J Wood Sci (2006) 52:20–24
DOI 10.1007/s10086-005-0732-9

ORIGINAL ARTICLE

**Yoshitaka Kubojima · Mario Tonosaki
Hiroshi Yoshihara**

Young's modulus obtained by flexural vibration test of a wooden beam with inhomogeneity of density

Received: January 17, 2005 / Accepted: March 28, 2005 / Published online: February 1, 2006

Abstract The object of this study was to investigate the inhomogeneity of density within a beam from a relationship between the dynamic Young's moduli from the Euler-Bernoulli elementary theory of bending (E_n) and resonance mode numbers (n), which is plotted as the "E–n" diagram in this article. Rectangular beams with dimensions of 300 (L) × 25 (R) × 5 mm (T) of Sakhalin spruce (*Picea glehnii* Mast.), Sitka spruce (*Picea sitchensis* Carr.), Japanese red pine (*Pinus densiflora* Zieb. et Zucc.) and white oak (*Cyclobalanopsis myrsinaefolia* Oerst.) were used for specimens. Small parts of beams were replaced with a small portion of another species to examine the influence of the inhomogeneity of density on E_n. A free–free flexural vibration test was undertaken and E_n was calculated by the Euler-Bernoulli theory. The resonance frequency of a specimen with inhomogeneity of density was simulated by modal analysis. The density distribution in the longitudinal direction of the specimen for which E_n did not decrease monotonically with n was obtained. From the modal analysis, the inhomogeneity of density was equivalent to a concentrated mass attached to a uniform beam. The pattern of the E–n diagram was changed by replacing a part of the specimen with another species. Specimens for which E_n did not decrease monotonically with n had a high density part because of indented rings, knots, or resin.

Key words Euler-Bernoulli bending theory · Flexural vibration test · Inhomogeneity of density · Resonance mode number · Young's modulus

Y. Kubojima (✉) M. Tonosaki
Forestry and Forest Products Research Institute, P.O. Box 16, Tsukuba Norin Kenkyu Danchi-nai, Ibaraki 305-8687, Japan
Tel. +81-29-873-3211; Fax +81-29-874-3720
e-mail: kubojima@ffpri.affrc.go.jp

H. Yoshihara
Faculty of Science and Engineering, Shimane University, Matsue 690-8504, Japan

Introduction

In a previous report,[1] we investigated the influence of a concentrated mass attached to a wooden beam on its Young's modulus (E_n) based on the Euler-Bernoulli elementary theory of bending. As a result, E_n decreased with an increase in the resonance mode number (n) without iron pieces used as additional mass, but there was no such tendency when the concentrated mass was bonded to the specimen: the relationship between E_n and n, which is called the "E–n diagram" in this article, corresponded to the position of the bonded mass. This occurred because the constants used in the Euler-Bernoulli theory were changed by the additional mass. Thus, an equation to correct the influence of the additional mass was developed. This equation was effective wherever concentrated mass existed in a beam.

Given these results, it is possible that the inhomogeneity of density within a wooden beam can be shown by the E–n diagram. In this study, a wooden rectangular beam that was partly replaced by a small piece of another species was used as the simplest model of the specimen with the inhomogeneity of density and E_n was investigated.

Experimental

Specimens

Sakhalin spruce (*Picea glehnii* Masters), Sitka spruce (*Picea sitchensis* Carriége), Japanese red pine (*Pinus densiflora* Ziebold et Zuccarini), and white oak (*Cyclobalanopsis myrsinaefolia* Oerstedt) were used for specimens. The dimensions of the base wood were 300 mm (longitudinal, L) in length, 25 mm (radial, R) in width, and 5 mm (tangential, T) in thickness. A 25 (R) × 5 (T) × 20 mm (L) piece was eliminated from the base wood, and a small piece of the same dimensions of another species was fixed with an epoxy resin adhesive between the two parts of the base wood. The small

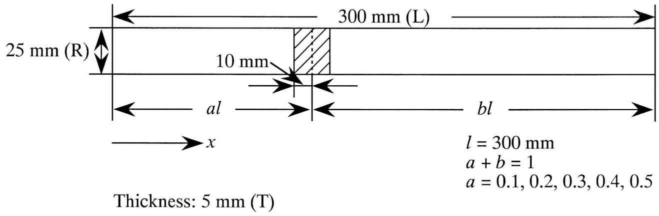

Fig. 1. Specimen with inhomogeneity of density. The *hatched* part shows inhomogeneity of density

piece was inserted at $x = 0.1l$, $0.2l$, $0.3l$, $0.4l$, and $0.5l$, where l is the total length of the specimen (Fig. 1).

