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ORIGINAL PAPER

Residual factor as a variable in slope reliability analysis

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Abstract In the past, residual factor R in strain-softening soil slopes has been included, either directly or indirectly, as a deterministic variable in both deterministic and probabilistic studies. This paper discusses the uncertainties associated with R and outlines a systematic approach for the reliability analysis of a natural slope in which shear strength parameters and pore pressure ratio are random variables, each assumed with a lognormal probability distribution. For the residual factor R, seven probability distribution options under the generalized beta-distribution system are considered. Slope reliability is computed based on the first order reliability method (FORM) and validated against Monte-Carlo simulation (MCS). Results obtained from two illustrative examples indicate that the probability of failure, with R as one of six random variables, can be orders of magnitude higher than that based on five random variables with R considered as a deterministic parameter. The magnitude of influence of R as a random variable is, however, highly dependent on its probability distribution, the left-skewed triangular distribution having the most significant influence in both the examples. Results of sensitivity analyses reveal that, for almost all of its assumed probability distributions, R is the most dominant among the six random variables. Effects of variation of some of the statistical and correlation properties of the other random variables, viz. the shear strength parameters and the pore pressure ratio, on the results of reliability analyses are also studied.

Keywords Slope reliability · Peak and residual strengths · Probability distribution · Pore water pressure · Coefficient of variation · Correlation coefficient

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Introduction

The processes of progressive failure are often associated with a decrease in the values of shear strength parameters in strain-softening soils. Skempton (1964, 1985) proposed a definition of residual factor at a point in a soil mass as the extent to which shear strength has decreased from its peak value to its residual value. The definition of local residual factor that has become accepted is the ratio $[(s_p - s)/(s_p - s_r)]$ in which s_p, s_r, and s denote the peak shear strength, the residual shear strength, and the current shear strength respectively at the concerned point in soil mass. If no decrease has occurred, the residual factor is equal to 0; if the strength has decreased to the residual value, the residual factor is 1; and in all other cases the residual factor lies between 0 and 1. It is very useful to consider an alternative definition of the residual factor which represents the whole of a potential slip surface. For a perfectly brittle soil, strain-softening will lead to one part of the slip surface being at residual shear strength and the remaining part at peak shear strength.

Skempton (1964) also proposed that the overall or average residual factor R for a slip surface in a slope could be represented as the proportion of slip surface length along which the shear strength has decreased to the residual, i.e.,

$R = L_r/L$ in which L is the total length of a slip surface of which the length L_r is at the residual shear strength, the remaining length ($L - L_r$) still being at the peak shear strength. The magnitude of the average residual factor represents the state of nature for a slope at a given point in time, being a consequence of the decrease in material strength parameters associated with processes of progressive failure. Considering average shear strength along a slip surface, the two definitions are found to be consistent with each other as shown in Appendix 1. The first definition is convenient for formulation of performance function to include the residual factor as a variable. The second definition is useful to visualize the change in shear strength along one or more sections of a potential slip surface in a slope when strain-softening process occurs.

For analysis of a potential first-time slope failure, it is generally assumed that peak shear strength is operative all along a slip surface. For analysis of potential reactivation of a landslide along a pre-existing discontinuity, on the other hand, residual shear strength is considered to be operative all along the slip surface. Yet, careful research over several decades has revealed that the shear strength may have decreased to the residual within parts of a slope with no observed history of sliding of the slope as a whole and thus part of a potential slip surface may already be at the residual shear strength (James 1971; Morgenstern 1977). Significant relative displacement is often considered as a precondition for decrease of shear strength to a residual value. However, research has revealed important exceptions with far-reaching implications. The shear strength of clays along bedding planes can fall from the intact to the residual condition as a result of quite modest displacements, and, on pre-existing shear surfaces, the residual condition may be reached at virtually zero displacement (Skempton 1966, 1985; Skempton and Petley 1967; Skempton and Vaughan 1995). The displacement required to reach the residual, however, is expected to depend on the initial degree of clay particle orientation parallel to the bedding plane, the thickness of the shear zone and the direction of shearing.

Mesri and Shahien (2003) carried out a comprehensive review of long-term stability of stiff clay and clay shale slopes and detailed re-analyses of 99 case histories of slopes in soft clays to stiff clays and clay shales. They concluded that, for first-time slope failures in stiff clays and clay-shales, the slip surface may be unsheared prior to the occurrence of a landslide but a part of the slip surface may be at the residual condition before the final slope is formed. They cited extensive published evidence in support of the view that, in geological settings other than reactivated landslides, a potential slip surface may incorporate a segment that is at the residual state. Shear strength along horizontal and sub-horizontal bedding planes, laminations, and weak seams can reduce to residual condition after relatively small shear displacements, measured in millimeters or centimeters rather than meters, as commonly assumed for homogeneous clays or across laminations. More recently, the view that the average shear strength along a potential slip surface may be at levels in between the peak and the residual has been supported during discussion of case histories of slope stability and landslides by Stark and Hussain (2010, 2011) and by Hamel and Adams (2011).

In the past, residual factor has been included, either directly or indirectly, as a deterministic variable in both deterministic and probabilistic studies (e.g., Lo and Lee 1973; Christian and Whitman 1969; Chowdhury et al. 1987; Chowdhury and Zhang 1993). However, because of uncertainties associated with R, it is very important to consider it as a random variable in reliability studies.

The main objectives of this paper are to highlight the importance of the residual factor R as an additional random variable for assessing the reliability of natural slopes, to update the analysis method, and to study the influence of the residual factor, relative to other variables, on slope reliability. Fulfilment of these objectives, however, involves the following essential steps: (1) discuss the uncertainties associated with the residual factor, (2) express the overall residual factor R for a slip surface in terms of average values of current shear strength, peak shear strength and residual shear strength, (3) demonstrate how the factor of safety F can be modified by including R as an additional geotechnical parameter, (4) discuss sensible alternatives for assuming the probability distribution of residual factor R, (5) carry out reliability analyses considering both symmetrical and asymmetrical (skewed) probability distributions of R, and discuss the relative influence of different distributions; (6) carry out sensitivity analyses in order to study the relative influence of different random variables including R. It is of real interest to know whether the influence of R is more or less significant than that of pore water pressure or that of peak and residual shear strength parameters, and to discuss how this influence may be related to the assumed distribution of R.

For the performance function, considered as ($F - 1$), an expression for the factor of safety F is developed for an 'infinite slope' analysis. Due to space limitations, analyses for natural slopes corresponding only to this assumption are included in this paper. The authors have already extended the concepts to slopes with curved slip surfaces using Bishop simplified and Spencer methods and including the search for critical slip surfaces (Metya et al. 2016a, b; Metya 2017). While full details are outside the scope of this paper, a modified expression for F, considering a curved slip surface and based on Bishop simplified method, is given in Appendix 2.

Uncertainties associated with the residual factor

Uncertainties associated with the residual factor R may be due to a number of factors. Natural slopes may have potential slip surfaces which include portions belonging to old landslides but the precise extent of that proportion may be difficult to estimate. Even in man-made slopes such as excavations and fills, the proportion of slip surface length already at the residual shear strength is not known accurately. Moreover, in both natural and man-made slopes, the location of the part of slip surface with residual shear strength is often unknown. In some slopes the rear part of a potential slip surface below the crest may already be at the residual strength whereas the remaining part may be at the peak strength. For example, such was the interpretation, based on some early studies, concerning the bi-planar slip surface of Kettleman Hills landfill (Gilbert et al. 1998). In contrast, the upper part of a slip surface in an excavated slope may still be at peak strength while the lower part is at the residual strength.

It is also important to highlight uncertainties associated with shear strengths along slip surfaces within both failed slopes and those currently stable. In brittle strain-softening soils uncertainties in shear strength, combined with other factors, lead to uncertainties in the proportion of slip surface at the residual and in the location of the residual part of the slip surface relative to the rest of the length. Back-calculated strengths for failed slopes are often subject to considerable uncertainty. Shear strengths measured in the laboratory may be significantly different from mobilized field shear strengths. Hamel and Adams (2011) state that shear strength could be higher by 20% to 50% of the "shear strengths measured in laboratory tests of practical duration on small samples" implying field residual factor between 0 and 1. They also concluded that amongst intermittently creeping colluvial slopes, some quasi-stable masses "have nominal strength reserve—perhaps on the order of 20% above field residual level". The percentages would, of course, be different in other regions with different geology, soil types, and environmental conditions, highlighting the significant uncertainties associated with such estimates.

With regard to the Kettleman Hills landfill case history mentioned earlier, uncertainties were explored by a number of investigators and summarized by Gilbert et al. (1998). This slope failure involved slippage along two distinct interfaces, a geotextile/geomembrane interface and a geomembrane/clay interface. A back analysis procedure was used to account for uncertainty through probability theory. Based on a number of assumptions and using Bayesian analysis, the probability distribution of a non-dimensional, average shear strength mobilization factor R_s was estimated. This factor is directly related to the average residual factor by the relationship: $R_s = 1 - R$. The outcome was a mildly skewed probability distribution for R_s with a mean of 0.44. As the relationship between R_s and R is linear, R will have the same distribution as that of R_s, with a mean of (1 – 0.44) or 0.56.

Overall or average residual factor

Slip surface of planar shape in an 'infinite slope' — special case

From the original definition of residual factor at a point in soil mass considered earlier, the overall or average residual factor R over a slip surface of length L could be expressed as follows for a simple slope assuming a slip surface parallel to the ground surface.

$$\boldsymbol{R} = \frac{s_p - s_{av}}{s_p - s_r} \tag{1}$$

In Eq. (1) above, s_p is the average peak shear strength, s_r is the average residual shear strength, and s_{av} is the average current shear strength along a slip surface. Assume that steady seepage also occurs parallel to the slope surface. Therefore, pore water pressure and, hence, the effective normal stress is constant along the slip surface. Both the peak shear strength s_p and the residual shear strength s_r are thus constant along the slip surface of length L. In a perfectly brittle strain-softening soil, any part of a potential slip surface will be either at the peak strength or at the residual strength. If L_r is the length of the portion of slip surface over which strength has decreased to the residual, the average current shear strength s_{av} over the whole of the slip surface is $[s_r L_r + s_p (L - L_r)]/L$ which, when substituted in Eq. (1), yields the average residual factor R as equal to the ratio L_r/L. Thus, in this special case of 'infinite slope' and planar slip surface, the definition of an overall residual factor for a slip surface in terms of the part of length of a slip surface at residual strength relative to the whole length (Skempton 1964) is consistent with that for a local residual factor in terms of average current shear strength, average peak strength and average residual strength. As outlined below and in Appendix 1, the same conclusion can be reached for the general case of a slip surface of arbitrary shape for which the shear strength may vary from point to point.

