{C}_h} \frac{h_K^2}{t^2+h_K^2} \|t^2 \boldsymbol{A}\boldsymbol{v} - \boldsymbol{v} - \nabla q + \boldsymbol{f}\|_{0,K}^2 \right)^{1/2} \Bigg\}. \tag{90}$$

Proof The proof is very similar to the proof of Theorem 5 and here we only consider the additional terms arising from the added stabilizing term.

For (90) to hold, all we need to bound is the stabilizing term

$$\sum_{K\in\mathcal{C}_h} \frac{h_K^2}{t^2+h_K^2} \big(t^2\boldsymbol{A}(\boldsymbol{u}-\boldsymbol{v}) - (\boldsymbol{u}-\boldsymbol{v}) - \nabla(p-q), t^2\boldsymbol{A}\boldsymbol{w} - \boldsymbol{w} - \nabla r\big)_K = I. \tag{91}$$

Assumption (82) gives

$$\begin{aligned} I &= \sum_{K\in\mathcal{C}_h} \frac{h_K^2}{t^2+h_K^2} (-t^2\boldsymbol{A}\boldsymbol{v} + \boldsymbol{v} + \nabla q - \boldsymbol{f}, t^2\boldsymbol{A}\boldsymbol{w} - \boldsymbol{w} - \nabla r)_K \\ &\le \left(\sum_{K\in\mathcal{C}_h} \frac{h_K^2}{t^2+h_K^2} \|t^2\boldsymbol{A}\boldsymbol{v} - \boldsymbol{v} - \nabla q + \boldsymbol{f}\|_{0,K}^2 \right)^{1/2} \\ &\quad \times \left(\sum_{K\in\mathcal{C}_h} \frac{h_K^2}{t^2+h_K^2} \|t^2\boldsymbol{A}\boldsymbol{w} - \boldsymbol{w} - \nabla r\|_{0,K}^2 \right)^{1/2}. \end{aligned} \tag{92}$$

Using the inverse inequality (84) we have

$$\begin{aligned} &\sum_{K\in\mathcal{C}_h} \frac{h_K^2}{t^2+h_K^2} \|t^2\boldsymbol{A}\boldsymbol{w} - \boldsymbol{w} - \nabla r\|_{0,K}^2 \\ &\le C \sum_{K\in\mathcal{C}_h} \left(\frac{t^2}{t^2+h_K^2} t^2 h_K^2 \|\boldsymbol{A}\boldsymbol{w}\|_{0,K}^2 + \frac{h_K^2}{t^2+h_K^2} \big(\|\boldsymbol{w}\|_{0,K}^2 + \|\nabla q\|_{0,K}^2 \big) \right) \\ &\le C(\|\boldsymbol{w}\|_t + |||r|||_{t,h}) \le C. \end{aligned} \tag{93}$$

The relations (91)–(93) prove (90). □

Again, standard interpolation estimates give

Theorem 11 *Assume that* $0 < \alpha < \min\{1/(2C_I), 1/2\}$ *and that the problem has a smooth solution. Then it holds*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_t + |||p - p_h|||_{t,h} = \mathcal{O}(h^k).$$

4.3 A posteriori estimate

The a posteriori estimator is defined exactly as for the mixed method, i.e. by (51).

Theorem 12 *Let Assumption 6 hold. Then there exist constants* $C_1, C_2 > 0$ *such that*

$$C_1\eta \le \|\boldsymbol{u}-\boldsymbol{u}_h\|_t + |||p-p_h|||_{t,h} \le C_2\eta. \tag{94}$$

Proof In addition to the terms estimated in Theorem 7 we get the term

$$\left|\alpha \sum_{K\in\mathcal{C}_h} \frac{h_K^2}{t^2+h_K^2}\big(-t^2\boldsymbol{A}\boldsymbol{u}_h+\boldsymbol{u}_h+\nabla p_h-\boldsymbol{f}, t^2\boldsymbol{A}(\boldsymbol{v}-\tilde{\boldsymbol{v}})-(\boldsymbol{v}-\tilde{\boldsymbol{v}})-\nabla(q-\tilde{q})\big)_K\right|. \tag{95}$$

Using the Schwarz inequality this is bounded by

$$\begin{aligned}&\left(\sum_{K\in\mathcal{C}_h}\frac{h_K^2}{t^2+h_K^2}\|t^2\boldsymbol{A}\boldsymbol{u}_h-\boldsymbol{u}_h-\nabla p_h+\boldsymbol{f}\|_{0,K}^2\right)^{1/2}\\ &\quad\times\left(\sum_{K\in\mathcal{C}_h}\frac{h_K^2}{t^2+h_K^2}\|t^2\boldsymbol{A}(\boldsymbol{v}-\tilde{\boldsymbol{v}})-(\boldsymbol{v}-\tilde{\boldsymbol{v}})-\nabla(q-\tilde{q})\|_{0,K}^2\right)^{1/2}.\end{aligned} \tag{96}$$

