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Spectral element discretization of the vorticity, velocity and pressure formulation of the Navier–Stokes problem

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Abstract The two-dimensional Navier–Stokes equations, when subject to non-standard boundary conditions which involve the normal component of the velocity and the vorticity, admit a variational formulation with three independent unknowns, the vorticity, velocity and pressure. We propose a discretization of this problem by spectral element methods. A detailed numerical analysis leads to optimal error estimates for the three unknowns and numerical experiments confirm the interest of the discretization.

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1 Introduction

Let Ω be a bounded connected domain in $\mathbb{R}^2$. We introduce the unit outward normal vector $\boldsymbol{n}$ to Ω on $\partial\Omega$ and we consider the nonlinear problem

$$\begin{cases} \nu\, \mathbf{curl}\, \omega + \omega \times \boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \omega = \operatorname{curl} \boldsymbol{u} & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \\ \omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Indeed it is readily checked that this system is equivalent to the full Navier–Stokes equations when subject to boundary conditions on the normal component of the velocity and the vorticity. Such conditions appear for a large number of flows, for instance, in the case of a fluid on both sides of a membrane or for the well-known Green–Taylor flow; see [16].

In system (1.1), the unknowns are the vorticity of the fluid ω, its velocity $\boldsymbol{u}$ and its pressure p. This formulation with three unknowns was first proposed in [11] and [17] (see also [12] and [1]) and seems the most appropriate for handling the type of boundary conditions we are interesred in, both for the Stokes and Navier–Stokes problems. We have decided to treat only the case of a two-dimensional domain. Indeed, the variational spaces are rather different in dimension 2 (where the vorticity is a scalar function) and in dimension 3 (where the vorticity is a vector field). Moreover, the existence of a solution in the three-dimensional case is, to our knowledge, only proved for a smooth domain Ω when the viscosity is sufficiently large enough; see [4]. Thus we first check the existence of a solution and its stability in the case of a possibly multiply-connected bidimensional domain.

We are interested in the spectral element discretization of system (1.1). The numerical analysis of discretizations of the Stokes problem relying on this formulation was first carried out for finite element methods; see [17] and its references. It has recently been extended to the case of spectral methods in [5] and of spectral element methods in the [2]. We also refer to [4] for the first work concerning the discretization of the Navier-Stokes equations (1.1) by spectral methods. The main idea of this paper is to extend these results to the case of spectral element methods.

We first describe the discrete problem and prove that it admits at least a solution. Next, relying on the arguments presented in [2], we perform its numerical analysis. By using the theory introduced in [8], we prove optimal error estimates for the three unknowns. It can be noted that this is a special property of the formulation that we use, since the approximation of the pressure for other formulations of the Stokes or Navier–Stokes problem is most often not optimal (see [7, §§24–26]).

We describe the Newton-type iterative algorithm that is used to solve the nonlinear discrete problem. Relying once more on the arguments in [8], we check its convergence. We also describe a possible algorithm for exhibiting an appropriate initial guess in order to initiate Newton's method. We conclude with numerical experiments which confirm the optimality of the discretization and justify the choice of this formulation.

An outline of the paper is as follows.

- In Sect. 2, we present the variational formulation of system (1.1) and recall from previous work the existence of a solution.
- Sect. 3 is devoted to the description of the spectral element discrete problem. We also prove the existence of a solution.
- Optimal error estimates are derived in Sect. 4.
- In Sect. 5, we describe the iterative algorithm that is used for solving the discrete problem and prove its convergence.
- Numerical experiments are presented in Sect. 6.

2 The velocity, vorticity and pressure formulation

In order to describe the variational formulation of problem (1.1) and for the sake of precision, we first recall the definition of the scalar and vector curl operators in dimension 2. For any vector field $\boldsymbol{v} = (v_x, v_y)$ and any scalar function φ,

$$\operatorname{curl} \boldsymbol{v} = \partial_x v_y - \partial_y v_x, \qquad \mathbf{curl}\, \varphi = \begin{pmatrix} \partial_y \varphi \\ -\partial_x \varphi \end{pmatrix}, \tag{2.1}$$

where all derivatives in the previous line are taken in the distribution sense. We also recall that, for any vector field $\boldsymbol{v} = (v_x, v_y)$ and any scalar function φ, the product $\varphi \times \boldsymbol{v}$ means the vector with components φv_y and $-\varphi v_x$.

We note that the boundary conditions in this problem are not sufficient to guarantee the uniqueness of the solution in the case of multiply-connected domains even for the Stokes problem, see [2, §2] for more details. We need the following notation.

Notation 2.1 Let Σ_j, $1 \le j \le J$, be connected open curves, called "cuts", such that

(i) each Σ_j is an open part of a smooth curve;
(ii) each Σ_j, $1 \le j \le J$, is contained in Ω and its two endpoints belong to two different connected components of $\partial\Omega$;
(iii) the intersection of Σ_j and $\Sigma_{j'}$, $1 \le j < j' \le J$, is empty;
(iv) the open set $\Omega^\circ = \Omega \setminus \cup_{j=1}^{J} \Sigma_j$ is simply-connected.

The existence of such Σ_j is clear. We make the further assumption that the domain Ω° is pseudo–Lipschitz, in the sense that, for each point $\boldsymbol{x}$ of $\partial\Omega^\circ$, the intersection of Ω° with a smooth neighborhood of $\boldsymbol{x}$ has one or two connected components and each of them has a Lipschitz–continuous boundary (we refer to [3, §3.a] for a more precise definition). Then, the further conditions read

$$\langle \boldsymbol{u} \cdot \boldsymbol{n}, 1\rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J, \tag{2.2}$$

where $\langle \cdot, \cdot\rangle_{\Sigma_j}$ stands for the duality pairing between $H^{-\frac{1}{2}}(\Sigma_j)$ and $H^{\frac{1}{2}}(\Sigma_j)$.

We consider the standard spaces $L^p(\Omega)$, $1 \leq p \leq +\infty$, and also the full scale of Sobolev spaces $H^s(\Omega)$ and $H_0^s(\Omega)$, $s \geq 0$. We introduce the domain $H(\mathrm{div}, \Omega)$ of the divergence operator, namely,

$$H(\mathrm{div}, \Omega) = \left\{\boldsymbol{v} \in L^2(\Omega)^2;\ \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega)\right\}. \tag{2.3}$$

Since the normal trace operator: $\boldsymbol{v} \mapsto \boldsymbol{v} \cdot \boldsymbol{n}$ can be defined from $H(\mathrm{div}, \Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$, see [13, Chap. I, Thm. (2.5)], we also consider its kernel

$$H_0(\mathrm{div}, \Omega) = \left\{\boldsymbol{v} \in H(\mathrm{div}, \Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\right\}. \tag{2.4}$$

Finally, let $L_0^2(\Omega)$ stand for the space of functions in $L^2(\Omega)$ with a null integral on Ω.

In view of conditions (2.2) and according to [6, §2.5], we introduce the space

$$\mathbb{D}(\Omega) = \left\{\boldsymbol{v} \in H_0(\mathrm{div}, \Omega);\ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1\rangle_{\Sigma_j} = 0,\ 1 \leq j \leq J\right\}. \tag{2.5}$$

We now consider the variational problem:

Find $(\omega, \boldsymbol{u}, p)$ *in* $H_0^1(\Omega) \times \mathbb{D}(\Omega) \times L_0^2(\Omega)$ *such that*

$$\begin{aligned} &\forall \boldsymbol{v} \in \mathbb{D}(\Omega), && a(\omega, \boldsymbol{u}; \boldsymbol{v}) + K(\omega, \boldsymbol{u}; \boldsymbol{v}) + b(\boldsymbol{v}, p) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle,\\ &\forall q \in L_0^2(\Omega), && b(\boldsymbol{u}, q) = 0,\\ &\forall \varphi \in H_0^1(\Omega), && c(\omega, \boldsymbol{u}; \varphi) = 0, \end{aligned} \tag{2.6}$$

where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between $H_0(\mathrm{div}, \Omega)$ and its dual space. The bilinear forms $a(\cdot, \cdot; \cdot)$, $b(\cdot, \cdot)$ and $c(\cdot, \cdot; \cdot)$ are defined by

$$\begin{aligned} a(\omega, \boldsymbol{u}; \boldsymbol{v}) &= \nu \int_\Omega (\mathbf{curl}\,\omega)(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x},\\ b(\boldsymbol{v}, q) &= -\int_\Omega (\mathrm{div}\,\boldsymbol{v})(\boldsymbol{x}) q(\boldsymbol{x})\, d\boldsymbol{x},\\ c(\omega, \boldsymbol{u}; \varphi) &= \int_\Omega \omega(\boldsymbol{x})\varphi(\boldsymbol{x})\, d\boldsymbol{x} - \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot (\mathbf{curl}\,\varphi)(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned} \tag{2.7}$$

The trilinear form $K(\cdot, \cdot; \cdot)$ is given by

$$K(\omega, \boldsymbol{u}; \boldsymbol{v}) = \int_\Omega (\omega \times \boldsymbol{u})(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\boldsymbol{x}. \tag{2.8}$$

As a consequence of the density of the space of infinitely differentiable functions with a compact support in Ω in $H_0(\mathrm{div}, \Omega)$ and $H_0^1(\Omega)$ (see [13, Chap. I, §2]), we derive the following statement. It involves the solutions q_j^t, $1 \le j \le J$, of the problem (see [3, Prop. 3.14] for more details on these functions)

$$\begin{cases} -\Delta q_j^t = 0 & \text{in } \Omega^\circ, \\ \partial_n q_j^t = 0 & \text{on } \partial\Omega, \\ \left[q_j^t\right]_{j'} = \text{constant}, & 1 \le j' \le J, \\ \left[\partial_n q_j^t\right]_{j'} = 0, & 1 \le j' \le J, \\ \langle \partial_n q_j^t, 1\rangle_{\Sigma_{j'}} = \delta_{jj'}, & 1 \le j' \le J, \end{cases} \tag{2.9}$$

where $[\cdot]_{j'}$ denotes the jump through $\Sigma_{j'}$ (making its sign precise is not needed in what follows). Note that each $\widetilde{\mathbf{grad}}\, q_j^t$ belongs to $H_0(\mathrm{div}, \Omega)$, where $\widetilde{\mathbf{grad}}$ stands for the gradient defined in the distribution sense on Ω°, and that $H_0(\mathrm{div}, \Omega)$ is the direct sum of $\mathbb{D}(\Omega)$ and of the space spanned by the $\widetilde{\mathbf{grad}}\, q_j^t$, $1 \le j \le J$.

