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ORIGINAL PAPER

An analytical probabilistic analysis of slopes based on limit equilibrium methods

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Abstract

Probabilistic analysis of slopes has been used as an effective tool to evaluate uncertainty that is so prevalent in variables. In this paper, the jointly distributed random variables (JDRV) method is used as an analytical method to compare the reliability of four widely used limit equilibrium methods for slope stability analysis. These methods include the simplified Bishop, simplified Janbu, Morgenstern–Price, and Spencer's methods. The selected stochastic parameters are angle of shearing resistance (φ), cohesion intercept (c), and unit weight (γ) of soil, which are modeled using a truncated normal probability distribution function. Geometric parameters such as height and angle of the slope relative to the horizontal are regarded as constant parameters. For reliability assessment, the reliability indices of the limit equilibrium methods for the critical surface with minimum factor of safety are determined by the particle swarm optimization (PSO) technique. It is shown that, among the assessed methods, the Janbu and Bishop methods are those with upper and lower probabilities of failure, respectively, in two conditions with and without considering cross correlation between c and φ.

Keywords Probabilistic analysis · Jointly distributed random variables method · Slope stability · Limit equilibrium method

Introduction

There are various methods used in slope stability analysis. Among these methods, the limit equilibrium method (LEM) of slices (Bishop 1955; Fellenius 1936; Janbu 1954, 1973; Moregenstern 1963; Morgenstern and Price 1965, 1967; Spencer 1967) has attracted considerable attention, because of its simplicity and accuracy. In this method, the ratio of resisting to driving forces on a potential sliding surface is defined as the factor of safety (FS).The limit equilibrium techniques are the most commonly used analytical methods to investigate the stability of landslides.

A slope is considered safe only if the calculated FS clearly exceeds unity. The LEM considers the material of the sliding body as a rigid body (Cheng and Zhou 2015; Zhou and Cheng 2013). However, due to the model and parameter uncertainties, even an FS greater than one does not confirm the safety against failure of the slope. Therefore, it is important to calibrate the deterministic methods considering the effect of different sources of model and parameter uncertainties. A common approach in the stochastic analysis of a slope by the LEM of slices is to determine the reliability index corresponding to the critical surface with the minimum FS.

The reliability analysis of slope stability has attracted considerable attention in the research community in the past few decades (Griffiths and Fenton 2004; Husein Malkawi et al. 2000). Many probabilistic methods have been used for slope stability analysis. These methods can be grouped into five main categories: approximate methods, Monte Carlo simulation (MCS), numerical methods, analytical methods, and artificial intelligence methods.

- Initial research works on the probabilistic evaluation of slope stability were done by using approximate methods. Most of the approximate methods are modified versions of two methods, namely, the first-order second-moment (FOSM) method (Ang and Tang 1984) and the point estimate method (PEM) (Rosenblueth 1975). These approaches require knowledge of the mean and variance of all input variables, as well as the performance function that

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defines the FS (e.g., Bishop's equation). Many attempts have been made to apply the PEM and FOSM method in the reliability analysis of slope stability. Some important researches by these methods are listed in Table 1.

- MCS (Metropolis and Ulam 1949) is a computational algorithm that relies on repeated random sampling to address risk and uncertainty in quantitative analysis and decision-making. This method provides a range of possible outcomes and the probabilities that will occur for any choice of action. Many attempts have been made to analyze the stability of slopes using MCS. Some important researches by this method are listed in Table 1.
- In numerical methods, a deterministic numerical method like the finite element method (FEM) has been merged by probabilistic approaches. These methods can be grouped into two main categories: random finite element method (RFEM) and stochastic finite element method (SFEM). RFEM combines elastoplastic finite-element analysis with random fields generated using the local average subdivision method. SFEM is an extension of the classical deterministic FE approach to the stochastic framework, i.e., to the solution of stochastic (static and dynamic) problems involving finite elements whose properties are random. A number of researches based on RFEM and SFEM are presented in Table 1.
- In analytical methods, the probability density functions (PDFs) of input variables are joined together to derive a mathematical expression for the density function of the FS. These approaches can be grouped into the jointly distributed random variables (JDRV) method (Hoel et al. 1971; Stirzaker 1999; Tijms 2007) and the first-order reliability method (FORM) (Hasofer and Lind 1973). Considerable researches have been done on the application of the FORM to slopes. Limited attempts have been made to apply the JDRV method in the reliability analysis of slope stability, which are listed in Table 1.
- Artificial intelligence is an approach based on the concepts of natural biological evolution to process information. This technique has the capability to respond to input stimuli, produce the corresponding response, and adapt to the changing environment by learning from experience. This method has been applied to the reliability analysis of slope stability. Some important researches using this approach are listed in Table 1.
- The response surface method (RSM) is an approach that models and analyzes by a finite element. The simulation is repeated a limited number of times to give a point estimate of the response corresponding to uncertainties in the model parameters. A graduating function is then fitted to these point estimates (Wong 1985). The approximating function is called the response surface.

In this study, the reliability of four widely used limit equilibrium-based methods [including simplified Bishop (Bishop 1955), simplified Janbu (Janbu 1954, 1973), Morgenstern–Price (Morgenstern and Price 1965, 1967), and Spencer's (Spencer 1967) methods] in the stability analysis of slopes is compared using the JDRV method. For this purpose, the FS relationships for PDFs of the above mentioned methods are derived analytically based on the selected stochastic parameters for any arbitrary slope. In numerical simulation methods such as MCS, the probability distribution of output parameters is obtain by a considerable number of iterations of deterministic analysis. Since iterative slope stability analysis is time-consuming, at the first step of this research, the JDRV method as a substitution method has been used, in which the results are approaching more accurately those of Monte Carlo in a lower computational time. In the next step, using the PDFs and mean values of the stochastic parameters, the critical surface with the minimum FS is determined by the particle swarm optimization (PSO) (Cheng et al. 2007; Kennedy 2010) technique. The reliability indices of the above four methods are calculated in two conditions with and without considering the correlation between c and φ.

Limit equilibrium methods

LEM is the most popular approach in slope stability analysis. This method is well known to be a statically indeterminate problem, and assumptions on the interslice shear forces are required to render the problem statically determinate.

In the LEM of slices, the sliding body is discretized into a number of columns with vertical interfaces (Zhou and Cheng 2015). The actual number of slices depends on the slope geometry and soil profile. Some methods assume a circular slip surface, while others assume an arbitrary noncircular slip surface. Procedures that assume a circular slip surface consider equilibrium of moments about the center of the circle for the entire free body composed of all slices. In contrast, the procedures that assume an arbitrary shape for the slip surface usually consider equilibrium in terms of the individual slices. In this paper, the slope stability analysis is evaluated by using simplified Bishop, simplified Janbu, Morgenstern–Price, and Spencer's (Fredlund and Krahn 1977) methods. These methods are presented in the Appendix.