Noticing that

$$\begin{aligned}&\sum_{K\in\mathcal{C}_h}\frac{h_K^2}{t^2+h_K^2}\|t^2\boldsymbol{A}(\boldsymbol{v}-\tilde{\boldsymbol{v}})-(\boldsymbol{v}-\tilde{\boldsymbol{v}})-\nabla(q-\tilde{q})\|_{0,K}^2\\ &\le C\sum_{K\in\mathcal{C}_h}\left(\frac{t^2}{t^2+h_K^2}t^2h_K^2\|\boldsymbol{A}(\boldsymbol{v}-\tilde{\boldsymbol{v}})\|_{0,K}^2+\frac{h_K^2}{t^2+h_K^2}\|\boldsymbol{v}-\tilde{\boldsymbol{v}}\|_{0,K}^2\right.\\ &\qquad\left.+\frac{h_K^2}{t^2+h_K^2}\|\nabla(q-\tilde{q})\|_{0,K}^2\right)\\ &\le C\big(t^2\|\nabla\boldsymbol{v}\|_0^2+\|\boldsymbol{v}\|_0^2+|||q|||_{t,h}^2\big)\le C\end{aligned} \tag{97}$$

completes the proof of the upper bound.

The proof of the lower bound does not use the bilinear form. Hence the proof of Theorem 8 also holds in the present case. □

5 Imposing boundary conditions using Nitsche's method

In this section we will outline the modified finite element methods when the Dirichlet boundary conditions are imposed in a weak sense using the technique of Nitsche [14]. Using this, we obtain formulations that uses the same finite element spaces both for $t>0$ and in the limit $t=0$. The finite element space Q_h used for the pressure is unaltered, i.e. (24) and (78). The spaces for the velocity are altered so that no boundary conditions are assumed; a spaces including "bubbles" for the mixed formulation:

$$\boldsymbol{V}_h = \{\boldsymbol{v}\in[C(\Omega)]^N \mid \boldsymbol{v}|_K\in[P_k(K)\cup B_{k+N}(K)]^N\ \forall K\in\mathcal{C}_h\}, \tag{98}$$

and a clean polynomial space for the stabilized method:

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in [C(\Omega)]^N \mid \boldsymbol{v}|_K \in [P_k(K)]^N \ \forall K \in \mathcal{C}_h\}. \tag{99}$$

The discrete variational formulations are modified by changing the bilinear form $a(\cdot,\cdot)$ in (18) to

$$\begin{aligned} a_h(\boldsymbol{u},\boldsymbol{v}) = t^2\Bigg(& (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})) \\ & + \sum_{E\in\Gamma_h} \big(-\langle \boldsymbol{\varepsilon}_n(\boldsymbol{u}),\boldsymbol{v}\rangle_E - \langle \boldsymbol{\varepsilon}_n(\boldsymbol{v}),\boldsymbol{u}\rangle_E + \gamma h_E^{-1}\langle \boldsymbol{u},\boldsymbol{v}\rangle_E\big)\Bigg) \\ & + (\boldsymbol{u},\boldsymbol{v}), \end{aligned} \tag{100}$$

where we denote with Γ_h the edges/faces on the boundary $\partial\Omega$. The right hand sides, given by (20) and (81), respectively, remain unaltered for homogenous boundary conditions.

This formulation is clearly consistent. For the analysis one uses the following norms for the velocity

$$\|\boldsymbol{v}\|_{t,h}^2 = t^2\Bigg(\|\nabla\boldsymbol{v}\|_0^2 + \sum_{E\in\Gamma_h} h_E^{-1}\|\boldsymbol{v}\|_{0,E}^2\Bigg) + \|\boldsymbol{v}\|_0^2, \tag{101}$$

$$|||\boldsymbol{v}|||_{t,h}^2 = \|\boldsymbol{v}\|_{t,h}^2 + t^2\sum_{E\in\Gamma_h} h_E\|\boldsymbol{\varepsilon}_n(\boldsymbol{v})\|_{0,E}^2. \tag{102}$$

By the discrete trace inequality (when $E\subset\partial K$ we have $h_E \approx h_K$)

$$h_K\|\boldsymbol{\varepsilon}_n(\boldsymbol{v})\|_{0,\partial K}^2 \le C_I'\|\nabla\boldsymbol{v}\|_{0,K}^2 \quad \forall \boldsymbol{v}\in \boldsymbol{V}_h|_K \tag{103}$$

the two norms are equivalent in $\boldsymbol{V}_h$. From which the coercivity of $a_h(\cdot,\cdot)$ easily follows using Schwartz and Young's inequalities [14, 17].

