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Multipoint Taylor interpolation

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Abstract. We construct new multivariate polynomial interpolation schemes of Hermite type. The interpolant of a function is obtained by specifying suitable discrete differential conditions on the restrictions of the function to algebraic hypersurfaces. The least space of a finite-dimensional space of analytic functions plays an essential role in the definition of these differential conditions.

1 Introduction

An n-dimensional Hermite (or Birkhoff) interpolation scheme of degree d is a collection $H = \{\mu_s : s \in S\}$ of discrete (differential) functionals μ_s such that for every suitably defined function f there exists a unique polynomial p of n variables and degree at most d satisfying $\mu_s(p) = \mu_s(f)$, $s \in S$. The polynomial p is then called the H-interpolation polynomial of f. Classical Lagrange-Hermite interpolation furnishes the most important general example of a specific univariate Hermite scheme. In the multivariate case it is generally difficult to check whether a given set of functionals H is a Hermite scheme, even when every $\mu_s \in H$ is a point-evaluation functional, $\mu_s(f) = f(u_s)$, which corresponds to ordinary Lagrange interpolation. Actually, the sole multivariate case for which the verification that H is a Hermite scheme is absolutely straightforward is obtained by taking $H = \{f \to D^\alpha(f)(a),\ |\alpha| \le d\}$. In that

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case the H-interpolation polynomial is simply the Taylor polynomial of f at a to the order d. Much work has been dedicated to the study of particular multivariate Hermite schemes. Noteworthy recent examples include the schemes introduced by Bojanov and Xu in [1] and further studied in [7] and [8]. Older results are referred to, for instance, in the work of Lorentz [9, 10] .

In this note we construct new Hermite schemes based on the following idea. We define local Taylor interpolation for the restrictions of an analytic function to a general irreducible algebraic hypersurface of $\mathbb{C}^n$. We then collect the corresponding interpolation conditions (which reflect the local behavior of the function restricted to the hypersurface) in a suitable manner to construct an n-dimensional Hermite scheme that we naturally call a multipoint Taylor interpolation. The same idea was previously used by Bos [5] to construct multivariate unisolvent arrays for Lagrange interpolation. A simple and probably well-known example of our multipoint Taylor interpolation scheme — for which the hypersurfaces are merely lines — is illustrated in Figure 1.

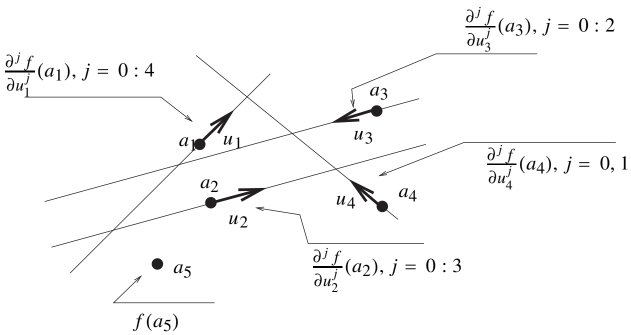

Fig. 1 A simple example of a bivariate multipoint Taylor interpolation scheme of degree 4

The paper is organized as follows. We first define and recall fundamental properties of the *least space* of a finite-dimensional space of analytic functions. It serves as a basic tool for defining the polynomial projectors that play the role of Taylor interpolation on hypersurfaces and provide the interpolation conditions which we want to consider. The definition and required properties of these projectors are given in §3. Our main theorem is then easily derived in the last section. We concentrate on the complex case but explain how to work with real hypersurfaces.

We use standard multi-index notation. In particular $|\alpha|$ denotes the length of $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $D^\alpha := \partial^{|\alpha|}/\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}$. The space of polynomials of degree at most d in n complex variables is denoted by $\mathbb{P}^d(\mathbb{C}^n)$. Recall that its dimension $N_d(n)$ is $\binom{n+d}{n}$. For $b \in \mathbb{C}^n$, a *b-homogeneous polynomial* of *b-degree d* is an element of $\mathbb{H}_b^d(\mathbb{C}^n) := \mathsf{span}\{\mathbb{C}^n \ni z \mapsto (z-b)^\alpha \ : \ |\alpha| = d\}$.

2 The least space

We collect a few basic facts on the least space of a finite-dimensional space of analytic functions. This space was introduced by de Boor and Ron and has proved to be very useful in multivariate interpolation theory; see, e.g., [2–4].

Let $n \geq 1, b \in \mathbb{C}^n$ and let F be a finite-dimensional vector space of analytic functions on a neighborhood of b. Any $f \in F$ can be uniquely expanded in a power series about b, that is, $f(z) = \sum_{|\alpha|=0}^{\infty} c_\alpha (z-b)^\alpha$. If f is nonzero, the order of f at b – or, for short, b-order –, denoted by $\mathrm{ord}_b(f)$, is the degree of the lowest non-vanishing b-homogeneous term of this series, that is,

$$\mathrm{ord}_b(f) = \min \left\{ k \ : \ \max_{|\alpha|=k} |c_\alpha| \neq 0 \right\}. \tag{1}$$

The corresponding term is called the *least (term)* of f at b and denoted by $f_{b,\downarrow}$; thus,

$$f_{b,\downarrow}(z) = \sum_{|\alpha|=\mathrm{ord}_b f} c_\alpha (z-b)^\alpha \in \mathbb{H}_b^{\mathrm{ord}_b f}(\mathbb{C}^n). \tag{2}$$

The least of the zero function at b is taken to be 0 and its b-order is $+\infty$.

The *least space* of F at b is the vector space defined by

$$F_{b,\downarrow} := \mathsf{span}\{f_{b,\downarrow} \ : \ f \in F\}. \tag{3}$$

Thus, $F_{b,\downarrow}$ is a space of polynomials spanned by b-homogeneous polynomials, precisely,

$$F_{b,\downarrow} = \bigoplus_{j \in J} F_{b,\downarrow} \cap \mathbb{H}_b^j(\mathbb{C}^n), \tag{4}$$

where $J = \{j \geq 0 \ : \ F_{b,\downarrow} \cap \mathbb{H}_b^j(\mathbb{C}^n) \neq \{0\}\}$.

By a theorem of de Boor and Ron [2, p. 291], F and $F_{b,\downarrow}$ have the same dimension,

$$\dim F_{b,\downarrow} = \dim F. \tag{5}$$

The following simple properties help in computing least terms.

