



Locally efficient and strategy-proof allocation mechanisms in exchange economies

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Received: 9 November 2021 / Accepted: 20 November 2022 / Published online: 1 December 2022
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Abstract

In this paper, we investigate whether efficiency and strategy-proofness of allocation mechanisms defined on a “local” preference set imply dictatorship. Although there is an extensive literature on the characterization of efficient and strategy-proof allocation mechanisms defined on the whole preference set, little attention has been given to a local characterization even in two-agent economies. This paper presents three results. First, we point out that locally efficient, strategy-proof, and nondictatorial allocation mechanisms exists even in two-agent economies, when boundary allocations can be efficient. Second, excluding such exceptional cases, we show that in economies where the number of goods equals or exceeds the number of agents, any efficient and strategy-proof allocation mechanism defined on any local preference domain is alternately dictatorial, that is, it always allocates the total amount of goods to some single agent, even if the receivers vary. Third, we clarify that the local characterization is generally an open question even with allocation conditions such as the minimum consumption guarantee, and show that efficiency and strategy-proofness are incompatible with allocation conditions when all agents have the same local preference set.

Keywords Social choice · Strategy-proofness · Pareto efficiency · Exchange economy · Local domain

JEL Classification D71

I would like to thank the two anonymous referees for their helpful comments and suggestions. This work was supported by JSPS KAKENHI Grant Number JP20K01568.

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1 Introduction

Following the seminal work of Hurwicz (1972), the manipulability and efficiency of allocation mechanisms in pure exchange economies have been studied intensively. In particular, many studies have been stimulated by the work of Zhou (1991), who established that any Pareto-efficient and strategy-proof allocation mechanism is dictatorial in exchange economies with two agents having classical (i.e., continuous, strictly monotonic, and strictly convex) preferences. Although the dictatorship result has been strengthened and extended in various directions, little attention has been paid to local characterization. Hence, in this paper, we aim to investigate this issue, responding to the following research question: Does the dictatorship result hold if an allocation mechanism is efficient and strategy-proof only on a local preference domain? To ask the question positively, does local efficiency and strategy-proofness of allocation mechanisms allow a nondictatorial allocation? This would be a natural question to ask not only from a theoretical perspective but also from a practical perspective because it is quite likely that a planner will have some information about the agents' preferences, which allows him/her to narrow the focus to a local preference set rather than the whole preference set.

Zhou's (1991) dictatorship result in two-agent economies has been strengthened by being proven in the domain of restricted preferences. See Schummer (1997), Ju (2003), Hashimoto (2008), and Momi (2013a).¹ Note that these studies restricted the types of preferences, and we will narrow the domain of classical preferences to a neighborhood of a given preference in this paper.

First, we emphasize that there exist Pareto-efficient, strategy-proof, and nondictatorial allocation mechanisms defined on a local preference domain even in the case of two-agent economies.² This is in sharp contrast to the abovementioned dictatorship result on the whole preference domain. In an economy with two agents and two goods, consider allocation where one agent's consumption is on the boundary of the agent's consumption set. For many reasonable preferences, the allocation will be Pareto efficient despite an inequality of marginal rates of substitution. The mechanism that always assigns this allocation is strategy-proof and nondictatorial. Furthermore, it is locally efficient because the allocation keeps being Pareto efficient for preferences in a neighborhood. In the Appendix, we provide an illustration of such an example. Although the example should be noted as a positive result, it is exceptional in that it ignores what Pareto efficiency usually imposes on allocation mechanisms, that is, the equality of the marginal rates of substitution of goods among agents. To exclude such cases, we assume that the upper contour set of preferences at any positive consumption is a strictly convex set included in the interior of the consumption set.

Recently, the dictatorship result of Zhou (1991) has been extended to many-agent economies. As pointed out by Satterthwaite and Sonnenschein (1981) and Kato and Ohseto (2002), there exist Pareto-efficient, strategy-proof, and nondictatorial allocation mechanisms in many-agent economies, but these are typically alternating

¹ However, Nicolò (2004) showed a Pareto-efficient, strategy-proof, and nondictatorial mechanism in the domain of Leontief preferences.

² Tierney (2019) showed the possibility result with quasilinear preferences.

dictatorships. Momi (2017) proved the alternating dictatorship result in many-agent economies under the condition that the number of goods equals or exceeds the number of agents, and subsequently proved the result without such a condition (Momi 2020).

In this paper, we extend Momi's (2017) approach and show that if the number of goods equals or exceeds the number of agents, any Pareto-efficient and strategy-proof allocation mechanism defined on any local preference domain is alternately dictatorial. We leave examining the issue without the condition on the numbers of agents and goods to future research.

It is important to note that we cannot directly extend the previous studies on the whole preference set to the local characterization problem. The difficulty is that most studies have used specific, carefully constructed preference profiles in their proofs, and an arbitrarily given local preference set would not include such a specific preference profile. In particular, most of the abovementioned studies of two-agent economies used a preference profile where both agents have the same preference. Although Hashimoto (2008) and Momi (2013a) proved the local characterization in two-agent economies with Cobb–Douglas preferences, an arbitrarily given local preference set of classical preferences might not include any Cobb–Douglas preferences.³ Replacing the Cobb–Douglas preferences used in Momi (2017) with preferences in an arbitrarily given preference set is one of the difficulties we overcome in this paper, as we will discuss later. We cannot apply the approach of Momi (2020) for the same reason. In the proof, Momi (2020) replaced all but two of the agents' preferences with a common preference, and such replacements are not possible in an arbitrarily given local preference set. If a local preference set includes such a specific preference profile, a local characterization result might be obtained as a straight extension. We will observe this in establishing the local incompatibility of Pareto efficiency and strategy-proofness of allocation mechanisms when agents' individual local preference sets are the same.