There were several specimens for which E_n did not decrease monotonically with n, as shown later. It is thought that such specimens had inhomogeneity of density. Hence, these specimens were divided into small pieces after measuring E_n and the change in density in the L-direction was examined.

The specimens were conditioned at 20°C, and 65% relative humidity (RH) for several months. The tests were conducted under the same conditions.

Vibration test

To obtain E_n by bending, free–free flexural vibration tests were conducted by the following procedure. The test beam was suspended by two threads at the nodal positions of the free–free vibration corresponding to its resonance mode. Vibration was excited in the direction of the thickness at one end by a hammer. Motion of the beam was detected by a microphone at the other end. The signal was processed through a fast Fourier transform (FFT) digital signal analyzer to yield high-resolution resonance frequencies.

Estimation of the pattern of the E–n diagram by vibration test

According to the Euler-Bernoulli elementary theory of bending, E_n is calculated as follows:

$$E_n = \frac{48\pi^2 \rho l^4}{m_n^4 h^2} f_n^2 \tag{1}$$

where ρ, f, m, and h are density, resonance frequency, constant, and thickness, respectively, and subscript n is resonance mode number.

The constant m_n of a beam with additional mass (M) satisfies the following equation:[1]

$$\begin{aligned}&(\cos m_n \cosh m_n - 1) - \frac{1}{2}\mu m_n \big[(\cos am_n \cosh am_n + 1)\\ &(\sin bm_n \cosh bm_n - \cos bm_n \sinh bm_n)\\ &+ (\cos bm_n \cosh bm_n + 1)(\sin am_n \cosh am_n\\ &- \cos am_n \sinh am_n)\big] = 0\end{aligned} \tag{2}$$

where a and b ($a + b = 1$) indicate the position of the inserted small piece (Fig. 1), $\mu = M/\rho Al$ (A is cross-sectional area) is the ratio of the additional mass to the mass of the beam. In this study, μ was used as the index of the inhomogeneity of density and was calculated as follows:

$$\mu = \left(W^{\mathrm{M}} - W^{\mathrm{U}}\right)/W^{\mathrm{U}} \tag{3}$$

where W^{U} and W^{M} are weights of a specimen before and after inserting the small piece, respectively. In other words, the change in weight caused by inserting the small piece was regarded as the additional mass. We used the density of the base wood before and after inserting the small piece supposing that the inserted small piece is composed of the base wood and additional mass. The superscripts "U" and "M" mean "uniform" and "additional mass," respectively, and do not represent mathematical exponents.

In the case of $\mu = 0$, Eq. 2 becomes:

$$\cos m_n \cosh m_n - 1 = 0. \tag{4}$$

This is the equation for a beam that has both ends free without the additional mass.

The E_n of wood with the inhomogeneity of density was estimated as follows. The resonance frequency of a uniform beam before dividing into two parts (f_n^{U}) was measured. Then, because only the resonance frequency and m_n in Eq. 1 were changed by the additional mass, the resonance frequencies of a beam with the inhomogeneity of density (f_n^{M}) were estimated by Eq. 5:

$$f_n^{\mathrm{M}} = \left(m_n^{\mathrm{M}}/m_n^{\mathrm{U}}\right)^2 f_n^{\mathrm{U}} \tag{5}$$

where, m_n^{U} and m_n^{M} are calculated from Eqs. 4 and 2, respectively. Then E_n is estimated from Eq. 1 using f_n^{M} and m_n^{U}. The estimated E_n was compared with the experimental value of a beam with the small piece from Eq. 1 using the measured resonance frequency and m_n^{U}.