Slip surface of arbitrary shape — general case

In accordance with Eq. (1), the overall residual factor must be expressed in terms of average values of shear strength — peak, residual, and current. Consider a slip surface of arbitrary shape and of length L along which shear strength varies in an arbitrary manner. Let part of the length L_r be at the residual shear strength and the remaining length $(L - L_r)$ be still at the peak shear strength. The expression for average residual factor, defined in terms of average shear strengths, is again shown to be the ratio (L_r /L). The derivation is given in Appendix-A.

Expression for factor of safety in terms of R

Average shear strength in terms of R

From Eq. (1), the average shear strength may be expressed in terms of the average or overall residual factor R as follows:

$$\boldsymbol{s_{av}} = \boldsymbol{R}\, s_r + (\boldsymbol{1-R})\, s_p \tag{2}$$

The Mohr-Coulomb shear strength equation relates shear strength to cohesive and frictional strength parameters through the normal effective stress as $s = c' + \sigma' \tan \phi'$. Thus, the peak and the residual shear strengths can be related to corresponding strength parameters as follows:

$$s_p = c'_p + \sigma' \tan\phi'_p \tag{3a}$$

$$s_r = c'_r + \sigma' \tan\phi'_r \tag{3b}$$

From Eqs. (2) and (3),

$$\boldsymbol{s_{av}} = \boldsymbol{R}\left[c'_r + \sigma' \tan\phi'_r\right] + (\boldsymbol{1-R})\left[c'_p + \sigma' \tan\phi'_p\right] \tag{4a}$$

or,

$$\boldsymbol{s_{av}} = \left[\boldsymbol{R}c'_r + (\boldsymbol{1-R})\, c'_p\right] + \sigma' \left[\boldsymbol{R} \tan\phi'_r + (\boldsymbol{1-R}) \tan\phi'_p\right] \tag{4b}$$

Factor of safety for 'infinite slope' analysis

Consider the simple 'infinite slope' model for the stability of a slope with a potential slip surface parallel to the ground surface assuming first that no strain-softening has occurred. Denote the ground surface inclination by i, the vertical depth to potential slip surface by z, the unit weight of the soil by γ, the shear strength parameters by c′ and tan ϕ′, and the dimensionless pore water pressure ratio by r_u (corresponding to pore water pressure u). The peak shear strength parameters are denoted by suffix p and residual shear strength parameters by suffix r. Assume now that strain-softening has occurred over part of the slip surface within such a slope and that the residual factor is represented by a variable R. The average shear strength along the potential slip surface may be obtained from Eq. (4) after substituting for normal effective stress as given by:

$$\sigma' = \gamma z \cos^2 i - u \tag{5a}$$

One may now use the dimensionless pore pressure ratio r_u ($r_u = u/\gamma z$), instead of u. Considering seepage parallel to slope surface, a value of $r_u = 0.5$ represents seepage occurring throughout the slope with top flow line at ground surface. A value of $r_u < 0.5$ indicates that the top flow line is below the ground surface. Thus, the effective normal stress, in terms of the dimensionless pore pressure ratio, is given by:

$$\sigma' = \gamma z \left(\cos^2 i - r_u\right) \tag{5b}$$

From Eqs. (4a) and (5b), the average shear strength along the potential slip surface is given by:

$$s_{av} = R\left[c'_r + \gamma z\left(\cos^2 i - r_u\right)\tan\phi'_r\right] + (1-R)\left[c'_p + \gamma z\left(\cos^2 i - r_u{}^u\right)\tan\phi'_p\right] \tag{6}$$

The average shear stress along the potential slip surface parallel to the slope surface is given by:

$$\tau_{av} = \gamma z \sin i . \cos i \tag{7}$$

The factor of safety may be defined simply as the ratio of average shear strength and average shear stress. Thus, from Eqs. (6) and (7), one obtains the factor of safety for a slope in which strain-softening has occurred over part of the slip surface, as the following expression.

$$F = \frac{R\left[c'_r + \gamma z\left(\cos^2 i - r_u\right)\tan\phi'_r\right] + (1-R)\left[c'_p + \gamma z\left(\cos^2 i - r_u{}^u\right)\tan\phi'_p\right]}{\gamma z \sin i \cos i} \tag{8}$$

Alternative derivation of expression for F

It is important to underline the importance of residual factor R as an independent field variable defined by the ratio of length of slip surface at residual strength L_r to the total length of slip surface. Therefore, an alternative way to derive Eq. (8) is to consider factor of safety F as a ratio of total resisting force to total driving force. Consider a long slope of finite length L with slip surface parallel to ground surface. (The slope is assumed to be long enough so that end effects can be neglected). A portion L_r of the slip surface is at residual strength and the remaining part $(L - L_r)$ still at peak strength. The total resisting force along the slip surface is: $L_r s_r + (L - L_r)\, s_p$. The total driving force is L (γz sin i cos i). The factor of safety F is the ratio of resisting force and driving force. Thus $F = [L_r s_r + (L - L_r)\, s_p] / [L\,(\gamma z \sin i \cos i)]$. Dividing numerator and denominator by L, noting that $R = (L_r/L)$, substituting for s_r and s_p from Eqs. (3a) and (3b), and substituting for the effective normal stress from Eq. (5b), the factor of safety expression is still given by Eq. (8).

Probability distribution assumptions for random variables

Choice of random variables and probability distributions

For slope stability, the most important parameters are the four shear strength parameters (two for peak strength and two for residual strength), the pore water pressure ratio and the residual factor. Thus, it is reasonable to consider a total of six random variables for a reliability analysis and these are: $\left[c'_p, c'_r, \tan\phi'_p, \tan\phi'_r, r_u \, and \;\; R\right]$.

In geotechnical reliability, the factor of safety is often assumed to have either a normal or a lognormal distribution. The probability distributions of shear strength parameters and pore pressure ratio have often been assumed to be normal (e.g., Li and Lumb 1987; Xue and Gavin 2007; Ji et al. 2012). However, for those random variables which cannot take negative values, such as the shear strength parameters and the pore pressure ratio, a lognormal distribution assumption may be considered preferable. For examples considered in this paper, a lognormal distribution assumption has been made for all the random variables except for the residual factor R.

For the residual factor R, a choice may be made between the assumption of a normal distribution and that of a generalized beta distribution. The choice of a normal distribution allows wide flexibility in accommodating the mean value of R and its standard deviation. However, errors will arise as the mean values approach the end points 0 and 1. Moreover, consideration of skewed distributions is excluded. A generalized beta distribution with the end points of 0 and 1 seems more appropriate. Both symmetrical and skewed distributions can be included with the assumption of a beta system. For given values of mean and standard deviation of R, a corresponding beta distribution can be obtained [see comments after Eqs. (11) and (12)]. Therefore, it is feasible to independently vary the mean of R and the standard deviation of R. For this paper, it was considered appropriate to study both symmetrical and skewed distributions and hence a generalized beta distribution was assumed for R. Moreover, given the nature of the variable with end points of 0 and 1, a beta distribution is conceptually appealing and has practical benefits for accuracy.

Assumed shapes of the beta distribution

The probability density function (PDF) for the generalized beta distribution representing a variable between given bounding values a and b is represented by the following (Harr 1977):

$$f(x) = \frac{1}{C}(x-a)^{q-1}(b-x)^{r-1} \tag{9}$$

where,

$$C = \frac{(q-1)!(r-1)!(b-a)^{q+r-1}}{(q+r-1)!} \tag{10}$$

The following seven probability distributions were chosen for analysis to cover both symmetrical and skewed shapes.

(i). Shape (1): Triangular skewed right (r > q) [q=1, r=2]
(ii). Shape (2): Triangular skewed left (q > r) [q=2, r=1]
(iii). Shape (3): Curved symmetrical about (a+b)/2 [q=2, r=2]
(iv). Shape (4): Curved symmetrical about (a+b)/2 [q=3, r=3]
(v). Shape (5): Curved symmetrical about (a+b)/2 [q=6, r=6]
(vi). Shape (6): Curved skewed right (r > q) [q=2, r=6]
(vii). Shape (7): Curved Skewed left (q > r) [q=6, r=2]

In the above, the numbers in parenthesis, (1), (2), (3),…etc. assigned to the different shapes of beta distributions are referred to as '*R-shapes*' in tabular and graphical forms of presentation of results. For the sake of visual impression, these are presented in Fig. 1 below.

The expected value and variance of the beta distribution [a, b] are given by:

$$E[x] = a + \frac{q}{q+r}(b-a) \tag{11}$$

and,

$$V[x] = \frac{qr(b-a)^2}{(q+r)^2(q+r-1)} \tag{12}$$

It may be noted that if expected value and variance of residual factor are known or assumed, q and r can be calculated from Eqs. (11) and (12) (with a = 0 and b = 1). With those values of q and r, the corresponding beta distribution is defined by Eq. (9). A symmetrical beta distribution (of a variable x with a = 0 and b = 1) results only when the expected value of the variable is 0.5. Substituting for expected value of 0.5 in Eq. (11) and with a = 0 and b = 1, yields q = r. Then Eq. (9) becomes a symmetrical distribution with the actual value of q and r corresponding to the variance of x. For expected values of x other than 0.5, the beta distribution is skewed. Since the residual factor R can have a wide range of expected values within the boundaries of 0 and 1, skewed distributions are most likely. Therefore, the assumption of a symmetrical distribution (such as a normal distribution) is unduly restrictive. The degree of skewness will be low to mild for expected values very close or close to 0.5. However, the beta distribution will be highly skewed for expected values much higher or lower than 0.5. In order to get a broad view or the range of slope reliability values in strain softening soils, four skewed distribution shapes have been adopted which range from a medium to high degree of skewness. The other three distributions adopted are symmetrical.

Statistical parameters of assumed shapes of beta distributions

The residual factor R varies from 0 to 1. Hence, a = 0 and b = 1. Assuming different values for q and r, Eqs. (11) and (12) yield values of mean and standard deviation for the seven shapes of PDF given above, and are listed in Table 1.