Proposition 2.2 *For any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div}, \Omega)$ satisfying*

$$\langle \boldsymbol{f}, \widetilde{\mathbf{grad}}\, q_j^t\rangle = 0, \qquad 1 \le j \le J, \tag{2.10}$$

problems (1.1) $-$ (2.2) *and* (2.6) *are equivalent, in the sense that any triple* $(\omega, \boldsymbol{u}, p)$ *in* $H^1(\Omega) \times H(\mathrm{div}, \Omega) \times L_0^2(\Omega)$ *is a solution of problem* (1.1) $-$ (2.2) *if and only if it is a solution of problem* (2.6).

We briefly recall from [17], [5, §2], [6, §2.5] and [4, §2] the main arguments for proving the existence of a solution of problem (2.6). It is readily checked that the kernel

$$V = \left\{\boldsymbol{v} \in \mathbb{D}(\Omega);\ \forall q \in L_0^2(\Omega),\ b(\boldsymbol{v}, q) = 0\right\} \tag{2.11}$$

coincides with the space of divergence-free functions in $\mathbb{D}(\Omega)$. Similarly, the kernel

$$\mathcal{W} = \left\{(\theta, \boldsymbol{w}) \in H_0^1(\Omega) \times V;\ \forall \varphi \in H_0^1(\Omega),\ c(\theta, \boldsymbol{w}; \varphi) = 0\right\} \tag{2.12}$$

coincides with the space of pairs $(\theta, \boldsymbol{w})$ in $H_0^1(\Omega) \times V$ such that θ is equal to $\mathrm{curl}\, \boldsymbol{w}$ in the distribution sense. We observe that, for any solution $(\omega, \boldsymbol{u}, p)$ of problem (2.6), the pair $(\omega, \boldsymbol{u})$ is a solution of the following reduced problem:

Find $(\omega, \boldsymbol{u})$ *in* $\mathcal{W}$ *such that*

$$\forall \boldsymbol{v} \in V, \quad a(\omega, \boldsymbol{u}; \boldsymbol{v}) + K(\omega, \boldsymbol{u}; \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle. \tag{2.13}$$

We recall from [5, Lemma 2.3] and [6, Props. 2.5.3 & 2.5.4] the following properties (which require the further conditions that appear in the definition of $\mathbb{D}(\Omega)$). There exists a positive constant α such that

$$\begin{aligned} &\forall \boldsymbol{v} \in V \setminus \{0\}, && \sup_{(\omega,\boldsymbol{u})\in\mathcal{W}} a(\omega,\boldsymbol{u};\boldsymbol{v}) > 0, \\ &\forall (\omega,\boldsymbol{u}) \in \mathcal{W}, && \sup_{\boldsymbol{v}\in V} \frac{a(\omega,\boldsymbol{u};\boldsymbol{v})}{\|\boldsymbol{v}\|_{L^2(\Omega)^2}} \geq \alpha \left(\|\omega\|^2_{H^1(\Omega)} + \|\boldsymbol{u}\|^2_{L^2(\Omega)^2}\right)^{\frac{1}{2}}. \end{aligned} \tag{2.14}$$

This last property is derived from the more precise inequality

$$(\omega,\boldsymbol{u}) \in \mathcal{W}, \quad a(\omega,\boldsymbol{u};\boldsymbol{u} + \mathbf{curl}\,\omega) \geq 2\alpha \left(\|\omega\|^2_{H^1(\Omega)} + \|\boldsymbol{u}\|^2_{L^2(\Omega)^2}\right),$$

which is used in the proof of the existence result below.

The next statement is an easy consequence of the imbedding of $\mathcal{W}$ into $H^1(\Omega) \times H^{\frac{1}{2}}(\Omega)^2$ (see [10]) and of the Sobolev imbeddings of $H^1(\Omega)$ into $L^q(\Omega)$ for any $q < +\infty$ and of $H^{\frac{1}{2}}(\Omega)$ into $L^4(\Omega)$.

Lemma 2.3 *The form $K(\cdot,\cdot;\cdot)$ is continuous on $\mathcal{W} \times L^2(\Omega)^2$. Moreover, for any $(\omega,\boldsymbol{u})$ in $\mathcal{W}$, the operators: $(\theta,\boldsymbol{w}) \mapsto \omega \times \boldsymbol{w}$ and $(\theta,\boldsymbol{w}) \mapsto \theta \times \boldsymbol{u}$ are compact from $\mathcal{W}$ into $L^2(\Omega)^2$.*

We note the further antisymmetry properties

$$\forall (\omega,\boldsymbol{u}) \in \mathcal{W}, \qquad K(\omega,\boldsymbol{u};\boldsymbol{u}) = K(\omega,\boldsymbol{u};\mathbf{curl}\,\omega) = 0, \tag{2.15}$$

which allow us to establish a priori estimates on any solution of problem (2.13). Thus, the existence of a solution for this problem is derived from Brouwer's fixed point theorem in a standard way; see [4, Prop. 2.5] for a detailed proof.

Proposition 2.4 *For any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div},\Omega)$, problem* (2.13) *has a solution $(\omega,\boldsymbol{u})$ in $\mathcal{W}$.*

We also recall the standard inf-sup condition on the form $b(\cdot,\cdot)$. There exists a positive constant β such that

$$\forall q \in L^2_0(\Omega), \qquad \sup_{\boldsymbol{v}\in H_0(\mathrm{div},\Omega)} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{H(\mathrm{div},\Omega)}} \geq \beta\,\|q\|_{L^2(\Omega)}.$$

When applying this result with Ω replaced by Ω°, we easily derive that

$$\forall q \in L^2_0(\Omega), \qquad \sup_{\boldsymbol{v}\in \mathbb{D}(\Omega)} \frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{H(\mathrm{div},\Omega)}} \geq \beta\,\|q\|_{L^2(\Omega)}. \tag{2.16}$$

Combining this with Proposition 2.4 leads to the main result for problem (2.6).

Theorem 2.5 *For any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div},\Omega)$, problem* (2.6) *has a solution $(\omega,\boldsymbol{u},p)$ in $H^1_0(\Omega) \times \mathbb{D}(\Omega) \times L^2_0(\Omega)$. Moreover this solution satisfies*

$$\|\omega\|_{H^1(\Omega)} + \|\boldsymbol{u}\|_{H(\mathrm{div},\Omega)} + \|p\|_{L^2(\Omega)} \leq c\,\|\boldsymbol{f}\|_{H_0(\mathrm{div},\Omega)'}. \tag{2.17}$$

Remark 2.6 As usual for the Navier-Stokes equations, the solution of problem (2.6) is unique only if the viscosity ν is sufficiently large as a function of the data, see [4, Thm 2.9]. We prefer to avoid this overly restrictive assumption in what follows.

We conclude with some regularity properties of the solution of problem (2.6) which can easily be derived from [2, §2] thanks to a boot-strap argument. The mapping: $\boldsymbol{f} \mapsto (\omega, \boldsymbol{u}, p)$, where $(\omega, \boldsymbol{u}, p)$ is the solution of problem (2.6) with data $\boldsymbol{f}$, is continuous from $H^s(\Omega)^2$ into $H^{s+1}(\Omega) \times H^s(\Omega)^2 \times H^{s+1}(\Omega)$ for:

(i) all $s \le \frac{1}{2}$ in the general case,
(ii) all $s \le 1$ when Ω is convex,
(iii) all $s < \frac{\pi}{\alpha}$ when Ω is a polygon with largest angle equal to α.

3 The spectral element discrete problem

From now on, we assume that Ω admits a partition without overlap into a finite number of subdomains

$$\overline{\Omega} = \cup_{k=1}^{K} \Omega_k \qquad \text{and} \qquad \Omega_k \cap \Omega_{k'} = \emptyset, \quad 1 \le k < k' \le K, \tag{3.1}$$

which satisfy the further conditions

(i) each Ω_k, $1 \le k \le K$, is a rectangle;
(ii) the intersection of two subdomains $\overline{\Omega}_k$ and $\overline{\Omega}_{k'}$, $1 \le k < k' \le K$, if not empty, is either a vertex or a whole edge of both Ω_k and $\Omega_{k'}$;
(iii) the $\overline{\Sigma}_j$, $1 \le j \le J$, introduced in Notation 2.1, are the union of whole edges of some Ω_k.

The discrete spaces are constructed from the finite elements proposed by Nédélec on cubic three-dimensional meshes; see [15, §2]. In order to describe them, for any pair (m, n) of nonnegative integers, we introduce the space $\mathbb{P}_{m,n}(\Omega_k)$ of restrictions to Ω_k of polynomials with degree $\le m$ with respect to x and $\le n$ with respect to y. When m is equal to n, this space is simply denoted by $\mathbb{P}_n(\Omega_k)$. Using these definitions, for an integer $N \ge 2$, we introduce the local spaces.

$$D_N^k = \mathbb{P}_{N,N-1}(\Omega_k) \times \mathbb{P}_{N-1,N}(\Omega_k), \quad C_N^k = \mathbb{P}_N(\Omega_k), \qquad M_N^k = \mathbb{P}_{N-1}(\Omega_k). \tag{3.2}$$

The space $\mathbb{D}_N$ which approximates $\mathbb{D}(\Omega)$ is then defined by

$$\mathbb{D}_N = \left\{ \boldsymbol{v}_N \in \mathbb{D}(\Omega);\ \boldsymbol{v}_N|_{\Omega_k} \in D_N^k,\ 1 \le k \le K \right\}. \tag{3.3}$$

The space $\mathbb{C}_N$ which approximates $H_0^1(\Omega)$ is defined by

$$\mathbb{C}_N = \left\{ \varphi_N \in H_0^1(\Omega);\ \varphi_N|_{\Omega_k} \in C_N^k,\ 1 \le k \le K \right\}. \tag{3.4}$$

Finally, for the approximation of $L_0^2(\Omega)$, we consider the space

$$\mathbb{M}_N = \left\{q_N \in L_0^2(\Omega);\ q_N|_{\Omega_k} \in M_N^k,\ 1 \le k \le K\right\}. \tag{3.5}$$

It can be noted that the functions in $\mathbb{D}_N$ have continuous normal traces through the interfaces $\overline{\Omega}_k \cap \overline{\Omega}_{k'}$ while the functions in $\mathbb{C}_N$ have continuous traces. Thanks to the previous choice, the discretization that we propose is perfectly conforming.