Hurwicz (1972) originally proved that, in two-agent economies, Pareto-efficient and strategy-proof allocation mechanisms are incompatible with the individual rationality condition, where agents possess their initial endowments and a mechanism is assumed to allocate consumption that benefits all agents. The incompatibility result has been extended to many-agent economies with various welfare lower bound (WLB) conditions. Serizawa (2002) showed the incompatibility with the individual rationality condition. Serizawa and Weymark (2003) showed the incompatibility with a minimum consumption guarantee condition, where the consumption of each agent is assumed to be away from zero with some minimum distance. Momi (2013b) relaxed the minimum consumption guarantee condition to a simple positive consumption condition, where a mechanism is assumed to allocate positive consumption to all agents.⁴ Again, the proofs of these papers relied on a preference profile where all agents have the same preference, which would not be included in an arbitrarily given local preference set. Therefore, it is an open question whether such incompatibility results hold locally. In this paper, we extend Momi's (2013b) proof and show that Pareto efficiency and

³ In fact, Cobb–Douglas preferences are topologically negligible.

⁴ Barberà and Jackson (1995) discarded Pareto efficiency and characterized strategy-proof mechanisms that satisfied the individual rationality condition. Refer to Goswami et al. (2014) for an incompatibility result when preferences are quasilinear.

strategy-proofness are incompatible with the positive consumption condition when all agents have the same local preference set

Note that the alternating dictatorship, where one agent receives all goods and the others receive zero consumption, violates the WLB conditions. Therefore, the alternating dictatorship result in this paper establishes the incompatibility of Pareto efficiency and strategy-proofness with the WLB conditions for any local preference set when the number of agents does not exceed the number of goods. However, it remains an open question whether a Pareto-efficient and strategy-proof allocation mechanism defined on an arbitrarily given local preference set can be compatible with the WLB conditions when the number of agents exceeds the number of goods.

The remainder of the paper is organized as follows. Section 2 describes the model and results. Section 3 reviews the approach by Momi (2017) and explains the difficulty that we face when a mechanism is defined only locally. Section 4 presents some technical results. Sections 5 and 6 provide the proofs of the main results. The Appendix contains an example of the positive result and the proofs of all lemmas and corollaries from Sects. 4 and 5.

2 Model and results

We consider an economy with N agents, indexed by $\mathbf{N} = \{1, \dots, N\}$, where $N \geq 2$, and L goods, indexed by $\mathbf{L} = \{1, \dots, L\}$, where $L \geq 2$. The consumption set for each agent is \mathbb{R}_+^L . A consumption bundle for agent $i \in \mathbf{N}$ is a vector $x^i = (x_1^i, \dots, x_L^i) \in \mathbb{R}_+^L$. The total endowment of goods for the economy is $\Omega = (\Omega_1, \dots, \Omega_L) \in \mathbb{R}_{++}^L$. An allocation is a vector $\mathbf{x} = (x^1, \dots, x^N) \in \mathbb{R}_+^{LN}$. Thus, the set of feasible allocations for the economy with N agents and L goods is $X = \{\mathbf{x} \in \mathbb{R}_+^{LN} : \sum_{i \in \mathbf{N}} x^i \leq \Omega\}$.

A preference R is a complete, reflexive, and transitive binary relation on \mathbb{R}_+^L . The corresponding strict preference P_R and indifference I_R are defined in the usual way. Given a preference R and a consumption bundle $x \in \mathbb{R}_+^L$, the upper contour set of R at x is $UC(x; R) = \{x' \in \mathbb{R}_+^L : x' R x\}$, and the lower contour set of R at x is $LC(x; R) = \{x' \in \mathbb{R}_+^L : x R x'\}$. We use $I(x; R) = \{x' \in \mathbb{R}_+^L : x' I_R x\}$ to denote the indifference set of R at x , and $P(x; R) = \{x' \in \mathbb{R}_+^L : x' P_R x\}$ denotes the strictly preferred set of R at x .

A preference R is continuous if $UC(x; R)$ and $LC(x; R)$ are both closed for any $x \in \mathbb{R}_+^L$. A preference R is strictly convex on \mathbb{R}_{++}^L if $UC(x; R)$ is a strictly convex set in \mathbb{R}^L for any $x \in \mathbb{R}_{++}^L$. A preference R is monotonic if, for any x and x' in \mathbb{R}_+^L , $x > x'$ implies that $x R x'$.⁵ A preference R is strictly monotonic on \mathbb{R}_{++}^L if, for any x and x' in \mathbb{R}_{++}^L , $x > x'$ implies that $x P_R x'$. Therefore, if R is continuous, strictly convex on \mathbb{R}_{++}^L , and strictly monotonic on \mathbb{R}_{++}^L , then $UC(x; R) \subset \mathbb{R}_{++}^L$ for any $x \in \mathbb{R}_{++}^L$ and the boundary $\partial \mathbb{R}_+^L$ is an indifference set. Note that the definition of strict convexity differs slightly from that of Zhou (1991), where $UC(x; R)$, $x \in \mathbb{R}_{++}^L$ might have an intersection with the boundary. As mentioned in the introduction, a mechanism that assigns a boundary allocation could be locally Pareto efficient, strategy-proof and

⁵ For vectors x and x' in \mathbb{R}^L , $x > x'$ denotes that $x_l \geq x'_l$ for any $l \in \mathbf{L}$ and $x \neq x'$.

nondictatorial. Therefore, the assumption that keeps the upper contour sets away from the boundary is crucial in this paper.