Lemma 2.1 *Let $\lambda \in \mathbb{C}$, $f, g \in F$. Then:*

A. $(\lambda f)_\downarrow = \lambda f_\downarrow$ *and* $(fg)_{b,\downarrow} = f_{b,\downarrow} \cdot g_{b,\downarrow}$. *In particular, if $g(b) \neq 0$, then* $(fg)_{b,\downarrow} = g(b) f_{b,\downarrow}$.
B. $\mathrm{ord}_b(f) = \mathrm{ord}_b(g) \implies (f+g)_{b,\downarrow} = f_{b,\downarrow} + g_{b,\downarrow}$ *except if* $f_{b,\downarrow} = -g_{b,\downarrow}$.
C. *If h is a nonzero b-homogeneous element of $F_{b,\downarrow}$ then there exists $f \in F$ such that $h = f_{b,\downarrow}$.*

Proof For the third claim we observe that, by definition (and (A)), we have $h = \sum_{i\in I} h^i$ with $h^i = f^i_{b,\downarrow}$ and I a finite set of indices. Since h is a nonzero b-homogeneous polynomial, we may remove all the h^i's whose b-degrees are different from that of h (for their sum has to vanish) and therefore assume that $\operatorname{ord} f^i$ is constant for $i \in I$. In view of (B), we then have $h = (\sum_{i\in I} f^i)_{b,\downarrow}$. □

Lemma 2.2 *If ϕ is a local complex diffeomorphism, that is, an analytic function on a neighborhood of b such that $\phi'(b)$, the first (Fréchet) derivative of ϕ at b, is a linear automorphism of $\mathbb{C}^n$, then*

$$f_{\phi(b),\downarrow} \circ T_\phi = (f \circ \phi)_{b,\downarrow}, \tag{6}$$

where T_ϕ is the affine automorphism defined by $T_\phi(x) = \phi(b) + \phi'(b)(x - b)$.

It follows that if F is a space of analytic functions on a neighborhood of $\phi(b)$ then

$$(F \circ \phi)_{b,\downarrow} = F_{\phi(b),\downarrow} \circ T_\phi. \tag{7}$$

Proof Let $\nu = \operatorname{ord}_{\phi(b)}(f)$ and $h_\nu(z - \phi(b)) = f_{\phi(b),\downarrow}(z)$ with $h_\nu \in \mathbb{H}_0^\nu(\mathbb{C}^n)$ so that

$$f(\phi(z)) = h_\nu(\phi(z) - \phi(b)) + \sum_{k>\nu} h_k(\phi(z) - \phi(b)), \quad h_k \in \mathbb{H}_0^k(\mathbb{C}^n).$$

Since $\phi - \phi(b)$ vanishes at b, the b-order of the second term is not smaller than $\nu + 1$. On the other hand, since

$$\phi(z) - \phi(b) = \phi'(b)(z - b) + (\text{terms of higher } b\text{-order}),$$

we have

$$(f \circ \phi)(z) = h_\nu(\phi'(b)(z - b)) + (\text{terms of } b\text{-order not smaller that } \nu + 1).$$

Now, since $\phi'(b)$ is a linear automorphism, the b-order of $h_\nu(\phi'(b)(z - b))$ is equal to that of h_ν. It follows that $h_\nu(\phi'(b)(z - b))$ is the least term of $f \circ \phi$ at b. □

3 Taylor projectors on an algebraic hypersurface

3.1 Algebraic hypersurfaces

Let q be a (nonzero) polynomial in $n + 1$ complex variables and $V = V(q)$ the complex algebraic hypersurface $\{q = 0\}$ in $\mathbb{C}^{n+1}$. We may always assume that q is square-free (that is, not divisible by the square of a non-constant polynomial). We denote by $\mathbb{P}(V)$ the ring of polynomial functions on V and by $\mathbb{P}^d(V)$ the subspace of polynomial functions on V of degree at most d,

$$\mathbb{P}^d(V) = \{p_{|V} \; : \; p \in \mathbb{P}^d(\mathbb{C}^{n+1})\}. \tag{8}$$

Use of the Nullstellensatz shows that, when q is square-free, the kernel of the linear map

$$\mathbb{P}^d(\mathbb{C}^{n+1}) \ni p \mapsto p_{|V} \in \mathbb{P}^d(V)$$

is $q\mathbb{P}^{d-r}(\mathbb{C}^{n+1})$, where $r = \deg q$; this readily implies the following well-known result.

Lemma 3.1 *If q is a nonzero square-free polynomial of degree $r \geq 1$ and $V = V(q)$ then the dimension $N_d(V)$ of $\mathbb{P}^d(V)$ is given by*

$$N_d(V) = N_d(n+1) - N_{d-r}(n+1) \tag{9}$$

with the convention that $N_{d-r}(n+1) = 0$ when $d < r$.

We denote by V^0 the set of regular (smooth) points of $V(q)$, that is, the set of points $a \in V(q)$ such that $q'(a)$ is not the zero linear form (not all the first partial derivatives of q at a are equal to 0). We recall that V^0 is canonically endowed with the structure of a complex analytic variety. The inverse of a coordinate mapping of V^0 is called a *local parametrization*. Here is a precise definition.

Definition 3.1 A *local parametrization* of V (and of V^0) at $a \in V^0$ is a 3-tuple $\mathcal{L} = (b, W, R)$, where $b \in \mathbb{C}^n$, W is an open connected neighborhood of b in $\mathbb{C}^n$ and $R : W \to \mathbb{C}^{n+1}$ an analytic function such that $R(b) = a$, $R(W) \subset V^0$ and R is an homeomorphism from W to $R(W)$.

Local parametrizations are furnished by an application of the implicit function theorem. Note that the function R above satisfies $\mathsf{rank}(R'(b)) = n$.

3.2 Local differential operators

We now define a class of operators acting on analytic functions on a neighborhood (in $\mathbb{C}^{n+1}$) of a point of V^0. These operators are to play the role of the usual partial derivatives in ordinary multivariate Taylor interpolation.

Given a local parametrization $\mathcal{L} = (b, W, R)$ of V at a, we consider the space of functions on W induced by the polynomials of degree at most d on V,

$$\mathbb{P}^d_{\mathcal{L}} := \mathbb{P}^d(\mathbb{C}^{n+1}) \circ R \;\; (= \mathbb{P}^d(V) \circ R). \tag{10}$$

This is a finite-dimensional space of analytic functions on a neighborhood of b in $\mathbb{C}^n$ and we may therefore consider its least space at b,

$$\mathbb{P}^d_{\mathcal{L}\downarrow} \subset \mathbb{P}(\mathbb{C}^n).$$

(We omit the subscript b in the notation of the least space as it is implied by $\mathcal{L}$.) The degree of a polynomial in $\mathbb{P}^d_{\mathcal{L}\downarrow}$ may be (and usually is) greater than d; see the examples in § 3.4.