Modal analysis

To simulate the resonance frequency, modal analysis was conducted using the finite element method program ANSYS 5.7 (ANSYS Inc.). The resonance frequencies of the first to fifth modes were calculated with the block Lanczos method. The dimensions of the specimen were 25mm in width, 5mm in thickness, and 300mm in length. Dividing the length into 20 two-dimensional isotropic elastic beam-type elements, the size of each element was 25mm in width, 5mm in thickness, and 15mm in length. The element numbers ranged from 1 at one end to 20 at the other end. The parameters for the modal analysis were as follows: density was 0.461g/cm^3 and Young's modulus was 14.0GPa. These values were the averages of the Sakhalin spruce used in this study. Simulating the case of the inhomogeneity of density at $x = 0.1l$, $0.2l$, $0.3l$, $0.4l$, and $0.5l$, the density of the adjacent elements (2, 3), (4, 5), (6, 7), (8, 9), and (10, 11) were all doubled ($\mu = 0.1$). The simulated resonance frequency was compared with the theoretical

Table 1. Theoretical and simulated resonance frequencies

μ	a	n	m_n	f_n (Hz)		
				Theoretical	Simulated	Ratio
0	(Initial)	1	4.730	315	315	1.000
		2	7.853	867	866	0.999
		3	10.996	1701	1697	0.998
		4	14.137	2811	2803	0.997
		5	17.279	4200	4182	0.996
0.1	0.1	1	4.630	301	300	0.997
		2	7.825	861	853	0.991
		3	10.993	1700	1673	0.984
		4	14.042	2774	2731	0.984
		5	16.967	4049	4035	0.997
0.1	0.2	1	4.726	314	313	0.997
		2	7.748	844	842	0.998
		3	10.649	1595	1603	1.005
		4	13.813	2684	2687	1.001
		5	17.180	4152	4081	0.983
0.1	0.3	1	4.699	311	310	0.997
		2	7.580	808	810	1.002
		3	10.871	1662	1654	0.995
		4	14.081	2789	2745	0.984
		5	16.724	3934	3970	1.009
0.1	0.4	1	4.620	300	300	1.000
		2	7.707	836	834	0.998
		3	10.901	1671	1658	0.992
		4	13.657	2624	2641	1.006
		5	17.264	4192	4089	0.975
0.1	0.5	1	4.582	295	296	1.003
		2	7.853	867	862	0.994
		3	10.577	1573	1582	1.006
		4	14.137	2811	2760	0.982
		5	16.705	3925	3966	1.010

The theoretical and simulated resonance frequencies were obtained from Eqs. 1–3 and the modal analysis, respectively. Ratios were obtained by the simulated / theoretical resonance frequencies. Inhomogeneity existed at $x = al$. m_n was obtained by Eq. 2. μ, Degree of inhomogeneity of density (Eq. 3); n, resonance mode number; f_n, resonance frequency

value calculated using density = 0.461 g/cm^3, Young's modulus = 14.0 GPa, and Eqs. 1 and 2.

Results and discussion

Table 1 shows the simulated resonance frequencies for the modal analysis, which were compared with the theoretical values. The results of the modal analysis were similar to the theoretical values. This indicates that the inhomogeneity of density can be equivalent to a concentrated mass attached to a uniform specimen in Eq. 2.

Figure 2 shows the influence of the inhomogeneity of density on the E–n diagram. Before inserting the small piece (crosses), E_n decreased with an increase in the resonance mode number.

The E_n after inserting the small piece was estimated by the procedure mentioned above. The averages of the Young's modulus and density of the base wood (Sakhalin spruce) were 0.461 g/cm^3 and 14.0 GPa, respectively, as mentioned above and those of the inserted small piece (white oak) were 0.842 g/cm^3 and 18.4 GPa, respectively; these were used for calculation.