A preference R is homothetic if, for any x and x' in \mathbb{R}_+^L and any $t > 0$, xRx' implies that $(tx)R(tx')$. A preference R is smooth if for any $x \in \mathbb{R}_{++}^L$, there exists a unique vector $p \in S_{++}^{L-1} \equiv \{x \in \mathbb{R}_+^L : \|x\| = 1\}$ such that p is the normal of a supporting hyperplane to $UC(x; R)$ at x . We call the vector p the gradient vector of R at x , and write $p = p(R, x)$. Note that if R is smooth, strictly convex on \mathbb{R}_{++}^L , and strictly monotonic on \mathbb{R}_{++}^L , then the gradient vector is positive in the positive orthant: $p(R, x) \in S_{++}^{L-1} \equiv \{x \in \mathbb{R}_{++}^L : \|x\| = 1\}$ for any $x \in \mathbb{R}_{++}^L$.

Let \mathcal{R} denote the set of preferences that are continuous, strictly convex on \mathbb{R}_{++}^L , strictly monotonic on \mathbb{R}_{++}^L , smooth, and homothetic. A preference profile is an N -tuple $\mathbf{R} = (R^1, \dots, R^N) \in \mathcal{R}^N$. We write the subprofile obtained by removing R^i from \mathbf{R} as $\mathbf{R}^{-i} = (R^1, \dots, R^{i-1}, R^{i+1}, \dots, R^N)$ and write the profile $(R^1, \dots, R^{i-1}, \bar{R}^i, R^{i+1}, \dots, R^N)$ as $(\bar{R}^i, \mathbf{R}^{-i})$. We also write $\mathbf{R}^{-\{i,j\}}$ to denote the subprofile obtained by removing R^i and R^j from \mathbf{R} .

A social choice function $f : \mathcal{R}^N \rightarrow X$ assigns a feasible allocation to each preference profile in \mathcal{R}^N . For a preference profile $\mathbf{R} \in \mathcal{R}^N$, the outcome chosen can be written as $f(\mathbf{R}) = (f^1(\mathbf{R}), \dots, f^N(\mathbf{R}))$, where $f^i(\mathbf{R})$ is the consumption bundle allocated to agent i by f . We let $\mathcal{B} \subset \mathcal{R}^N$ be a subset of \mathcal{R}^N . In this paper, we deal with a case where a social choice function is defined on \mathcal{B} , or it satisfies desirable properties only on \mathcal{B} .

The following three definitions are standard.

Definition 1 A social choice function $f : \mathcal{R}^N \rightarrow X$ is strategy-proof on $\mathcal{B} \subset \mathcal{R}^N$ if $f^i(\mathbf{R})R^i f^i(\bar{R}^i, \mathbf{R}^{-i})$ for any $i \in \mathbf{N}$, any $\mathbf{R} \in \mathcal{B}$, and any $\bar{R}^i \in \mathcal{R}$ such that $(\bar{R}^i, \mathbf{R}^{-i}) \in \mathcal{B}$.

Definition 2 A social choice function $f : \mathcal{R}^N \rightarrow X$ is Pareto efficient on $\mathcal{B} \subset \mathcal{R}^N$ if $f(\mathbf{R})$ is Pareto efficient for any $\mathbf{R} \in \mathcal{B}$.

Definition 3 A social choice function $f : \mathcal{R}^N \rightarrow X$ is dictatorial on \mathcal{B} if there exists $i \in \mathbf{N}$ such that $f^i(\mathbf{R}) = \Omega$ for any $\mathbf{R} \in \mathcal{B}$.

We say that a social choice function is alternately dictatorial if it always allocates the total endowment to some single agent though not always the same agent. Note that under an alternately dictatorial social choice function, the identity of the receiver of the total endowment may vary depending on preference profiles.

Definition 4 A social choice function $f : \mathcal{R}^N \rightarrow X$ is alternately dictatorial on $\mathcal{B} \subset \mathcal{R}^N$ if, for any $\mathbf{R} \in \mathcal{B}$, there exists $i_{\mathbf{R}} \in \mathbf{N}$ such that $f^{i_{\mathbf{R}}}(\mathbf{R}) = \Omega$.

Taking the same approach as previous studies, including Serizawa (2002) and Momi (2013b), we follow Kannai (1970) and introduce the Kannai metric into \mathcal{R} to discuss continuity. For $x \in \mathbb{R}_+^L \setminus 0$, we use $[x]$ to denote the ray starting from the origin and passing through x : $[x] = \{y \in \mathbb{R}_+^L : y = tx, t \geq 0\}$. We define $\mathbf{1} \equiv (1, \dots, 1) \in \mathbb{R}_+^L$

so that $[1]$ denotes the principal diagonal of R^L_+ . Using these definitions, the Kannai metric $d(R, R')$ for continuous and monotonic preferences R and R' is defined as

$$d(R, R') = \max_{x \in R^L_+} \frac{\| I(x; R) \cap [1] - I(x; R') \cap [1] \|}{1 + \| x \|^2},$$

where $\| \cdot \|$ denotes the Euclid norm in \mathbb{R}^L . With the Kannai metric, \mathcal{R} is a metric space. See Kannai (1970) for details.⁶

