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Regular matrix transformations and rates of convergence of positive linear operators

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Abstract This paper investigates the effects of matrix summability methods on the A-statistical approximation of sequences of positive linear operators defined on the space of all 2π-periodic and continuous functions on the whole real axis. The two main tools used in this paper are A-statistical convergence and the modulus of continuity.

Keywords Regular infinite matrices, A-statistical convergence, rates of A-statistical convergence, positive linear operators, the Korovkin theorem, modulus of continuity.

Mathematics Subject Classification (2000): 41A25, 41A36

1 Introduction

In classical approximation theory, Korovkin [14] (see [1] also) studied the problem of whether a sequence $\{L_n(f)\}$ of positive linear operators converges uniformly to the function f if f is a continuous function in an interval $[a, b]$ in the algebraic case or a continuous function with period 2π in the trigonometric case. He also investigated another problem: how quickly the difference $L_n(f; x) - f(x)$ tends to zero if uniform convergence holds. Later many authors investigated these problems on various function spaces (see, e.g., [2,3, 11,12,16]). Recently, it has been shown that regular (non-matrix) summability transformations are also effective for the approximation of positive linear operators (see [4,5,9]). In particular, using the concept of A-statistical convergence,

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where A is a non-negative regular matrix, instead of ordinary convergence in the approximation theory gives us many advantages since A-statistical convergence is stronger than the usual convergence.

If $A := (a_{jn})$ is an infinite summability matrix, then the A-transform of a sequence $x := (x_n)$, denoted by $Ax := ((Ax)_j)$, is defined by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn}x_n$ provided the series converges for each $j \in \mathbb{N}$, the set of all natural numbers [10]. Recall that A is regular if $\lim Ax = L$ whenever $\lim x = L$. Assume that A is a non-negative regular summability matrix. Then $x = (x_n)$ is called A-statistically convergent to L provided that, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0,$$

and we write $\text{st}_A\text{-}\lim x = L$ (see [7,13,15]). Observe that, if we replace A with the identity matrix, then we get ordinary convergence, and also if $A = C_1$, the Cesáro matrix of order one, then the concept coincides with that of statistical convergence [6,8]. Furthermore, Kolk [13] proved that A-statistical convergence is stronger than convergence if the condition $\lim_j \max_n\{a_{jn}\} = 0$ holds.

Let C^* be the space of all 2π-periodic and continuous functions on $\mathbb{R}$, the set of all real numbers. This space is equipped with the usual supremum norm

$$\|f\|_{C^*} = \sup_{x \in \mathbb{R}} |f(x)|, \quad f \in C^*.$$

With this terminology the author [4] proved the following theorem, which corresponds to the A-statistical sense of the first problem above studied by Korovkin.

Theorem A. *Let $A = (a_{nk})$ be a non-negative regular summability matrix, and let $\{L_n\}$ be a sequence of positive linear operators mapping C^* to C^*. Then, for all $f \in C^*$,*

$$\text{st}_A\text{-}\lim \|L_n(f) - f\|_{C^*} = 0$$

if and only if

$$\text{st}_A\text{-}\lim \|L_n(f_\upsilon) - f_\upsilon\|_{C^*} = 0, \quad \upsilon = 0, 1, 2,$$

where $f_0(y) = 1,\ f_1(y) = \cos y,\ f_2(y) = \sin y$.

The main goal of the present paper is to compute the rates of A-statistical approximation in Theorem A. So, by using the A-statistical rates introduced in [5], we investigate the problem of how quickly the difference

$$\|L_n(f) - f\|_{C^*}$$

is A-statistically convergent to zero.

2 A-statistical rates

In [5] various ways of defining rates of convergence in the A-statistical sense were introduced as follows.

Let $A = (a_{jn})$ be a non-negative regular summability matrix and let (p_n) be a positive non-increasing sequence of real numbers. Then:

(A) A sequence $x = (x_n)$ is A-statistically convergent to the number L with rate $o(p_n)$ if, for every $\varepsilon > 0$,

$$\lim_j \frac{1}{p_j} \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0.$$

In this case we write $x_n - L = \text{st}_A\text{-}o(p_n)$, as $n \to \infty$.

(B) If, for every $\varepsilon > 0$,

$$\sup_j \frac{1}{p_j} \sum_{n:|x_n| \geq \varepsilon} a_{jn} < \infty,$$

then x is A-statistically bounded with rate $O(p_n)$, denoted by $x_n = \text{st}_A\text{-}O(p_n)$, as $n \to \infty$.