To every polynomial Q in this least space we associate a (local) differential operator $\overline{Q}_{\mathcal{L}}(D)$ defined on the space of analytic functions on a neighborhood of a by

$$\overline{Q}_{\mathcal{L}}(D)(f) := \overline{Q}(D)(f \circ R), \tag{11}$$

where the right-hand term is

$$\sum_{\alpha} \overline{a}_{\alpha}\, D^{\alpha}(f \circ R)(b) \quad \text{if } Q(z) = \sum_{\alpha} a_{\alpha}(z-b)^{\alpha}.$$

Definition 3.2 An operator $\overline{Q}_{\mathcal{L}}(D)$ with $Q \in \mathbb{P}^d_{\mathcal{L}\downarrow}$ is called an *$\mathcal{L}$-differential operator* at a and the linear space spanned by these operators is denoted by $\mathsf{Dif}(\mathcal{L}, d)$.

Since the spaces $\mathbb{P}^d_{\mathcal{L}\downarrow}$ increase with d, we have

$$d_1 < d_2 \implies \mathsf{Dif}(\mathcal{L}, d_1) \subset \mathsf{Dif}(\mathcal{L}, d_2). \tag{12}$$

Note also that $\mathsf{Dif}(\mathcal{L}, 0) = \{f \to \lambda f(a),\ \lambda \in \mathbb{C}\}$.

The restriction of an $\mathcal{L}$-local differential operator to $\mathbb{P}_d(V)$ is a linear form on $\mathbb{P}_d(V)$, that is, an element of its dual $[\mathbb{P}^d(V)]^{\star}$.

Lemma 3.2 *If the hypersurface V is irreducible (defined by an irreducible polynomial q), then the map*

$$Q \in \mathbb{P}^d_{\mathcal{L}\downarrow} \longrightarrow \overline{Q}_{\mathcal{L}}(D) \in [\mathbb{P}^d(V)]^{\star} \tag{13}$$

is a linear isomorphism. Equivalently,

$$\mathsf{Dif}(\mathcal{L}, d)_{|\mathbb{P}_d(V)} = [\mathbb{P}^d(V)]^{\star}. \tag{14}$$

Moreover,

$$\dim \mathsf{Dif}(\mathcal{L}, d) = N_d(V). \tag{15}$$

Proof We first show that both spaces have the same dimension. By (5), we just need to verify that $\dim \mathbb{P}^d_{\mathcal{L}} = \dim \mathbb{P}^d(V)$ or, equivalently, that the linear map $p \in \mathbb{P}_d(V) \to p \circ R \in \mathbb{P}^d_{\mathcal{L}}$ is one-to-one. This can be seen as follows. Since q is irreducible the complex variety V^0 is connected [12, Chap. II, §8] and we may apply the theorem of uniqueness of analytic continuation [12, Chap. II, §5]. Thus, if $p \circ R = 0$ on W, p vanishes on an open set of V^0, hence on the whole of V^0. Since the set of singular points are nowhere dense in V [12, Chap. II, §8], the continuity of p implies that $p = 0$ on V.

Now to prove the lemma, it suffices to show that the map (13) is one-to-one. We assume that for some nonzero $Q \in \mathbb{P}^d_{\mathcal{L}\downarrow}$ we have

$$\overline{Q}(D)(p \circ R) = 0, \quad p \in \mathbb{P}^d(V), \tag{16}$$

and we look for a contradiction. Let

$$Q^+(z) = \sum_{|\alpha|=k} a_\alpha (z-b)^\alpha$$

be the b-homogeneous part of Q of b-degree k. By (4) this is an element of $\mathbb{P}^d_{\mathcal{L},\downarrow}$ and, in view of Lemma 2.1 (C), there exists $T \in \mathbb{P}^d(V)$ such that $Q^+ = (T \circ R)_\downarrow$. Applying (16) with $p = T$, we obtain

$$\begin{aligned} 0 &= \overline{Q}(D)(T \circ R) = \overline{Q}(D)\big(Q^+ + \text{(terms of higher } b\text{-order)}\big) \\ &= \overline{Q^+}(D)(Q^+) = \sum_{|\alpha|=k} \alpha!|a_\alpha|^2. \end{aligned}$$

Hence, all the coefficients of Q^+ are equal to zero and this contradicts $Q \neq 0$. This concludes the proof of (13). Next, using the definition for the first inequality and (13) for the third one, we have

$$\begin{aligned} \dim \mathsf{Dif}(\mathcal{L}, d) \le \dim \mathbb{P}^d_{\mathcal{L}\downarrow} &= N_d(V) \\ &= \dim(\mathsf{Dif}(\mathcal{L}, d)_{|\mathbb{P}_d(V)}) \le \dim \mathsf{Dif}(\mathcal{L}, d), \end{aligned} \tag{17}$$

from which we deduce (15). □

Examples of $\mathsf{Dif}(\mathcal{L}, d)$ spaces are given in § 3.4.

3.3 $\mathcal{L}$-Taylor interpolation

Lemma 3.2 and elementary linear algebra now lead to the definition of our $\mathcal{L}$-Taylor interpolation polynomials.

Theorem 3.1 *Let q an irreducible non-constant polynomial and $V = V(q)$. If a is a smooth point of V and $\mathcal{L}$ a local parametrization of V at a then, for every analytic function f on a neighborhood (in $\mathbb{C}^n$) of a, there exists a unique polynomial p in $\mathbb{P}^d(V)$ such that*

$$(f - p) \perp \mathsf{Dif}(\mathcal{L}, d), \tag{18}$$

that is, $\mu(f) = \mu(p)$ for every $\mu \in \mathsf{Dif}(\mathcal{L}, d)$.

This polynomial is called the *$\mathcal{L}$-Taylor interpolation polynomial* of f at a to the order d and denoted by $\mathbf{T}^d_{\mathcal{L}}(f)$. Here is short list of immediate properties.

A. The Taylor polynomial of order 0 at a is just the constant polynomial $f(a)$.
B. For every $p \in \mathbb{P}^d(\mathbb{C}^{n+1})$, $\mathbf{T}^d_{\mathcal{L}}(p) = p_{|V}$.
C. $\mathcal{L}$-Taylor projectors are invariant with respect to affine automorphisms.