We write $B_\epsilon(\bar{R}) \subset \mathcal{R}$ to denote the open ball of preferences in \mathcal{R} , with center \bar{R} and radius $\epsilon > 0$: $B_\epsilon(\bar{R}) = \{ R \in \mathcal{R} : d(R, \bar{R}) < \epsilon \}$. For a preference profile $\mathbf{R} = (R^1, \dots, R^N)$, we write $B_\epsilon(\mathbf{R})$ to denote the product set of $B_\epsilon(R^i)$, $i = 1, \dots, N$: $B_\epsilon(\mathbf{R}) = \prod_{i=1}^N B_\epsilon(R^i) = B_\epsilon(R^1) \times \dots \times B_\epsilon(R^N)$.

We often write $B^i \subset \mathcal{R}$ to denote an open ball of agent i 's preferences without a specified center or radius and $\mathbf{B} = \prod_{i=1}^N B^i$ to denote the product of such open balls over agents.

We define various open balls in a similar manner. For example, we write $B_\epsilon(\bar{p}) \subset S^{L-1}_{++}$ to denote the open ball set of price vectors p : $B_\epsilon(\bar{p}) = \{ p \in S^{L-1}_{++} : \| p - \bar{p} \| < \epsilon \}$. We write $B_\epsilon(\bar{y}) \subset \mathbb{R}^L$ to denote the open ball set of L -dimensional vectors y : $B_\epsilon(\bar{y}) = \{ y \in \mathbb{R}^L : \| y - \bar{y} \| < \epsilon \}$.

Our main result is as follows.

Theorem 1 *Assume that $L \geq N$. If a social choice function $f : \mathcal{R}^N \rightarrow X$ is Pareto efficient and strategy-proof on a product set of open balls $\mathbf{B} = \prod_{i=1}^N B^i$, then f is alternately dictatorial on \mathbf{B} .*

Alternating dictatorship violates the individual rationality condition of Serizawa (2002), the minimum consumption guarantee condition of Serizawa and Weymark (2003), and the positive consumption condition of Momi (2013b). Therefore, the incompatibility of the Pareto-efficient and strategy-proof mechanism with these WLB conditions is ensured by this theorem for the case where $L \geq N$. If we assume that all agents have the same local preference sets, we can show the incompatibility without the condition on the numbers of agents and goods. In this paper, we prove the incompatibility with the positive consumption condition.

Theorem 2 *If a social choice function $f : \mathcal{R}^N \rightarrow X$ is Pareto efficient and strategy-proof on a product set of open balls $\mathbf{B} = \prod_{i=1}^N B^i$, where $B^i = B$ for all i , then there exist some j and $\mathbf{R}' \in \mathbf{B}$ such that $f^j(\mathbf{R}') = 0$.*

3 Preliminary results

Momi (2017) proved the alternating dictatorship result for a social choice function f defined on the whole domain \mathcal{R}^N ; when $L \geq N$, any Pareto-efficient and strategy-proof social choice function $f : \mathcal{R}^N \rightarrow X$ is alternately dictatorial. In this section,

⁶ See Hildenbrand (1974) and Grodal (1974) for further information on the topology of this space.

referring to this result, we explain the difficulties faced in the case of local preference domains.

We consider a social choice function f that is Pareto efficient and strategy-proof on a product set of open balls $\mathbf{B} = \prod_{i=1}^N B^i$.

As in Momi (2017), we define the option set as follows. For agent i , when the other agents' preferences $\bar{\mathbf{R}}^{-i} \in \mathbf{B}^{-i} \equiv \prod_{j \neq i} B^j$ are fixed, we define the option set, $G^i(\bar{\mathbf{R}}^{-i}) \subset \mathbb{R}_+^L$, as the union of the agent's consumption bundles given by f over his/her preferences in B^i :

$$G^i(\bar{\mathbf{R}}^{-i}) = \bigcup_{R^i \in B^i} f^i(R^i, \bar{\mathbf{R}}^{-i}).$$

The key feature of the option set is that, because of the strategy-proofness on \mathbf{B} , $f^i(R^i, \bar{\mathbf{R}}^{-i})$ should be the most preferred consumption bundle in $G^i(\bar{\mathbf{R}}^{-i})$ with respect to $R^i \in B^i$.

Because the boundary of the consumption set $\partial \mathbb{R}_+^L$ is an indifference set of each agent's preferences, each agent's consumption assigned by a Pareto-efficient social choice function f is not on the boundary except for the origin.⁷ Then, because of the Pareto efficiency, at least one agent i has positive consumption $f^i(\mathbf{R}) \in \mathbb{R}_{++}^L$, and the agent's gradient vector at the consumption is well defined and $p(R^i, f^i(\mathbf{R})) \in S_{++}^{L-1}$. The Pareto efficiency also implies that all agents who are assigned positive consumption have the same gradient vector. We call the gradient vector the price vector at the allocation and write $p(\mathbf{R}, f)$. Thus, the price vector $p(\mathbf{R}, f) \in S_{++}^{L-1}$ is well defined for the economy even if the gradient vector of an agent who is assigned zero consumption is not well defined.