(C) $x = (x_n)$ is A-statistically convergent to L with rate $o_m(p_n)$, denoted by $x_n - L = \text{st}_A\text{-}o_m(p_n)$, as $n \to \infty$, if, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon p_n} a_{jn} = 0.$$

(D) $x = (x_n)$ is A-statistically bounded with rate $O_m(p_n)$ provided that there is a positive number M such that

$$\lim_j \sum_{n:|x_n| \geq M p_n} a_{jn} = 0.$$

In this case we write $x_n = \text{st}_A\text{-}O_m(p_n)$, as $n \to \infty$.

Note that unfortunately no single definition has become the "standard" for the comparison of rates of summability transforms. The situation becomes even more uncharted when one considers rates of A-statistical convergence. Actually, in definitions (A) and (B), the "rate" is more controlled by the entries of the summability method rather than by the terms of the sequence $x = (x_n)$. For instance, when one takes the identity matrix I, if $a_{nn} = o(p_n)$ then $x_n - L = \text{st}_A\text{-}o(p_n)$, as $n \to \infty$, for any convergent sequence $(x_n - L)$ regardless of how slowly it goes to zero. To avoid such an unfortunate situation one may borrow the concept of convergence in measure from measure theory to define the rate of convergence as in definitions (C) and (D).

Now let $f \in C^*$. As usual, the modulus of continuity of f, denoted by $w(f, \delta)$, is defined to be

$$w(f, \delta) = \sup_{|y-x|<\delta} |f(y) - f(x)|.$$

It is well-known that a necessary and sufficient condition for a function f to belong to C^* is $\lim_{\delta \to 0} w(f, \delta) = 0$. Then we have the following main result.

Theorem 2.1 *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}$ be a sequence of positive linear operators mapping C^* into C^*. Assume that (p_n) and (q_n) are any non-increasing sequences of positive real numbers. If the conditions*

(i) $\|L_n(f_0) - f_0\|_{C^*} = \mathrm{st}_A\text{-}o(p_n)$ *as* $n \to \infty$ *with* $f_0(y) = 1$,
(ii) $w(f, \delta_n) = \mathrm{st}_A\text{-}o(q_n)$ *as* $n \to \infty$ *with* $\delta_n := \sqrt{\|L_n(\varphi_x)\|_{C^*}}$, *where* $\varphi_x(y) = \sin^2(\frac{y-x}{2})$

hold, then, for all $f \in C^*$,

$$\|L_n(f) - f\|_{C^*} = \mathrm{st}_A\text{-}o(r_n) \quad \text{as } n \to \infty, \tag{2.1}$$

where $r_n := \max\{p_n, q_n\}$. Furthermore, a similar result holds when little "o" is replaced by big "O".

Proof Let $f \in C^*$ and fix $x \in [-\pi, \pi]$. Then by Theorem 2.4 of [3, pp. 29–30], we may write, for all $n \in \mathbb{N}$, that

$$\begin{aligned} |L_n(f; x) - f(x)| &\le |f(x)|\,|L_n(f_0; x) - f_0(x)| \\ &\quad + w\left(f, \sqrt{L_n(\varphi_x; x)}\right)\left\{L_n(f_0; x) + \pi\sqrt{L_n(f_0; x)}\right\}. \end{aligned}$$

Hence

$$\begin{aligned} \|L_n(f) - f\|_{C^*} &\le \|f\|_{C^*}\,\|L_n(f_0) - f_0\|_{C^*} + (1+\pi)w(f, \delta_n) \\ &\quad + w(f, \delta_n)\,\|L_n(f_0) - f_0\|_{C^*} \\ &\quad + \pi\, w(f, \delta_n)\sqrt{\|L_n(f_0) - f_0\|_{C^*}} \end{aligned},$$

where $\delta_n := \sqrt{\|L_n(\varphi_x)\|_{C^*}}$. Letting $M := \max\{\|f\|_{C^*}, 1+\pi\}$, we get

$$\begin{aligned} \frac{1}{M}\,\|L_n(f) - f\|_{C^*} &\le \|L_n(f_0) - f_0\|_{C^*} + w(f, \delta_n) \\ &\quad + w(f, \delta_n)\,\|L_n(f_0) - f_0\|_{C^*} \\ &\quad + w(f, \delta_n)\sqrt{\|L_n(f_0) - f_0\|_{C^*}} \end{aligned}. \tag{2.2}$$