The JDRV method

The JDRV method is an analytical probabilistic method. In this method, the PDFs of input variables are expressed

Table 1 Literature review of the various methods

Method		Literature review
Approximate methods	PEM	Li (1992); Thornton (1994); Wang and Huang (2012)
	FOSM	Alonso (1976); Christian et al. (1994); Duncan (2000); Hong and Roh (2008); Suchomel and Mašín (2010); Tang et al. (1976); Vanmarcke (1977); Xue and Gavin (2007)
Monte Carlo simulation		Abbaszadeh et al. (2011); Au and Beck (2001, 2003); Au et al. (2010); Cho (2007, 2010); Dai et al. (1993); Duncan (2000); El-Ramly et al. (2002, 2005, 2006); Hassan and Wolff (1999); Husein Malkawi et al. (2000); Salgado and Kim (2014); Tobutt (1982); Wang (2012); Wang et al. (2011)
Numerical methods		Griffiths et al. (2009, 2011); Hicks and Samy (2002); Huang et al. (2010, 2017); Jiang et al. (2014); Liu et al. (2014, 2015); Phoon and Kulhawy (1999); Xu and Low (2006); Zhou et al. (2018)
Analytical methods	FORM	Bhattacharya et al. (2017); Cho (2007, 2013); Goh and Zhang (2012); Hong and Roh (2008); Ji (2014); Low and Tang (1997a, b, 2004, 2007); Low (2007); Low et al. (1998); Wu (2013); Zeng and Jimenez (2014); Zhang and Goh (2012)
	JDRV	Johari and Javadi (2012); Johari and Khodaparast (2013); Johari et al. (2013)
Artificial intelligence methods		Ahangar-Asr et al. (2012); Cho (2009); Cui and Sheng (2005); Ma et al. (2017); Xue and Gavin (2007)
Response surface method		Huang and Zhou (2017); Li et al. (2011, 2016); Low and Tang (2007); Zhang et al. (2013); Zhou and Huang (2018)

mathematically and joined together by statistical relations. By integrating into the adopted model, a mathematical expression of the PDF of the output parameter is derived (Hoel et al. 1971; Johari and Javadi 2012; Johari and Khodaparast 2013; Johari et al. 2013; Stirzaker 1999; Tijms 2007). If the joint PDF of continuous random variables $k_1, k_2,\ldots, k_n$ is $f_{K1,K2,\ldots,Kn}(k_1, k_2,\ldots, k_n)$, the PDF of the output parameter (FS) is:

$$f_{FS}(FS) = \iint_{R_{Xi}} \cdots \int f_{K_1,K_2,\ldots,K_n}(k_1,k_2,\ldots,k_n)\, dk_1\, dk_2 \ldots dk_{i-1}\, dk_{i+1} \ldots dk_n \tag{1}$$

where:

$$f_{K_1,K_2,\ldots,K_n}(k_1,k_2,\ldots,k_n) = |J(u_1,u_2,\ldots,u_n)|.f_{K_1,K_2,\ldots,K_n}(h_1(u_1,u_2,\ldots,u_n),\ldots,h_n(u_1,u_2,\ldots,u_n)) \tag{2}$$

where $u_1, u_2,\ldots, u_n$ are change of variables $k_1, k_2,\ldots, k_n$, h_i is a function of u_i, and $|J(u_1, u_2,\ldots, u_n)|$ is the determinant of $J(u_1, u_2,\ldots, u_n)$:

$$J(u_1,u_2,\ldots,u_n) = \left|\begin{matrix} \frac{\partial k_1}{\partial u_1} & \frac{\partial k_1}{\partial u_2} & \cdots & \frac{\partial k_1}{\partial u_n} \\ \frac{\partial k_2}{\partial u_1} & \frac{\partial k_2}{\partial u_2} & \cdots & \frac{\partial k_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial k_n}{\partial u_1} & \frac{\partial k_n}{\partial u_2} & \cdots & \frac{\partial k_n}{\partial u_n} \end{matrix}\right| \tag{3}$$

Stochastic parameters

To account for the uncertainties in slope stability, three input parameters, including the angle of shearing resistance (φ), cohesion intercept (c), and unit weight (γ), have been defined as stochastic variables. The statistical distributions of these uncertainties have been studied by many researchers. Numerous researchers emphasized that the normal, truncated normal, and lognormal distributions are more compatible with the behavior of soil parameters (Brejda et al. 2000; Fenton and Griffiths 2003; Lumb 1966, 1970; Tobutt 1982). However, other distributions, such as triangular, Gumbel, Weibull, versatile beta, and generalized gamma, are also reported (Christian and Baecher 2002). In this paper, for simplicity in analytical calculations, the truncated normal distributions are used for modeling of the stochastic soil parameters. The parameters related to the geometry of a slope are regarded as constant parameters. The PDFs of truncated normal distributions for the stochastic parameters are as follows (Olive 2008):

$$f_c(c) = \frac{1}{\sigma_c\sqrt{2\pi}}\exp\left(-0.5\left(\frac{c-\mu_c}{\sigma_c}\right)^2\right) \quad c_{\min}\le c\le c_{\max} \tag{4}$$

$$f_\varphi(\varphi) = \frac{1}{\sigma_\varphi\sqrt{2\pi}}\exp\left(-0.5\left(\frac{\varphi-\mu_\varphi}{\sigma_\varphi}\right)^2\right) \quad \varphi_{\min}\le\varphi\le\varphi_{\max} \tag{5}$$

$$f_\gamma(\gamma) = \frac{1}{\sigma_\gamma\sqrt{2\pi}}\exp\left(-0.5\left(\frac{\gamma-\mu_\gamma}{\sigma_\gamma}\right)^2\right) \quad \gamma_{\min}\le\gamma\le\gamma_{\max} \tag{6}$$

where:

$$\begin{cases}\varphi_{\min}=\varphi_{mean}-4\sigma_{\varphi}\\ \varphi_{\max}=\varphi_{mean}+4\sigma_{\varphi}\\ c_{\min}=c_{mean}-4\sigma_{c}\\ c_{\max}=c_{mean}+4\sigma_{c}\\ \gamma_{\min}=\gamma_{mean}-4\sigma_{\gamma}\\ \gamma_{\max}=\gamma_{mean}+4\sigma_{\gamma}\end{cases} \tag{7}$$

where:

φ_{min}	Minimum values of soil angle of shearing resistance
φ_{max}	Maximum values of soil angle of shearing resistance
σ_{φ}	Standard deviation of soil angle of shearing resistance
c_{min}	Minimum values of soil cohesion intercept
c_{max}	Maximum values of soil cohesion intercept
σ_c	Standard deviation of soil cohesion intercept
γ_{min}	Minimum values of soil unit weight
γ_{max}	Maximum values of soil unit weight
σ_{γ}	Standard deviation of soil unit weight

By considering the stochastic variables within the range of their mean plus or minus four times the standard deviation [Eq. (7)], 99.994% of the area beneath the normal density curve is covered. It should be noted that, for choosing the initial data, the following conditions must be observed for the angle of shearing resistance, cohesion intercept, and unit weight of soil in the sliding surface:

$$\begin{cases}\varphi_{mean}-4\sigma_{\varphi}>0\\ c_{mean}-4\sigma_{c}>0\\ \gamma_{mean}-4\sigma_{\gamma}>0\end{cases} \tag{8}$$

Probabilistic analysis

For reliability assessment of the FS of slopes using the JDRV method, the suggested FS equations of simplified Bishop, simplified Janbu, Morgenstern–Price, and Spencer's methods are rewritten into terms of k_1 to k_4. The terms k_1 to k_4 are introduced in Eq. (9). The PDFs of each term and for each method are derived separately.

$$\begin{cases}k_1=c\\ k_2=\tan\varphi\\ k_3=\gamma\\ k_4=FS\end{cases} \tag{9}$$

Using the new form of independent input parameters, the PDFs of k_1 to k_3 are obtained by Eqs. (10) to (12):

$$f_{K_1}(k_1)=\frac{1}{\sigma_c\sqrt{2\pi}}\exp\left(-0.5\left(\frac{k_1-c_{mean}}{\sigma_c}\right)^2\right)\quad c_{\min}\le k_1\le c_{\max} \tag{10}$$

$$f_{K_2}(k_2)=\frac{1}{(1+k_2{}^2)\sigma_{\varphi}\sqrt{2\pi}}\exp\left(-0.5\left(\frac{\tan^{-1}(k_2)-\varphi_{mean}}{\sigma_{\varphi}}\right)^2\right)\quad \tan\varphi_{\min}\le k_2\le \tan\varphi_{\max} \tag{11}$$

$$f_{K_3}(k_3)=\frac{1}{\sigma_{\gamma}\sqrt{2\pi}}\exp\left(-0.5\left(\frac{k_3-\gamma_{mean}}{\sigma_{\gamma}}\right)^2\right)\gamma_{\min}\le k_3\le \gamma_{\max} \tag{12}$$