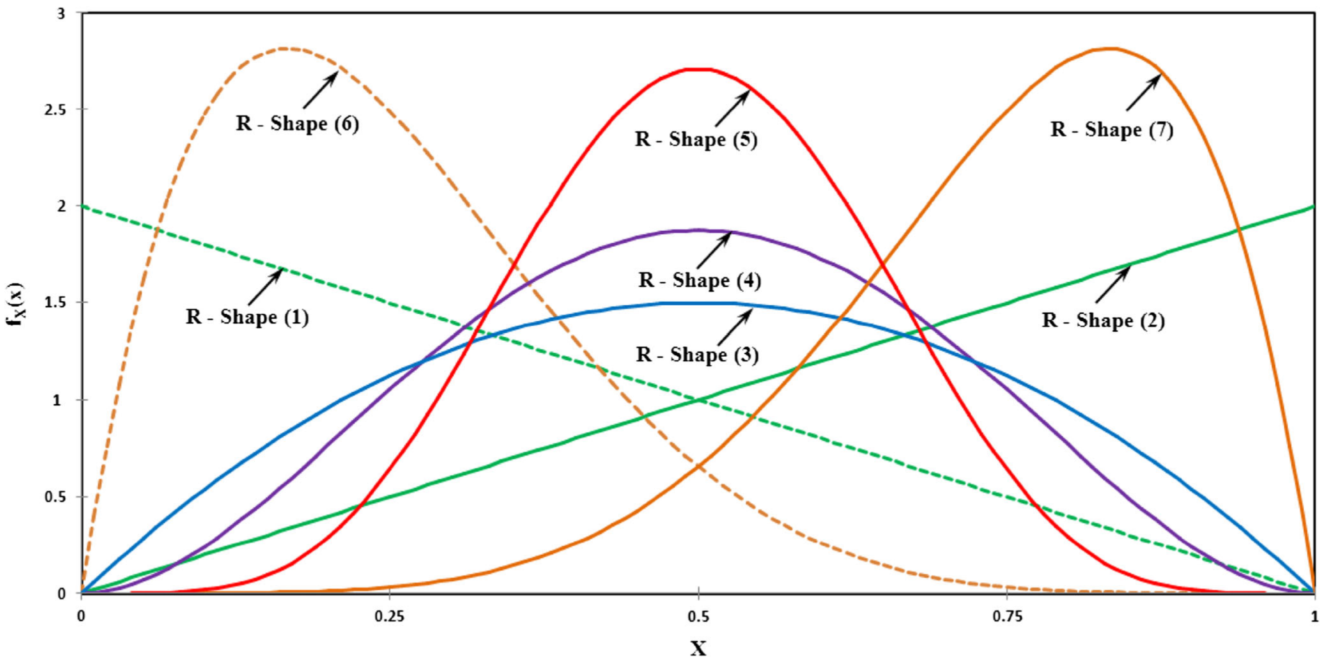

Fig. 1 Assumed beta distributions for the residual factor R (R-shapes)

Methodology for reliability analysis

Choice of reliability analysis method

The first order reliability method is widely regarded as a rigorous method of reliability analysis. The method is free from the shortcomings (Ang and Tang 1984; Metya and Bhattacharya 2012) of the mean value first order second moment (MVFOSM) method used by early geotechnical researchers (Hassan and Wolff 1999; Duncan 2000; Bhattacharya et al. 2003). The FORM is also regarded as a versatile method in view of the fact that it can handle non-normal probability distributions of the basic random variables of a system (Haldar and Mahadevan 2000). Keeping the above in mind, the FORM has been adopted in this study. In this method, the reliability index β_{HL} is defined as the minimum distance from the origin to the failure surface in the standard normal space, using a linearization of the performance function around the design point as originally proposed by Hasofer and Lind (1974). The performance function, also called the limit state function, for the slope stability analysis is usually defined as g(X) = F – 1, X being the vector of basic state (or design) variables of the system consisting of the uncertain geotechnical parameters.

Reliability index, β_{HL}

A widely used expression for the reliability index β_{HL} (Low and Tang 2004; Huang and Griffiths 2011) is given by Eq. (13).

$$\beta_{HL} = \min_{g(X)=0} \sqrt{\left\{\frac{X_i-\mu_i^N}{\sigma_i^N}\right\}^T [C]^{-1} \left\{\frac{X_i-\mu_i^N}{\sigma_i^N}\right\}} \qquad (13)$$

where, μ_i^N and σ_i^N are the equivalent normal mean and standard deviation respectively of the ith random variable X_i and [C] is the matrix of correlation coefficients between the standard normal variables. The determination of the reliability index β_{HL} is, thus, a problem of optimization, and, as indicated by Wang et al. (2011), the successful application of FORM relies on the selection of a robust optimization algorithm for the multi-dimensional minimization. Keeping this in mind, the sequential quadratic programming (SQP) (Rao 2009) in the MATLAB environment is employed to solve this problem. The solution yields the design point on the failure surface and the corresponding reliability index β_{HL}. The adoption of the SQP technique is based on its recommendations in the literature. For example, Hong and Roh (2008) reported that 'an extensive comparative study of nonlinear programming codes presented by Schittkowski (1980) ranked the performance of the SQP method to be the highest'.

Probability of failure, p_F

Having obtained the computed value of the reliability index, β_{HL}, the probability of failure can be obtained from Eq. (14) on the assumption that the performance function follows a normal distribution.

$$p_F = \Phi(-\beta_{HL}) \qquad (14)$$

Table 1 Statistical properties of the residual factor R for various shapes of beta distributions

Sl. No.	Geometrical shape	Beta distribution parameters		Designation	Statistical properties		
		q	r		Mean μ_R	Standard deviation σ_R	Coefficient of variation (COV) δ_R
1.	Triangular skewed right (r > q)	1	2	R-shape (1)	0.333	0.236	0.707
2.	Triangular skewed left (q > r)	2	1	R-shape (2)	0.667	0.236	0.354
3.	Curved symmetrical about (a+b)/2	2	2	R-shape (3)	0.500	0.224	0.447
4.	Curved symmetrical about (a+b)/2	3	3	R-shape (4)	0.500	0.189	0.378
5.	Curved symmetrical about (a+b)/2	6	6	R-shape (5)	0.500	0.139	0.277
6.	Curved skewed right (r > q)	2	6	R-shape (6)	0.250	0.144	0.577
7.	Curved Skewed left (q > r)	6	2	R-shape (7)	0.750	0.144	0.193

where Φ (.) denotes the standard normal cumulative distribution function.

Direction cosine as sensitivity index

In the reliability analysis using FORM, the unit vector $\alpha^{'}$ normal to the limit state surface at the design point is a measure of the sensitivity of the reliability index with respect to variations in each of the standard normal variates given by (Cho 2007),

$$\alpha^{'} = \nabla_X \beta_{HL} \qquad (15)$$

where the notation ∇_X stands for the operators $\partial/\partial X_1{'}_1$, $\partial/\partial X_2'$,….,$\partial/\partial X_n'$. X_1', X_2',….X_n' being the standard normal variates. The elements (α'_i) of the vector $\alpha^{'}$ are the direction cosines along the reduced co-ordinate axes X'_i corresponding to the ith random variable, X_i.

Illustrative examples on natural slopes

The above formulation for the analysis of strain-softening slopes is illustrated below with the help of two example problems [example 1 in Fig. 2(a) and example 2 in Fig. 2(b)] of natural slopes which can be analyzed on the basis of the 'infinite slope' model. Both the example problems assume homogeneous slope in cohesive soil with a slip surface parallel to the ground surface and with seepage occurring parallel to the slope surface. They, however, differ in the data on slope inclination, depth to the potential failure plane, and the shear strength parameters.

Assumed slope data

In example 1 [Fig. 2(a)] the slope is assumed to have an inclination i = 15°, depth to potential failure surface z = 5 m, and bulk unit weight of soil γ = 20 kN/m^3. In example 2 [Fig. 2(b)] the corresponding values are taken as i = 12°, z = 3 m, and γ = 20 kN/m^3. These three parameters are considered to be deterministic and constant in both the deterministic and the probabilistic studies.

For a deterministic analysis, shear strength parameters (peak and residual) and pore pressure ratio, are each single-valued constants.

For a probabilistic analysis, each shear strength parameter is regarded as a random variable and the pore pressure ratio is also regarded as a random variable. At least two statistical parameters of each random variable must be known or assumed. These are the mean of the random variable and its standard deviation (or the coefficient of variation COV). Moreover, advanced methods of reliability analysis, such as the FORM, also makes use of the information regarding the probability distribution of the in Choice of random variables and probability distributions, a lognormal distribution is used for all the reliability analyses to guard against occurrence of negative values. For convenience and comparison between deterministic and probabilistic analyses, the mean value of each random variable is selected as the single-valued constant for each deterministic analysis.

Fluctuation of pore water pressure ratio, r_u

Reliability of natural slopes is often influenced significantly by rainfall-induced seepage. The top flow line within a slope may be located at any depth below the surface of the slope and above the potential slip surface. The pore pressure ratio is a non-dimensional parameter and, as defined earlier, is the ratio of pore water pressure u to the product of γ and z. Where pore pressure varies over a slip surface, an average value of r_u representing the whole slip surface is often used for parametric studies. Rainfall-induced seepage pore pressure reduces slope stability. Thus, a landslide caused by seepage induced by rainfall is often a consequence of pore water pressures increasing to critical values.

An average value of r_u = 0.5 represents seepage throughout the slope in a direction parallel to the ground surface and the

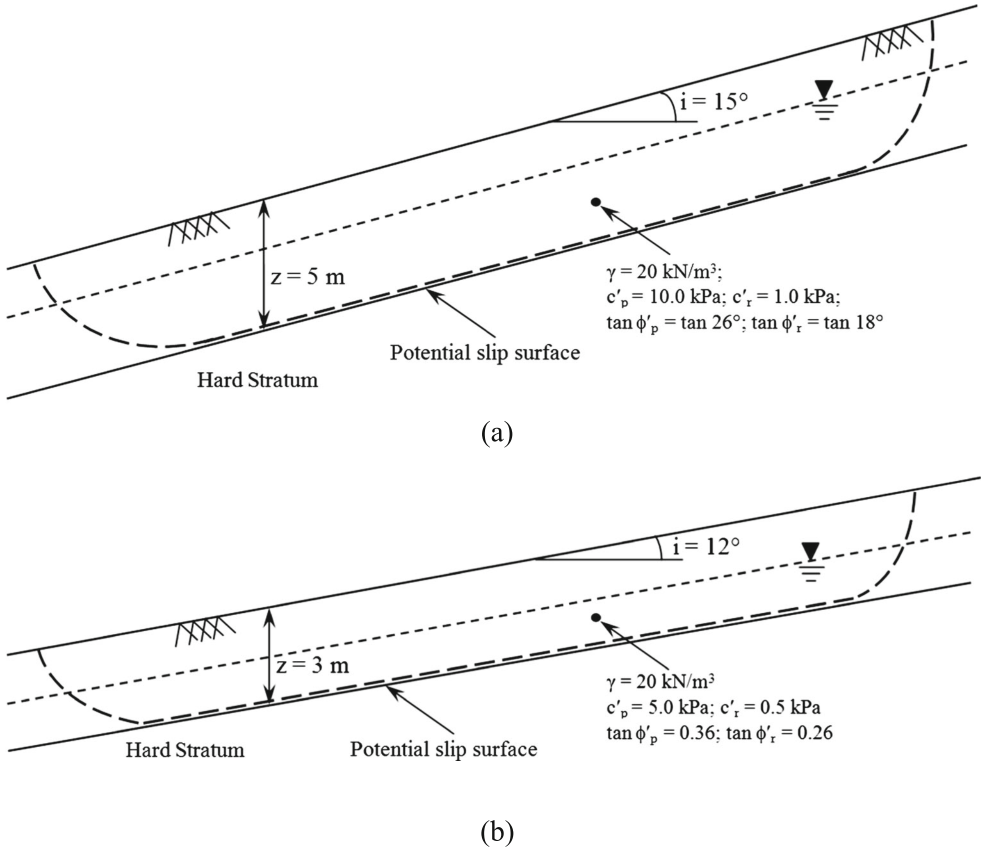

Fig. 2 Slope sections considered in (a) Example 1 and (b) Example 2. Note: The top seepage line is shown as a small dashed line. Its location changes with the rate of rainfall infiltration

corresponding value of factor of safety F will be a minimum for a given set of strength parameters and a given value of the residual factor R. A value of $r_u < 0.5$ indicates that the top seepage line is below the ground surface. In deterministic analysis, a value of r_u for which F = 1 is considered to be the critical value representing overall slope failure.