Table 2. Nodal points of the free–free vibration

Resonance mode				
1st	2nd	3rd	4th	5th
0.2242l	0.1321l	0.0944l	0.0735l	0.0601l
0.7758l	0.5000l	0.3558l	0.2768l	0.2265l
	0.8679l	0.6442l	0.5000l	0.4091l
		0.9506l	0.7232l	0.5909l
			0.9265l	0.7735l
				0.9399l

Values calculated from Eq. 4
l, length of a beam

It can be considered that E_n is not subject to the additional mass if the small piece is inserted at a nodal position of the specimen. Hence, from Table 2, the position of the inserted small piece and the estimated E_n, which is similar to the initial value, are supposed to be the following cases: $x = 0.1l$: E_2 and E_3; $x = 0.2l$: E_1 and E_5; $x = 0.3l$: E_1, E_3, and E_4; $x = 0.4l$: E_3 and E_5; $x = 0.5l$: E_2 and E_4. In other cases, the Young's modulus will be decreased by the inhomogeneity of density: $x = 0.1l$: E_1, E_4, and E_5; $x = 0.2l$: E_2, E_3, and E_4; $x = 0.3l$: E_2 and E_5; $x = 0.4l$: E_1, E_2, and E_4; $x = 0.5l$: E_1, E_3, and E_5. The estimated E_n (unfilled circles in Fig. 2) showed

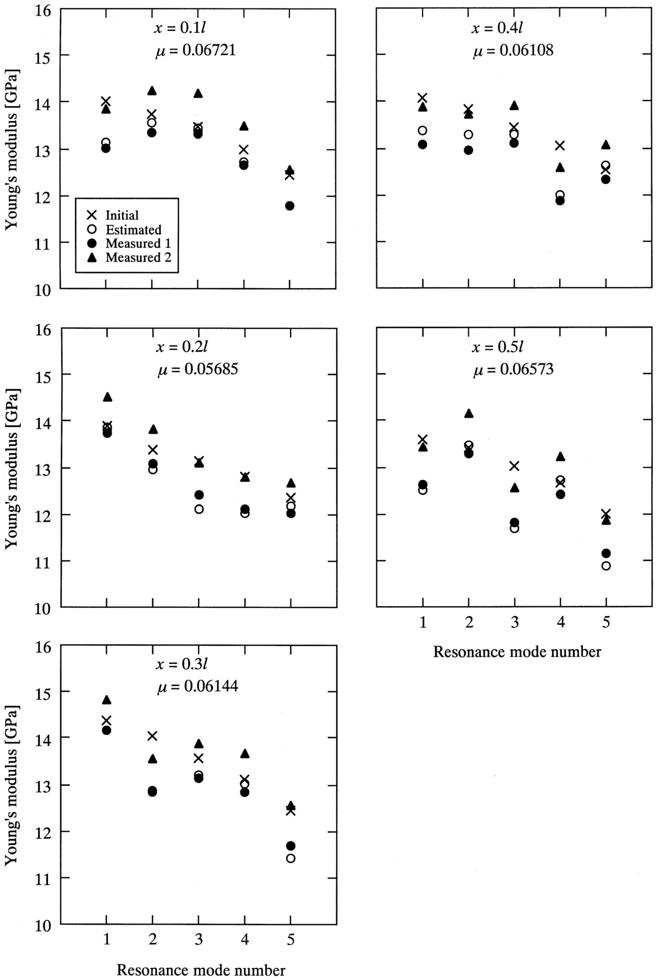

Fig. 2. Influence of the inhomogeneity of density on the E–n diagram. The base wood and small piece were Sakhalin spruce and white oak, respectively. The positions of the inhomogeneity of density were x = 0.1l, 0.2l, 0.3l, 0.4l, and 0.5l. *Crosses*, initial values before inserting a small piece; *unfilled circles*, values estimated from Eq. 1 using f_n^M and m_n^U; *filled circles*, measured values using the density before inserting a small piece; *triangles*, measured values using the density after inserting a small piece