According to the approach suggested in [14] and in order to handle the nonlinear term, we use over-integration. For a fixed real number μ, $0 < \mu \le 1$, we associate with each value of N the quantity $m(N)$ equal to the integer part of $(1+\mu)N$. Setting $\xi_0 = -1$ and $\xi_{m(N)} = 1$, we introduce the $m(N) - 1$ nodes ξ_j, $1 \le j \le m(N) - 1$, and the $m(N) + 1$ weights ρ_j, $0 \le j \le m(N)$, of the Gauss-Lobatto quadrature formula on $[-1, 1]$. Denoting by $\mathbb{P}_n(-1, 1)$ the space of restrictions to $[-1, 1]$ of polynomials with degree $\le n$, we recall that the following equality holds:

$$\forall \Phi \in \mathbb{P}_{2m(N)-1}(-1,1), \quad \int_{-1}^{1} \Phi(\zeta)\, d\zeta = \sum_{j=0}^{m(N)} \Phi(\xi_j)\, \rho_j. \tag{3.6}$$

We also recall [7, (13.20)] the following property, which is useful in what follows:

$$\forall \varphi_N \in \mathbb{P}_{m(N)}(-1,1), \quad \|\varphi_N\|^2_{L^2(-1,1)} \le \sum_{j=0}^{m(N)} \varphi_N^2(\xi_j)\, \rho_j \le 3\, \|\varphi_N\|^2_{L^2(-1,1)}. \tag{3.7}$$

Denoting by F_k the affine mapping that sends $]-1,1[^2$ onto Ω_k, we introduce the local discrete products, defined on continuous functions u and v on $\overline{\Omega}_k$ by

$$(u, v)_N^k = \frac{\text{meas}(\Omega_k)}{4} \sum_{i=0}^{m(N)} \sum_{j=0}^{m(N)} u \circ F_k(\xi_i, \xi_j) v \circ F_k(\xi_i, \xi_j)\, \rho_i \rho_j. \tag{3.8}$$

The global product is then defined on continuous functions u and v on $\overline{\Omega}$ by

$$((u, v))_N = \sum_{k=1}^{K} (u|_{\Omega_k}, v|_{\Omega_k})_N^k. \tag{3.9}$$

We also need the local Lagrange interpolation operators $\mathcal{I}_N^k$. For each function φ continuous on $\overline{\Omega}_k$, $\mathcal{I}_N^k \varphi$ belongs to $\mathbb{P}_{m(N)}(\Omega_k)$ and is equal to φ at all nodes $F_k(\xi_i, \xi_j)$, $0 \le i, j \le m(N)$. Finally, for each function φ continuous on $\overline{\Omega}$, $\mathcal{I}_N \varphi$ denotes the function equal to $\mathcal{I}_N^k \varphi|_{\Omega_k}$ on each Ω_k, $1 \le k \le K$.

The discrete problem is now constructed from (2.6) by using the Galerkin method combined with numerical integration. It reads:

Find $(\omega_N, \boldsymbol{u}_N, p_N)$ *in* $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$ *such that*

$$\begin{aligned}
\forall \boldsymbol{v}_N \in \mathbb{D}_N, \quad & a_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N) + K_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N) \\
& \qquad + b_N(\boldsymbol{v}_N, p_N) = ((\boldsymbol{f}, \boldsymbol{v}_N))_N, \\
\forall q_N \in \mathbb{M}_N, \quad & b_N(\boldsymbol{u}_N, q_N) = 0, \\
\forall \varphi_N \in \mathbb{C}_N, \quad & c_N(\omega_N, \boldsymbol{u}_N; \varphi_N) = 0,
\end{aligned} \tag{3.10}$$

where the bilinear forms $a_N(\cdot,\cdot;\cdot)$, $b_N(\cdot,\cdot)$ and $c_N(\cdot,\cdot;\cdot)$ are defined by

$$\begin{aligned}
a_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N) &= \nu\, ((\mathbf{curl}\, \omega_N, \boldsymbol{v}_N))_N, \\
b_N(\boldsymbol{v}_N, q_N) &= -((\operatorname{div} \boldsymbol{v}_N, q_N))_N, \\
c_N(\omega_N, \boldsymbol{u}_N; \varphi_N) &= ((\omega_N, \varphi_N))_N - ((\boldsymbol{u}_N, \mathbf{curl}\, \varphi_N))_N,
\end{aligned} \tag{3.11}$$

while the trilinear form $K_N(\cdot,\cdot;\cdot)$ is now given by

$$K_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = ((\omega_N \times \boldsymbol{u}_N, \boldsymbol{v}_N))_N. \tag{3.12}$$

As a consequence of the exactness property (3.6), the forms $a(\cdot,\cdot;\cdot)$ and $a_N(\cdot,\cdot;\cdot)$, and also $c(\cdot,\cdot;\cdot)$ and $c_N(\cdot,\cdot;\cdot)$ coincide on $\big(\mathbb{C}_N \times \mathbb{D}_N\big) \times \mathbb{D}_N$ and $\big(\mathbb{C}_N \times \mathbb{D}_N\big) \times \mathbb{C}_N$, respectively, when $m(N) > N$. Moreover, the forms $b(\cdot,\cdot)$ and $b_N(\cdot,\cdot)$ coincide on $\mathbb{D}_N \times \mathbb{M}_N$ even for $m(N) = N$. In any case, it follows from (3.7) combined with the Cauchy–Schwarz inequality that the forms $a_N(\cdot,\cdot;\cdot)$, $b_N(\cdot,\cdot)$ and $c_N(\cdot,\cdot;\cdot)$ are continuous on $\big(\mathbb{C}_N \times \mathbb{D}_N\big) \times \mathbb{D}_N$, $\mathbb{D}_N \times \mathbb{M}_N$ and $\big(\mathbb{C}_N \times \mathbb{D}_N\big) \times \mathbb{C}_N$, respectively, with norms bounded independently of N.

In order to perform the numerical analysis of problem (3.10), we first recall from the finite element analogous result [15] that the range of $\mathbb{D}_N$ by the divergence operator is contained in $\mathbb{M}_N$. So, if V_N denotes the kernel

$$V_N = \big\{\boldsymbol{v}_N \in \mathbb{D}_N;\ \forall q_N \in \mathbb{M}_N,\ b_N(\boldsymbol{v}_N, q_N) = 0\big\}, \tag{3.13}$$

it is readily checked by taking q_N equal to $\operatorname{div} \boldsymbol{v}_N$ in the previous line that V_N is the space of divergence-free functions in $\mathbb{D}_N$, i.e., coincides with $\mathbb{D}_N \cap V$. Similarly, we introduce the discrete kernel

$$\mathcal{W}_N = \big\{(\theta_N, \boldsymbol{w}_N) \in \mathbb{C}_N \times V_N;\ \forall \varphi_N \in \mathbb{C}_N,\ c_N(\theta_N, \boldsymbol{w}_N; \varphi_N) = 0\big\}. \tag{3.14}$$

We observe that, for any solution $(\omega_N, \boldsymbol{u}_N, p_N)$ of problem (3.10), the pair $(\omega_N, \boldsymbol{u}_N)$ is a solution of the reduced problem:

Find $(\omega_N, \boldsymbol{u}_N)$ *in* $\mathcal{W}_N$ *such that*

$$\forall \boldsymbol{v}_N \in V_N, \quad a_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N) + K_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = ((\boldsymbol{f}, \boldsymbol{v}_N))_N. \tag{3.15}$$

We first recall the next result which is proved in [2, Prop. 3.3].

Lemma 3.1 *For each $\boldsymbol{v}_N$ in V_N, there exists a unique ψ_N in $\mathbb{C}_N$ such that $\boldsymbol{v}_N = \mathbf{curl}\, \psi_N$ and which satisfies*

$$\|\psi_N\|_{L^2(\Omega)} \leq c\, \|\boldsymbol{v}_N\|_{L^2(\Omega)^2}. \tag{3.16}$$

Using (3.16), we prove the following property.

Lemma 3.2 *There exists a constant α_* such that*

$$\forall (\omega_N, \boldsymbol{u}_N) \in \mathcal{W}_N, \quad a_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{u}_N) \geq \alpha_* \left(\|\omega_N\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_N\|^2_{L^2(\Omega)^2}\right). \tag{3.17}$$

Proof From the definitions of the form $a_N(\cdot, \cdot; \cdot)$ and the space $\mathcal{W}_N$, we have

$$a_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{u}_N) = \nu\, ((\mathbf{curl}\, \omega_N, \boldsymbol{u}_N))_N = \nu\, ((\omega_N, \omega_N))_N,$$

whence, owing to (3.7),

$$a_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{u}_N) \geq \nu\, \|\omega_N\|^2_{L^2(\Omega)}. \tag{3.18}$$

On the other hand, associating with $\boldsymbol{u}_N$ the function ψ_N exhibited in Lemma 3.1 and using once more the definition of $\mathcal{W}_N$, we have

$$((\boldsymbol{u}_N, \boldsymbol{u}_N))_N = ((\mathbf{curl}\, \psi_N, \boldsymbol{u}_N))_N = ((\omega_N, \psi_N))_N.$$

Combining (3.7) with (3.16) yields

$$\|\boldsymbol{u}_N\|^2_{L^2(\Omega)^2} \leq 9\, \|\omega_N\|_{L^2(\Omega)} \|\psi_N\|_{L^2(\Omega)} \leq 9c\, \|\omega_N\|_{L^2(\Omega)} \|\boldsymbol{u}_N\|_{L^2(\Omega)^2},$$

whence

$$\|\boldsymbol{u}_N\|_{L^2(\Omega)^2} \leq 9c\, \|\omega_N\|_{L^2(\Omega)}.$$

This last inequality and (3.18) give the desired property.