The derivation of probabilistic relations based on Morgenstern–Price's method is presented as follows (the other methods are presented in the Appendix). The Morgenstern–Price method assumes that the shear forces between slices are related to the normal forces as (Morgenstern and Price 1965, 1967):

$$X=\lambda.f(x).E \tag{13}$$

where X and E are the vertical and horizontal forces between slices, respectively, λ is an unknown scaling factor that is solved for as part of the unknowns, and f(x) is an assumed function that has prescribed values at each slice boundary. In Morgenstern–Price's method, the FS is determined by the following equation (Zhu et al. 2005):

$$FS=\frac{\sum_{i=1}^{n-1}\left(R_i\prod_{j=i}^{n-1}\psi_j\right)+R_n}{\sum_{i=1}^{n-1}\left(T_i\prod_{j=i}^{n-1}\psi_j\right)+T_n} \tag{14}$$

where R_i is the sum of the shear resistances contributed by all the forces acting on the slices except the normal shear interslice

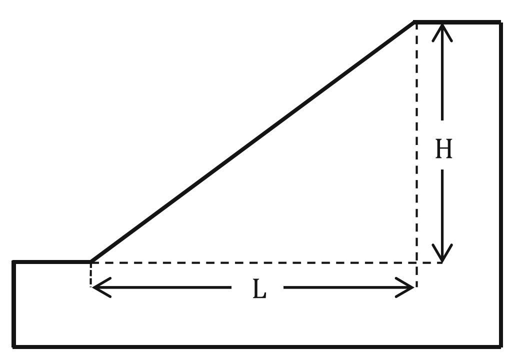

Fig. 1 A typical slope

Table 2 Arbitrary stochastic parameters

Parameters	Mean	Standard deviation	Minimum	Maximum
c (kPa)	10.0	2.0	2.0	18.0
ϕ (°)	28.0	3.0	16.0	40.0
γ (kN/m^3)	18.0	1.0	14.0	22.0

Table 3 Arbitrary deterministic parameters

Height of slope (m)	Horizontal length of slope (m)	γ_{water} (kN/m^3)
12.0	12.0	10.0

forces and T_i is the sum of the components of these forces tending to cause instability (Zhu et al. 2005).

$$\psi_i = [(\sin\alpha_{i+1} - \lambda.f_i.\cos\alpha_{i+1}).\tan\varphi + (\cos\alpha_{i+1} + \lambda.f_i.\sin\alpha_{i+1}).FS]/\phi_i \quad (15)$$

$$\phi_i = (\sin\alpha_i - \lambda.f_i.\cos\alpha_i).\tan\varphi + (\cos\alpha_i + \lambda.f_i.\sin\alpha_i).FS \quad (16)$$

where α_i is the base inclination and f_i is $f(x_i)$ of the ith slice. According to variable conversion, u_1, u_2, and u_3 are defined as independent and arbitrary parameters of variables such as k_1, k_2, and k_3 as follows:

$$\begin{cases} u_1 = g_1(k_1, k_2, k_3) = FS \\ u_2 = g_2(k_2) = k_2 \\ u_3 = g_3(k_3) = k_3 \end{cases} \quad (17)$$

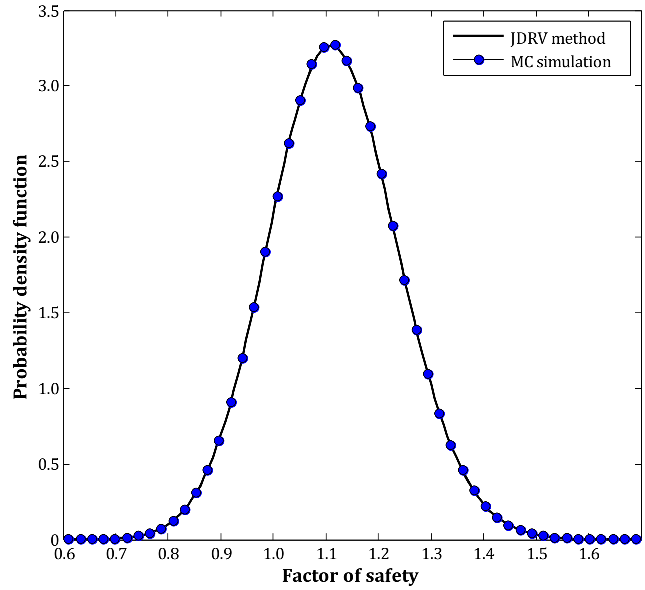

Fig. 2 Probability density function (PDF) of the factor of safety (FS) by simplified Bishop's method

As the function between two series of points (k_1, k_2, k_3) and (u_1, u_2, u_3) is considered as an injective function, the following functions are defined as below. In this case, Eq. (14) is defined by the independent parameter k_1:

$$\begin{cases} k_1 = h_1(u_1, u_2, u_3) = c = \dfrac{u_1.\left(\sum_{i=1}^{n-1}\left[u_3.b_i.h_i.\sin\alpha_i.\prod_{j=i}^{n-1}\psi_j\right] + T_n\right) - \sum_{i=1}^{n-1}\left[u_3.b_i.h_i.\cos\alpha_i.u_2.\prod_{j=i}^{n-1}\psi_j\right] - w_n.\cos\alpha_n.u_2}{\sum_{i=1}^{n-1}\left[b_i.\sec\alpha_i.\prod_{j=i}^{n-1}\psi_j\right] + b_n.\sec\alpha_n} \\ k_2 = h_2(u_2) = u_2 \\ k_3 = h_3(u_3) = u_3 \end{cases} \quad (18)$$

Consequently, according to Eq. (1), the PDF of u_1, u_2, and u_3 is calculated as below:

$$f_{K_4}(k_4) = \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} |J(u_1, u_2, u_3)|.f_{K_1}(h_1(u_1, u_2, u_3)).f_{K_2}(h_2(u_2)).f_{K_3}(h_3(u_3))\ dk_2.dk_3 \quad (19)$$

$$|J(u_1, u_2, u_3)| = \frac{(M_1 + M_6).(M_3 + b_n.\sec\theta_n) - M_7.(FS.(M_1 + w_n.\sin\theta_n) - M_2 - w_n.\cos\theta_n.k_2)}{(M_3 + b_n.\sec\theta_n)^2} \quad (20)$$

where:

$$M_1 = \sum_{i=1}^{n-1}\left[w_i.\sin\theta_i.\prod_{j=i}^{n-1}\psi_j\right] \tag{21}$$

$$M_2 = \sum_{i=1}^{n-1}\left[w_i.cos\theta_i.k_2.\prod_{j=i}^{n-1}\psi_j\right] \tag{22}$$

$$M_3 = \sum_{i=1}^{n-1}\left[w_i.\sec\theta_i.\prod_{j=i}^{n-1}\psi_j\right] \tag{23}$$

$$M_4 = \sum_{j=i}^{n-1}\left[\frac{\left(\frac{d\psi}{dx}\right)_j.\prod_{k=i}^{n-1}\psi_k}{\psi_j}\right] \tag{24}$$

$$M_5 = M_1 + FS.\sum_{i=1}^{n-1}[w_i.\sin\theta_i.M_4] + w_n.\sin\theta_n \tag{25}$$

$$M_6 = -\sum_{i=1}^{n-1}[w_i.\cos\theta_i.k_2.M_4] \tag{26}$$

$$M_7 = \sum_{i=1}^{n-1}[b_i.\sec\theta_i.M_4] \tag{27}$$

Equation (19) is the PDF of the slope stability safety factor, while the integral bounds are specified as below:

$$\begin{cases}\alpha_1 = \min(k_3)\\ \beta_1 = \max(k_3)\end{cases} \tag{28}$$

$$\begin{cases}\alpha_2 = \min(k_2)\\ \beta_2 = \max(k_2)\end{cases} \tag{29}$$

Using the mathematical functions for k_1 to k_3 [Eqs. (10) to (12)] and $fK_1(k_1)$ to $fK_3(k_3)$, a computer program was developed (coded in MATLAB) to determine the PDF curve for the safety factor of slope stability. In addition, for comparison, determination of the safety factor using MCS was also coded in the same computer program. To show the capabilities of the proposed method, an example with arbitrary data is presented in the following sections.