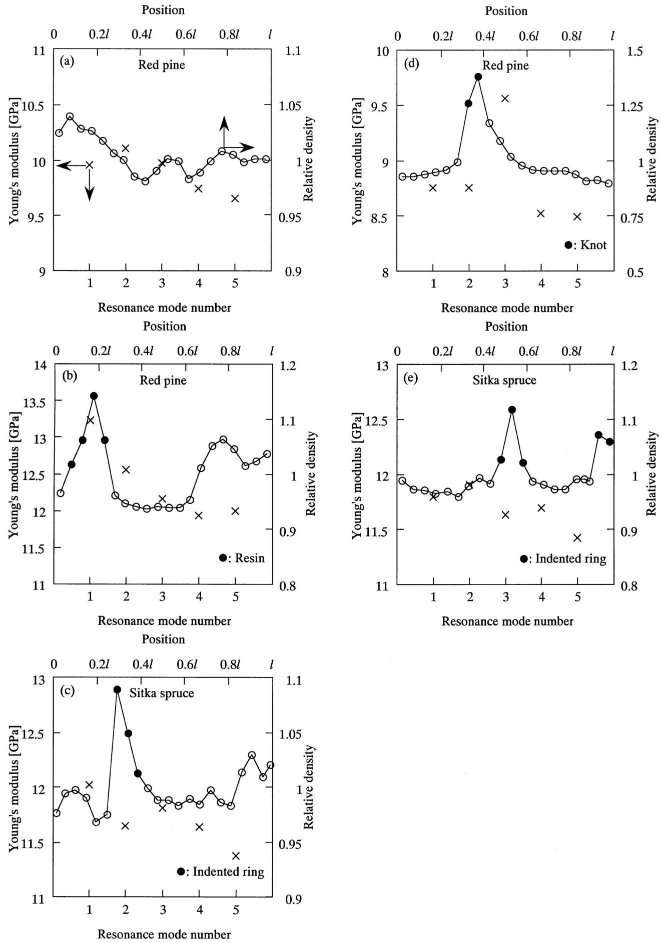

Fig. 3. Estimation of the inhomogeneity of density by the E–n diagram for different samples. *Crosses*, E_n; *circles*, relative density (each small piece / initial specimen)

such tendencies. The E_n was compared with the measured value (filled circles in Fig. 2). Both Young's moduli were very similar. The unfilled and filled circles of $x = 0.3l$ were overlapped. Hence, Eq. 2 is effective when considering that the inserted small piece consists of the base wood and additional mass. The influence of adhesive layers was not thought to be serious.

In practical cases, because it is difficult to guess the degree of the inhomogeneity of density within a beam, that is to say, to estimate μ by its looks, the average density of the beam is used for Eq. 1. Thus, E_n using the average density after inserting the small piece is also shown in Fig. 2 (filled triangles). Although the Young's moduli were changed by the density used as a matter of course, the patterns of the E–n diagram themselves were the same.

The position of the inhomogeneity was predicted based on the E–n diagram shown in Fig. 2 as follows. Figure 3 shows typical examples of the density change in the L-direction of the specimens for which E_n did not decrease monotonically with n. A small piece was not inserted in the case of Fig. 3. Although the ratio of the density of each small piece to that of the initial specimen is different from μ, this value is useful to express the density distribution.

In the case of Fig. 3a, because E_1 is smaller than E_2, the density around $x = 0.1l$ would be larger than that at other positions. As expected, the density around $x = 0.1l$ is the highest. This is a similar pattern to that seem in Fig. 2, $x = 0.1l$. In the case of Fig. 3b, it was supposed that inhomogeneity of density existed at $x = 0.2l$ because E_5 is larger than E_4. The density is largest around $x = 0.2l$ and resin existed there (Fig. 2, $x = 0.2l$). In the cases of Fig. 3c,d, the large density part is around $x = 0.3l$ because $E_2 < E_3$. The prediction was correct and there were indented rings[2–4] in the case of Fig. 3c and a knot in the case of Fig. 3d around $x = 0.3l$ (Fig. 2, $x = 0.3l$ and $x = 0.4l$). In the case of Fig. 3e, given that E_n values of even-resonance modes increased, the high density part would be around $x = 0.5l$. Such parts were around $x = 0.5l$ and 0.9l and there were indented rings there. We think that the influence on E_n was larger around $x = 0.5l$ than around $x = 0.9l$ (Fig. 2, $x = 0.5l$).