We are now in a position to prove the existence of a solution to problem (3.15). □

Proposition 3.3 *For any function $\boldsymbol{f}$ continuous on $\overline{\Omega}$, problem* (3.15) *has a solution $(\omega_N, \boldsymbol{u}_N)$ in $\mathcal{W}_N$. Moreover this solution satisfies, for a constant c, independent of N,*

$$\|\omega_N\|_{L^2(\Omega)} + \|\boldsymbol{u}_N\|_{L^2(\Omega)^2} \leq c\, \|\mathcal{I}_N \boldsymbol{f}\|_{L^2(\Omega)^2}. \tag{3.19}$$

Proof We introduce the mapping Φ_N defined from $\mathcal{W}_N$ into its dual space by

$$\forall (\omega_N, \boldsymbol{u}_N) \in \mathcal{W}_N,\ \forall (\theta_N, \boldsymbol{w}_N) \in \mathcal{W}_N,$$
$$\langle \Phi_N(\omega_N, \boldsymbol{u}_N), (\theta_N, \boldsymbol{w}_N) \rangle = a_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{w}_N) + K_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{w}_N) - (\boldsymbol{f}, \boldsymbol{w}_N)_N.$$

The space $\mathcal{W}_N$ is here provided with the weak norm

$$\left(\|\omega_N\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_N\|^2_{L^2(\Omega)^2}\right)^{\frac{1}{2}}.$$

Since $\mathcal{W}_N$ is finite-dimensional, it is readily checked that Φ_N is continuous. Next, noting that $K_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{u}_N)$ is zero (indeed, the product $(\omega_N \times \boldsymbol{u}_N) \cdot \boldsymbol{u}_N$ vanishes at all nodes $F_k(\xi_i, \xi_j)$), we have

$$\langle \Phi_N(\omega_N, \boldsymbol{u}_N), (\omega_N, \boldsymbol{u}_N) \rangle = a_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{u}_N) - (\boldsymbol{f}, \boldsymbol{u}_N)_N,$$

whence, owing to Lemma 3.2,

$$\langle \Phi_N(\omega_N, \boldsymbol{u}_N), (\omega_N, \boldsymbol{u}_N) \rangle \geq \alpha_* \left(\|\omega_N\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_N\|^2_{L^2(\Omega)^2} \right) - (\boldsymbol{f}, \boldsymbol{u}_N)_N.$$

On the other hand, we derive from (3.7) that

$$(\boldsymbol{f}, \boldsymbol{u}_N)_N = (\mathcal{I}_N \boldsymbol{f}, \boldsymbol{u}_N)_N \leq 3\, \|\mathcal{I}_N \boldsymbol{f}\|_{L^2(\Omega)^2} \|\boldsymbol{u}_N\|_{L^2(\Omega)^2},$$

which leads to

$$\langle \Phi_N(\omega_N, \boldsymbol{u}_N), (\omega_N, \boldsymbol{u}_N) \rangle \geq \frac{\alpha_*}{2} \left(\|\omega_N\|^2_{L^2(\Omega)} + \|\boldsymbol{u}_N\|^2_{L^2(\Omega)^2} \right) - \frac{9}{2\alpha^*} \|\mathcal{I}_N \boldsymbol{f}\|^2_{L^2(\Omega)^2}.$$

Thus, setting

$$\mu_N = \frac{3}{\alpha_*} \|\mathcal{I}_N \boldsymbol{f}\|_{L^2(\Omega)^2},$$

we observe that $\langle \Phi_N(\omega_N, \boldsymbol{u}_N), (\omega_N, \boldsymbol{u}_N) \rangle$ is nonnegative on the sphere of $\mathcal{W}_N$ with radius μ_N. So applying Brouwer's fixed point theorem (see [13, Chap. IV, Cor. 1.1]) gives the existence result together with estimate (3.19).

To go further, we recall from [2, Lemma 3.8] the inf-sup condition on the form $b_N(\cdot, \cdot)$. There exists a positive constant β_* independent of N such that the form $b_N(\cdot, \cdot; \cdot)$ satisfies the inf-sup condition

$$\forall q_N \in \mathbb{M}_N, \qquad \sup_{\boldsymbol{v}_N \in \mathbb{D}_N} \frac{b_N(\boldsymbol{v}_N, q_N)}{\|\boldsymbol{v}_N\|_{H(\mathrm{div},\Omega)}} \geq \beta_* \|q_N\|_{L^2(\Omega)}. \tag{3.20}$$

The final existence result is derived from this condition and Proposition 3.3 in a standard way; see, e.g., [13, Chap. I, Lemma 4.1].

Theorem 3.4 *For any function $\boldsymbol{f}$ continuous on $\overline{\Omega}$, problem* (3.10) *has a solution $(\omega_N, \boldsymbol{u}_N, p_N)$ in $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$. Moreover the part $(\omega_N, \boldsymbol{u}_N)$ of this solution satisfies* (3.19).

Note that all the results in this section hold without over-integration, i.e., for $\mu = 0$ and $m(N) = N$. However, the choice of a $\mu > 0$ is needed in what follows.

4 Error estimates

As already hinted, the error analysis of the discrete problem relies on the theory of Brezzi, Rappaz and Raviart [8]. In order to apply it, we first express both problems (2.13) and (3.15) in a different form.

We set

$$\mathcal{X} = H_0^1(\Omega) \times V.$$

Owing to the characterization of V, this space is equipped with the norm

$$\|(\theta, \boldsymbol{w})\|_{\mathcal{X}} = \big(|\theta|^2_{H^1(\Omega)} + \|\boldsymbol{w}\|^2_{L^2(\Omega)^2}\big)^{\frac{1}{2}}. \tag{4.1}$$

Let $\mathcal{S}$ denote the following Stokes operator. For any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div}, \Omega)$, $\mathcal{S}\boldsymbol{f}$ denotes the solution $(\omega, \boldsymbol{u})$ of the reduced problem:

Find $(\omega, \boldsymbol{u})$ *in* $\mathcal{W}$ *such that*

$$\forall \boldsymbol{v} \in V, \quad a(\omega, \boldsymbol{u}; \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle. \tag{4.2}$$

The fact that $\mathcal{S}$ is well-defined is easily derived from properties (2.14). We also introduce the mapping G defined from $\mathcal{W}$ into the dual space of $H_0(\mathrm{div}, \Omega)$ by

$$\forall (\omega, \boldsymbol{u}) \in \mathcal{W}, \forall \boldsymbol{v} \in H_0(\mathrm{div}, \Omega), \quad \langle G(\omega, \boldsymbol{u}), \boldsymbol{v}\rangle = K(\omega, \boldsymbol{u}; \boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v}\rangle. \tag{4.3}$$

Then, problem (2.13) can equivalently be written as

$$(\omega, \boldsymbol{u}) + \mathcal{S}G(\omega, \boldsymbol{u}) = 0. \tag{4.4}$$

Similarly, we set

$$\mathcal{X}_N = \mathbb{C}_N \times V_N,$$

and note, since V_N is contained in V, $\mathcal{X}_N$ is a finite-dimensional subspace of $\mathcal{X}$. It is still provided with the norm defined in (4.1). We thus define the discrete Stokes operator. For any data $\boldsymbol{f}$ in the dual space of $H_0(\mathrm{div}, \Omega)$, $\mathcal{S}_N\boldsymbol{f}$ denotes the solution $(\omega_N, \boldsymbol{u}_N)$ of the problem:

Find $(\omega_N, \boldsymbol{u}_N)$ *in* $\mathcal{W}_N$ *such that*

$$\forall \boldsymbol{v}_N \in V_N, \quad a_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N) = \langle \boldsymbol{f}, \boldsymbol{v}_N\rangle. \tag{4.5}$$

The well-posedness of such a problem is proved in [2, Cor. 3.6] for a slightly different right-hand side. Finally we consider the mapping G_N defined from $\mathcal{X}_N$ into the dual space of $\mathbb{D}_N$ by

$$\begin{aligned}&\forall (\omega_N, \boldsymbol{u}_N) \in \mathcal{X}_N, \forall \boldsymbol{v}_N \in \mathbb{D}_N,\\ &\qquad \langle G_N(\omega_N, \boldsymbol{u}_N), \boldsymbol{v}_N\rangle = K_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N) - ((\boldsymbol{f}, \boldsymbol{v}_N))_N.\end{aligned} \tag{4.6}$$

Then, problem (3.15) can equivalently be written as

$$(\omega_N, \boldsymbol{u}_N) + \mathcal{S}_N G_N(\omega_N, \boldsymbol{u}_N) = 0. \tag{4.7}$$

Using analogous arguments to those in [2, Cors. 3.6 & 4.9], we easily derive the following results.

(i) The operator $\mathcal{S}_N$ satisfies the stability property

$$\|\mathcal{S}_N \boldsymbol{f}\|_{\mathcal{X}} \leq c \sup_{\boldsymbol{v}_N \in V_N} \frac{\langle \boldsymbol{f}, \boldsymbol{v}_N \rangle}{\|\boldsymbol{v}_N\|_{L^2(\Omega)^2}}. \tag{4.8}$$

(ii) The following error estimate holds for all $\boldsymbol{f}$ such that $\mathcal{S}\boldsymbol{f}$ belongs to $H^{s+1}(\Omega) \times H^s(\Omega)^2$, $s \geq 0$,

$$\|(\mathcal{S} - \mathcal{S}_N)\boldsymbol{f}\|_{\mathcal{X}} \leq c\, N^{-s}\, \|\mathcal{S}\boldsymbol{f}\|_{H^{s+1}(\Omega)\times H^s(\Omega)^2}. \tag{4.9}$$

We need further properties of the form $K(\cdot, \cdot; \cdot)$. The following result is derived in [4, Lemma 3.4] in the case of one subdomain Ω_k and relies on the inverse inequality

$$\forall \varphi_N \in \mathbb{P}_N(\Omega_k), \quad \|\varphi_N\|_{L^\infty(\Omega_k)} \leq c\, |\log N|^{\frac{1}{2}}\, \|\varphi_N\|_{H^1(\Omega_k)}. \tag{4.10}$$

Applying the same arguments on each Ω_k leads to the next statement.

Lemma 4.1 *The following property holds:*

$$\begin{aligned}
&\forall \omega_N \in \mathbb{C}_N, \forall \boldsymbol{u}_N \in \mathbb{D}_N, \forall \boldsymbol{v}_N \in \mathbb{D}_N,\\
&\quad |K(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N)| \leq c\, |\log N|^{\frac{1}{2}}\, \|\omega_N\|_{H^1(\Omega)} \|\boldsymbol{u}_N\|_{L^2(\Omega)^2} \|\boldsymbol{v}_N\|_{L^2(\Omega)^2}.
\end{aligned} \tag{4.11}$$

Remark 4.2 Similar arguments yield that estimate (4.11) still holds when at most two of the three functions ω_N, $\boldsymbol{u}_N$ and $\boldsymbol{v}_N$ are replaced by their analogues ω in $H^1_0(\Omega)$, $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{D}(\Omega)$.