The JDRV method assumes that the parameters are uncorrelated. In this method, the governing mathematical equations cannot be solved by considering the correlation coefficient between the cohesion and the friction angle. To overcome this limitation, in this study, the two parameters c_1 and φ are considered independent with truncated normal distributions and the distribution of parameter c was determined with the correlation coefficient ρ using the following equation:

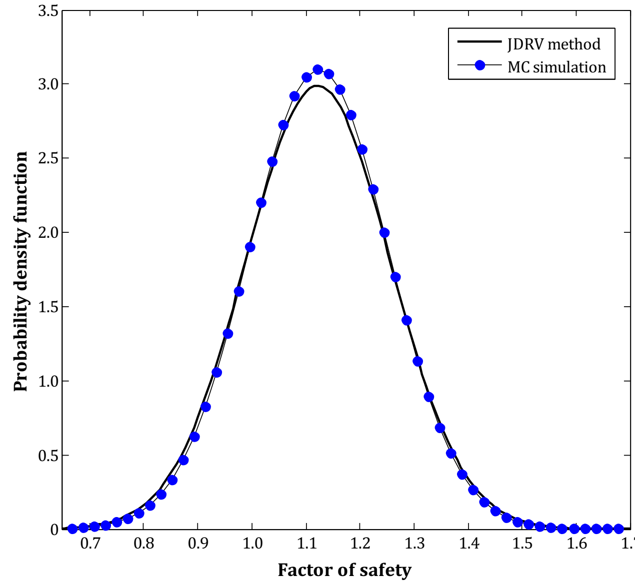

Fig. 4 PDF of the FS by Spencer's method

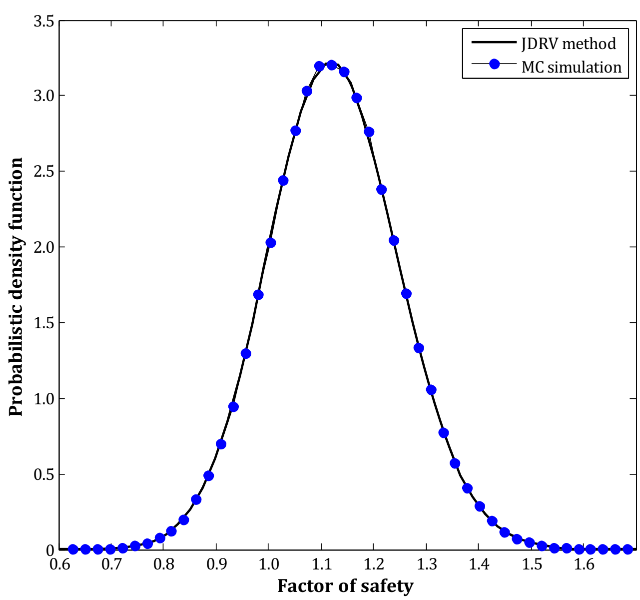

Fig. 3 PDF of the FS by simplified Janbu's method

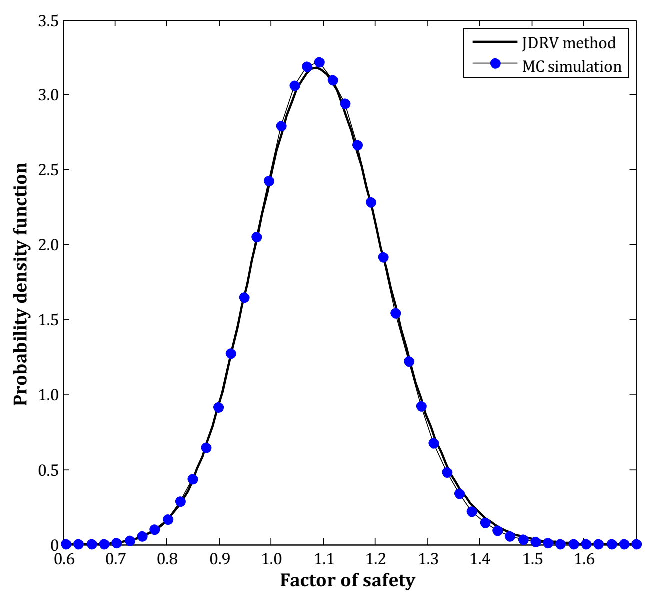

Fig. 5 PDF of the FS by Morgenstern–Price's method

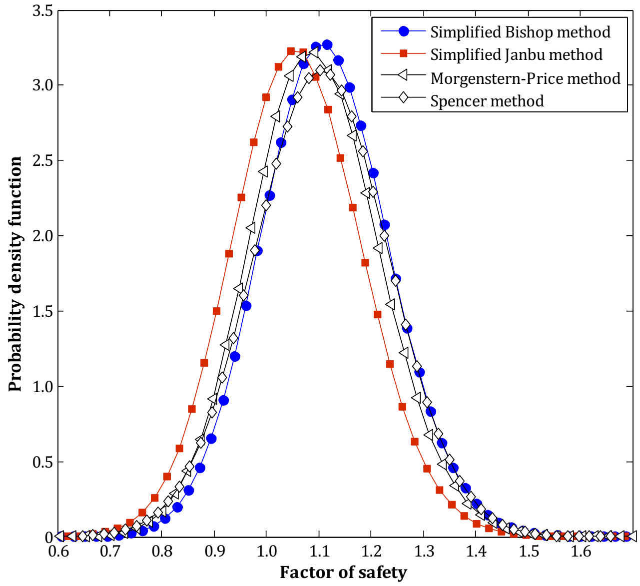

Fig. 6 Comparison of the PDFs of the FS using the four methods

$$c = \rho \times \varphi + \sqrt{1-\rho^2} c_1 \tag{30}$$

Using Eq. (18), it can be seen that the parameter k_1 is a function of u_1, u_2, and u_3. In this equation, the values of k_1 were determined using the numeric values defined for k_2 and k_3 and other input parameters. Accordingly, the values given for k_1 and k_2 in this step can be considered as c_1 and φ in the above equation, respectively. Consequently, the probabilistic distribution of the cohesion can be defined by the given correlation coefficient of ρ with the internal friction angle.

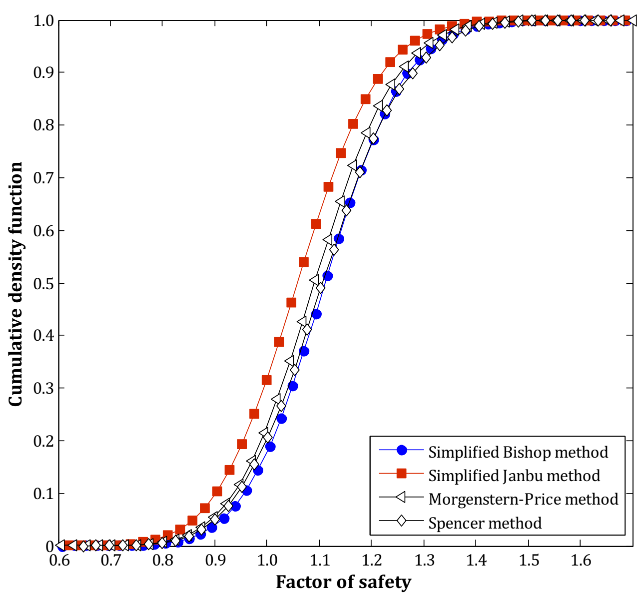

Fig. 7 Comparison of the cumulative distribution functions (CDFs) of the FS using the four methods