On the other hand, for a preference $R \in \mathcal{R}$ and a price vector $p \in S_{++}^{L-1}$, we define the consumption-direction vector $g(R, p) \in S_{++}^{L-1}$ as the normalized consumption vector where the gradient vector of R is p . It should be noted that the consumption-direction vector is well defined because the preference is homothetic and strictly convex in \mathbb{R}_{++}^L . The proof of the next lemma is provided in the Appendix.

Lemma 1 For $R \in \mathcal{R}$ and $p \in S_{++}^{L-1}$, the consumption-direction vector $g(R, p) \in S_{++}^{L-1}$ is uniquely determined. Furthermore, $g(\cdot, p)$ is a continuous function.

Thus, $g(R^i, p(\mathbf{R}, f))$ is agent i 's consumption-direction vector at the preference profile \mathbf{R} under f . Note that this is well defined even for an agent who is assigned zero consumption. To simplify notation, we write $g^i(\mathbf{R}, f) = g(R^i, p(\mathbf{R}, f))$. Agent i 's consumption $f^i(\mathbf{R})$ assigned by f should be on the ray $[g^i(\mathbf{R}, f)]$, and we can write $f^i(\mathbf{R}) = \|f^i(\mathbf{R})\| g^i(\mathbf{R}, f)$.

Momi (2017) focused on a preference profile $\bar{\mathbf{R}} \in \mathbf{B}$ where the consumption-direction vectors are independent. The role of this independence should be clear. Because consumption vectors $f^i(\bar{\mathbf{R}})$, $i = 1, \dots, N$, are on the rays $[g^i(\mathbf{R}, f)]$, $i = 1, \dots, N$, respectively, and they sum to the total endowment Ω , the consumption vectors should be determined uniquely if the consumption-direction vectors are

⁷ See Momi (2017, p. 1275) for the proof.

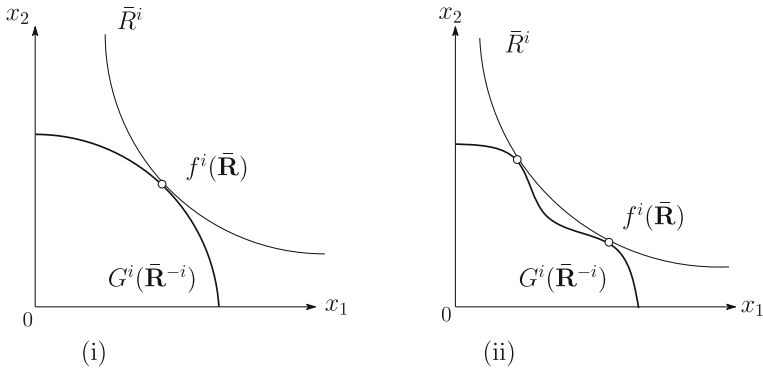


Fig. 1 The option set

independent. Momi (2017) showed that in a neighborhood of $f^i(\bar{\mathbf{R}})$, where the consumption-direction vectors are independent at $\bar{\mathbf{R}}$, the option set $G^i(\bar{\mathbf{R}}^{-i})$ is the $L - 1$ -dimensional smooth surface of a strictly convex set, as drawn in Fig. 1(i).

The role of this strict convexity and smoothness is clear. If the option set satisfies such properties, $f^i(R^i, \bar{\mathbf{R}}^{-i})$, which is the most preferred consumption bundle in the option set with respect to R^i , is a continuous function of R^i . Based on these topological properties of the option set, we can prove the following proposition. See Momi (2017, Proposition 6) for the proof.

Proposition 1 *Suppose that f is a social choice function that is Pareto efficient and strategy-proof on a product set of open balls $\mathbf{B} = \prod_{i=1}^N B^i$. If $g^i(\mathbf{R}, f)$, $i = 1, \dots, N$, are independent at a preference profile $\bar{\mathbf{R}} = (\bar{R}^1, \dots, \bar{R}^N) \in \mathbf{B}$, then $f^i(\bar{\mathbf{R}}) \in \{0, \Omega\}$ for any $i \in \mathbf{N}$.*

This proposition ensures the alternating dictatorship at a preference profile where the consumption-direction vectors are independent. If the consumption-direction vectors are independent at a preference profile, independence holds for a preference profile in a neighborhood because of the continuity of f , and the alternating dictatorship also holds in the neighborhood. However, in general, it is difficult to know whether the independence of the consumption-direction vectors holds for a given preference profile. It depends not only on the preference profile \mathbf{R} but also on the price vector $p(\mathbf{R}, f)$ determined by the social choice function f , the behavior of which is unknown. Without the independence of the consumption directions, the option set might not be either strictly convex or smooth, and hence f might not even be a continuous function. Figure 1 (ii) depicts an example of such an option set.

A key to overcome this difficulty is that there exists a preference profile $\mathbf{R}^* \in \mathcal{R}^N$ that ensures the independence of the consumption-direction for any price vector. Momi (2017) constructed such a preference profile \mathbf{R}^* using Cobb–Douglas utility functions. Then, through preference exchanges between two preference profiles, the alternating dictatorship result at \mathbf{R}^* was extended to any preference profile.

However, in a small local domain \mathbf{B} , we cannot expect the existence of such a preference profile that ensures the independence of the consumption-direction vectors

for any price vectors. This is the difficulty that we face in this paper. In an arbitrarily given domain \mathbf{B} , we have to find a preference profile that satisfies the condition of Proposition 1.