The latter statement is to be understood as follows. Let A be an affine automorphism of $\mathbb{C}^{n+1}$ and $V_A = A(V) = \{q \circ A^{-1} = 0\}$, where $V = \{q = 0\}$. If $\mathcal{L} = (b, W, R)$ is a local parametrization of V at a then $\mathcal{L}_A = (b, w, A \circ R)$ is a local parametrization of V_A at $A(a)$. Since A is an automorphism we have $\mathbb{P}^d(\mathbb{C}^{n+1}) = \mathbb{P}^d(\mathbb{C}^{n+1}) \circ A$ and hence $\mathbb{P}^d_{\mathcal{L}} = \mathbb{P}^d_{\mathcal{L}_A}$ and

$$\mathbb{P}^d_{\mathcal{L},\downarrow} = \mathbb{P}^d_{\mathcal{L}_A,\downarrow},$$

from which we readily deduce

$$\mathsf{Dif}(\mathcal{L}_A, d) = A * \mathsf{Dif}(\mathcal{L}, d) \quad \text{and} \quad \mathbf{T}^d_{\mathcal{L}_A} = A * \mathbf{T}^d_{\mathcal{L}}, \tag{19}$$

where the notation $A * \Xi$ is used for the operator defined by $(A * \Xi)(f) = \Xi(f \circ A)$, f being an analytic function on a neighborhood of $A(a) \in V_A$.

As emphasized by the notation, $\mathbf{T}^d_{\mathcal{L}}$ in general depends on $\mathcal{L}$. However it is easy to go from one parametrization to another. If $\mathcal{L}_1 = (b_1, W_1, R_1)$ and $\mathcal{L}_2 = (b_2, W_2, R_2)$ are two local parametrizations of V at $a \in V^0$ then $\phi = R_1^{-1} \circ R_2$ is a complex diffeomorphism from a neighborhood of b_2 onto a neighborhood of b_1 and the spaces $\mathbb{P}^d_{\mathcal{L}_1,\downarrow}$ and $\mathbb{P}^d_{\mathcal{L}_2,\downarrow}$ are related by Eq. (7). Namely, in the notation of (7), if $\phi = R_2^{-1} \circ R_1$ then

$$\mathbb{P}^d_{\mathcal{L}_1} = \mathbb{P}^d_{\mathcal{L}_2} \circ \phi \quad \text{and} \quad \mathbb{P}^d_{\mathcal{L}_1,\downarrow} = \mathbb{P}^d_{\mathcal{L}_2,\downarrow} \circ T_\phi. \tag{20}$$

In particular, the integer

$$\max\{\deg p \,:\, p \in \mathbb{P}^d_{\mathcal{L}\downarrow}\}$$

does not depend on $\mathcal{L}$. This integer gives the order of the highest derivatives of the function f needed in order to compute $\mu(f)$ for every $\mu \in \mathsf{Dif}(\mathcal{L}, d)$. We call it the *d-order* of V at a and denote it by $\mathcal{O}(a, d, V)$.

Note that, while the relation between the least spaces only involves the first derivatives of ϕ, the relation between $\mathsf{Dif}(\mathcal{L}_1, d)$ and $\mathsf{Dif}(\mathcal{L}_2, d)$ generally depends on the first $\mathcal{O}(a, d, V)$ derivatives of ϕ. However, in some particular but interesting cases (see below), the $\mathcal{L}$-Taylor polynomials are intrinsic, that is to say, independent of the particular parametrization we use in their definition.

An algebraic formulation of the classical Leibniz formula for the computation of the derivatives of a product of two functions reads, in terms of Taylor polynomials, as follows

$$\mathbf{T}^d_a(fg) \equiv \mathbf{T}^d_a(f) \cdot \mathbf{T}^d_a(g) \quad \mod (z-a)^\alpha, \quad |\alpha| = d + 1.$$

As a (weak) consequence, if f is a polynomial of degree at most d, $g(a) \neq 0$ and $\mathbf{T}^d_a(fg) = 0$, then $f = 0$. We prove a similar result for our $\mathcal{L}$-Taylor projectors.

Lemma 3.3 (Weak Leibniz formula) *Let V be an irreducible algebraic hypersurface, $a \in V^0$ and $\mathcal{L} = (b, W, R)$ a local parametrization of V at a. If $p \in \mathbb{P}^d(\mathbb{C}^{n+1})$ and g is an analytic function on a neighborhood of a then $\mathbf{T}^d_{\mathcal{L}}(pg) = 0$ and $g(a) \neq 0$ imply that $p = 0$ on V.*

Proof To say that $\mathbf{T}^d_{\mathcal{L}}(pg) = 0$ means that $\overline{Q}_{\mathcal{L}}(D)(pg) = 0$ for every $\overline{Q}_{\mathcal{L}}(D) \in \mathsf{Dif}(\mathcal{L}, d)$, or equivalently,

$$\overline{Q}(D)(pg \circ R) = 0, \quad Q \in \mathbb{P}^d_{\mathcal{L},\downarrow}. \tag{21}$$

On the other hand, in order to prove that $p = 0$ on V, it is enough to show that $(p \circ R) = 0$ on W (see the proof of Lemma 3.2) for which it suffices to show that $(p \circ R)_{b,\downarrow} = 0$. We use (21) with

$$Q = (p \circ R)_{b,\downarrow} \in \mathbb{P}^d_{\mathcal{L}}.$$

Recall that, by assumption, $\deg p \leq d$. Then, using $g(a) \neq 0$ and Lemma 2.1 (A) (on the third line), we have

$$\begin{aligned}
0 &= \overline{Q}(D)(pg \circ R) \\
&= \overline{Q}(D)\big((pg \circ R)_{b,\downarrow} + (\text{terms of higher } b\text{-order})\big) \\
&= \overline{Q}(D)\big(g(a)(p \circ R)_{b,\downarrow} + (\text{terms of higher } b\text{-order})\big) \\
&= g(a)\overline{Q}(D)\big((p \circ R)_{b,\downarrow} + (\text{terms of higher } b\text{-order})\big) \\
&= g(a)\overline{Q}(D)\big((p \circ R)_{b,\downarrow}\big) \text{ (since } Q \in \mathbb{H}_b \text{ and } \deg Q = \deg(p \circ R)_{b,\downarrow}) \\
&= g(a)\overline{Q}(D)\big(Q\big).
\end{aligned}$$

Using $g(a) \neq 0$ again, we deduce from the last equation (see the proof of Lemma 3.2) that $Q = (p \circ R)_{b,\downarrow} = 0$, as was to be proved.