Lemma 4 *For $\gamma > C_I'$ it holds*

$$a_h(\boldsymbol{v},\boldsymbol{v}) \ge C\|\boldsymbol{v}\|_{t,h}^2 \quad \forall \boldsymbol{v}\in\boldsymbol{V}_h. \tag{104}$$

The proofs of the stability of the original methods carry over the present modifications with the norm $\|\cdot\|_t$ changed to $\|\cdot\|_{t,h}$.

Theorem 13 *Assume that the stability parameters satisfy $\gamma > C_I'$ and $0<\alpha<\min\{1/(2C_I), 1/2\}$. Then there exists a constant $C>0$ such that for all $(\boldsymbol{w},r)\in\boldsymbol{V}_h\times Q_h$*

$$\sup_{(\boldsymbol{v},q)\in\boldsymbol{V}_h\times Q_h} \frac{\mathcal{B}_h(\boldsymbol{w},r;\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{t,h} + |||q|||_{t,h}} \ge C\big(\|\boldsymbol{w}\|_{t,h} + |||r|||_{t,h}\big). \tag{105}$$

The previous a priori estimates are now valid with $\|\boldsymbol{u}-\boldsymbol{u}_h\|_t$ replaced by $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{t,h}$ on the left hand sides, and with $\|\boldsymbol{u}-\boldsymbol{v}\|_{t,h}$ replaced by $|||\boldsymbol{u}-\boldsymbol{v}|||_{t,h}$ on the right hand side, respectively. As before, for a smooth solution we obtain an $\mathcal{O}(h^k)$ convergence rate.

The modification needed for the a posteriori estimate is to add the term $t^2 h_K^{-1}\|\boldsymbol{u}_h\|^2_{0,\partial K\cap\partial\Omega}$ to $E_K(\boldsymbol{u}_h, p_h)^2$.

References

1. Allaire, G.: Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes. Arch. Ration. Mech. Anal. **113**(3), 209–259 (1990)
2. Allaire, G.: Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. II. Noncritical sizes of the holes for a volume distribution and a surface distribution of holes. Arch. Ration. Mech. Anal. **113**(3), 261–298 (1990)
3. Arbogast, T., Lehr, H.L.: Homogenization of a Darcy-Stokes system modeling vuggy porous media. Comput. Geosci. **10**(3), 291–302 (2006)
4. Arnold, D.N., Brezzi, F., Fortin, M.: A stable finite element for the Stokes equations. Calcolo **21**(4), 337–344 (1985)
5. Badia, S., Codina, R.: Unified stabilized finite element formulations for the stokes and the Darcy problems. SIAM J. Numer. Anal. **47**(3), 1971–2000 (2009)
6. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods, 2nd edn. Texts in Applied Mathematics, vol. 15. Springer, New York (2002)
7. Clément, P.: Approximation of finite element functions using local regularization. RAIRO Anal. Numer. **9**, 77–84 (1975)
8. Franca, L.P., Stenberg, R.: Error analysis of some Galerkin least-squares methods for the elasticity equations. SIAM J. Numer. Anal. **28**(6), 1680–1699 (1991)
9. Hannukainen, A., Juntunen, M., Stenberg, R.: Computations with finite element methods for the Brinkman problem, Helsinki University of Technology, Institute of Mathematics Research Report, A569 (2009)
10. Hansbo, P., Juntunen, M.: Weakly imposed Dirichlet boundary conditions for the Brinkman model of porous media flow. Appl. Numer. Math. **59**(6), 1274–1289 (2009)
11. Juntunen, M., Stenberg, R.: A residual based a posteriori estimator for the reaction-diffusion problem. C. R. Math. Acad. Sci. Paris **347**, 555–558 (2009)
12. Lévy, T.: Loi de Darcy ou loi de Brinkman? C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers Sci. Terre **292**(12), 871–874 (1981). Erratum (17), 1239 (1981)
13. Mardal, K.A., Tai, X.-C., Winther, R.: A robust finite element method for Darcy-Stokes flow. SIAM J. Numer. Anal. **40**(5), 1605–1631 (2002) (electronic)
14. Nitsche, J.: Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. Abh. Math. Semin. Univ. Hamb. **36**, 9–15 (1970/1971)
15. Pitkäranta, J.: Boundary subspaces for the finite element method with Lagrange multipliers. Numer. Math. **33**, 273–289 (1979)
16. Rajagopal, K.R.: On a hierarchy of approximate models for flows of incompressible fluids through porous solids. Math. Models Methods Appl. Sci. **17**(2), 215–252 (2007)
17. Stenberg, R.: On some techniques for approximating boundary conditions in the finite element method. J. Comput. Appl. Math. **63**(1–3), 139–148 (1995)
18. Verfürth, R.: Error estimates for a mixed finite element approximation of the Stokes equations. RAIRO Anal. Numer. **18**, 175–182 (1984)
19. Verfürth, R.: A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques. Teubner and Wiley, Stuttgart (1996)