4 Technical results

As mentioned in the previous section, we must deal with the case where $g^i(\mathbf{R}, f)$, $i = 1, \dots, N$, are dependent. In the next section, starting from such a preference profile, we construct a preference profile in any neighborhood where the independence of the consumption-direction vectors holds. In this section, we present some technical results that we use for the proof. Throughout this section, we assume that the social choice function f is Pareto efficient and strategy-proof on a product set \mathbf{B} and we deal with preference profiles in \mathbf{B} , although we do not repeat them in each lemma. Proofs of all lemmas and corollaries are provided in the Appendix.

For a preference $R \in \mathcal{R}$ and a consumption bundle $x \in \mathbb{R}^L_+$, a preference \bar{R} is called a Maskin monotonic transformation (MMT, hereafter) of R at x if $\bar{x} \in UC(x; \bar{R})$ and $\bar{x} \neq x$ implies that $\bar{x} P_R x$.⁸ It is well known that if an agent receives x at a preference profile \mathbf{R} , strategy-proofness implies that this agent receives the same consumption bundle x when his/her preference is subject to an MMT at x . Note that \bar{R} and R share the same price vector at x . As shown in Momi (2013b, Lemma 4), for a preference $R \in \mathcal{R}$ and a consumption bundle $x \in \mathbb{R}^L_{++}$, there exists a preference that is an MMT of R at x in any neighborhood of R .

Because $f^i(\mathbf{R})$ is the most preferred consumption bundle in $G^i(\mathbf{R}^{-i})$ with respect to $R^i \in B^i$, the upper contour set $UC(f^i(\mathbf{R}); R^i)$ intersects with the option set $G^i(\mathbf{R}^{-i})$ at $f^i(\mathbf{R})$. As mentioned in the previous section, this might not be a unique intersection, and then $f^i(\cdot, \mathbf{R}^{-i})$ might not be a continuous function of R^i . Therefore, we define $F(R^i; G^i(\mathbf{R}^{-i}))$ as the intersection between the upper contour set of R^i at $f^i(\mathbf{R})$ and the option set: $F(R^i; G^i(\mathbf{R}^{-i})) = UC(f^i(\mathbf{R}); R^i) \cap G^i(\mathbf{R}^{-i})$. It is clear from the definition that $f^i(\mathbf{R}) \in F(R^i; G^i(\mathbf{R}^{-i}))$ and $F(R^i; G^i(\bar{\mathbf{R}}^{-i})) \subset I(f^i(\mathbf{R}); R^i)$.

The next lemma implies that if $R^{i'}$ is close to R^i , then any element of $F(R^{i'}; G^i(\mathbf{R}^{-i}))$ is close to $F(R^i; G^i(\mathbf{R}^{-i}))$. For a set $A \subset \mathbb{R}^L$, we let $B_\delta(A)$ denote the union of open balls with radius δ and center $x \in A$: $B_\delta(A) = \bigcup_{x \in A} B_\delta(x)$.

Lemma 2 *For any $\delta > 0$, there exists $\epsilon > 0$ such that if $R^{i'} \in B_\epsilon(R^i)$, then $F(R^{i'}; G^i(\mathbf{R}^{-i})) \subset B_\delta(F(R^i; G^i(\mathbf{R}^{-i})))$.*

If the consumption $f^i(\mathbf{R})$ is the unique intersection between the upper contour set and the option set, that is, if $f^i(\mathbf{R}) = F(R^i; G^i(\mathbf{R}^{-i}))$, then as $R^{i'}$ converges to R^i , $F(R^{i'}; G^i(\mathbf{R}^{-i}))$ converges to $f^i(\mathbf{R})$ as in Lemma 2. Then, the consumption $f^i(R^{i'}, \mathbf{R}^{-i})$, which is in $F(R^{i'}; G^i(\mathbf{R}^{-i}))$, also converges to $f^i(\mathbf{R})$, and hence the social choice function $f^i(\cdot, \mathbf{R}^{-i})$ is continuous at R^i .

⁸ Note that these are usually called strict MMTs, whereas standard MMTs require only $\bar{x} R x$.

We write $p(R^i; G^i(\mathbf{R}^{-i}))$ to denote the set of gradient vectors at consumption bundles in $F(R^i; G^i(\mathbf{R}^{-i}))$: $p(R^i; G^i(\mathbf{R}^{-i})) = \{p(R^i, x) \in S_{++}^{L-1} : x \in F(R^i; G^i(\mathbf{R}^{-i}))\}$. It is clear from the definition that $p(\mathbf{R}, f) = p(R^i, f^i(\mathbf{R})) \in p(R^i; G^i(\mathbf{R}^{-i}))$ when $f^i(\mathbf{R}) \in \mathbb{R}_{++}^L$.

When \hat{R}^i is an MMT of R^i at $f^i(\mathbf{R})$, we have $f^i(\mathbf{R}) = f^i(\hat{R}^i, \mathbf{R}^{-i}) = F(\hat{R}^i; G^i(\mathbf{R}^{-i}))$. Therefore, $F(R^{i'}; G^i(\mathbf{R}^{-i}))$ is in a neighborhood of $f^i(\mathbf{R})$ and $p(R^{i'}; G^i(\mathbf{R}^{-i}))$ is in a neighborhood of $p(\mathbf{R}, f)$ when $R^{i'}$ is close to \hat{R}^i . The next lemma considers the case where the other agents' preferences change.