3.4 Examples

Hyperplanes. Let q be a linear polynomial of $n+1$ complex variables, $q(z) = \alpha \cdot z - b$, where $\alpha \neq 0$ and $u \cdot v = \sum_{i=1}^{n+1} u_i v_i$, and V the hyperplane $V(q)$. Every point a of V is regular and a standard (global) parametrization at a is $\mathcal{L} = (0, \mathbb{C}^n, R)$ with $R(z_1, \ldots, z_n) = a + \sum_{i=1}^{n} z_i v_i$, where $v = (v_1, \ldots, v_n)$ is a basis of the subspace $\{\alpha \cdot x = 0\}$. We have $\mathbb{P}^d_{\mathcal{L}} = \mathbb{P}^d_{\mathcal{L}\downarrow} = \mathbb{P}^d(\mathbb{C}^n)$, and

$$\mathsf{Dif}(\mathcal{L}, d) = \mathsf{span}\left\{f \to (\partial^{|\alpha|} f / \partial v^\alpha)(a) \,:\, |\alpha| \leq d\right\}, \tag{22}$$

where $\partial^{|\alpha|} f / \partial v^\alpha = \partial^{|\alpha|} f / \partial v_1^{\alpha_1} \partial v_2^{\alpha_2} \ldots \partial v_n^{\alpha_n}$ and $\partial f / \partial v_i$ is the usual derivative along the vector v_i. We therefore have $\mathcal{O}(a, d, V) = d$ as in the classical case. In fact,

$$\mathbf{T}^d_{\mathcal{L}}(f) = \mathbf{T}^d_a(f_{|V}),$$

where $\mathbf{T}^d_a(f_{|V})$ is to be understood as the classical Taylor polynomial at a to the order d of the restriction of f to the n-dimensional affine space V.

First-order local derivatives. Let $q \in \mathbb{P}(\mathbb{C}^{n+1})$ be an irreducible polynomial of degree $d \geq 2$, $V = V(q)$ and $a \in V^0$. We let $\mathcal{L} = (b, W, R)$ denote any local parametrization of V at a. Since $N_1(V) = n + 2$, by Lemma 3.2, $\mathsf{Dif}(\mathcal{L}, 1)$ is spanned by $n+2$ linearly independent operators. These are easily described. Let $R = (R_1, \ldots, R_{n+1})$ and let $\nabla R_i(b)$ be the gradient of R_i at b so that

$$R'(b)(h) = (\nabla R_1(b) \cdot h, \ldots, \nabla R_{n+1}(b) \cdot h), \qquad h \in \mathbb{C}^n.$$

Since $\mathsf{rank}(R'(b)) = n$, the space $\{\lambda \in \mathbb{C}^{n+1} : \sum_{i=1}^{n+1} \lambda_i \nabla R_i'(b) = 0\}$ is of dimension 1 and therefore spanned by a nonzero $n + 1$ tuple μ. With this notation, we have

$$\mathbb{P}^1_{\mathcal{L},\downarrow} = \mathsf{span}\left\{1, z_1, \ldots, z_n, \sum_{i=1}^{n+1} \mu_i R_i^{(k)}(b)(z-b, \ldots, z-b)\right\}, \tag{23}$$

where $R_i^{(k)}(b)$ denote the kth (total) derivative of R_i at b and $k \geq 2$ is the smallest integer such that $\sum \mu_i R_i^{(k)}(b)(z-b, \ldots, z-b)$ does not vanish. Since the range of $R'(b)$ coincides with the kernel of $q'(a)$, it is not difficult to see that the operators of $\mathsf{Dif}(\mathcal{L}, 1)$ produced by the linear (non-affine) polynomials which are members of the space (23) are the *tangential derivatives*

$$f \to f'(a)(t), \quad t \in \ker q'(a).$$

It is important to note that, as soon as $\deg q > 1$, $\mathcal{O}(a, 1, V(q)) > 1$.

The curve $V = \{w = z^m\}$, $m \geq 2$. Using the canonical parametrization at $a = (0, 0) \in V^0$, $\mathcal{L} = (0, \mathbb{C}, z \to (z, z^m))$, we find $\mathbb{P}^d_{\mathcal{L}} = \mathbb{P}^d_{\mathcal{L}\downarrow} = \mathsf{span}\left\{z^{\alpha_1 + m\alpha_2} : \alpha_1 + \alpha_2 \leq d\right\}$ and

$$\mathsf{Dif}(\mathcal{L}, d) = \mathsf{span}\left\{f \to \frac{d^k}{dz^k} f(z, z^m)\Big|_{z=0} : k = \alpha_1 + m\alpha_2,\ \alpha_1 + \alpha_2 \leq d\right\}. \tag{24}$$

In particular $\mathcal{O}(0, d, V) = md$. We note the presence of gaps in the basis of $\mathbb{P}^d_{\mathcal{L},\downarrow}$ (and in the corresponding basis of operators) as soon as $m > 2$ for every $d > m$. A precise description of the space $\mathsf{span}\left\{z^{\alpha_1 + m\alpha_2} : \alpha_1 + \alpha_2 \leq d\right\}$ is given in [6, Prop. 1].

Irreducible quadratic curves. Let $V = \{q = 0\}$ be a quadratic irreducible curve in $\mathbb{C}^2$. All points of such a curve are smooth. We describe $\mathbb{P}^d_{\mathcal{L}}$ (hence $\mathsf{Dif}(d, \mathcal{L})$), where $\mathcal{L}$ is any local parametrization of V at a. First, in view of the invariance property (3.3 (C)), we may restrict ourselves to the cases for which $q(z, w) = w - z^2$ or $q(z, w) = z^2 + w^2 - 1$. We only give the calculations in the second case.