In reliability analysis, both the mean and the COV of the pore pressure ratio are required to estimate the reliability index and probability of failure. The critical pore pressure may be defined in relation to zero reliability index ($\beta = 0$) which corresponds to a probability of failure $p_F = 50\%$. However, it may be desired to set a higher p_F value as the benchmark for critical r_u, especially to define catastrophic failure. One must remember, however, that values of $p_F > 50\%$ correspond to negative values of the reliability index. An inconsistency may be observed in the computed values of a negative reliability index β if the COV is varied. For example, with a given value of the mean of the performance function F, an increase in its standard deviation would tend to decrease the negative value of β, which means an increase in algebraic value. This is opposite to how the computed value changes in the positive domain of β (an increase in standard deviation of F decrease reliability index). Such an inconsistency has little practical significance because the negative range of reliability indicates a state of failure. Moreover, with a fixed number of variables (either 5 or 6), such an inconsistency does not arise. In other words, the change in reliability index or probability of failure with increasing pore pressure ratio shows a consistent trend.

Assumed data on (a) pore water pressure ratio and (b) shear strength parameters

Assumed value for pore water pressure ratio

Because of uncertainties related to pore water pressure, it is important to consider pore water pressure ratio as a random variable in slope reliability analysis. Most of the analyses in this paper have been carried out for pore pressure ratio with a mean of 0.2 and a coefficient of variation of 0.1. However, to check whether a change in the value of mean r_u (from the selected value of 0.2) results in a change in the trend of reliability analysis results, a parametric study has been carried out by considering different values of mean r_u, specifically, 0.1, 0.15, 0.25, 0.3, 0.4, and 0.45 (besides 0.2), while keeping the value of COV of r_u 0.1. Further, the effect of variation in the value of COV of r_u within its range (in combination with variation in the values of COV's of the other random variables within their respective ranges) has also been studied through another parametric study in which the mean of r_u remains 0.2. The assumed range of mean and COV values of the pore pressure ratio is shown in Table 2 at the bottom.

Assumed values of shear strength parameters

The assumed mean values of the peak and the residual shear strength parameters are as given in Table 2. As far as the uncertainty or variability of each random shear strength parameter is concerned, there is a range within which its coefficient of variation (COV) is known to vary. These ranges are collected from the literature (Hong and Roh 2008), and presented in Table 2 for the sake of ready reference. As stated before, all analyses are carried out with the assumption of lognormal distribution for each of these five random variables.

Table 2 Statistical properties of the random variables except R

Random Variable	Probability distribution	Mean		Range of COV*
		Example 1	Example 2	
c'_p	Lognormal	10.0 kPa	5.0 kPa	0.2–0.5
c'_r	Lognormal	1.0 kPa	0.5 kPa	0.2–0.5
$\tan\phi'_p$	Lognormal	tan 26°	0.36	0.1–0.2
$\tan\phi'_r$	Lognormal	tan 18°	0.26	0.1–0.2
r_u	Lognormal	(0.1–0.45)	(0.1–0.45)	0.1–0.2

Note: *Range of COV for the shear strength parameters are collected from Hong and Roh (2008); the pore pressure ratio r_u is assumed in this study

Correlations between random variables

To highlight the versatility of the adopted method of analysis, some studies reported here include assumed correlations between peak and residual shear strength parameters. These studies were found to be useful. However, in order to concentrate on the main aims of this paper, correlations between parameters were ignored in other studies reported in this paper. It must be emphasized that, provided data on correlation between any pair of variables become available, methods developed in this paper are still applicable and the analyses can be extended accordingly.

Correlation coefficients between shear strength parameters c and ϕ have been presented in a number of published papers such as Lumb (1970), Yuceman et al. (1973), Matsuo and Kuroda (1974), Wolff (1985), Low and Tang (1997), Khajehzadeh et al. (2010) etc. The magnitude of this correlation coefficient depends on type of soil, and on whether the data for shear strength parameters are obtained from drained, undrained, or consolidated undrained shear tests. Some researchers mention a negative correlation between c and ϕ (or tan ϕ) from drained shear tests relevant to slope stability problems (e.g., Lumb 1970; Yuceman et al. 1973; Wolff 1985); others mention little or negligible correlation

It is important to note that significant data are not available concerning correlations between shear strength parameters. The evidence is inconsistent in the few reported cases where correlations have been investigated. Consequently, it is not surprising that slope reliability assessments are often based on the assumption of uncorrelated random variables (e.g., Duncan 2000, Xue and Gavin 2007, Hong and Roh 2008; Metya and Bhattacharya 2014; Metya 2017; Metya et al. 2017).

The value of the proposed new variable R over a slip surface may depend on a number of factors and it is reasonable, therefore, to regard it as an independent random variable. Moreover, no evidence has been presented in the vast geotechnical literature concerning correlation between R and any other geotechnical parameter.

As stated above, the effect of correlation between the peak and residual shear strength parameters has been examined based on a range of assumed values of the correlation coefficients. The reason for considering only positive values of the correlation coefficients must, therefore, be stated. During the strain-softening process the composition and mineralogy of a soil remain unchanged. Therefore, it is entirely reasonable to stipulate that any correlation between peak and residual friction angles, and that between peak and residual cohesion, will be positive rather than negative. Reliable evidence is needed before a credible value can be adopted in design problems or case studies.

Deterministic analyses and results

Table 3 presents a series of results of deterministic analyses of the slopes in example 1 and example 2 in the form of values of factor of safety F when the values of the shear strength parameters (peak and residual) are taken to be equal to their respective mean values and the value of the pore pressure ratio is varied within the range of its mean values (0.1 - 0.45) as given in Table 2. In addition, the residual factor is varied from R = 0 (when the entire slip surface is at peak strength) to R = 1 (when the entire slip surface is at residual strength). In between these two limiting values, factor of safety has also been evaluated considering seven discrete values of R equal to its mean values corresponding to the seven shapes of beta distributions (R-Shapes) to be considered for the reliability analyses of the slopes (Table 1). From Table 3, the following can be observed for both the example problems:

Variation in residual factor R and/or pore pressure ratio r_u leads to very large variation in the value of factor of safety F. For instance, as value of R varies from 0 to 1 together with r_u which varies from 0.1 to 0.45, the value of F varies from 2.025 to 0.668 for example 1, and from 1.926 to 0.689 for example 2. Further, for any value of pore pressure ratio r_u, as the value of residual factor R increases, the value of factor of safety decreases. Again, for any value of residual factor R, as the value of pore pressure ratio r_u increases, the value of factor of safety decreases. Both these observations are as per expectation, and, therefore, the results of the deterministic analyses

Table 3 Variation of factor of safety F with variation of R and r_u for examples 1 and 2

Residual factor R	Pore pressure ratio r_u							Remarks
	0.1	0.15	0.2	0.25	0.3	0.4	0.45	
Example 1								
0.000	2.025	1.928	1.830	1.733	1.635	1.440	1.342	Entire slip surface at peak strength
0.250	1.800	1.710	1.621	1.531	1.442	1.263	1.174	R=mean of R-shape (6)
0.333	1.724	1.638	1.551	1.464	1.378	1.204	1.117	R=mean of R-shape (1)
0.500	1.574	1.493	1.411	1.330	1.249	1.086	1.005	R=mean of any of R-shape (3),(4),(5)
0.666	1.423	1.348	1.272	1.196	1.120	*0.968*	*0.893*	R=mean of R-shape (2)
0.750	1.348	1.275	1.202	1.129	1.056	*0.910*	*0.836*	R=mean of R-shape (7)
1.000	1.123	1.058	*0.993*	*0.928*	*0.863*	*0.733*	*0.668*	Entire slip surface at residual strength
Example 2								
0.000	1.926	1.838	1.749	1.661	1.572	1.395	1.307	Entire slip surface at peak strength
0.250	1.729	1.647	1.564	1.488	1.399	1.235	1.152	R=mean of R-shape (6)
0.333	1.663	1.583	1.502	1.422	1.342	1.181	1.101	R=mean of R-shape (1)
0.500	1.531	1.455	1.379	1.303	1.227	1.074	*0.998*	R=mean of any of R-shape (3),(4),(5)
0.666	1.400	1.328	1.255	1.183	1.111	*0.967*	*0.895*	R=mean of R-shape (2)
0.750	1.334	1.264	1.194	1.124	1.054	*0.913*	*0.843*	R=mean of R-shape (7)
1.000	1.136	1.072	1.008	*0.945*	*0.881*	*0.753*	*0.689*	Entire slip surface at residual strength

are seen to be consistent. Further, quite a few combinations of R and r_u lead to $F < 1.0$, which indicates overall slope failure. These values are shown in italics in Table 3.

In these two examples of relatively flat slopes the critical pore pressure ratio r_u (which leads to $F = 1$) is reached only at high values of the residual factor R. In both examples, these critical values of r_u are below the maximum of 0.5 which represents seepage occurring throughout the slope.

Reliability analyses and results

Probability distributions used in the analyses

FORM based reliability analysis has been carried out considering all six random variables including the residual factor R for which the seven probability distributions [R-shapes (1) through (7)] listed in Table 1 (and Fig. 1) have been used in turn. Thus, for a given set of statistical properties (mean and COV) of the random variables other than R, the results of reliability analysis comprise a total of seven sets of values of reliability index β_{HL} and probability of failure p_F, corresponding to the seven R-shapes. A comparison of these seven sets of values of β_{HL} and p_F then reveals the critical R-shape which yields the lowest value of β_{HL} or the highest value of p_F.

All the results presented in section Results — reliability index and probability of failure are based on the assumption of a lognormal probability distribution for all random variables except R (four shear strength parameters and one pore pressure ratio, a total of 5 variables).

Reliability analysis: general features

Case I and case II analysis

Further, as a means to quantify the effect of including the residual factor R as one of the random variables on the reliability results, reliability analysis has also been carried out considering the residual factor as a deterministic variable in which case there are only five random variables as listed in Table 2. For the sake of convenience this is referred to as the *case I* analysis while the one considering R as random (with six random variables) is referred to as the *case II* analysis. The effect of treating R as a random variable is then brought out from the difference between the values of the reliability index (and probability of failure) obtained from the *case I and case II* analyses. In order for such a difference to be meaningful, the deterministic value of R in *case I* analysis has been taken as equal to the mean value of R considered in the corresponding *case II* analysis.