We need the analogous result for the form $K_N(\cdot, \cdot; \cdot)$.

Lemma 4.3 *The following property holds:*

$$\begin{aligned}
&\forall \omega_N \in \mathbb{C}_N, \forall \boldsymbol{u}_N \in \mathbb{D}_N, \forall \boldsymbol{v}_N \in \mathbb{D}_N,\\
&|K_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N)| \leq c\, |\log N|^{\frac{1}{2}}\, \|\omega_N\|_{H^1(\Omega)} \|\boldsymbol{u}_N\|_{L^2(\Omega)^2} \|\boldsymbol{v}_N\|_{L^2(\Omega)^2}.
\end{aligned} \tag{4.12}$$

Proof We have, with obvious notation,

$$\begin{aligned}
K_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N) &= ((\omega_N u_{Ny}, v_{Nx}))_N - ((\omega_N u_{Nx}, v_{Ny}))_N\\
&= \big((\mathcal{I}_N(\omega_N u_{Ny}), v_{Nx})\big)_N - \big((\mathcal{I}_N(\omega_N u_{Nx}), v_{Ny})\big)_N.
\end{aligned}$$

By combining the Cauchy–Schwarz inequalities with (3.7), we obtain

$$|K_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N)| \leq c\, \|\mathcal{I}_N(\omega_N \boldsymbol{u}_N)\|_{L^2(\Omega)^2} \|\boldsymbol{v}_N\|_{L^2(\Omega)^2}.$$

The following result can easily derived from its one-dimensional analogue (see [7, (13.28)]).

$$\forall \varphi_M \in \mathbb{P}_M(\Omega_k), \quad \|\mathcal{I}_N^k \varphi_M\|_{L^2(\Omega_k)} \leq c\,(1 + \frac{M}{m(N)})^2\, \|\varphi_M\|_{L^2(\Omega_k)}.$$

Since both products $(\omega_N u_{Nx})_{|\Omega_k}$ and $(\omega_N u_{Ny})_{|\Omega_k}$ belong to $\mathbb{P}_{2N}(\Omega_k)$ and the ratio $\frac{2N}{m(N)}$ is smaller than 2, this gives

$$|K_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N)| \leq c\, \|\omega_N \boldsymbol{u}_N\|_{L^2(\Omega)^2} \|\boldsymbol{v}_N\|_{L^2(\Omega)^2}.$$

We conclude by using the inequality

$$\|\omega_N \boldsymbol{u}_N\|_{L^2(\Omega)^2} \leq \|\omega_N\|_{L^\infty(\Omega)} \|\boldsymbol{u}_N\|_{L^2(\Omega)^2},$$

together with (4.10).

We are led to make the following assumptions. Here, D stands for the differential operator.

Assumption 4.4 The triple $(\omega, \boldsymbol{u}, p)$ is a solution of problem (2.6) such that the operator $\mathrm{Id} + \mathcal{S}DG(\omega, \boldsymbol{u})$ is an isomorphism of $\mathcal{X}$.

Note that this assumption can equivalently be written as follows (this requires the inf-sup condition (2.16)). For any data $\boldsymbol{g}$ in the dual space of $H_0(\mathrm{div}, \Omega)$, the linearized problem

Find $(\theta, \boldsymbol{w}, r)$ *in* $H_0^1(\Omega) \times H_0(\mathrm{div}, \Omega) \times L_0^2(\Omega)$ *such that*

$$\begin{aligned}
&\forall \boldsymbol{v} \in H_0(\mathrm{div}, \Omega), && a(\theta, \boldsymbol{w}; \boldsymbol{v}) + K(\omega, \boldsymbol{w}; \boldsymbol{v}) + K(\theta, \boldsymbol{u}; \boldsymbol{v}) + b(\boldsymbol{v}, r) = \langle \boldsymbol{g}, \boldsymbol{v}\rangle,\\
&\forall q \in L_0^2(\Omega), && b(\boldsymbol{w}, q) = 0,\\
&\forall \varphi \in H_0^1(\Omega), && c(\theta, \boldsymbol{w}; \varphi) = 0,
\end{aligned} \tag{4.13}$$

has a unique solution with norm bounded by a constant times $\|\boldsymbol{g}\|_{H_0(\mathrm{div},\Omega)'}$. It yields the local uniqueness of the solution $(\omega, \boldsymbol{u}, p)$ but is much less restrictive than the conditions for its global uniqueness; see Remark 2.6.

Assumption 4.5 The solution $(\omega, \boldsymbol{u}, p)$ of problem (2.6) introduced in Assumption 4.4 belongs to $H^{s+1}(\Omega) \times H^s(\Omega)^2 \times H^s(\Omega)$, $s > 0$.

Relying on this last assumption and taking $\tilde{N}$ equal to the integer part of $2\mu\, N - 1$, we can also construct from the arguments in [2, §4] a pair $(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)$ in $\mathbb{C}_{\tilde{N}} \times V_{\tilde{N}}$ (with obvious definitions for these new spaces) which satisfies

$$\|(\omega - \tilde{\omega}_N, \boldsymbol{u} - \tilde{\boldsymbol{u}}_N)\|_{\mathcal{X}} \leq c\, \tilde{N}^{-s}\, \|(\omega, \boldsymbol{u})\|_{H^{s+1}(\Omega)\times H^s(\Omega)^2}. \tag{4.14}$$

Note that estimate (4.14) makes sense only when $\tilde{N} \geq 2$.

Let $\mathcal{L}(\mathcal{X}_N)$ denote the space of endomorphisms on $\mathcal{X}_N$. We are now in a position to state and prove the following lemma.

Lemma 4.6 *If Assumptions* 4.4 *and* 4.5 *are satisfied, there exists an integer* N_0 *such that, for all* $N \geq N_0$*, the operator* $\mathrm{Id} + \mathcal{S}_N DG_N(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)$ *is an isomorphism of* $\mathcal{X}_N$*. Moreover the norm of its inverse operator is bounded independently of* N*.*

Proof We write the expansion

$$\begin{aligned} &\mathrm{Id} + \mathcal{S}_N DG_N(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N) = \mathrm{Id} + \mathcal{S} DG(\omega, \boldsymbol{u}) - \big(\mathcal{S} - \mathcal{S}_N\big) DG(\omega, \boldsymbol{u}) \\ &- \mathcal{S}_N \left(DG(\omega, \boldsymbol{u}) - DG(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)\right) - \mathcal{S}_N \left(DG(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N) - DG_N(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)\right). \end{aligned} \tag{4.15}$$

Moreover, it follows from the definition of G and G_N that, for all $(\theta_N, \boldsymbol{w}_N)$ in $\mathcal{X}_N$ and $\boldsymbol{v}_N$ in V_N,

$$\begin{aligned} \langle DG(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N) \cdot (\theta_N, \boldsymbol{w}_N), \boldsymbol{v}_N \rangle &= K(\tilde{\omega}_N, \boldsymbol{w}_N; \boldsymbol{v}_N) + K(\theta_N, \tilde{\boldsymbol{u}}_N; \boldsymbol{v}_N), \\ \langle DG_N(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N) \cdot (\theta_N, \boldsymbol{w}_N), \boldsymbol{v}_N \rangle &= K_N(\tilde{\omega}_N, \boldsymbol{w}_N; \boldsymbol{v}_N) + K_N(\theta_N, \tilde{\boldsymbol{u}}_N; \boldsymbol{v}_N). \end{aligned}$$

Owing to the choice of $(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)$, all products $\big((\tilde{\omega}_N \times \boldsymbol{w}_N) \cdot \boldsymbol{v}_N\big)_{|\Omega_k}$ and $\big((\theta_N \times \tilde{\boldsymbol{u}}_N) \cdot \boldsymbol{v}_N\big)_{|\Omega_k}$ belong to $\mathbb{P}_{2m(N)-1}(\Omega_k)$, so that the last term in (4.15) vanishes. By combining (4.8) and Lemma 4.1, we also have

$$\begin{aligned} &\|\mathcal{S}_N \left(DG(\omega, \boldsymbol{u}) - DG(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)\right) \cdot (\theta_N, \boldsymbol{w}_N)\|_{\mathcal{X}} \\ &\leq c\, |\log N|^{\frac{1}{2}} \left(\|\omega - \tilde{\omega}_N\|_{H^1(\Omega)} \|\boldsymbol{w}_N\|_{L^2(\Omega)^2} + \|\theta_N\|_{H^1(\Omega)} \|\boldsymbol{u} - \tilde{\boldsymbol{u}}_N\|_{L^2(\Omega)^2}\right). \end{aligned}$$

Thus, applying estimate (4.14) yields

$$\lim_{N \to +\infty} \|\mathcal{S}_N \left(DG(\omega, \boldsymbol{u}) - DG(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)\right)\|_{\mathcal{L}(\mathcal{X}_N)} = 0. \tag{4.16}$$

Finally, it follows from Assumption 4.5 that, when $(\theta, \boldsymbol{w})$ runs through the unit ball of $\mathcal{X}$, $DG(\omega, \boldsymbol{u}).(\theta, \boldsymbol{w})$ belongs to a compact subset of $L^2(\Omega)^2$, so that the next property is derived from (4.8) and (4.9) by standard arguments

$$\lim_{N \to +\infty} \|\big(\mathcal{S} - \mathcal{S}_N\big) DG(\omega, \boldsymbol{u})\|_{\mathcal{L}(\mathcal{X}_N)} = 0. \tag{4.17}$$

Thanks to Assumption 4.4, if γ denotes the norm of the inverse of $\mathrm{Id} + \mathcal{S} DG(\omega, \boldsymbol{u})$, choosing N large enough for the quantities in (4.16) and (4.17) to be smaller than $\frac{1}{4\gamma}$ gives the desired property with the norm of the inverse of $\mathrm{Id} + \mathcal{S}_N DG_N(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)$ smaller than 2γ.