Now let $\varepsilon > 0$. Then define the following subsets of the natural numbers:

$$\begin{aligned} D &:= \left\{ n \in \mathbb{N} : \|L_n(f) - f\|_{C^*} \geq \varepsilon \right\}, \\ D_1 &:= \left\{ n \in \mathbb{N} : \|L_n(f_0) - f_0\|_{C^*} \geq \frac{\varepsilon}{4M} \right\}, \\ D_2 &:= \left\{ n \in \mathbb{N} : w(f, \delta_n) \geq \frac{\varepsilon}{4M} \right\}, \\ D_3 &:= \left\{ n \in \mathbb{N} : w(f, \delta_n)\, \|L_n(f_0) - f_0\|_{C^*} \geq \frac{\varepsilon}{4M} \right\}, \\ D_4 &:= \left\{ n \in \mathbb{N} : w(f, \delta_n) \sqrt{\|L_n(f_0) - f_0\|_{C^*}} \geq \frac{\varepsilon}{4M} \right\}. \end{aligned}$$

In this case it follows from (2.2) that

$$D \subseteq D_1 \cup D_2 \cup D_3 \cup D_4. \tag{2.3}$$

Furthermore, consider the sets

$$\begin{aligned} D_3^{'} &:= \left\{ n \in \mathbb{N} : w(f, \delta_n) \geq \sqrt{\frac{\varepsilon}{4M}} \right\}, \\ D_3^{''} &:= \left\{ n \in \mathbb{N} : \|L_n(f_0) - f_0\|_{C^*} \geq \sqrt{\frac{\varepsilon}{4M}} \right\}. \end{aligned}$$

Then it easy to see that $D_3 \subseteq D_3' \cup D_3''$ and $D_4 \subseteq D_1 \cup D_3'$, and so

$$D \subseteq D_1 \cup D_2 \cup D_3' \cup D_3''. \tag{2.4}$$

The inclusion (2.4) implies that, for all $j \in \mathbb{N}$,

$$\frac{1}{r_j} \sum_{n \in D} a_{jn} \leq \frac{1}{r_j} \sum_{n \in D_1} a_{jn} + \frac{1}{r_j} \sum_{n \in D_2} a_{jn} + \frac{1}{r_j} \sum_{n \in D_3'} a_{jn} + \frac{1}{r_j} \sum_{n \in D_3''} a_{jn}.$$

Since $r_n = \max\{p_n, q_n\}$, we immediately get that, for each $j \in \mathbb{N}$,

$$\frac{1}{r_j} \sum_{n \in D} a_{jn} \leq \frac{1}{p_j} \sum_{n \in D_1} a_{jn} + \frac{1}{q_j} \sum_{n \in D_2} a_{jn} + \frac{1}{q_j} \sum_{n \in D_3'} a_{jn} + \frac{1}{p_j} \sum_{n \in D_3''} a_{jn}. \tag{2.5}$$

Now, letting $j \to \infty$ in (2.5) and using the hypotheses (i) and (ii), we obtain

$$\lim_j \frac{1}{r_j} \sum_{n \in D} a_{jn} = 0,$$

which gives (2.1). The proof is complete. □

The above proof can easily be modified to prove the following analog.

Theorem 2.2 *Let* $A = (a_{jn})$, $\{L_n\}$, (δ_n), (p_n) *and* (q_n) *be the same as in Theorem* 2.1. *If the conditions*

(i) $\|L_n(f_0) - f_0\|_{C^*} = \mathrm{st}_A\text{-}o_m(p_n)$ *as* $n \to \infty$ *with* $f_0(y) = 1$
(ii) $w(f, \delta_n) = \mathrm{st}_A\text{-}o_m(q_n)$ *as* $n \to \infty$

hold, then, for all $f \in C^*$

$$\|L_n(f) - f\|_{C^*} = \mathrm{st}_A\text{-}o_m(s_n) \quad \text{as } n \to \infty,$$

where $s_n := \max\{p_n, q_n, p_n q_n\}$. *Furthermore, a similar result holds when little "*o_m*" is replaced by big "*O_m*".*

Concluding remarks

1) Specializing the sequences (p_n) and (q_n) in Theorem 2.1 or Theorem 2.2, we can easily get Theorem A. Thus, Theorems 2.1 and 2.2 give us the rates of A-statistical convergence of the operators L_n from C^* to C^*.
2) Replacing the matrix A by the identity matrix and taking $p_n = q_n = 1$ for all $n \in \mathbb{N}$, we get the ordinary rate of convergence of the operators L_n.

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