If $a = (\alpha, \beta)$ we take $\mathcal{L} = (b, W, R)$, where $R(z) = (\cos z, \sin z)$, b, is such that $\exp(ib) = \alpha + i\beta$ (and $\exp(-ib) = \alpha - i\beta$) since $1 = \alpha^2 + \beta^2 = (\alpha + i\beta)(\alpha - i\beta)$) and W is a sufficiently small disc centered at b. We claim that

$$\mathbb{P}^d_{\mathcal{L}} = \mathsf{span}\{1, (z - b), \cdots, (z - b)^{2d}\} = \mathbb{P}^{2d}(\mathbb{C}).$$

Since $\mathbb{P}^d_{\mathcal{L}}$ is known to be spanned by $\dim \mathbb{P}^d(V) = 2d + 1$ monomials, the assertion is proved if we establish that, for $k \geq 1$,

$$(z - b)^k \in \mathbb{P}^d_{\mathcal{L}} \implies (z - b)^{k-1} \in \mathbb{P}^d_{\mathcal{L}}.$$

But, if $(z - b)^k \in \mathbb{P}^d_{\mathcal{L}}$, one can find $p \in \mathbb{P}^d$, $p(x, y) = \sum_{i+j \leq d} c_{ij} x^i y^j$, such that

$$(z - b)^k + (\text{terms of higher } b\text{-degree}) = p(\cos z, \sin z).$$

Differentiating both sides, we get

$$\begin{aligned} &k(z - b)^{k-1} + (\text{terms of higher } b\text{-order}) \\ &\qquad = \sum_{i+j \leq d} \left\{-i c_{ij} \cos^{i-1} z \sin^{j+1} z + j c_{ij} \cos^{i+1} z \sin^{j-1} z\right\}. \quad (25) \end{aligned}$$

Since the left-hand side is still a polynomial of degree at most d in $(\cos z, \sin z)$, it follows that $(z - b)^{k-1} \in \mathbb{P}^d_{\mathcal{L}}$.

Now, if $\mathcal{L}'(c, X, S)$ is another parametrization of V at a, in view of (20), for the affine automorphism T_ϕ of $\mathbb{C}$, $\phi = R^{-1} \circ S$, we have

$$\mathbb{P}^d_{\mathcal{L}',\downarrow} = \mathbb{P}^d_{\mathcal{L},\downarrow} \circ T_\phi = \mathbb{P}^{2d}(\mathbb{C}) \circ T_\phi = \mathbb{P}^{2d}(\mathbb{C}).$$

Next, let $Q(z) = z^k$, $1 \leq k \leq 2d$, and let f be analytic on a neighborhood of (α, β); use of the (ordinary) Leibniz formula yields

$$\overline{Q}_{\mathcal{L}'}(D)(f) = \overline{Q}(D)(f \circ S) = \overline{Q}(D)(f \circ R \circ \phi) = \sum_{i=1}^{k} \lambda_i \frac{d^i}{dz^i}(f \circ R)\Big|_{z=b},$$

where the λ_i's only depend on the derivatives of ϕ at c. This shows that $\overline{Q}_{\mathcal{L}'}(D) \in \mathsf{Dif}(d, \mathcal{L})$ and, since both spaces have the same dimension, it follows that $\mathsf{Dif}(d, \mathcal{L}') = \mathsf{Dif}(d, \mathcal{L})$. Summing up we have proved the following result.

Theorem 3.2 *Let V be an irreducible quadratic hypersurface in $\mathbb{C}^2$. For every $a \in V$ and every $d \geq 0$, the Taylor projector $T^d_{\mathcal{L}}$ depends only on a, V and d, that is to say, does not depend on the particular parametrization $\mathcal{L}$ used for constructing it.*

One may check that all the examples given in this section satisfy the same property except the curve $y = x^m$ for $m > 2$ when $a = (0, 0)$. This property, which is equivalent in the bivariate case to

$$\mathbb{P}^d_{\mathcal{L},\downarrow} = \mathbb{P}^{N_d(V)-1}(\mathbb{C}),$$

leads to other interesting results. We hope to return to this question in a further paper.

3.5 Real case

For practical applications the real case is more interesting. It requires only a few other observations. First, by "real case" we mean that we consider $q \in \mathbb{P}(\mathbb{R}^{n+1})$, $\deg q \geq 1$ and

$$V_{\mathbb{R}} = V_{\mathbb{R}}(q) = \{x \in \mathbb{R}^{n+1} \,:\, q(x) = 0\}.$$

Lemma 3.1 remains true (for the space $\mathbb{P}^d(V_{\mathbb{R}})$ of real polynomials on $V_{\mathbb{R}}$) provided that q is irreducible and V^0 contains at least one point, since the divisibility property used in the proof extends to this case [11, Lemma 2.5, p. 14]. Here "irreducible" is to be understood as irreducible over $\mathbb{C}$ rather than over $\mathbb{R}$. Indeed, if q is irreducible over $\mathbb{R}$ but not over $\mathbb{C}$, then there exists a non-trivial polynomial $r \in \mathbb{P}(\mathbb{C}^{n+1})$ such that $q = r\overline{r}$, where $\overline{r}(z) = \sum \overline{a}_\alpha z^\alpha$ if $r(z) = \sum a_\alpha z^\alpha$. We then have $V_{\mathbb{R}}(q) \subset V_{\mathbb{C}}(p) \cap V_{\mathbb{C}}(\overline{p})$ and we readily check by differentiating $q = r\overline{r}$ that $V_{\mathbb{R}}(q)$ contains no regular point, which is contrary to assumption.

In the proof of Lemma 3.2 we needed the facts that $V^0(q)$ is connected and everywhere dense in $V(q)$. Neither property remains true in the real case. For example, if $q(x, y) = x^3 - x^2 + y^2$ then $V^0_{\mathbb{R}}(q) = V_{\mathbb{R}}(q) \setminus \{(0, 0)\}$ and has three distinct components. On the other hand, if $q(x, y) = x^3 + x^2 + y^2$ then $V^0_{\mathbb{R}}(q)$ is connected but $(0, 0)$ is an isolated (singular) point of $V_{\mathbb{R}}(q)$ so that $V^0_{\mathbb{R}}(q)$ fails to be everywhere dense in $V_{\mathbb{R}}(q)$. To circumvent this difficulty, we use the probably well-known Lemma 3.4 below.

Note that, as soon as it is non-empty, $V^0_{\mathbb{R}}(q)$ is a real analytic variety of dimension n (and so are its connected components), (see [11, Theorem 2.3, p. 11]) and, substituting the word "analytic" by "real analytic" in Definition 3.1, we obtain the definition of a local (real analytic) parametrization of $V_{\mathbb{R}}$ at $a \in V^0_{\mathbb{R}}$.