Sensitivity analysis

As stated earlier, in a reliability analysis using FORM, the direction cosines (α_i', i=1,n) for the n nos. of random variables, whose values are obtained as part of the solution, indicate sensitivity indices for the random variables (Haldar and Mahadevan 2000). Taking advantage of this feature, a sensitivity study has been carried out in order to compare the contribution of the residual factor relative to that of the other random variables.

Effect of variation of statistical properties and correlation

In regard to the input data for the reliability analysis, it is noted in Table 2 that while the COV's of the random shear strength parameters are known to vary within their respective ranges, the mean and COV of the random pore pressure ratio r_u are also considered here to vary within some assumed ranges. It would thus be of interest to investigate the effect of varying (i) the COV's of the random variables other than R, as well as (ii) the mean value of the pore pressure ratio r_u. It would also be of interest to investigate the effect of any kind of correlation between the shear strength parameters.

Four studies — study 1 to study 4

The analyses mentioned above (in sub-sections Case I and case II analysis, Sensitivity analysis, Effect of variation of statistical properties and correlation) have been covered under the following four studies:

Study 1: Study 1 comprises case I (R-deterministic) and case II (R-random) analyses assuming (i) the random variables as uncorrelated, (ii) mean r_u = 0.2, and (iii) the COV's of random variables (other than R) equal to the lowest values in their respective ranges (Table 2). For this initial study, FORM based results in terms of probability of failure are also compared with those from Monte Carlo simulation (MCS).
Study 2: Study 2 is similar to study 1 except that the random variables for the peak and residual components of shear strength parameters are assumed to be correlated.
Study 3: Study 3 is also similar to study 1 except that the COV's of the random variables (other than R) are varied within their respective ranges (Table 2).
Study 4: Study 4 is also similar to study 1 except that mean r_u is varied within its range (Table 2). Specifically, $\overline{r}_u$ = 0.1,0.2,0.3,0.4, and 0.45 have been used.

Results — reliability index and probability of failure

Results of study 1

For this initial reliability analyses, while the residual factor R is assumed to have all seven forms of beta distributions [R-shapes (1) to (7)] detailed in Fig. 1 and Table 1, the statistical properties of the other five lognormally distributed random variables are taken as in Table 2. Based on the data in Table 2, the pore pressure ratio r_u is assumed to have a mean value of 0.2, and the COVs of the random variables are taken as the base values in their respective ranges. Further, all the six random variables are assumed to be uncorrelated.

Table 4 presents the values of reliability index obtained from case I and case II studies of both the example problems. It is seen that for the case II analysis of both the example problems, R-shape (2) results in the lowest value of β_{HL} (and therefore, the highest value of p_F). Thus, for the residual factor, the assumption of triangular left-skewed beta distribution [R-shape (2)] is found to be the critical distribution, followed by R-shape (7), also a left skewed distribution. This is not surprising because these distributions correspond to the highest mean values of R among the seven distributions selected for analysis. For right skewed distributions, mean values of R are low and, therefore, the reliability indexes are relatively higher. Thus R-shape (6) results in the highest value of β_{HL}. In general, the reliability index will decrease as mean value of R increases if the standard deviation is the same. An increase in COV for given mean value decreases the reliability index. This is clear from the results for R-shapes (3), (4), and (5). It may also be noted that the correspondence between the ranks of the intermediate values of β_{HL} and R-shapes is identical for both the example problems.

As stated earlier, the difference between the β_{HL} values obtained for case I and case II analyses for each of the seven cases of R distributions quantifies the effect of treating R as a random variable. Table 4 shows that the amount of reduction in the value of β_{HL} from case I to case II varies significantly with the assumed beta distribution. The reduction is largest for R-shape (2) and smallest for R-shape (6). The largest reduction is as high as 65% for example 1 and 61% for example 2, while the smallest reduction is 36% for example 1 and 32% for example 2. Thus, even the smallest reductions are also substantial. Consequently, the probability of failure based on residual factor as a random variable can be one or more orders of magnitude higher than that based on assumption of residual factor as deterministic.

For the sake of validation through comparison, values of p_F for the case II analyses have also been obtained based on the direct MCS with 10^6 number of simulations and a seed of 28061987. Table 5 shows that there is a reasonably good agreement between the two sets of values (being of the same order of magnitude); further, the FORM based values of p_F are, for all R-shapes, a little higher than the MCS based values, and, hence, the former is on the conservative side.

Results of study 2

In study 1 described above, all the random variables were assumed to be uncorrelated. In this study, however, correlations are assumed to exist between the peak and the residual cohesion parameters c_p' and c_r' as well as between the peak and the residual friction parameters $\tan\varphi_p'$ and $\tan\varphi_r'$.

In absence of published data, a parametric study has been carried out with assumed values of the correlation coefficients between c_p' and c_r' and between $\tan\varphi_p'$ and $\tan\varphi_r'$. For simplicity, these two correlation coefficients are assumed to be of

Table 4 Summary of results of reliability analyses in study 1

Statistical properties of residual factor, R			Reliability index, β_{HL}					
			Example 1			Example 2		
R- shape (1)	Mean(2)	COV(3)	Case I (4)	Case II (5)	Difference (%) (6)	Case I (7)	Case II (8)	Difference (%) (9)
(1)	0.333	0.707	5.681	2.403	57.70	5.319	2.415	54.61
(2)	0.667	0.354	3.335	1.184	64.52	3.157	1.229	61.06
(3)	0.500	0.447	4.743	2.026	57.29	4.457	2.042	54.18
(4)	0.500	0.378	4.743	2.356	50.33	4.457	2.357	47.12
(5)	0.500	0.277	4.743	2.961	37.56	4.457	2.924	34.39
(6)	0.250	0.577	6.000	3.858	35.69	5.605	3.812	31.98
(7)	0.750	0.193	2.461	1.309	46.80	2.358	1.346	42.91

equal value and denoted by ρ. A parametric study has been conducted considering values of ρ as 0.25, 0.5, 0.75, and 1.0. The cross correlation coefficients between the different strength parameters are, however, assumed to be zero.

Table 6 presents, for example 1, a summary of values of reliability index for all seven R-shapes. By comparing the results of study 2 with those from study 1 above [i.e., for $\rho = 0$ in Table 4], it is observed that the trend of results in study 2 remains essentially the same as in study 1. For any non-zero value of ρ, like study 1, R-shape (2) results in the lowest value of β_{HL} and the R-shape (6) results in the highest value of β_{HL}, in a case II analysis. For any situation, as ρ increases, β_{HL} decreases, which is expected in view of the assumed positive correlation. Moreover, as the value of ρ increases from 0.0 to 1.0, the value of β_{HL} decreases by nearly 5% and 9% in the case II analysis with R-shape (2) and R-shape (6) respectively. It is seen that the trend of results in terms of the reduction in the values of reliability index from case I analyses (R-deterministic) to case II analyses (R-random) remains the same in Table 4 and Table 6. Specifically for $\rho = 1.0$, the largest and the smallest reductions corresponding to R-shape (2) and R-shape (6) respectively, are nearly 55% and 32% in place of 65% and 36% noted in study 1. Details of corresponding results for example 2 are not presented here due to space limitations. On the whole it can be stated that correlation between the peak and residual strength parameters has no major influence on the trend in reliability results although the influence can be significant for a specific analysis relevant to a given example with another set of data. In view of such an observation from this study, in the subsequent studies, e.g., study 3 and study 4, no correlation has been considered between the random variables to save space. Figure 3 presents a plot of probability of failure vs ρ for study 2 on example 1. The plot of p_F vs ρ for study 2 on example 2 has been found to be very similar to that for example 1.

Table 5 Comparison between FORM and MCS results in study 1

R-shape	Probability of failure, p_F			
	Example 1		Example 2	
	FORM	MCS	FORM	MCS
(1)	$8.12\text{x}10^{-3}$	$6.10\text{x}10^{-3}$	$7.88\text{x}10^{-3}$	$5.90\text{x}10^{-3}$
(2)	$1.18\text{x}10^{-1}$	$8.99\text{x}10^{-2}$	$1.09\text{x}10^{-1}$	$8.12\text{x}10^{-2}$
(3)	$2.14\text{x}10^{-2}$	$1.66\text{x}10^{-2}$	$2.06\text{x}10^{-2}$	$1.56\text{x}10^{-2}$
(4)	$9.24\text{x}10^{-3}$	$7.40\text{x}10^{-3}$	$9.22\text{x}10^{-3}$	$7.50\text{x}10^{-3}$
(5)	$1.53\text{x}10^{-3}$	$1.10\text{x}10^{-3}$	$1.73\text{x}10^{-3}$	$1.30\text{x}10^{-3}$
(6)	$5.71\text{x}10^{-5}$	$3.00\text{x}10^{-5}$	$6.88\text{x}10^{-5}$	$4.00\text{x}10^{-5}$
(7)	$9.52\text{x}10^{-2}$	$7.70\text{x}10^{-2}$	$8.91\text{x}10^{-2}$	$7.17\text{x}10^{-2}$

Results of study 3

In the study 1 and study 2 described above, the COVs of the shear strength parameters were taken equal to their base values (the lowest values) in their respective ranges (Table 2) and the effect of treating R as a random variable has been brought out. It, however, remains to be seen how the effect changes for higher level of uncertainty (variability), and, therefore, the same has been undertaken here in study 3. The COVs for the different random variables except R are assumed to vary linearly within their ranges in terms of a parameter η which varies from 0 to 1. Thus, the COVs of the five random variables are represented by $\delta_{c'p} = 0.2 + 0.3\,\eta$; $\delta_{c'r} = 0.2 + 0.3\,\eta$; $\delta_{\tan\phi'p} = 0.1 + 0.1\,\eta$; $\delta_{\tan\phi'r} = 0.1 + 0.1\,\eta$ and $\delta_{ru} = 0.1 + 0.1\,\eta$, and several values of the parameter η such as $\eta = 0.0$, 0.25, 0.50, 0.75, and 1.0 have been considered ($\eta = 0.0$ corresponds to the base values of the COVs considered in study 1 and study 2). For the sixth random variable R, different shapes of the probability distribution are considered one by one, as in study 1 and study 2. The results for reliability index are tabulated in Table 7 for example 1.