Lemma 4.7 *The following Lipschitz property holds:*

$$\forall(\omega_N^*, \boldsymbol{u}_N^*) \in \mathcal{X}_N,$$
$$\|\mathcal{S}_N\big(DG_N(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N) - DG_N(\omega_N^*, \boldsymbol{u}_N^*)\big)\|_{\mathcal{L}(\mathcal{X}_N)} \le c\,|\log N|^{\frac{1}{2}}\,\|(\tilde{\omega}_N - \omega_N^*, \tilde{\boldsymbol{u}}_N - \boldsymbol{u}_N^*)\|_{\mathcal{X}}. \tag{4.18}$$

Proof We have

$$\langle\big(DG_N(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N) - DG_N(\omega_N^*, \boldsymbol{u}_N^*)\big)\cdot(\theta_N, \boldsymbol{w}_N), \boldsymbol{v}_N\rangle$$
$$= K_N(\tilde{\omega}_N - \omega_N^*, \boldsymbol{w}_N; \boldsymbol{v}_N) + K_N(\theta_N, \tilde{\boldsymbol{u}}_N - \boldsymbol{u}_N^*; \boldsymbol{v}_N).$$

So combining (4.8) and Lemma 4.3 leads to the desired property.

Lemma 4.8 *Assume that the data* $\boldsymbol{f}$ *belong to* $H^\sigma(\Omega)^2$, $\sigma > 1$. *If Assumption* 4.5 *is satisfied, the following estimate holds:*

$$\|(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N) + \mathcal{S}_N G_N(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)\|_{\mathcal{X}}$$
$$\le c(\omega, \boldsymbol{u})\left(N^{-s}\,\|(\omega, \boldsymbol{u})\|_{H^{s+1}(\Omega)\times H^s(\Omega)^2} + N^{-\sigma}\,\|\boldsymbol{f}\|_{H^\sigma(\Omega)^2}\right) \tag{4.19}$$

for a constant $c(\omega, \boldsymbol{u})$ *only depending on the solution* $(\omega, \boldsymbol{u})$.

Proof From Eq. (4.4), we derive

$$\|(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N) + \mathcal{S}_N G_N(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)\|_{\mathcal{X}} \le \|(\omega - \tilde{\omega}_N, \boldsymbol{u} - \tilde{\boldsymbol{u}}_N)\|_{\mathcal{X}} + \|\big(\mathcal{S} - \mathcal{S}_N\big)G(\omega, \boldsymbol{u})\|_{\mathcal{X}}$$
$$+ \|\mathcal{S}_N\left(G(\omega, \boldsymbol{u}) - G(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)\right)\|_{\mathcal{X}} + \|\mathcal{S}_N\left(G(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N) - G_N(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)\right)\|_{\mathcal{X}}.$$

The bound for the first term in the right-hand side obviously follows from (4.14). By combining estimate (4.9) and Assumption 4.5, we also derive

$$\|\big(\mathcal{S} - \mathcal{S}_N\big)G(\omega, \boldsymbol{u})\|_{\mathcal{X}} \le c\,N^{-s}\,\|(\omega, \boldsymbol{u})\|_{H^{s+1}\Omega)\times H^s(\Omega)^2}.$$

On the other hand,

$$K(\omega, \boldsymbol{u}; \boldsymbol{v}_N) - K(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N; \boldsymbol{v}_N)$$
$$= K(\omega - \tilde{\omega}_N, \boldsymbol{u}; \boldsymbol{v}_N) + K(\omega, \boldsymbol{u} - \tilde{\boldsymbol{u}}_N; \boldsymbol{v}_N) - K(\omega - \tilde{\omega}_N, \boldsymbol{u} - \tilde{\boldsymbol{u}}_N; \boldsymbol{v}_N).$$

Moreover it follows from Assumption 4.5 that $(\omega, \boldsymbol{u})$ belongs to $L^\infty(\Omega) \times L^q(\Omega)^2$ for some $q > 2$. So, combining (4.8) and (4.14) with Remark 4.2 and a modified version of it taking into account this further regularity yields, with obvious notation for $c(\omega, \boldsymbol{u})$,

$$\|\mathcal{S}_N\left(G(\omega, \boldsymbol{u}) - G(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)\right)\|_{\mathcal{X}} \le c(\omega, \boldsymbol{u})\,N^{-s}$$
$$(1 + N^{-s}\,|\log N|^{\frac{1}{2}})\,\|(\omega, \boldsymbol{u})\|_{H^{s+1}\Omega)\times H^s(\Omega)^2}.$$

Finally, it follows from the exactness property (3.6) and the choice of $(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N)$ that, for all $\boldsymbol{v}_N$ in $\mathbb{D}_N$, the quantities $K(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N; \boldsymbol{v}_N)$ and $K_N(\tilde{\omega}_N, \tilde{\boldsymbol{u}}_N; \boldsymbol{v}_N)$

coincide. Thus, if Π_{N-1} denotes the orthogonal projection operator from $L^2(\Omega)$ onto the space of functions such that their restrictions to all Ω_k, $1 \leq k \leq K$, belong to $\mathbb{P}_{N-1}(\Omega_k)$, adding and subtracting the quantity $\Pi_{N-1}\boldsymbol{f}$ in the last term and using (4.8) and (3.7) lead to

$$\|\mathcal{S}_N\left(G(\tilde{\omega}_N,\tilde{\boldsymbol{u}}_N)-G_N(\tilde{\omega}_N,\tilde{\boldsymbol{u}}_N)\right)\|_{\mathcal{X}} \leq c\left(\|\boldsymbol{f}-\Pi_{N-1}\boldsymbol{f}\|_{L^2(\Omega)^2}+\|\boldsymbol{f}-\mathcal{I}_N\boldsymbol{f}\|_{L^2(\Omega)^2}\right).$$

The standard approximation properties of the operators Π_{N-1} and $\mathcal{I}_N$ [7, Thms. 7.1 & 14.2] yield

$$\|\mathcal{S}_N\left(G(\tilde{\omega}_N,\tilde{\boldsymbol{u}}_N)-G_N(\tilde{\omega}_N,\tilde{\boldsymbol{u}}_N)\right)\|_{\mathcal{X}} \leq c\,N^{-\sigma}\;\|\boldsymbol{f}\|_{H^\sigma(\Omega)^2}.$$

The desired bound is then derived by combining the previous estimates.

We are now in a position to prove the error estimate.

Theorem 4.9 *Assume that the data $\boldsymbol{f}$ belong to $H^\sigma(\Omega)^2$, $\sigma > 1$, and that the solution $(\omega, \boldsymbol{u}, p)$ of problem* (2.6) *satisfies Assumptions* 4.4 *and* 4.5. *Then, there exist an integer $N_\diamond$ and a constant $c_\diamond$ such that, for all $N \geq N_\diamond$, problem* (3.10) *has a unique solution $(\omega_N, \boldsymbol{u}_N, p_N)$ such that*

$$\|\omega-\omega_N\|_{H^1(\Omega)}+\|\boldsymbol{u}-\boldsymbol{u}_N\|_{H(\mathrm{div},\Omega)} \leq c_\diamond\,|\log N|^{-\frac{1}{2}}. \tag{4.20}$$

Moreover this solution satisfies the following error estimate:

$$\begin{aligned}&\|\omega-\omega_N\|_{H^1(\Omega)}+\|\boldsymbol{u}-\boldsymbol{u}_N\|_{H(\mathrm{div},\Omega)}+\|p-p_N\|_{L^2(\Omega)}\\ &\leq c(\omega,\boldsymbol{u})\left(N^{-s}\left(\|\omega\|_{H^{s+1}(\Omega)}+\|\boldsymbol{u}\|_{H^s(\Omega)^2}+\|p\|_{H^s(\Omega)}\right)+N^{-\sigma}\;\|\boldsymbol{f}\|_{H^\sigma(\Omega)^2}\right)\end{aligned} \tag{4.21}$$

for a constant $c(\omega, \boldsymbol{u})$ only depending on the solution $(\omega, \boldsymbol{u})$.

Proof Combining Lemmas 4.6–4.8 with the Brezzi–Rappaz–Raviart theorem [8] (see also [13, Chap. IV, Thm 3.1]) yields that, for N sufficiently large, problem (3.15) has a unique solution $(\omega_N, \boldsymbol{u}_N)$ which satisfies (4.20) and the first part of (4.21). Moreover, thanks to the discrete inf-sup condition (3.20), there exists a unique p_N in $\mathbb{M}_N$ such that

$$\forall \boldsymbol{v}_N \in \mathbb{D}_N, \quad b_N(\boldsymbol{v}_N, p_N) = (\boldsymbol{f}, \boldsymbol{v}_N)_N - a_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N) - K_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N),$$

whence the existence and local uniqueness result follows. Moreover, we have, for any q_N in $\mathbb{M}_N$,

$$\begin{aligned}b_N(\boldsymbol{v}_N, p_N-q_N) &= b(\boldsymbol{v}_N, p-q_N) - \langle \boldsymbol{f}, \boldsymbol{v}_N\rangle + (\boldsymbol{f}, \boldsymbol{v}_N)_N\\ &\quad + a(\omega-\omega_N, \boldsymbol{u}-\boldsymbol{u}_N; \boldsymbol{v}_N) + (a-a_N)(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N)\\ &\quad + K(\omega, \boldsymbol{u}; \boldsymbol{v}_N) - K_N(\omega_N, \boldsymbol{u}_N; \boldsymbol{v}_N),\end{aligned}$$

so that the estimate for $\|p-p_N\|_{L^2(\Omega)}$ follows from (3.20), a triangle inequality and the same arguments as in the proof of Lemma 4.8.

Estimate (4.21) is fully optimal and justifies both the choice of the discretization and the use of over-integration.