Lemma 3.4 *Let q be a real non-constant and irreducible polynomial such that $V^0_{\mathbb{R}}(q)$ contains at least one point. If $p \in \mathbb{P}(\mathbb{R}^{n+1})$ is equal to zero on a connected component of $V^0_{\mathbb{R}}(q)$ then it must also be equal to* 0 *on the whole of $V_{\mathbb{R}}(q)$.*

Proof Let a be any point in the component C of $V^0_{\mathbb{R}}(q)$ on which p is known to vanish. We also have $a \in V^0_{\mathbb{C}}(q)$. We assume, without loss of generality, that $0 \neq \partial p(a)/\partial x_{n+1} (= \partial p(a)/\partial z_{n+1})$. We let x' denote $(x_1, \ldots, x_n)$ so that $x = (x', x_{n+1})$. The implicit function theorem furnishes a real analytic local parametrization $R_{\mathbb{R}} : x' \to (x', \rho_{\mathbb{R}}(x'))$ of C at a and also a complex analytic local parametrization $R_{\mathbb{C}} : z' \to (z', \rho_{\mathbb{C}}(z'))$ of $V^0_{\mathbb{C}}(q)$ at a defined on a real, respectively complex, neighborhood of a'. Since the coefficients of the power series expansion of both $\rho_{\mathbb{R}}$ and $\rho_{\mathbb{C}}$ depend only on the coefficients of q which are real, on a real neighborhood $\mathcal{U}$ of a', we must have

$$R_{\mathbb{C}|\mathbb{R}} = R_{\mathbb{R}}. \tag{26}$$

From $p = 0$ on C we get $p \circ R_{\mathbb{R}} = 0$ on $\mathcal{U}$ and hence all the (real) derivatives of $p \circ R_{\mathbb{R}}$ at a' vanish. But, since p is a real polynomial, in view of (26), the real derivatives of $p \circ R_{\mathbb{R}}$ at a' are equal to the complex derivatives of $p \circ R_{\mathbb{C}}$ at a'. Now, the usual uniqueness theorem for complex analytic functions gives $p \circ R_{\mathbb{C}} = 0$ on a complex open neighborhood of a'; hence $p = 0$ on an open neighborhood of a in $V^0_{\mathbb{C}}(q)$. Since q is irreducible over $\mathbb{C}$, $V^0_{\mathbb{C}}(q)$ is connected and it follows that $p = 0$ on $V^0_{\mathbb{C}}(q)$, hence also on its closure $V_{\mathbb{C}}(q)$, a fortiori, on its subset $V_{\mathbb{R}}(q)$. □

The other properties we use in the complex case remain valid when we work with real analytic functions and the modification of the definitions are straightforward. For example, we change $\overline{Q}$ to Q in Definition 3.2 to obtain $\mathsf{Dif}_{\mathbb{R}}(\mathcal{L}, d)$. From Lemma 3.4, we readily obtain a real version of Lemma 3.2 and the real version of Theorem 3.1 is as follows.

Theorem 3.3 *Let q be a non-constant irreducible real polynomial such that $V^0_{\mathbb{R}}(q)$ is non-empty. Further, let $a \in V^0_{\mathbb{R}}$ and $\mathcal{L}$ be a local parametrization of $V_{\mathbb{R}}$ at a. For every function f differentiable on a neighborhood of a, there exists a unique polynomial p in $\mathbb{P}^d(V_{\mathbb{R}})$ such that $(f - p) \perp \mathsf{Dif}_{\mathbb{R}}(\mathcal{L}, d)$.*

In the rest of the paper, apart from the illustrations, we no longer consider the real case. With the same kind of modifications as above all the results remain valid.

4 Multipoint Taylor interpolation

4.1 The main theorem

As explained in the introduction, the idea is to collect sufficiently many interpolation conditions in spaces $\mathsf{Dif}(\mathcal{L}, d)$ in order to obtain an n-dimensional Hermite scheme.

Theorem 4.1 *Let $n \geq 1$, $m \geq 2$ and, for $i = 1, 2, \ldots, m$, let V_i be the hypersurface $\{q_i = 0\}$, where q_i is an irreducible polynomial of degree $r_i \geq 1$ in $\mathbb{P}(\mathbb{C}^{n+1})$. Let $A = \{a_i, i = 1, \ldots m\}$ be a set of m pairwise distinct points such that $a_i \in V_i^0$. Each a_i is associated to a local parametrization $\mathcal{L}_i$ of V_i. We assume that $a_i \notin V_j$ for $j < i$.*

Let $d \in \mathbb{N}$ be such that

$$r_1 + r_2 + \cdots + r_{m-1} < d \leq r_1 + r_2 + \cdots + r_{m-1} + r_m. \tag{27}$$

We define the integers s_i by the relation

$$\begin{cases} s_1 = d \\ s_i = d - r_1 - r_2 - \cdots - r_{i-1} \ (i = 2, \ldots, m). \end{cases} \tag{28}$$

A) *In the case where*

$$r_1 + r_2 + \cdots + r_{m-1} < d < r_1 + r_2 + \cdots + r_{m-1} + r_m, \tag{29}$$

for every suitably defined function f, there exists a unique polynomial $p \in \mathbb{P}^d(\mathbb{C}^{n+1})$ such that

$$T^{s_i}_{\mathcal{L}_i}(f - p) = 0 \quad (i = 1, \ldots, m). \tag{30}$$

B) *In the case where*

$$r_1 + r_2 + \cdots + r_{m-1} < d = r_1 + r_2 + \cdots + r_{m-1} + r_m, \tag{31}$$

if a is any supplementary point outside the V_i's ($i = 1, \ldots, m$), then we may conclude that, for every suitably defined function f, there exists a unique polynomial $p \in \mathbb{P}^d(\mathbb{C}^{n+1})$ such that $p(a) = f(a)$ and (30) holds true.

We give some very simple bivariate examples prior to the proof.

4.2 Examples

We take $n = 2$ and $d = 4$ so that $N_d(n) = 15$. For $i = 1, \cdots, 4$, V_i is a line of direction u_i which passes through a_i such that $a_i \notin V_j$ for $j < i$. We assume that the four lines are pairwise distinct. The conditions associated to the computation of $\mathbf{T}^s_{\mathcal{L}}$ are given in § 3.4. The following table indicates the various parameters involved in the theorem,

i	r_i	s_i	conditions for $T^{s_i}_{\mathcal{L}}$
1	1	4	5 conditions : $\frac{\partial^j f}{\partial^j u_1}(a_1)$, $j = 0 \cdots 4$
2	1	3	4 conditions : $\frac{\partial^j f}{\partial^j u_2}(a_2)$, $j = 0 \cdots 3$
3	1	2	3 conditions : $\frac{\partial^j f}{\partial^j u_3}(a_3)$, $j = 0 \cdots 2$
4	1	1	2 conditions : $\frac{\partial^j f}{\partial^j u_4}(a_4)$, $j = 0, 1$
$\sum_{i=1}^{4} r_i = d \rightarrow$			1 extra condition : $f(a_5)$
			A total of $15 = N_2(4)$ conditions.