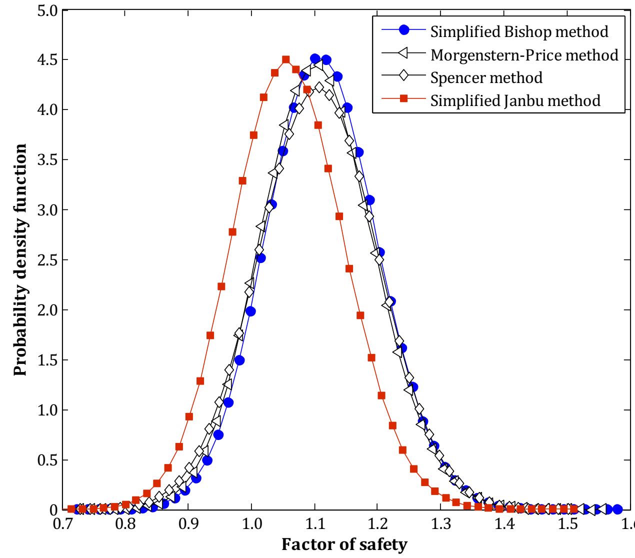

Fig. 8 Comparison of the PDFs of the safety factor of the methods considering the correlation coefficient between c and φ

Illustrative example

To examine the accuracy of the proposed method in determining the PDF of the FS, an illustrative example with arbitrary parameter values is demonstrated. A typical slope shape for this example is shown in Fig. 1. The stochastic parameters with truncated normal distributions are given in Table 2 and the deterministic parameters are given in Table 3.

Probabilistic analysis of slope stability

Using the selected deterministic and mean of stochastic parameters, the slip surface with minimum FS is determined by

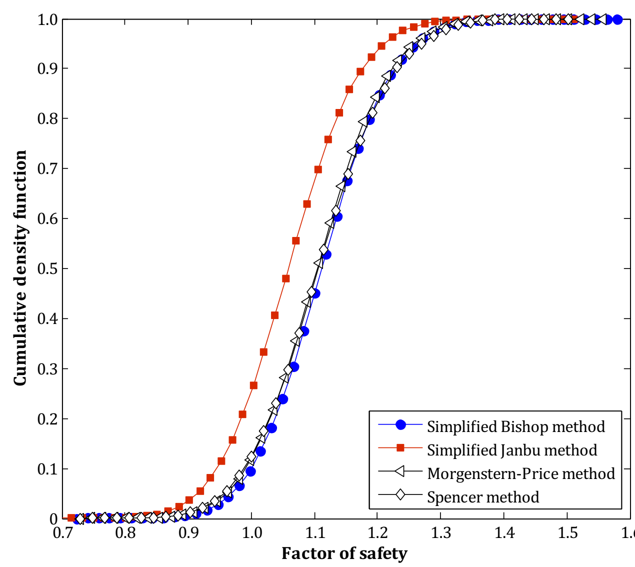

Fig. 9 Comparison of the CDFs of the safety factor of the methods considering the correlation coefficient between c and φ

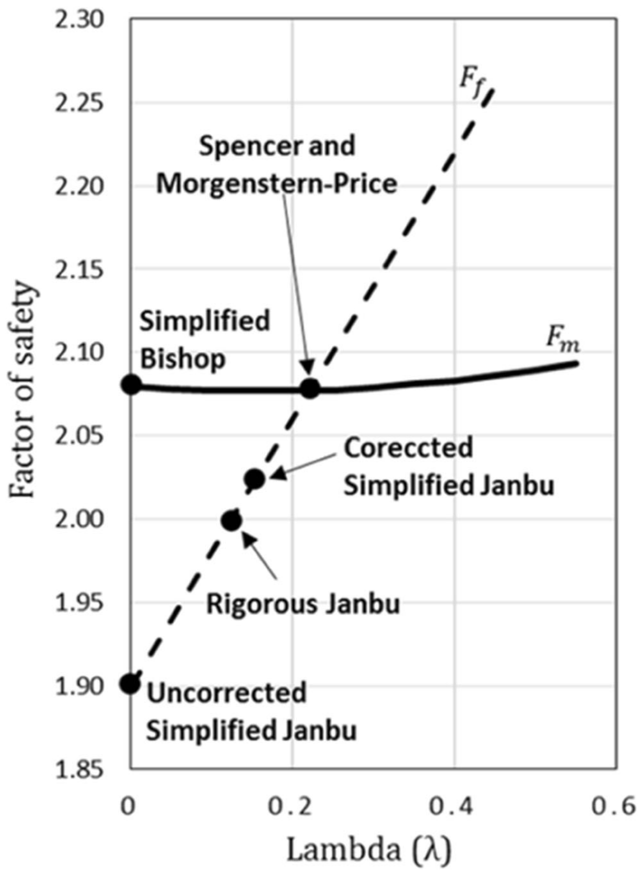

Fig. 10 Comparison of the limit equilibrium methods (LEMs) of slices (Fredlund and Krahn 1977)

the PSO algorithm (Cheng et al. 2007; Kennedy 2010). Using Eqs. (A.1) to (A.26), a computer program was developed (coded in MATLAB) to determine the PDF of slope stability FS. In order to verify the results of the presented methods against those of MCS, the final PDFs for the FS are determined using the same data for both methods. For this purpose, 2,000,000 generations are used for MCS and for the four methods.

The results are shown in Figs. 2, 3, 4, and 5 for simplified Bishop, simplified Janbu, Spencer, and Morgenstern–Price's methods. As can be seen in these figures, the results obtained using the developed methods are very close to those obtained using MCS.

To compare the four slope stability methods (i.e., Bishop, Janbu, Spencer, and Morgenstern–Price), the predictions of the PDF and cumulative distribution function (CDF) of the

Table 4 Comparison of the reliability indices of the four methods

Method	Simplified Bishop	Simplified Janbu	Spencer	Morgenstern–Price
Reliability index (β)	0.9247	0.4850	0.8463	0.8576

Table 5 Comparison of the reliability indices of the four methods when considering a correlation coefficient of − 0.5 between c and φ

Method	Simplified Bishop	Simplified Janbu	Spencer	Morgenstern–Price
Reliability index (β)	1.2797	0.6722	1.1535	1.1774

FS by the proposed method are plotted in Figs. 6 and 7, respectively. It can be seen that the simplified Janbu's method predicted the upper probability of failure with respect to the other assessed methods. Additionally, for assessing the influence of the correlation coefficient between c and φ, the PDF and CDF of the FS are determined with the correlation coefficient − 0.5. Figures 8 and 9 show these curves for the above methods. It can be seen that, again, the simplified Janbu's method predicted the upper probability of failure with respect to the other assessed methods.

Based on the governing assumptions of the simplified Janbu method, the predicted average slope stability safety factor by this method is lower than the corresponding values by the other methods (Fig. 10). In this method, the shear force between the components is not directly considered; however, the correction coefficient is used to account for this force. Comparison of the reliability coefficient with different LEMs of slices has been presented in the literature (Fredlund and Krahn 1977). In this figure, lambda (λ) is a ratio of interslice forces for slices.