Lemma 3 *Suppose that \hat{R}^i is an MMT of R^i at \bar{x}^i and the gradient vector at \bar{x}^i is \bar{p} . For any $\varepsilon' > 0$, there exists $\phi' > 0$ such that if $f^i(R^i, \mathbf{R}^{-i}) \neq 0$ and $p((R^i, \mathbf{R}^{-i}), f) \in B_{\phi'}(\bar{p})$ for some \mathbf{R}^{-i} , then $p(\hat{R}^i; G^i(\mathbf{R}^{-i})) \subset B_{\varepsilon'}(\bar{p})$.*

Lemma 3 implies that if \hat{R}^i is an MMT of R^i at \bar{x}^i with price vector \bar{p} , and if $p((R^i, \mathbf{R}^{-i}), f)$ is sufficiently close to \bar{p} , then $p(\hat{R}^i; G^i(\mathbf{R}^{-i}))$ is close to \bar{p} , and hence $p((\hat{R}^i, \mathbf{R}^{-i}), f)$ is close to \bar{p} . In other words, the lemma implies that if $f^i(R^i, \mathbf{R}^{-i})$ is sufficiently close to the ray $[\bar{x}^i]$, then $F(\hat{R}^i; G^i(\mathbf{R}^{-i}))$, the intersection between the upper contour set of \hat{R}^i and the option set, is close to $[\bar{x}^i]$, and hence $f^i(\hat{R}^i, \mathbf{R}^{-i})$ is close to $[\bar{x}^i]$.

The next lemma implies that if the consumption-direction vectors are independent among some agents, then the independence holds after slight changes in their preferences and the price vector. Furthermore, if the total endowments are assigned among the agents, then the positive consumption receivers remain such receivers after the changes.

Lemma 4 *Suppose that $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \dots, K$, where $K < N$, are independent at $\bar{\mathbf{R}} = (\bar{R}^1, \dots, \bar{R}^N)$. There exist scalars $\bar{\varepsilon} > 0$ and $\bar{\varepsilon} > 0$ satisfying the following properties.*

- (1) $g^i(R^i, p)$, $i = 1, \dots, K$, are independent for any $R^i \in B_{\bar{\varepsilon}}(\bar{R}^i)$ and $p \in B_{\bar{\varepsilon}}(p(\bar{\mathbf{R}}, f))$.
- (2) Let $\mathbf{R} = (R^1, \dots, R^N)$ be another preference profile. If $f^j(\bar{\mathbf{R}}) = f^j(\mathbf{R}) = 0$ for $j \geq K + 1$, $p(\mathbf{R}, f) \in B_{\bar{\varepsilon}}(p(\bar{\mathbf{R}}, f))$, and $R^i \in B_{\bar{\varepsilon}}(\bar{R}^i)$, $i = 1, \dots, K$, then $f^i(\bar{\mathbf{R}}) > 0$ implies that $f^i(\mathbf{R}) > 0$, for any $i = 1, \dots, K$.

In the proof of the theorem, we change the agents' preferences slightly and increase the number of agents whose consumption-direction vectors are independent. In the process, we have to change the preferences of an agent who receives positive consumption. When agent i 's consumption is positive, we exchange the agent's preference R^i with a preference in a neighborhood of an MMT of R^i at the consumption. Then, the price changes only slightly, as shown in Lemma 2.

The next lemma shows that if the consumption-direction vectors are independent among some agents who are not assigned zero consumption or the total consumption, then we can change their preferences slightly such that another agent receives neither zero nor the total consumption and the price vector is sufficiently close to the original price vector.

Lemma 5 *Suppose that $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \dots, K$, where $K < N$, are independent and $f^i(\bar{\mathbf{R}}) \notin \{0, \Omega\}$ for any $i = 1, \dots, K$, at $\bar{\mathbf{R}} = (\bar{R}^1, \dots, \bar{R}^N)$. For any $\epsilon > 0$ and $\varepsilon > 0$, there exist $(R^1, \dots, R^K) \in B_\epsilon(\bar{R}^1) \times \dots \times B_\epsilon(\bar{R}^K)$ and $j \geq K + 1$ such that $p((R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N), f) \in B_\varepsilon(p(\bar{\mathbf{R}}, f))$ and $f^j(R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N) \notin \{0, \Omega\}$.*

The next corollary is an immediate consequence of Lemma 5.

Corollary 1 *Suppose that $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \dots, K$, where $K < N$, are independent and $f^i(\bar{\mathbf{R}}) \notin \{0, \Omega\}$ for any $i = 1, \dots, K$, at $\bar{\mathbf{R}} = (\bar{R}^1, \dots, \bar{R}^N)$. There exists $j \geq K + 1$ such that for any $\epsilon > 0$ and $\varepsilon > 0$, some $(R^1, \dots, R^K) \in B_\epsilon(\bar{R}^1) \times \dots \times B_\epsilon(\bar{R}^K)$ satisfies $p((R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N), f) \in B_\varepsilon(p(\bar{\mathbf{R}}, f))$ and $f^j(R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N) \notin \{0, \Omega\}$.*

The next lemma relaxes the condition of Lemma 5. If the consumption-direction vectors are independent among some agents and one of them is assigned neither zero nor the total consumption, then we can find a slight change in their preferences such that another agent receives positive consumption and the price vector is sufficiently close to the original price vector.