5 The iterative algorithm and its convergence

Applying Newton's method to problem (4.7) consists in solving iteratively the equation

$$(\omega_N^\ell, \boldsymbol{u}_N^\ell) = (\omega_N^{\ell-1}, \boldsymbol{u}_N^{\ell-1}) \\ - \big(\mathrm{Id} + \mathcal{S}_N DG_N(\omega_N^{\ell-1}, \boldsymbol{u}_N^{\ell-1})\big)^{-1} \big((\omega_N^{\ell-1}, \boldsymbol{u}_N^{\ell-1}) + \mathcal{S}_N G_N(\omega_N^{\ell-1}, \boldsymbol{u}_N^{\ell-1})\big). \tag{5.1}$$

Multiplying both sides of this equation by $\mathrm{Id}+\mathcal{S}_N DG_N(\omega_N^{\ell-1}, \boldsymbol{u}_N^{\ell-1})$ and using the inf-sup condition (3.20), we observe that this equation can equivalently be written as follows. Given an initial guess $(\omega_N^0, \boldsymbol{u}_N^0)$ in $\mathbb{C}_N \times \mathbb{D}_N$, we solve the following problem for $\ell \geq 1$:

Find $(\omega_N^\ell, \boldsymbol{u}_N^\ell, p_N^\ell)$ *in* $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$ *such that*

$$\begin{aligned} \forall \boldsymbol{v}_N \in \mathbb{D}_N, \quad & a_N(\omega_N^\ell, \boldsymbol{u}_N^\ell; \boldsymbol{v}_N) + K_N(\omega_N^{\ell-1}, \boldsymbol{u}_N^\ell; \boldsymbol{v}_N) + K_N(\omega_N^\ell, \boldsymbol{u}_N^{\ell-1}; \boldsymbol{v}_N) \\ & -K_N(\omega_N^{\ell-1}, \boldsymbol{u}_N^{\ell-1}; \boldsymbol{v}_N) + b_N(\boldsymbol{v}_N, p_N^\ell) = ((\boldsymbol{f}, \boldsymbol{v}_N))_N, \\ \forall q_N \in \mathbb{M}_N, \quad & b_N(\boldsymbol{u}_N^\ell, q_N) = 0, \\ \forall \varphi_N \in \mathbb{C}_N, \quad & c_N(\omega_N^\ell, \boldsymbol{u}_N^\ell; \varphi_N) = 0. \end{aligned} \tag{5.2}$$

It is readily checked that, for each value of ℓ, problem (5.2) results into a square linear system.

The convergence of this method can easily be derived from [8] (see [13, Chap. IV, Thm. 6.3]) owing to Lemmas 4.6 and 4.7.

Theorem 5.1 *Assume that the solution* $(\omega, \boldsymbol{u}, p)$ *of problem* (2.6) *satisfies Assumption* 4.4. *Then there exist an integer* N_* *and a constant* c_* *such that, for all* $N \geq N_*$ *and for any initial guess* $(\omega_N^0, \boldsymbol{u}_N^0)$ *in* $\mathbb{C}_N \times \mathbb{D}_N$ *such that*

$$\|\omega - \omega_N^0\|_{H(\mathbf{curl},\Omega)} + \|\boldsymbol{u} - \boldsymbol{u}_N^0\|_{H(\mathrm{div},\Omega)} \leq c_* \,|\log N|^{-\frac{1}{2}}, \tag{5.3}$$

problem (5.2) *for each* $\ell \geq 1$ *has a unique solution* $(\omega_N^\ell, \boldsymbol{u}_N^\ell, p_N^\ell)$. *Moreover the sequence* $(\omega_N^\ell, \boldsymbol{u}_N^\ell, p_N^\ell)_\ell$ *converges in a quadratic way towards the unique solution* $(\omega_N, \boldsymbol{u}_N, p_N)$ *of problem* (3.10) *satisfying* (4.20), *in the sense that*

$$\|(\omega_N^\ell, \boldsymbol{u}_N^\ell) - (\omega_N, \boldsymbol{u}_N)\|_{\mathcal{X}} \leq c\, \|(\omega_N^{\ell-1}, \boldsymbol{u}_N^{\ell-1}) - (\omega_N, \boldsymbol{u}_N)\|_{\mathcal{X}}^2, \tag{5.4}$$

for a constant $c < 1$.

As standard for Newton's method, the key point is to exhibit an initial guess $(\omega_N^0, \boldsymbol{u}_N^0)$ satisfying (5.3). In order to do that, we decided to use a continuation method. For simplicity, we set $\lambda = \frac{1}{\nu}$ and define the modified bilinear form

$$\tilde{a}_N(\omega_N^\ell, \boldsymbol{u}_N^\ell; \boldsymbol{v}_N) = \frac{1}{\nu}\, a_N(\omega_N^\ell, \boldsymbol{u}_N^\ell; \boldsymbol{v}_N).$$

We also introduce a pseudo-pressure $\tilde{p}_N^\ell = \frac{1}{\nu}\, p_N^\ell$. Thus, problem (5.2) can equivalently be written as:

Find $(\omega_N^\ell, \boldsymbol{u}_N^\ell, \tilde{p}_N^\ell)$ *in* $\mathbb{C}_N \times \mathbb{D}_N \times \mathbb{M}_N$ *such that*

$$\begin{aligned}
\forall \boldsymbol{v}_N \in \mathbb{D}_N, \quad & \tilde{a}_N(\omega_N^\ell, \boldsymbol{u}_N^\ell; \boldsymbol{v}_N) + \lambda\, K_N(\omega_N^{\ell-1}, \boldsymbol{u}_N^\ell; \boldsymbol{v}_N) + \lambda\, K_N(\omega_N^\ell, \boldsymbol{u}_N^{\ell-1}; \boldsymbol{v}_N) \\
& \quad -\lambda\, K_N(\omega_N^{\ell-1}, \boldsymbol{u}_N^{\ell-1}; \boldsymbol{v}_N) + b_N(\boldsymbol{v}_N, \tilde{p}_N^\ell) = \lambda\, ((\boldsymbol{f}, \boldsymbol{v}_N))_N, \\
\forall q_N \in \mathbb{M}_N, \quad & b_N(\boldsymbol{u}_N^\ell, q_N) = 0, \\
\forall \varphi_N \in \mathbb{C}_N, \quad & c_N(\omega_N^\ell, \boldsymbol{u}_N^\ell; \varphi_N) = 0.
\end{aligned} \tag{5.5}$$

Next, we fix a sample of parameters $(\lambda_m)_{0 \le m \le M}$ such that

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_M = \lambda. \tag{5.6}$$

We denote by $(\omega_N(\lambda_m)^\ell, \boldsymbol{u}_N(\lambda_m)^\ell, \tilde{p}_N(\lambda_m)^\ell)$ the solution of problem (5.5) with λ replaced by λ_m. We also fix an integer $L \geq 1$. Next, we use the following algorithm.

(i) Initial step. For $\lambda_0 = 0$, we observe that the solution $(\omega_N(0), \boldsymbol{u}_N(0), \tilde{p}_N(0))$ of the Stokes problem is zero.
(ii) Iterative step. Assuming that $(\omega_N(\lambda_{m-1})^L, \boldsymbol{u}_N(\lambda_{m-1})^L)$ is known, we take the intermediate initial guess $(\omega_N(\lambda_m)^0, \boldsymbol{u}_N(\lambda_m)^0)$ equal to $(\omega_N(\lambda_{m-1})^L, \boldsymbol{u}_N(\lambda_{m-1})^L)$ and solve problem (5.5) with λ equal to λ_m for $\ell = 1, \ldots, L$.

The iterative step is performed until $m = M - 1$ and the initial guess $(\omega_N^0, \boldsymbol{u}_N^0)$ is taken equal to $(\omega_N(\lambda_{M-1})^L, \boldsymbol{u}_N(\lambda_{M-1})^L)$.

The mapping: $\lambda \mapsto (\omega_N(\lambda), \boldsymbol{u}_N(\lambda), p_N(\lambda))$, where $(\omega_N(\lambda), \boldsymbol{u}_N(\lambda), p_N(\lambda))$ is the solution of problem (3.10) with $\nu = \frac{1}{\lambda}$, is clearly Lipschitz-continuous on any bounded interval of $\mathbb{R}_+$. So it can be checked that, when $\max_{1 \le m \le M} \lambda_m - \lambda_{m-1}$ is sufficiently small and for L sufficiently large, the previous algorithm provides an initial guess $(\omega_N^0, \boldsymbol{u}_N^0)$ satisfying (5.3) for one of the solutions $(\omega, \boldsymbol{u}, p)$ of problem (2.6). However, for the numerical experiments, we work with low values of M and L, and also smaller values of N than the final one. But, in any case, convergence seems likely.

6 Numerical experiments

Problem (5.2) is very similar to a discrete Stokes problem. So we refer to [2, §5] for a detailed description of its implementation. As also explained in this reference, the global system (which is not symmetric) is solved via a GMRES method with local preconditioners, so that it need not be assembled.

We first check the convergence of the discretization in the case of Taylor–Green flow. The domain is the square $\Omega =]-1,1[^2$ divided into two rectangles $\Omega_1 =]-1,0[\times]-1,1[$ and $\Omega_2 =]0,1[\times]-1,1[$. The exact solution is given by

$$\boldsymbol{u}(x,y) = \begin{pmatrix} -\sin(\pi x)\,\cos(\pi y) \\ \cos(\pi x)\,\sin(\pi y) \end{pmatrix}, \qquad p(x,y) = \cos^2(\pi x) + \cos^2(\pi y), \tag{6.1}$$

and the viscosity ν is taken equal to 10^{-2}. Figure 1 presents the log of the three errors

$$\|\omega - \omega_N\|_{H^1(\Omega)}, \qquad \|\boldsymbol{u} - \boldsymbol{u}_N\|_{H(\mathrm{div},\Omega)}, \qquad \|p - p_N\|_{L^2(\Omega)},$$

as a function of N, for N varying from 5 to 20, after $L = 6$ Newton iterations. When compared with finite element results for a similar test (see [9]), these curves confirm the exponential accuracy of spectral element methods for smooth solutions.