By comparing the results of study 3 with those from study 1 [i.e., for $\eta = 0$ in Table 4], it is observed that the trend of results in study 3 remains essentially the same as in study 1. For any non-zero value of η, like study 1, R-shape (2) again results in the lowest value of β_{HL} and the R-shape (6) again results in

Table 6 Values of reliability index β_{HL} for study 2 on example 1

R-shape	Analysis case	Reliability index, β_{HL}				
		$\rho = 0.0$	$\rho = 0.25$	$\rho = 0.5$	$\rho = 0.75$	$\rho = 1.0$
(1)	Case I	5.681	5.357	5.092	4.871	4.689
	Case II	2.403	2.372	2.342	2.312	2.284
	Difference (%)	57.70	55.71	54.01	52.53	51.29
(2)	Case I	3.335	3.053	2.832	2.653	2.509
	Case II	1.184	1.169	1.156	1.142	1.129
	Difference (%)	64.52	61.69	59.19	56.94	54.99
(3)	Case I	4.743	4.377	4.086	3.848	3.656
	Case II	2.026	1.994	1.963	1.933	1.905
	Difference (%)	57.29	54.44	51.95	49.76	47.88
(4)	Case I	4.743	4.377	4.086	3.848	3.656
	Case II	2.356	2.308	2.261	2.217	2.175
	Difference (%)	50.33	47.28	44.66	42.40	40.50
(5)	Case I	4.743	4.377	4.086	3.848	3.656
	Case II	2.961	2.868	2.780	2.698	2.624
	Difference (%)	37.56	34.48	31.97	29.89	28.22
(6)	Case I	6.000	5.731	5.509	5.322	5.168
	Case II	3.858	3.761	3.669	3.581	3.502
	Difference (%)	35.69	34.37	33.41	32.72	32.24
(7)	Case I	2.461	2.269	2.116	1.992	1.892
	Case II	1.309	1.282	1.256	1.231	1.208
	Difference (%)	46.80	43.49	40.66	38.22	36.16

the highest value of β_{HL} in a case II analysis. For any situation, as η increases, β_{HL} decreases, which is expected. Moreover, as the value of η increases from 0 to 1, the value of β_{HL} decreases by nearly 31% in the case II analysis both for R-shape (2) and R-shape (6), which are substantial. It is seen that the trend of results in terms of the reduction in the values of reliability index from case I analyses (R-deterministic) to case II analyses (R random) remains the same in Table 4 and Table 8. Specifically, for $\eta = 0.75$, the largest and the smallest reductions, corresponding to R-shape (2) and R-shape (6) respectively, are nearly 46% and 4% in place of 65% and 36% noted in study 1. However, details of such results for example 2 are not presented here to save space. That the amount of reduction is less in study 3 compared to study 1 is also expected because, while the level of uncertainty for the other five random variables (considered in case I analysis) increases with non-zero value of η, there is no corresponding increase in the level of uncertainty in the residual factor R. Figure 4 presents a plot of probability of failure vs η for study 3 on example 1. The plot of p_F vs η for study 3 on example 2 has been found to be very similar to that for example 1.

Results of study 4

In all three studies described above, the effect of treating the residual factor R as a random variable has been brought out for a medium level of pore water pressure, considering the pore pressure ratio r_u as having a mean value of 0.2. It would be interesting to study how the trend of results presented above changes with change in the level of pore pressure. Keeping this in mind, study 4 has been undertaken which is similar to study 1 except that here the mean of r_u is varied from 0.1 to 0.45.

Table 8 presents, for example 1, a summary of the values of reliability index obtained from case II analyses. It is seen that for mean $r_u = 0.1$, 0.15, and 0.25, the trend of results is the same as for study 1, 2, and 3 with mean $r_u = 0.2$. In other words, R-shape (2) again results in the lowest value of β_{HL}

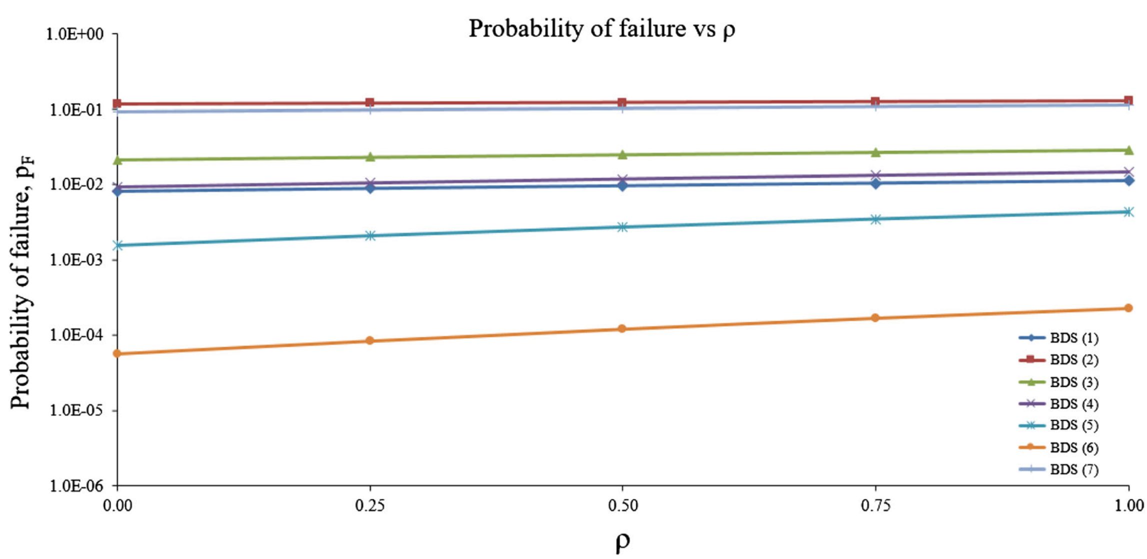

Fig. 3 Variation of probability of failure with increasing correlations between the peak and the residual strength parameters (study 2 and case II analyses on example 1)

(highest value of p_F), while R-shape (6) results in the highest value of β_{HL} (lowest value of p_F). It is also seen that the largest reductions in the value of β_{HL} [for R-shape (2)] are 59.29%, 61.83%, and 67.47% for mean r_u = 0.1, 0.15, and 0.25 respectively, compared to 64.52% for mean r_u = 0.2. Similarly, the smallest reductions in the value of β_{HL} [for R-shape (6)] are 37.52%, 36.92%, and 33.68% for mean r_u = 0.1, 0.15, and 0.25 respectively, compared to 35.69% for mean r_u = 0.2. For mean r_u = 0.3, however, it is seen that R-shape (2) does not result in the numerically lowest value of β_{HL}. It yields a β_{HL} of 0.389, which is higher than the value of 0.246 corresponding to R-shape (7) which, again, is a left-skewed distribution. For mean r_u = 0.4 and 0.45, the assumption of R-shape (2) gives case II analysis results to be higher (less negative) than those from the corresponding case I analysis. This is the apparent inconsistency to which reference was made in section Fluctuation of pore water pressure ratio, r_u concerning the effect of change in COV on computed value of reliability index. For fixed number of random variables (either 5 or 6), this inconsistency does not arise. There is a consistent trend for change of reliability index or probability of failure with increasing pore pressure ratio.

Finally, a computed negative value of reliability index is very useful for estimating the highest value of probability of failure due to a combination of high residual factor (or adverse probability distribution) and high pore pressure ratio. As an illustration of one such combination, for R-shape 7 and pore pressure ratio of 0.45, reliability index is –1.225 for case II analysis (see Table 8). The corresponding probability of failure from computation is 88.4%.

Figure 5 presents graphical variation of the probability of failure obtained from case II analysis of example 1 with variation in the mean of pore pressure ratio for the seven probability distributions of the residual factor. From Fig. 5, it is observed that (i) the relationships are nonlinear and the shape of each curve is influenced by the probability distribution of R; (ii) the probability of failure increases at a somewhat decreasing rate as the mean pore water pressure increases; and (iii) the probability of failure at any mean pore pressure ratio can vary by several orders of magnitude depending on the probability distribution of R.

Observations from the results obtained from example 2 being very similar to those for example 1, are not presented here.

Study on sensitivity index of the random variables

As stated earlier, in a reliability analysis using FORM, the direction cosines (α_i', i = 1 to n) for n number of random variables whose values are obtained as part of the solution,

Table 7 Values of reliability index β_{HL} for study 3 on example 1

R-shape	Analysis case	Reliability index, β_{HL}				
		η = 0.0	η = 0.25	η = 0.5	η = 0.75	η = 1.0
(1)	Case I	5.681	4.434	3.622	3.049	2.621
	Case II	2.403	2.253	2.120	1.997	1.882
	Difference (%)	57.70	49.19	41.48	34.49	28.20
(2)	Case I	3.335	2.606	2.121	1.772	1.508
	Case II	1.184	1.074	0.978	0.894	0.817
	Difference (%)	64.52	58.80	53.86	49.57	45.86
(3)	Case I	4.743	3.705	3.024	2.540	2.177
	Case II	2.026	1.866	1.726	1.599	1.483
	Difference (%)	57.29	49.64	42.92	37.03	31.89
(4)	Case I	4.743	3.705	3.024	2.540	2.177
	Case II	2.356	2.150	1.970	1.806	1.657
	Difference (%)	50.33	41.96	34.86	28.87	23.90
(5)	Case I	4.743	3.705	3.024	2.540	2.177
	Case II	2.961	2.644	2.366	2.118	1.897
	Difference (%)	37.56	28.63	21.75	16.61	12.86
(6)	Case I	6.000	4.680	3.824	3.221	2.772
	Case II	3.858	3.545	3.247	2.952	2.655
	Difference (%)	35.69	24.26	15.07	8.34	4.22
(7)	Case I	2.461	1.920	1.557	1.293	1.093
	Case II	1.309	1.145	1.008	0.889	0.786
	Difference (%)	46.80	40.36	35.27	31.25	28.07

Table 8 Summary of reliability results due to variation of mean r_u (Study 4 on Example 1)

R-shape	Analysis case	Reliability Index, β_{HL}						
		$r_u = 0.10$	$r_u = 0.15$	$r_u = 0.20$	$r_u = 0.25$	$r_u = 0.30$	$r_u = 0.40$	$r_u = 0.45$
(1)	Case I	7.489	6.625	5.681	4.689	3.696	1.839	1.001
	Case II	3.225	2.808	2.403	2.018	1.654	0.961	0.610
	Difference (%)	56.93	57.62	57.70	56.96	55.24	47.74	39.11
(2)	Case I	5.209	4.299	3.335	2.352	1.387	–0.406	–1.219
	Case II	2.121	1.641	1.184	0.765	0.389	–0.284	–0.611
	Difference (%)	59.29	61.83	64.52	67.47	71.94	29.93	49.86
(3)	Case I	6.648	5.732	4.743	3.720	2.710	0.844	0.005
	Case II	2.926	2.473	2.026	1.594	1.181	0.394	0.002
	Difference (%)	55.98	56.86	57.29	57.15	56.42	53.33	51.05
(4)	Case I	6.648	5.732	4.743	3.720	2.710	0.844	0.005
	Case II	3.311	2.837	2.356	1.875	1.399	0.466	0.003
	Difference (%)	50.19	50.51	50.33	49.60	48.37	44.77	42.56
(5)	Case I	6.648	5.732	4.743	3.720	2.710	0.844	0.005
	Case II	4.050	3.520	2.961	2.380	1.783	0.586	0.003
	Difference (%)	39.08	38.59	37.56	36.03	34.21	30.51	28.83
(6)	Case I	7.717	6.898	6.000	5.046	4.080	2.250	1.417
	Case II	4.822	4.351	3.858	3.347	2.819	1.723	1.159
	Difference (%)	37.52	36.92	35.69	33.68	30.91	23.42	18.24
(7)	Case I	4.250	3.376	2.461	1.532	0.620	–1.086	–1.867
	Case II	2.378	1.848	1.309	0.772	0.246	–0.751	–1.225
	Difference (%)	44.06	45.27	46.80	49.63	60.30	30.83	34.39

directly indicate the values of sensitivity index for the random variables (Haldar and Mahadevan 2000). Taking advantage of this feature, a sensitivity study has been carried out in order to compare the contribution of the residual factor R to the reliability index β_{HL} relative to that of the other random variables. Based on studies 1 through 4 described above, the following four sets of observations have been made on the sensitivity indexes of the six random variables.