In order to precisely determine whether inhomogeneity of density exists in a beam, and if so, where its position is, a huge number of beams should be examined. However, it is safe to state that inhomogeneity of density exists at some place in a beam when E_n does not decrease with n. There is some research on the influence of partial density variation on the flexural vibrational properties.[5–8]

Additionally, Fig. 3 indicates that the indented rings as well as the resin and knot were causes of the inhomogeneity that were significant enough to change E_n. Although it is natural that wooden beams with resin and knots are not used as a clear specimen, beams with indented rings also cannot be used. The existence of the indented rings should be checked, especially when the shear modulus of a thin wooden beam is measured with the flexural vibration test.[9–12] If E_n does not decrease with an increase in n because of the indented rings, the shear modulus cannot be calculated adequately with the Goens-Hearmon regression method[13,14] based on Timoshenko theory of bending.[15]

In the case of lumber for actual use, this simplest model may not be useful because such long lumber has inhomogeneity of density at various positions in it. However, specimens without any defect are needed in basic experiments: when some treatments to wood, such as chemical treatments and compressing are investigated with high accuracy, small clear specimens are used. In these cases, unless properties of the specimens before the treatment can be obtained properly, the effect and mechanism of the treatment cannot be understood. Thus, it is useful to guess the inhomogeneity of density of a small clear specimen. Specimens with inhomogeneity of density can be eliminated before the treatment from the pattern of the E–n diagram.

Conclusions

The inhomogeneity of density within a beam was investigated. Simulating the resonance frequency with the modal analysis, the inhomogeneity of density was equivalent to a concentrated mass attached at a uniform beam. The pattern of the E–n diagram was changed by replacing a part of the specimen with another species. The E_n of a specimen with inhomogeneity of density did not decrease monotonically with n. Therefore, we think that the E–n diagram is useful for finding the inhomogeneity of density.

References

1. Kubojima Y, Tonosaki M, Yoshihara H (2005) Effect of additional mass on Young's modulus of a wooden beam. J Test Eval (in press)
2. Fukazawa K, Ohtani J (1984) Indented rings in sitka spruce. Proceedings of Pacific Regional Wood Anatomy Conference, Tsukuba, pp 28–30
3. Ohtani J, Fukazawa K, Fukumorita T (1987) SEM observations on indented rings. IAWA Bull 8:113–124
4. Larson PR (1994) The vascular cambium. Springer, Berlin Heidelberg New York, pp 422–423
5. Yang X, Ishimaru Y, Iida I (2002) Application of modal analysis by the transfer function to nondestructive testing of wood III. Detection of knots and estimation of elastic modulus distribution in wood by the curvature ratio of the flexural vibration wave shape. Mokuzai Gakkaishi 48:16–22
6. Kodama Y, Zhang Z (2000) Evaluation of wood quality by wavelet analysis I. Wavelet analysis of the tapping tone of wood with a knot (in Japanese). Mokuzai Gakkaishi 46:197–202
7. Kodama Y, Zhang Z, Kawabata H (2001) Evaluation of wood quality by wavelet analysis II. Estimation of knot size (in Japanese). Mokuzai Gakkaishi 47:473–478
8. Kodama Y, Zhang Z, Kawabata H (2004) Study on evaluation of wood quality using wavelet transform (in Japanese). T Jpn Soc Mech Eng 70:184–191
9. Kubojima Y, Yoshihara H, Ohta M, Okano T (1996) Examination of the method of measuring the shear modulus of wood based on the Timoshenko theory of bending. Mokuzai Gakkaishi 42:1170–1176
10. Kubojima Y, Yoshihara H, Ohta M, Okano T (1997) Accuracy of the shear modulus of wood obtained by Timoshenko's theory of bending. Mokuzai Gakkaishi 43:439–443
11. Yoshihara H, Matsumoto S (1999) Examination of the proper span/depth ratio range in measuring the bending Young's modulus of wood based on the elementary beam theory (in Japanese). Wood Ind 54:269–272
12. Yoshihara H, Kubojima Y, Nagaoka K, Ohta M (1998) Measurement of the shear modulus of wood by static bending tests. J Wood Sci 44:15–20
13. Goens E (1931) Über die Bestimmung des Elastizitätsmodulus von Stäben mit Hilfe von Biegungsschwingungen. Ann Phys 5F 11:649–678
14. Hearmon RFS (1958) The influence of shear and rotatory inertia on the free flexural vibration of wooden beams. Brit J Appl Phys 9:381–388
15. Timoshenko SP (1921) On the correction for shear of the differential equation for transverse vibrations of prismatic bars. Philos Mag 6th Ser 41:744–746