Based on the PDF of the FS, the reliability indices of the four methods are determined using the following equation (Husein Malkawi et al. 2000):

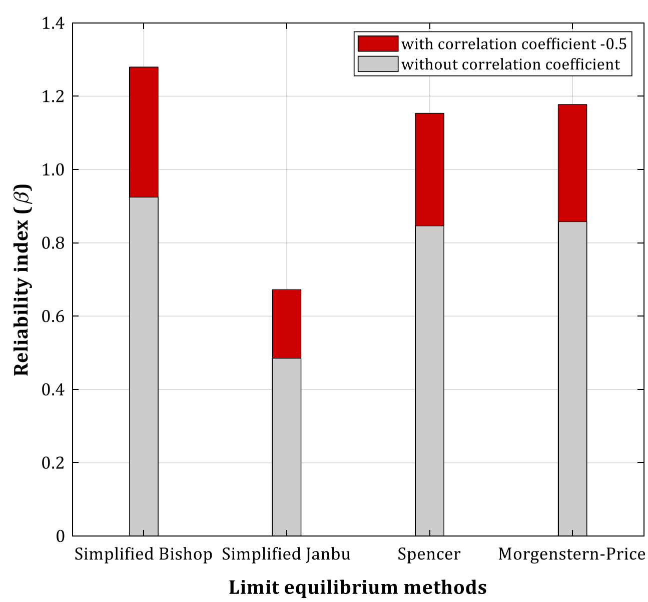

Fig. 11 Comparison of the reliability indices obtained by the four methods

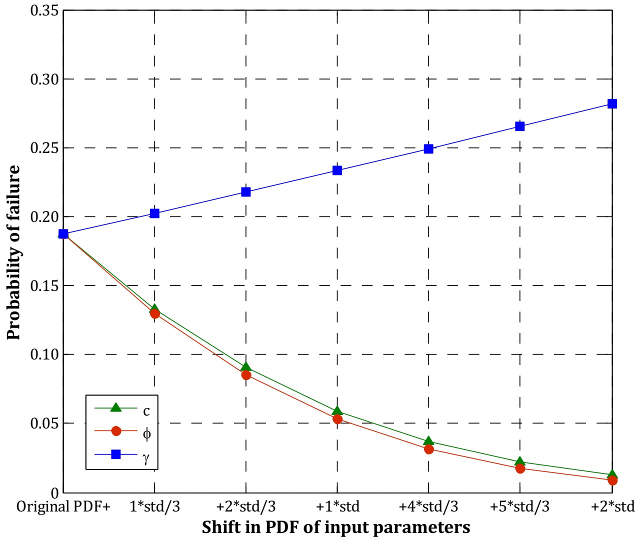

Fig. 12 Parametric analysis of the probability of failure with respect to change of the PDFs of the input parameters

$$\beta = \frac{E(FS)-1}{\sigma(FS)} \tag{31}$$

where β is the reliability index, E(FS) is the mean value of the FS, and σ(FS) is the standard deviation of the FS.

Comparisons of reliability indices for the different methods without and with considering the correlation coefficient are given in Tables 4 and 5, respectively. It can be seen that the simplified Janbu's method shows the lower reliability index or upper probability of failure with respect to the other methods in both conditions with and without considering the correlation coefficient between c and φ. However, the reliability indices of the LEMs is greater for the cases where the correlation coefficient is considered compared with those without considering cross correlation.

For direct comparison, the reliability indices determined by the methods are plotted using a bar chart in Fig. 11.

Parametric analysis

For further verification of the proposed model, a parametric analysis is performed using Janbu's method. The main goal is to determine how each parameter affects the stability of slopes. Figure 12 presents the predicted values of the probability of failure (instability) as a function of each parameter, with the others remaining constant. For this purpose, in six steps, the PDF of each stochastic input parameter is increased based on their standard deviation (new pdf = old pdf + 1/3 × std). For further explanation, the values used for this analysis are listed in Table 6. The results of the parametric analysis indicate that, as expected, the probability of failure (instability) continuously increases due to increasing unit weight. The probability of failure decreases with increase in the internal friction angle and cohesion. Also, it can be seen that the curve of change in the internal friction angle with respect to the probability of failure has a steeper slope than the others, indicating that it is the most influential parameter.

Comparison of the JDRV method and MCS

To compare the proposed method and MCS in predicting the probability of failure, Janbu's method is selected. Figures 13 and 14 indicate the variation of the probability of failure with respect to the number of generations while Janbu's method is used by JDRV and MCS, respectively. From these figures, it can be understood that, for reaching the same probability of failure, more generations (samples) is required in MCS compared with the JDRV method. Additionally, the required computational time for the two approaches is compared in Tables 7 and 8. As demonstrated in these tables, the time required to reach the same probability of failure is greater for MCS than the JDRV method. The analysis was performed using a desktop computer with a Core i7 CPU @3.50 GHz and 24.0 GB of RAM.

Table 6 Inputs for the distribution of soil parameters

Parameter selected for parametric analysis	Parameter	std.	Mean					
			+1/3 × std.	+2/3 × std.	+1 × std.	+4/3 × std.	+5/3 × std.	+2 × std.
c	c (kPa)	2.0	10.7	11.3	12.0	12.7	13.3	14.0
	ϕ (°)	3.0	28.0	28.0	28.0	28.0	28.0	28.0
	γ (kN/m^3)	1.0	18.0	18.0	18.0	18.0	18.0	18.0
ϕ	c (kPa)	2.0	10.0	10.0	10.0	10.0	10.0	10.0
	ϕ (°)	3.0	29.0	30.0	31.0	32.0	33.0	34.0
	γ (kN/m^3)	1.0	18.0	18.0	18.0	18.0	18.0	18.0
γ	c (kPa)	2.0	10.0	10.0	10.0	10.0	10.0	10.0
	ϕ (°)	3.0	28.0	28.0	28.0	28.0	28.0	28.0
	γ (kN/m^3)	1.0	18.3	18. 7	19.0	19. 3	19.7	20.0

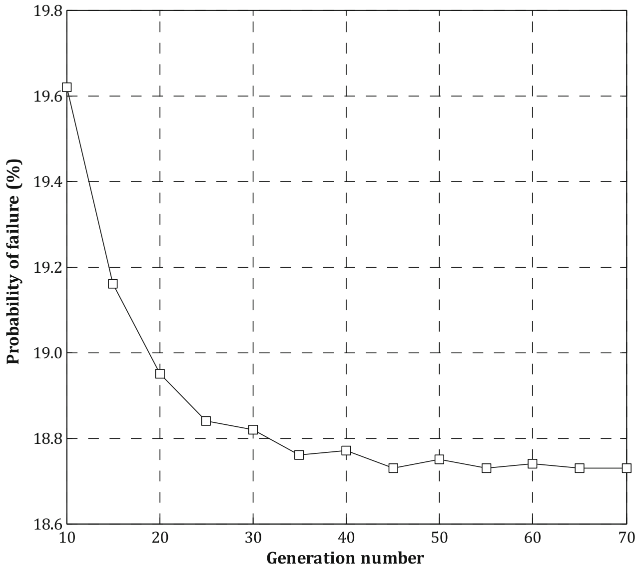

Fig. 13 Variation of the generation number and probability of failure using the jointly distributed random variables (JDRV) method by Janbu's method

The number of required MCS iterations is dependent on the desired level of confidence in the solution and the number of variables. It can be estimated using the following equation (Harr 1987):

$$N = \left[\frac{d^2}{4\,(1-E)^2}\right]^n \tag{32}$$

where N is the number of Monte Carlo simulations, d is the standard normal deviate corresponding to the level of confidence, E is the desired level of confidence (0 to 100%) expressed in decimal form, and n is the number of variables.