Study 1: Based on the results of Study 1 on example 1, values of the sensitivity indexes, in terms of direction cosines, of the six random variables are presented in the form of a histogram in Fig. 6. From Fig. 6 it is observed that the residual factor R has the highest value of the direction cosine, and therefore, has the single largest contribution or influence on the reliability index for all the R-shapes except for R-shape (7) in which case R has a joint largest influence along with $\tan\varphi_r'$. From the results of study 1 of example 2, the observations are similar.

Study 2: From the results of study 2 on both the examples, the observations are the same as in study 1.

Study 3: From the results of study 3 on both the examples, the observations are, however, somewhat different from those from study 1 or study 2. It is seen that the rank of the sensitivity index for R is heavily dependent on its distribution (R-shape). For all of R-shapes (1), (2), (3), and (4), R has the single largest sensitivity index. For R-shapes (5) and (7), R has a joint largest sensitivity index along with $\tan\varphi_r'$. For R-shape (6), R has the single largest sensitivity index for low to medium level of uncertainty of the other five random variables, i.e., $\eta = 0.0$ and 0.25. However, at higher level of uncertainty of the other five random variables, e.g., $\eta = 0.5$, 0.75, and 1.0, it is seen that $\tan\varphi_p'$ has the highest sensitivity index, followed by c_p', $\tan\varphi_r'$, r_u, R and c_r' in decreasing order.

Study 4: From the results of study 4 on both the examples, the observations are the same as in study 1 or study 2.

Discussion

Practical applications and challenges

The proposed method is ready for assessing reliability of slopes in practice. The application may be for analysis of natural slopes, assessment of landslide areas as well as the reliability

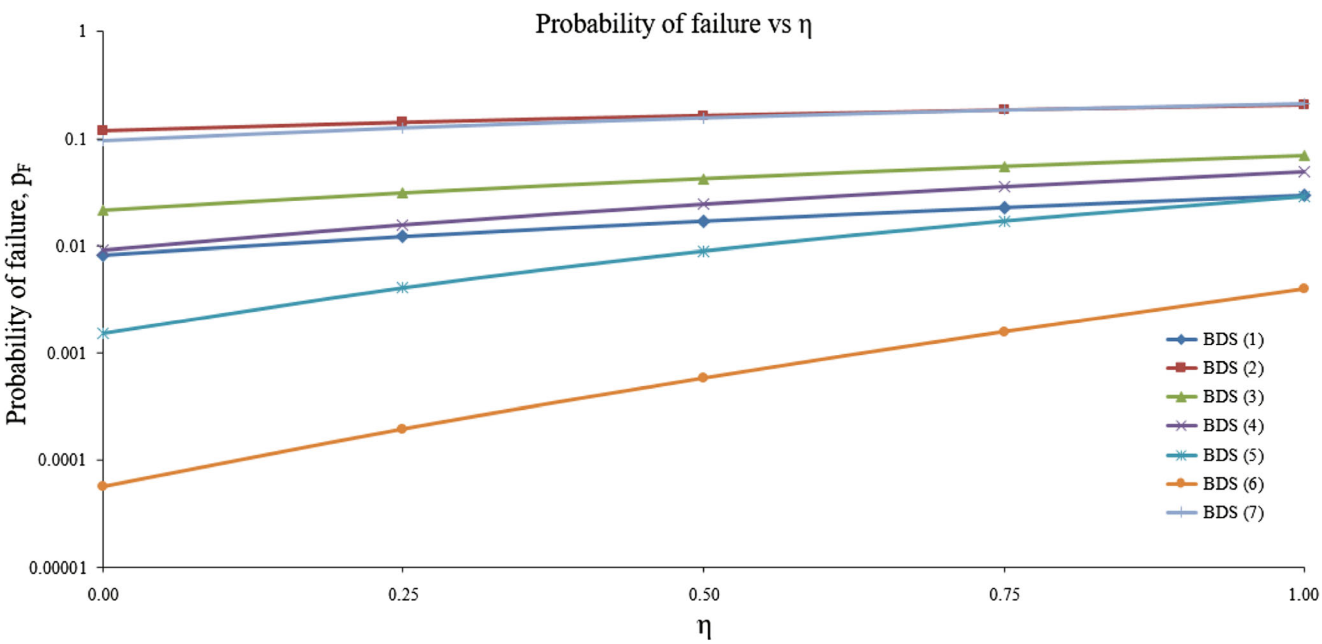

Fig. 4 Variation of probability of failure with increasing variability of the random variables except R (study 3 and case II analyses on example 1)

of excavated slopes. Based on the results discussed in this paper, inclusion of R as a random variable will lead to more accurate assessment of reliability for any slope. Ignoring R or treating it as a deterministic variable is likely to overestimate reliability and underestimate the probability of failure even if other important parameters are considered as random variables.

The main challenge in reliability analysis relates to availability of data and, in particular, the distributions of parameters assumed as random variables. In this respect the inclusion of residual factor R offers both a challenge and an opportunity to any researcher or geotechnical practitioner. As noted earlier, one group of researchers has shown how a probabilistic approach may be used for back-analysis of a significant landslide in such a way that the probability distribution of R can be assessed.

As stated above, for some cases, the mean and standard deviation of R would have to be assessed on the basis of experience or judgment in the form of some simple rules or procedures. Such procedures have been proposed for assessing shear strength parameters for reliability analysis (e.g., Duncan 2000). Further research may allow similar procedures to be developed for selecting the PDF of R or, at least, the COVs of R in different situations.

Another challenge that often arises in slope reliability is the question of spatial variability of shear strength parameters (Jiang et al. 2014; Jiang and Huang 2016; Metya and Bhattacharya 2016a b; Metya 2017). Consideration of spatial variability of peak and residual shear strength parameters, while also including the residual factor, would be a major extension which clearly is outside the scope of the paper. That extension is, however, desirable so that clear guidelines can be developed for different types of projects.

Additional comments on probability distribution for R

Good reasons have already been advanced for choosing a beta distribution system for the residual factor R. It has been shown that choosing a normal distribution is inconsistent with the boundary values of 0 and 1, and that restriction to symmetrical distribution implies a mean of 0.5 while the mean R actually can vary from 0 to 1. Then the question arises about how to arrive at the specific beta distribution. That is not as difficult as it may seem at first consideration. The Kettleman Hills study (Gilbert et al. 1998) already points the way in which a consistent approach may be used to derive the probability distribution of R based on observational data. More generally, deterministic back analysis already allows estimation of average shear strength at failure and hence the mean residual factor. The choice may then be made, based on engineering judgment and any other evidence, about the COV of R for a particular study or project. With a set of mean and COV of R, the particular beta distribution can be found from the basic equations adiscussed in earlier sections of this paper. Space does not permit a discussion of the other approaches that may be used for choosing a specific probability distribution of R; in particular, one can extend the Monte-Carlo simulation technique for adopting or fine tuning the probability distribution of R.

Further research including consideration of progressive failure

Several areas of research and extension can be pursued for reliability studies for slopes and landslides which include the residual factor as a random variable. The following would seem to be the most important areas:

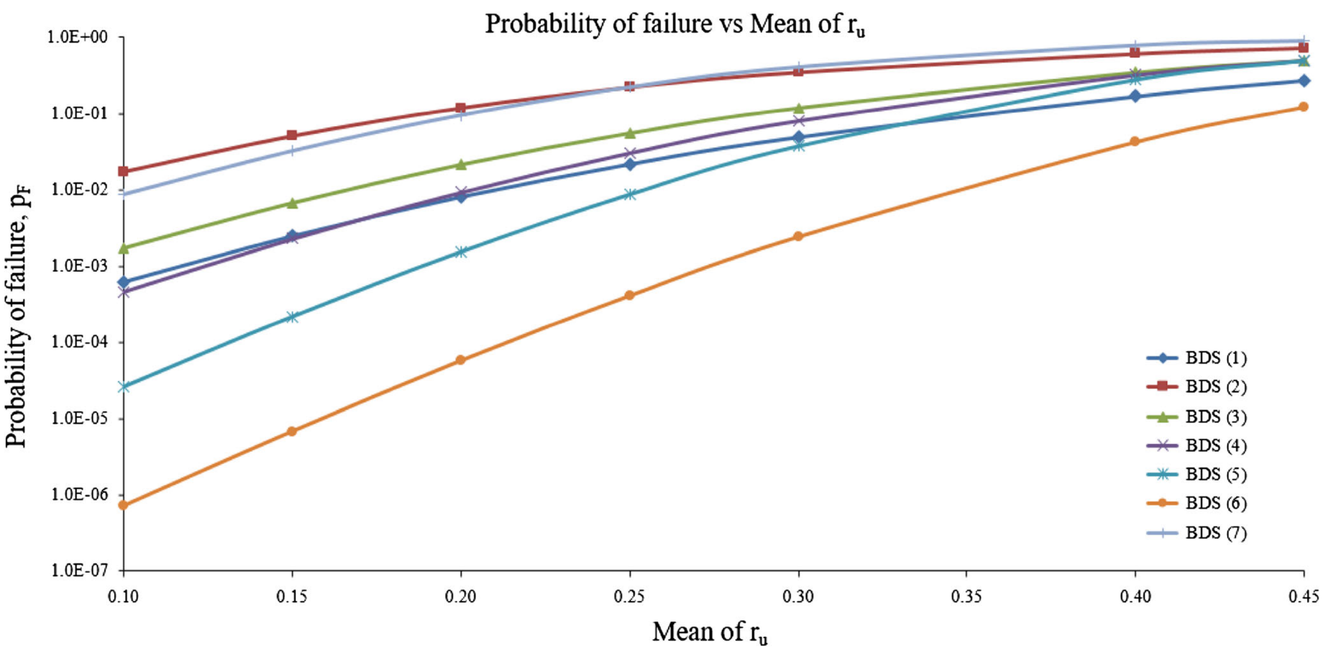

Fig. 5 Effect of varying pore water pressures on the probability of failure (study 4 and case II analyses on example 1)

(a) The proposed approach can be used to develop methods and procedures for reliability of slopes with slip surfaces of curved or arbitrary shape. In fact initial work on 2-D reliability analysis including residual factor R as a random variable has been carried out successfully by the authors (Metya et al. 2016a, b; Metya 2017). The limit equilibrium framework preceding the reliability procedure has been developed on the basis of Bishop simplified method as well as the Spencer method. The search or a critical slip surfaces is part of the reliability solution. The procedures need further development and more attention must be devoted to slopes with slip surfaces of general shape. Moreover, as mentioned in the previous paragraphs, consideration of spatial variability for shear strength parameters will require an intensive research effort.