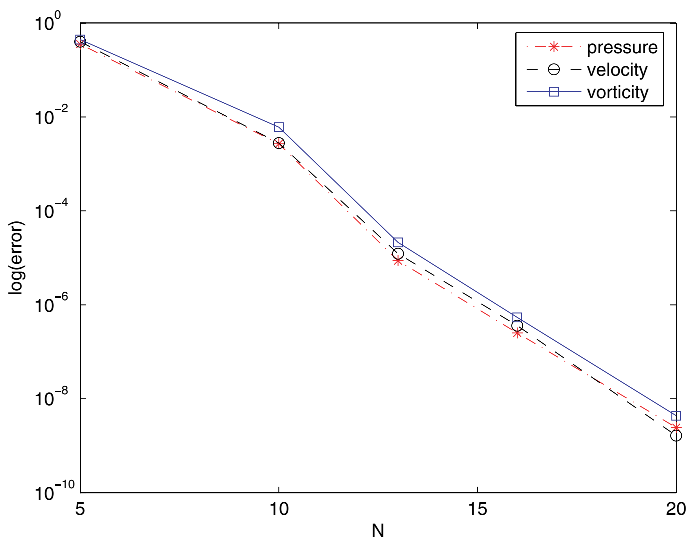

Fig. 1 The error curves for a Taylor–Green flow

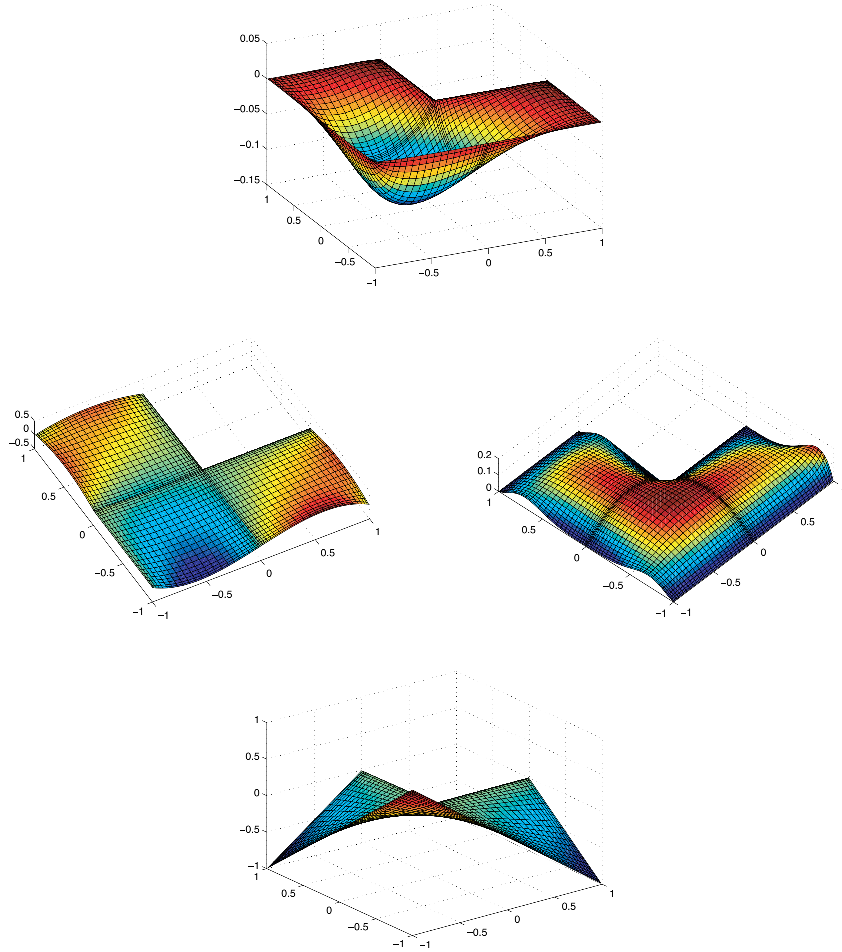

Fig. 2 The isovalues of the vorticity, velocity and pressure for the L-shaped domain

In the numerical experiments that we now present, the homogeneous boundary conditions on the velocity are replaced by

$$\boldsymbol{u} \cdot \boldsymbol{n} = k \quad \text{on } \partial\Omega, \tag{6.2}$$

where the datum k satisfies the compatibility condition

$$\int_{\partial\Omega} k(\tau)\, d\tau = 0. \tag{6.3}$$

In the discrete case, this condition becomes

$$\boldsymbol{u}_N \cdot \boldsymbol{n} = k_N \quad \text{on } \partial\Omega, \tag{6.4}$$

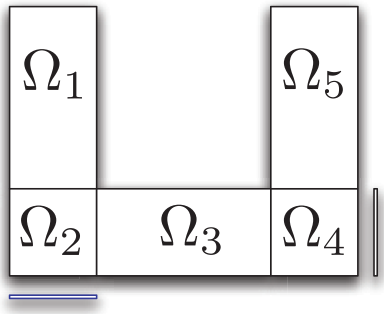

Fig. 3 The U-shaped domain and its partition

where k_N is defined in the following way: $\partial\boldsymbol{\Omega}$ is the union of several segments Γ_ℓ, and each Γ_ℓ is the union of one or several edges of the $\boldsymbol{\Omega}_k$ that we denote by $\Gamma_{\ell,i}$, $1 \le i \le I(\ell)$; then each $k_{N|\Gamma_\ell}$ is defined as the image of $k_{|\Gamma_\ell}$ by the orthogonal projection operator from $L^2(\Gamma_\ell)$ onto the space

$$\mathbb{T}(\Gamma_\ell) = \big\{g_N \in \mathscr{C}^0(\overline{\Gamma}_\ell);\ g_{N|\Gamma_{\ell,i}} \in \mathbb{P}_{N-1}(\Gamma_{\ell,i}),\ 1 \le i \le I(\ell)\big\}. \tag{6.5}$$

A detailed analysis of the corresponding discrete problem is given in [5, §5] for the Stokes problem. In particular, it is explained in [5, Rem. 5.4] that the function k_N still satisfies condition (6.3), so that the corresponding discrete velocity $\boldsymbol{u}_N$ is exactly divergence-free. Of course, extending this analysis to the Navier–Stokes equations requires a Hopf lemma, which is rather technical (see [13, Chap. IV, Lemma 2.3] for instance). We prefer to skip this analysis for brevity.

We now consider the L-shaped domain $\Omega =]-1,1[^2 \setminus [0,1[^2$, divided into three equal squares in the obvious way. We denote by Γ_1 the segment $\{-1\} \times [-1,1]$ and by Γ_2 the segment $\{1\} \times [-1,0]$, and we take the data $\boldsymbol{f}$ and k defined by

$$\boldsymbol{f}(x,y) = (y,0), \quad k(-1,y) = \begin{cases} -y(1-y) & \text{if } \ 0 \le y \le 1, \\ 0 & \text{if } -1 \le y \le 0, \end{cases} \quad \text{on } \Gamma_1,$$
$$k(1,y) = -y(1+y) \quad \text{on } \Gamma_2, \quad k = 0 \ \text{ on } \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2). \tag{6.6}$$

Note that the function k satisfies (6.3) and moreover that its restriction to each Γ_ℓ belongs to the $\mathbb{T}(\Gamma_\ell)$ introduced in (6.5), so that k_N is equal to k on $\partial\boldsymbol{\Omega}$. We take the viscosity ν equal to 10^{-2}. Figure 2 presents, from top to bottom, the values of the vorticity, the two components of the velocity, and the pressure for the discrete solution obtained with $N = 23$.

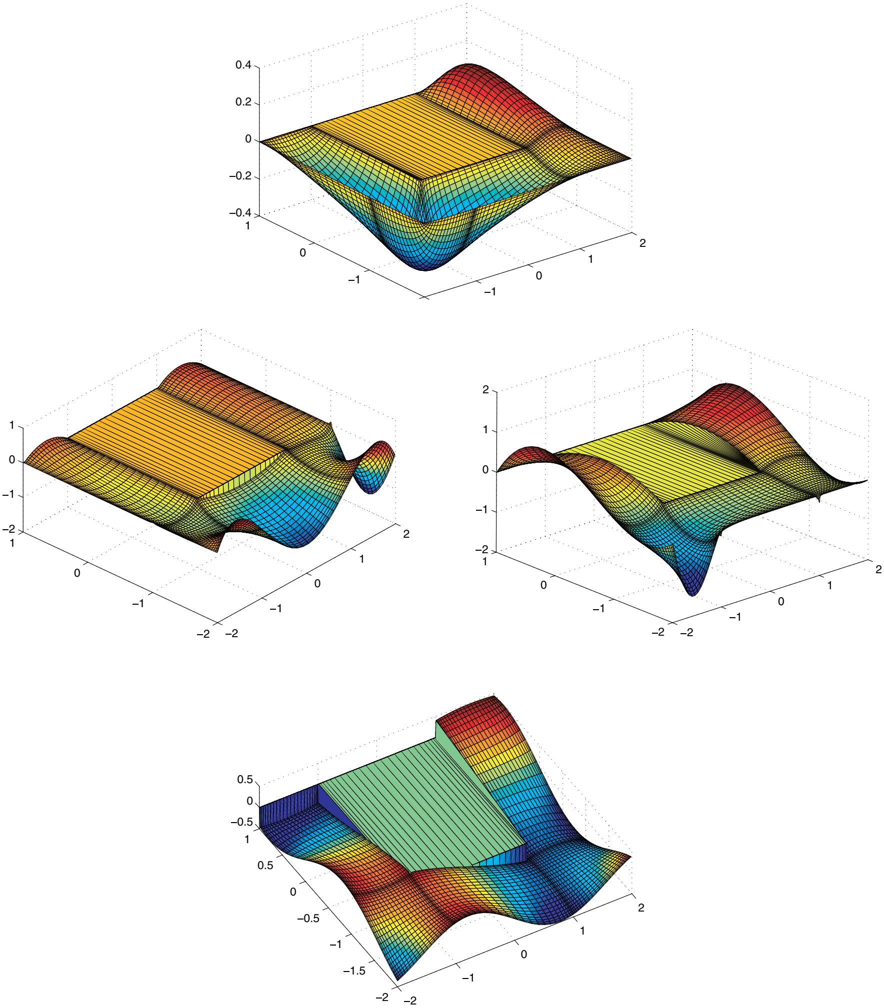

Fig. 4 The isovalues of the vorticity, velocity and pressure for the U-shaped domain

Finally, we consider the U-shaped domain $\Omega =]-2, 2[\times]-2, 1[\setminus[-1, 1[^2$, partitioned into two squares and three rectangles (see Fig. 3); when turning counterclockwise,

$$\Omega_1 =]-2, -1[\times]-1, 1[, \quad \Omega_2 =]-2, -1[\times]-2, -1[, \quad \Omega_3 =]-1, 1[\times]-2, -1[,$$
$$\Omega_4 =]1, 2[\times]-2, -1[, \quad \Omega_5 =]1, 2[\times]-1, 1[.$$

The datum $\boldsymbol{f}$ is taken equal to zero, while the datum k is given by

$$k(x, y) = \begin{cases} -x\ \sin(\pi x) & \text{when } -2 \le x \le -1,\ y = -2, \\ y\ \sin(\pi y) & \text{when } x = 2,\ -2 \le y \le -1, \\ 0 & \text{elsewhere.} \end{cases} \tag{6.7}$$

Accordingly it still satisfies (6.3) and vanishes on $\partial\Omega$ but in parts of $\partial\Omega_2$ and $\partial\Omega_4$ (as indicated in Fig. 3). Still with ν equal to 10^{-2}, Fig. 4 presents, from top to bottom, the values of the vorticity, the two components of the velocity, and the pressure for the discrete solution obtained with $N = 23$.

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