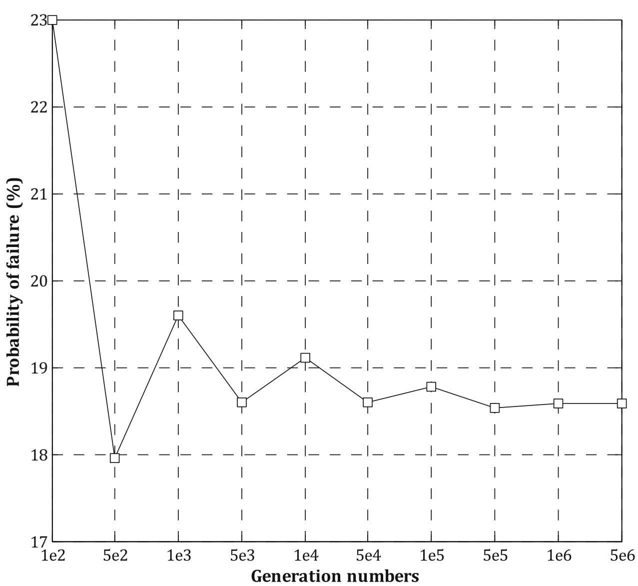

Fig. 14 Variation of the generation number and probability of failure using Monte Carlo simulation (MCS) by Janbu's method

Table 7 Computational time required to obtain a constant probability of failure by the jointly distributed random variables (JDRV) method

Generation number	10	20	30	40	50	60	70
Time (second)	0.16	0.66	2.01	4.62	8.88	15.34	24.32
Probability of failure	19.62	18.95	18.82	18.77	18.75	18.74	18.74

Conclusion

In this paper, the jointly distributed random variables (JDRV) method was used to compare the reliability of four limit equilibrium methods (LEMs), including the simplified Bishop, simplified Janbu, Morgenstern–Price, and Spencer's methods, in the slope stability analysis of slices. The selected soil stochastic parameters were internal friction angle, cohesion, and unit weight, which were modeled using a truncated normal probability density function (PDF). The parameters related to the geometry, height, and angle of the slope were regarded as constant parameters.

The factor of safety (FS) relationships for the PDFs of the mentioned methods were derived analytically based on the selected stochastic parameters and for an arbitrary slope. For this purpose, first using the mean value of the stochastic parameters, the critical surface with the minimum FS was determined by the particle swarm optimization (PSO) technique. Then, by considering the soil parameters' uncertainty, the PDFs of the FS of the methods were obtained by the JDRV method.

For reliability assessment, the reliability indices of the LEMs were calculated. It was shown that the Janbu's method is the method with the upper probability of failure with respect to the assessed methods in two conditions with and without considering the correlation coefficient between c and φ. However, the reliability

Table 8 Computational time required to obtain a constant probability of failure by Monte Carlo simulation (MCS)

Generation numbers	1e2	1e3	1e4	1e5	1e6
Time (s)	1.71	2.40	12.27	212.74	1461.3
Probability of failure	23.00	19.60	19.11	18.75	18.58

indices of the LEMs is greater for the cases where the correlation coefficient is considered compared with those without considering cross correlation.

In another part of the paper, to assess the efficiency of the proposed method with respect to Monte Carlo simulation (MCS), the time required to reach the same probability of failure by the JDRV method and MCS was compared. The results show that the time required by MCS is several times greater than the JDRV method.

Furthermore, the results of the parametric analysis indicate that the probability of failure continuously increases due to increasing unit weight. The probability of failure decreases with increase in the internal friction angle and cohesion. Also, it can be seen that the curve of change in the internal friction angle with respect to the probability of failure has a steeper slope than the others, indicating that it is the most influential parameter.

Appendix

The slope stability methods and derivations of mathematical functions k_1 to k_4 and factor of safety (FS) for all of the methods are presented in this appendix:

Simplified Bishop's method

In the simplified Bishop's method, the forces on the sides of the slice are assumed to be horizontal (i.e., there are no shear stresses between slices). This method considers equilibrium of moments about the center of the circle. The FS is determined by the following equation (Bishop 1955; Duncan and Wright 2005):

$$\mathrm{FS} = \frac{\sum_{i=1}^{n}\left[\frac{c.\Delta l_i.\cos\alpha_i + w_i.\tan\varphi}{\cos\alpha_i + (\sin\alpha_i.\tan\varphi)/\mathrm{FS}}\right]}{\sum_{i=1}^{n} w_i.\sin\alpha_i} \tag{A.1}$$

where:

w_i $\gamma.b_i.h_i$
c Cohesion intercept
Δl_i Area of the base of the slice for a slice of unit thickness
α_i Angle of the base of the slice
w_i Weight of the slice
γ Unit weight of soil
b_i Width of the slice
h_i Height of the slice at the centerline
φ Internal friction angle
FS Factor of safety

By the change of variables:

$$\begin{cases} k_1 = c \\ k_2 = \tan\varphi \\ k_3 = \gamma \\ k_4 = \mathrm{FS} \end{cases} \tag{A.2}$$

the following equations can be written:

$$\begin{cases} u_1 = g_1(k_1, k_2, k_3) = \mathrm{FS} = \dfrac{\sum_{i=1}^{n}\left[\frac{k_1.\Delta l_i.\cos\alpha_i + k_3.b_i.h_i.k_2}{\cos\alpha_i + (\sin\alpha_i.k_2)/\mathrm{FS}}\right]}{\sum_{i=1}^{n} k_3.b_i.h_i.\sin\alpha_i} \\ u_2 = g_2(k_1, k_2, k_3) = k_2 \\ u_3 = g_3(k_1, k_2, k_3) = k_3 \end{cases} \tag{A.3}$$

$$\begin{cases} k_1 = h_1(u_1, u_2, u_3) = c = \dfrac{u_1.\sum_{i=1}^{n}(u_3.b_i.h_i.\sin\alpha_i) - \sum_{i=1}^{n}\left[\frac{u_3.b_i.h_i.u_2}{\cos\alpha_i + (\sin\alpha_i.u_2)/u_1}\right]}{\sum_{i=1}^{n}\left[\frac{\Delta l_i.\cos\alpha_i}{\cos\alpha_i + (\sin\alpha_i.u_2)/u_1}\right]} \\ k_2 = h_2(u_1, u_2, u_3) = u_2 \\ k_3 = h_3(u_1, u_2, u_3) = u_3 \end{cases} \tag{A.4}$$

The PDF of the FS can be obtained by Eq. (A.5). This equation is developed by Eq. (1) directly.

$$f_{X_i}(x_i) = \iint_{R_{Xi}} \ldots \int f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)dx_1dx_2\ldots dx_{i-1}dx_{i+1}\ldots dx_n \tag{A.5}$$

where:

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = |J(x_1, x_2, \ldots, x_n)|.f_{X_1,X_2,\ldots,X_n}(h_1(x_1, x_2, \ldots, x_n), \ldots, h_n(x_1, x_2, \ldots, x_n)) \tag{A.6}$$

and

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1).f_{X_2}(x_2)\ldots.f_{X_n}(x_n) \tag{A.7}$$

$$J(u_1, u_2, u_3) = \left|\begin{matrix} \frac{\partial k_1}{\partial u_1} & \frac{\partial k_1}{\partial u_2} & \frac{\partial k_1}{\partial u_3} \\ \frac{\partial k_2}{\partial u_1} & \frac{\partial k_2}{\partial u_2} & \frac{\partial k_2}{\partial u_3} \\ \frac{\partial k_3}{\partial u_1} & \frac{\partial k_3}{\partial u_2} & \frac{\partial k_3}{\partial u_3} \end{matrix}\right| \tag{A.8}$$

$$J=\frac{\sum_{i=1}^{n}k_3.b_i.h_i.\sin\alpha_i-\sum_{i=1}^{n}\left[\frac{k_3.b_i.h_i.\sin\alpha_i.k_2^2}{k_4^2.(B_1)^2}\right]}{\sum_{i=1}^{n}\left[\frac{\Delta l_i.\cos\alpha_i}{B_1}\right]}+\frac{\left(\sum_{i=1}^{n}\left[\frac{\Delta l_i.\cos\alpha_i.\sin\alpha_i.k_2}{k_4^2.(B_1)^2}\right]\right)\times\left(\sum_{i=1}^{n}\left[\frac{k_3.b_i.h_i.k_2}{B_1}\right]-k_4.\sum_{i=1}^{n}(k_3.b_i.h_i.\sin\alpha_i)\right)}{\left(\sum_{i=1}^{n}\left[\frac{\Delta l_i.\cos\alpha_i}{B_1}\right]\right)^2} \tag{A.9}$$

$$B_1=\cos\alpha_i+(\sin\alpha_i.k_2)/k_4 \tag{A.10}$$

Simplified Janbu's method

The simplified Janbu's method is based on the assumption that the interslice forces are horizontal. This assumption alone almost always produces FS that are smaller than those obtained by more rigorous procedures that satisfy complete equilibrium (Janbu 1954). Janbu proposed some correction factors based on a number of slope stability computations (Janbu 1973). The FS is determined as (Duncan and Wright 2005):

$$FS=f_0.\left(\frac{\sum_{i=1}^{n}\left[\frac{c.\Delta l_i.\cos\alpha_i+w_i.\tan\varphi}{\cos^2\alpha_i+(\sin\alpha_i.\cos\alpha_i.\tan\varphi)/FS}\right]}{\sum_{i=1}^{n}w_i.\tan\alpha_i}\right) \tag{A.11}$$

where

$$\text{For } c,\varphi>0 \qquad f_0=1+0.5\left[\frac{d}{L}-1.4\left(\frac{d}{L}\right)^2\right] \tag{A.12}$$

where:

- f_0 Correction factors
- L The length joining the left and right exit points
- d The maximum thickness of the failure zone with reference to this line

$$\begin{cases} u_1=g_1(k_1,k_2,k_3)=FS=f_0.\left(\frac{\sum_{i=1}^{n}\left[\frac{k_1.\Delta l_i.\cos\alpha_i+k_3.b_i.h_i.k_2}{\cos^2\alpha_i+(\sin\alpha_i.\cos\alpha_i.k_2)/FS}\right]}{\sum_{i=1}^{n}k_3.b_i.h_i.\tan\alpha_i}\right)\\ u_2=g_2(k_1,k_2,k_3)=k_2\\ u_3=g_3(k_1,k_2,k_3)=k_3\end{cases} \tag{A.13}$$

$$\begin{cases} k_1=h_1(u_1,u_2,u_3)=c=\frac{\frac{u_1}{f_0}.\sum_{i=1}^{n}(u_3.b_i.h_i.\tan\alpha_i)-\sum_{i=1}^{n}\left[\frac{u_3.b_i.h_i.u_2}{\cos^2\alpha_i+(\sin\alpha_i.\cos\alpha_i.u_2)/u_1}\right]}{\sum_{i=1}^{n}\left[\frac{\Delta l_i.\cos\alpha_i}{\cos^2\alpha_i+(\sin\alpha_i.\cos\alpha_i.u_2)/u_1}\right]}\\ k_2=h_2(u_1,u_2,u_3)=u_2\\ k_3=h_3(u_1,u_2,u_3)=u_3\end{cases} \tag{A.14}$$

$$J=\frac{\sum_{i=1}^{n}\left[\frac{k_3.b_i.h_i.\tan\alpha_i}{f_0}\right]-\sum_{i=1}^{n}\left[\frac{k_3.b_i.h_i.\sin\alpha_i.\cos\alpha_i.k_2^2\varphi}{k_4^2.(J_1)^2}\right]}{\sum_{i=1}^{n}\left[\frac{\Delta l_i.\cos\alpha_i}{J_1}\right]}+\frac{\left(\sum_{i=1}^{n}\left[\frac{\Delta l_i.\cos^2\alpha_i.\sin\alpha_i.k_2}{k_4^2.(J_1)^2}\right]\right)\times\left(\sum_{i=1}^{n}\left[\frac{k_3.b_i.h_i.k_2}{(J_1)}\right]-\frac{k_4}{f_0}.\sum_{i=1}^{n}(k_3.b_i.h_i.\tan\alpha_i)\right)}{\left(\sum_{i=1}^{n}\left[\frac{\Delta l_i.\cos\alpha_i}{J_1}\right]\right)^2} \tag{A.15}$$

$$J_1=\cos^2\alpha_i+(\sin\alpha_i.\cos\alpha_i.k_2)/k_4 \tag{A.16}$$

Spencer's method

Spencer's method is based on the assumption that the interslice forces are parallel (i.e., all interslice forces have the same inclination). The specific inclinations of the interslice forces are unknown and are computed as one of the unknowns in the solution of the equilibrium equations. Spencer originally presented this procedure for circular slip surfaces, but the procedure is easily extended to noncircular slip surfaces (Duncan and Wright 2005; Spencer 1967).

The equation for force equilibrium can be written as:

$$\sum_{i=1}^{n}Q_i=0 \tag{A.17}$$

where Q_i is the resultant of the interslice forces. For moment equilibrium, moments can be summed about any arbitrary point. Taking moments about the origin ($x=0$, $y=0$) of a

Cartesian coordinate system, the equation for moment equilibrium is expressed as:

$$\sum_{i=1}^{n} Q_i.\left(x_{b_i}.\sin\theta - y_{b_i}.\cos\theta\right) = 0 \quad (A.18)$$

where x_b is the x (horizontal) coordinate of the center of the base of the slice and y_b is the y (vertical) coordinate of the point on the line of action of the force, Q_i, directly above the center of the base of the slice. Q_i is determined by following equation:

$$Q_i = \frac{w_i.\sin\alpha_i - c.\Delta l_i + w_i.\cos\alpha_i.\tan\varphi/FS}{\cos(\alpha_i-\theta) + \sin(\alpha_i-\theta).\tan\varphi/FS} \quad (A.19)$$

where θ is the interslice force inclination. By change in variation:

$$\begin{cases} u_1 = g_1(k_1, k_2, k_3) = FS \\ u_2 = g_2(k_1, k_2, k_3) = k_2 \\ u_3 = g_3(k_1, k_2, k_3) = k_3 \end{cases} \quad (A.20)$$

$$\begin{cases} k_1 = h_1(u_1, u_2, u_3) = c = \dfrac{\sum_{i=1}^{n}\left[\dfrac{(u_3.b_i.h_i.\sin\alpha_i - u_3.b_i.h_i.\cos\alpha_i.u_2/u_1)}{\cos(\alpha_i-\theta)+\sin(\alpha_i-\theta).u_2/u_1}.\left(1+\left(x_{b_i}.\sin\theta - y_{b_i}.\cos\theta\right)\right)\right]}{\sum_{i=1}^{n}\left[\dfrac{(\Delta l_i/u_1)}{\cos(\alpha_i-\theta)+\sin(\alpha_i-\theta).u_2/u_1}.\left(1+\left(x_{b_i}.\sin\theta - y_{b_i}.\cos\theta\right)\right)\right]} \\ k_2 = h_2(u_1, u_2, u_3) = u_2 \\ k_3 = h_3(u_1, u_2, u_3) = u_3 \end{cases} \quad (A.21)$$

$$J = \frac{\sum_{i=1}^{n}\left[\dfrac{k_3.b_i.h_i.\cos\alpha_i.k_2.S_3}{{k_4}^2.S_2}\right] + \sum_{i=1}^{n}\left[\dfrac{\sin(\alpha_i-\theta).k_2.S_1.S_3}{{k_4}^2.(S_2)^2}\right]}{S_4} + \frac{\left(\sum_{i=1}^{n}\left[\dfrac{\Delta l_i.S_3}{{k_4}^2.S_2}\right] - \sum_{i=1}^{n}\left[\dfrac{\Delta l_i.\sin(\alpha_i-\theta).k_2.S_3}{{k_4}^3.(S_2)^2}\right]\right)\times\left(\sum_{i=1}^{n}\left[\dfrac{S_1.S_3}{S_2}\right]\right)}{(S_4)^2} \quad (A.22)$$

where:

$$S_1 = k_3.b_i.h_i.\sin\alpha_i - k_3.b_i.h_i.\cos\alpha_i.k_2/k_4 \quad (A.23)$$

$$S_2 = \cos(\alpha_i-\theta) + \sin(\alpha_i-\theta).k_2/k_4 \quad (A.24)$$

$$S_3 = \left(x_{b_i}.\sin\theta - y_{b_i}.\cos\theta\right) \quad (A.25)$$

$$S_4 = \sum_{i=1}^{n}\frac{\Delta l_i.S_3}{k_4.S_2} \quad (A.26)$$

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