(b) Strategies and methods must be developed for research investigation of the residual factor as a variable in slopes and landslides. Such research would include the discovery of data

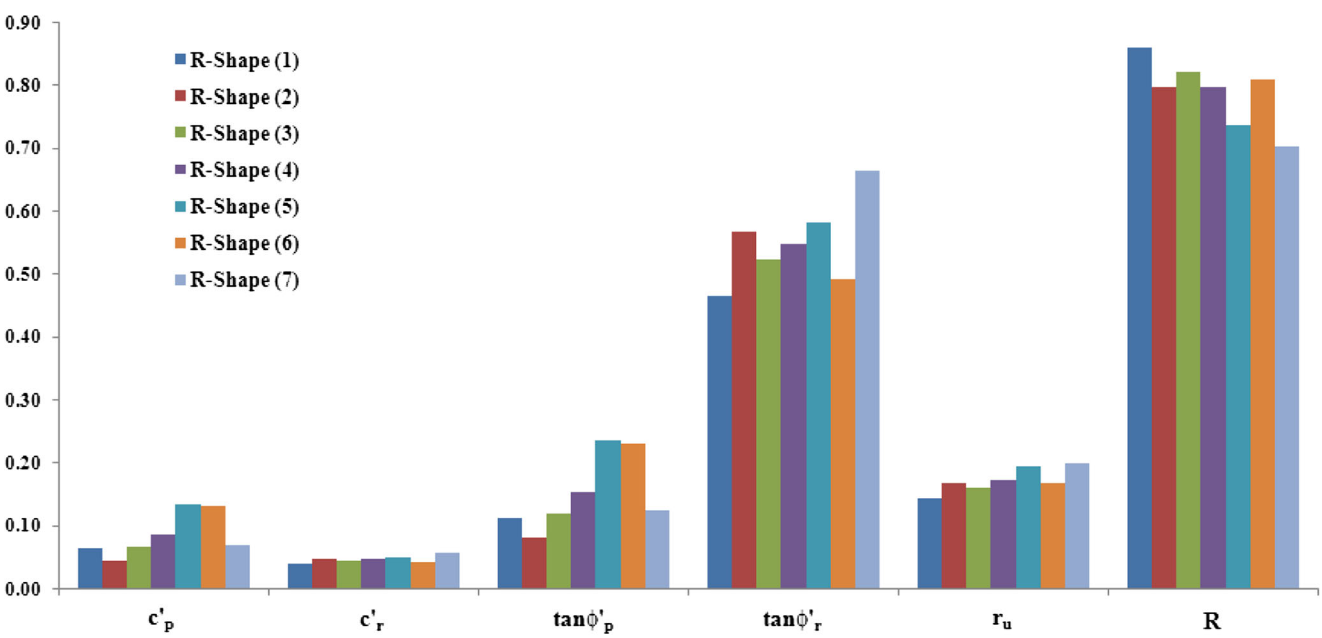

Fig. 6 Results of sensitivity analysis based on the FORM method (study 1 on example 1)

from past investigations which could be assembled and analyzed to obtain evidence about the probability distribution of R in real cases.

(c) Numerical models and computer simulation may be developed in order to understand the probability distribution of the residual factor for specific slopes under given conditions in order to investigate correlations between variables. Insight may also be gained for any correlation between random variables.

(d) Most importantly, selected case studies must be analyzed in detail in order to validate the methods and procedures. For example a large number of case studies have been investigated by Mesri and Shahien (2003) from a deterministic perspective. Some of these may be particularly appropriate for study from a probabilistic perspective and also including the residual factor as a random variable. Moreover, fresh insights may be gained by careful reconsideration, within a probabilistic framework, of classical case studies such as those considered by Skempton (1964, 1966) and his co-researchers in later years.

(e) The inclusion of residual factor as a random variable is a recognition of progressive failure processes having reached a certain stage which may or may not be close to the critical. Therefore, it is pertinent to consider the probability for continuation of the progressive failure process. It might also be feasible to formulate procedures to study the probability of successive failures where R is one of the random variables. Conditional probabilities of failure that are required in such extensions may be estimated based on procedures such as those proposed by Chowdhury et al. (1987) and Chowdhury and Zhang (1993). The procedures would have to be updated to include R as a random variable.

Conclusions

The paper concerns reliability analysis of a natural slope in a homogeneous strain-softening c′-ϕ′ soil modeled as an 'infinite slope' subjected to seepage parallel to the ground surface. A systematic approach for the reliability analysis of a natural slope in a homogeneous strain-softening soil by including the average residual factor R as one of the random variables has been outlined. Reliability analyses have been carried out based on the first order reliability method and validated against direct Monte-Carlo simulation. To include the residual factor R as a random variable altogether seven forms of beta distributions, both symmetrical and skewed, have been tried. For each trial probability distribution of R, two sets of reliability analyses have been carried out, one set considering R as deterministic and the other set considering R as random. The results of these analyses have clearly shown that including R as one of the random variables has significant influence on reliability index, and, therefore, on probability of failure. The influence of R as a random variable is, however, highly dependent on the assumption made regarding its probability distribution. The range of assumptions made in this investigation results in the variation of the probability of failure by several order of magnitude. For the particular illustrative examples studied in this paper, it is revealed that the residual factor with left-skewed triangular distribution [R-shape (2)] with mean of 0.667 and standard deviation of 0.236 (COV of 0.354) has the most significant influence on the reliability index and, therefore, on the probability of failure of a natural slope. For the particular situation in which the COVs of the other (five) random variables are at the base values in their ranges and a mean pore pressure ratio of 0.2, the probability of failure is found to vary from nearly 10^{-5} to nearly 10^{-1} as the assumption regarding the probability distribution of R is varied [R-shape (1) to R-shape (7)].

Moreover, sensitivity studies based on the FORM indicate that R can be the dominating random variable relative to the other five random variables. For most of the distribution assumptions R was the single most significant and for others the joint most significant among the six random variables, except for a situation in which the other random variables considered in the analysis have, simultaneously, high to very high level of uncertainty (COVs).

Appendix 1

Overall or average residual factor for a slip surface of arbitrary shape — general case

In general, shear strength, being proportional to the normal effective stress, would vary from point to point along a potential slip surface, and hence, the local residual factor, would also vary. Therefore, it is very useful to consider an expression for average residual factor, R, which represents the whole of a potential slip surface, as follows:

$$R = \frac{s_p - s_{av}}{s_p - s_r} \tag{A.1}$$

in which

s_p Average peak strength
s_r Average residual strength, and
s_{av} Average current shear strength

Let us consider an arbitrary slip surface of total length L being subdivided into n infinitesimally small segments of lengths Δl_i (i = 1, 2, …, n) such that the corresponding values of residual factor R_i (i = 1, 2, …, n) do not vary within a segment of length Δl_i. For a slope in perfectly brittle soils, the most general case would be when strain-softening to residuals have taken place at m nos. of segments (m<n) whose

total length is L_r, while the remaining (n – m) segments of total length ($L - L_r$) are still at their peak shear strengths. This means, at the current state, $s_i = s_{ri}$ for i = 1 to m, while, $s_i = s_{pi}$ for i = (m+1) to n. In such a situation, the three average strengths in Eq. (A.1), namely, s_r, s_p, and s_{av} are obtained as follows:

$$s_r = \frac{\sum_{i=1}^{m} s_{ri}\,\Delta l_i}{\sum_{i=1}^{m}\Delta l_i} = \frac{\sum_{i=1}^{m} s_{ri}\,\Delta l_i}{L_r} = \frac{SUM1}{L_r}\ (say) \tag{A.2}$$

$$s_p = \frac{\sum_{i=m+1}^{n} s_{pi}\,\Delta l_i}{\sum_{i=1}^{m}\Delta l_i} = \frac{\sum_{i=m+1}^{n} s_{pi}\,\Delta l_i}{L-L_r} = \frac{SUM2}{L-L_r}\ (say) \tag{A.3}$$

$$s_{av} = \frac{\sum_{i=1}^{n} s_i\,\Delta l_i}{\sum_{i=1}^{n}\Delta l_i} = \frac{\sum_{i=1}^{n} s_i\,\Delta l_i}{L} = \frac{\sum_{i=1}^{m} s_i\,\Delta l_i + \sum_{i=m+1}^{n} s_i\,\Delta l_i}{L} = \frac{SUM_1 + SUM_2}{L} \tag{A.4}$$

Substituting the above in Eq. (A.1),

$$R_{av} = \frac{s_p - s_{av}}{s_p - s_r} = \frac{SUM2/(L-L_r) - (SUM1+SUM2)/L}{SUM2/(L-L_r) - SUM1/L_r} = \frac{L_r}{L} \tag{A.5}$$

which agrees with Skempton's definition.

Appendix 2

Factor of safety for a curved slip surface — modified Bishop simplified method

The expression for the factor of safety, F, associated with a curved slip surface of circular shape for a simple slope, based on the Bishop simplified method, has been modified for a strain-softening soil, by including the residual factor R. The modified expression is as follows:

$$F = \frac{\sum\left[\left\{c'_{rf}\,b + W(1-r_u)\times\tan\varphi'_{rf}\right\}/m_{\alpha rf}\right]}{\sum W\sin\alpha} \tag{B.1}$$

where, b is the slice width, W is the slice weight, r_u is the non-dimensional pore water pressure ratio at slice base, and α is the inclination of slice base. Further,

$$c'_{rf} = Rc'_r + (1-R)c'_p \tag{B.2}$$

$$\tan\varphi'_{rf} = R\tan\varphi'_r + (1-R)\tan\varphi'_p \tag{B.3}$$

where, R is the overall or average residual factor for the entire length of the curved slip surface (assumed to be an arc of a circle in this case).

The factor $m_{\alpha rf}$ is given by:

$$m_{\alpha rf} = \left(1 + \frac{\tan\alpha\tan\varphi'_{rf}}{F}\right)\cos\alpha \tag{B.4}$$

The commonly used expression for factor of safety based on the Bishop simplified method (no strain-softening) is given by:

$$F = \frac{\sum\left[\left\{c'b + W(1-r_u)\times\tan\varphi'\right\}/m_{\alpha}\right]}{\sum W\sin\alpha} \tag{B.5a}$$

where, $m_\alpha = \left(1 + \frac{\tan\alpha\tan\varphi'}{F}\right)\cos\alpha$.(B.5b)

It may be noted that Eq. (B.1) is analogous to Eq. (B.5a) except that c′ is replaced by c'_{rf} given by Eq. (B.2), tan ϕ′ is replaced by tan φ'_{rf} given by Eq. (B.3), and m_α is replaced by $m_{\alpha rf}$ given by Eq. (B.4).

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