The scheme given in Figure 1 corresponds to the (real version) of this table. Note that the four lines need not be taken in general position. In Figure 1, u_2 and u_3 are collinear. We work with $n = 2$ and $d = 5$ so that $N_d(n) = 21$ and, for $i = 1, 2, 3$, V_i is an irreducible quadratic curve passing through a_i with $a_i \notin V_j$ for $j < i$. The conditions associated to the computation of $\mathbf{T}^s_{\mathcal{L}}$ are given in § 3.4. The corresponding table is as follows.

i	r_i	s_i	conditions for $T^{s_i}_{\mathcal{L}}$
1	2	5	$2 \times 5 + 1 = 11$ conditions
2	2	3	$2 \times 3 + 1 = 7$ conditions
3	2	1	$2 \times 1 + 1 = 3$ conditions
$\sum_{i=1}^{4} r_i > d \rightarrow$			no extra condition
			A total of $21 = N_2(5)$ conditions.

An example of a scheme corresponding to (the real version of) this table is given in Figure 2.

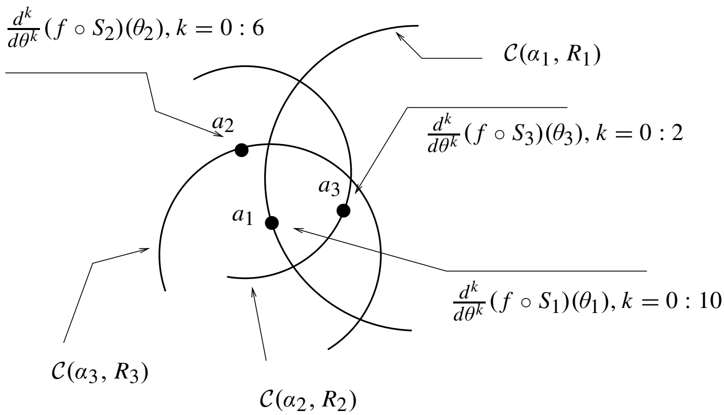

Fig. 2 An example of a bivariate multipoint Taylor interpolation scheme of degree 5. Here, $\mathcal{C}(\alpha_i, R_i)$ denotes the circle of radius R_i and center α_i, $a_i = \alpha_i + R_i(\cos\theta_i, \sin\theta_i)$ and $S_i(\theta) = \alpha_i + R_i(\cos\theta, \sin\theta)$.

4.3 Proof of Theorem 4.1

We first compute the number N of conditions imposed on p by the requirement (30). Since $T^{s_i}_{\mathcal{L}_i}(f - p) = 0$ gives $N_{s_i}(V_i)$ conditions, we have

$$\begin{aligned} N &= \textstyle\sum_{i=1}^m N_{s_i}(V_i) \\ &= \textstyle\sum_{i=1}^m \big(N_{s_i}(n+1) - N_{s_i - r_i}(n+1)\big) \quad \text{(by Lemma 3.1)} \\ &= \Big(\textstyle\sum_{i=1}^{m-1} N_{s_i}(n+1) - N_{s_{i+1}}(n+1)\Big) \\ &\qquad\qquad + N_{s_m}(n+1) - N_{s_m - r_m}(n+1) \\ &= N_{s_1}(n+1) - N_{s_m - r_m}(n+1) \\ &= \begin{cases} N_d(n+1) - 0 = N_d(n+1) & \text{if (29) holds true,} \\ N_d(n+1) - N_0(n+1) = N_d(n+1) - 1 & \text{if (31) holds true.} \end{cases} \end{aligned}$$

Thus, in both cases, the number of conditions imposed on p matches the dimension of $\mathbb{P}_d(\mathbb{C}^{n+1})$. To establish the theorem, it is therefore sufficient to show that, if $p \in \mathbb{P}_d(\mathbb{C}^{n+1})$ is such that $\mathbf{T}^{s_i}_{\mathcal{L}_i}(p) = 0$ for every i — and also, in the case (31), such that $p(a) = 0$, then p must be the zero polynomial.

Let p be such a polynomial. Since p is of degree at most $s_1 (= d)$ and $\mathbf{T}^{s_1}_{\mathcal{L}_1}(p) = 0$, we must have $p_{|V_1} = 0$. Since q_1 is irreducible this implies by the Nullstellensatz that q_1 divides p; thus, $p = q_1 h_1$ with $\deg h_1 = \deg p -$

$\deg q_1 \leq s_1 - r_1 = s_2$, that is, $h_1 \in \mathbb{P}_{s_2}(\mathbb{C}^{n+1})$. Now the second condition $\mathbf{T}^{s_2}_{\mathcal{L}_2}(p) = 0$ translates into $\mathbf{T}^{s_2}_{\mathcal{L}_2}(q_1 h_1) = 0$. Since $h_1 \in \mathbb{P}_{s_2}(\mathbb{C}^{n+1})$ and, by hypothesis, $q_1(a_2) \neq 0$, we may apply the weak Leibniz formula (Lemma 3.3) to get $h_1 = 0$ on V_2 so that $h_1 = h_2 q_2$ with $h_2 \in \mathbb{P}_{s_3}(\mathbb{C}^{n+1})$. The third condition now translates into $\mathbf{T}^{s_3}_{\mathcal{L}_3}(h_2 q_1 q_2) = 0$. Again, since $q_1(a_3)q_2(a_3) \neq 0$ (here we use $a_i \notin V_j$ for $j < i$) and $\deg h_2 \leq s_3$, an application of the weak Leibniz formula yields $h_2 = 0$ on V_3, hence $h_2 = h_3 q_3$ Continuing in this way we arrive at $p = q_1 q_2 \ldots q_m h_m$ for a polynomial h_m. When (29) holds, comparing the degree of both sides, we deduce at once that h_m must be zero which gives in turn $p = 0$, whereas when (31) holds, h_m must be a constant polynomial and the use of the condition $p(a) = 0$ forces this constant to be zero (for no $q_i(a)$ vanishes) which again permits us to conclude that $p = 0$ and this finishes the proof.

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