Lemma 6 *Suppose that $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \dots, K$, where $K < N$, are independent and $f^i(\bar{\mathbf{R}}) \notin \{0, \Omega\}$ for some $i \leq K$ at $\bar{\mathbf{R}} = (\bar{R}^1, \dots, \bar{R}^N)$. For any $\epsilon > 0$ and $\varepsilon > 0$, there exists $(R^1, \dots, R^K) \in B_\epsilon(\bar{R}^1) \times \dots \times B_\epsilon(\bar{R}^K)$ and $j \geq K + 1$ such that $p((R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N), f) \in B_\varepsilon(p(\bar{\mathbf{R}}, f))$ and $f^j(R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N) \notin \{0, \Omega\}$.*

Finally, the next corollary is to Lemma 6 as Corollary 1 is to Lemma 5.

Corollary 2 *Suppose that $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \dots, K$, where $K < N$, are independent and $f^i(\bar{\mathbf{R}}) \notin \{0, \Omega\}$ for some $i \leq K$ at $\bar{\mathbf{R}} = (\bar{R}^1, \dots, \bar{R}^N)$. There exists $j \geq K + 1$ such that for any $\epsilon > 0$ and $\varepsilon > 0$, some $(R^1, \dots, R^K) \in B_\epsilon(\bar{R}^1) \times \dots \times B_\epsilon(\bar{R}^K)$ satisfies $p((R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N), f) \in B_\varepsilon(p(\bar{\mathbf{R}}, f))$ and $f^j(R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N) \notin \{0, \Omega\}$.*

5 Proof of Theorem 1

We first explain how we use constant elasticity substitution (CES) utility functions to achieve the independence of the consumption-direction vectors and then prove the theorem. All proofs of the lemmas are in the Appendix. In this section, we consider preferences represented by CES utility functions. Note that the indifference curve of CES utility functions intersects with the boundary of the consumption set, and hence the preferences are not elements of \mathcal{R} . It should not cause a confusion that we sometimes extend definitions in Sect. 2 to such preferences.

Assume that a preference profile $(\bar{R}^1, \dots, \bar{R}^N) \in \mathcal{R}^N$ and a price vector $\bar{p} \in S_{++}^L$ are given. We write $\bar{g}^i = g(\bar{R}^i, \bar{p})$ to denote agent i 's consumption-direction vector for the price \bar{p} and the preference \bar{R}^i .

(1) CES utility function: We explain the CES utility function that we use. Let $U_{\alpha,\rho} : \mathbb{R}_+^L \rightarrow \mathbb{R}$ denote the CES utility function with parameters $\rho < 1$ and $\alpha = (\alpha_1, \dots, \alpha_L) \in S_{++}^{L-1}$:

$$U_{\alpha,\rho}(x_1, \dots, x_L) = (\alpha_1(x_1)^\rho + \dots + \alpha_L(x_L)^\rho)^{1/\rho}. \tag{1}$$

Abusing notation, we let $U_{\alpha,\rho}$ denote not only the utility function but also the preference represented by the utility function. It is straightforward to calculate the gradient vector of the utility function $U_{\alpha,\rho}$ at a consumption bundle $x \in \mathbb{R}_+^L$ and to observe that it is parallel to $(\alpha_1(x_1)^{\rho-1}, \dots, \alpha_L(x_L)^{\rho-1})$.⁹ That is,

$$\frac{\partial U_{\alpha,\rho}}{\partial x}(x) \parallel (\alpha_1(x_1)^{\rho-1}, \dots, \alpha_L(x_L)^{\rho-1}), \tag{2}$$

where $y \parallel z$ denotes that vectors $y \in \mathbb{R}^L$ and $z \in \mathbb{R}^L$ are parallel. Therefore, we can calculate the parameter α of the CES utility function such that the gradient vector of the represented preference at a given consumption bundle x equals $p \in S_{++}^L$ by solving

$$(\alpha_1(x_1)^{\rho-1}, \dots, \alpha_L(x_L)^{\rho-1}) \parallel p, \tag{3}$$

with respect to α for fixed ρ, x , and p . In addition, we can calculate the consumption-direction vector of a given CES utility function for a price $p \in S_{++}^L$ by solving (3) with respect to x for fixed values of α, ρ and p .

We set α^i as the parameter such that the gradient vector at \bar{g}^i is parallel to \bar{p} : $\frac{\partial U_{\alpha^i,\rho}}{\partial x}(\bar{g}^i) \parallel \bar{p}$.¹⁰ Then, the preferences $U_{\alpha^i,\rho}$ and \bar{R}^i have the same gradient vector \bar{p} at \bar{g}^i . Note that α^i thus depends on ρ .

It is well known that the CES utility function defined by (1) converges to a Leontief utility function as $\rho \rightarrow -\infty$. Therefore, with a sufficiently small ρ , in a neighborhood of \bar{g}^i , any consumption bundle except \bar{g}^i itself in the upper contour set of $U_{\alpha^i,\rho}$ at \bar{g}^i is strictly preferred to \bar{g}^i with respect to \bar{R}^i . We fix ρ such that $U_{\alpha^i,\rho}$ satisfies this property for any $i = 1, \dots, N$. We write the CES utility function as U^i with the fixed ρ and the α^i determined by the ρ , as in the previous paragraph.

As mentioned above, for a given price vector $p = (p_1, \dots, p_L)$, the consumption-direction vector $g(U^i, p) \in S_{++}^{L-1}$ of the CES preference is determined by

$$g(U^i, p) \parallel \left((p_1/\alpha_1^i)^{1/(\rho-1)}, \dots, (p_L/\alpha_L^i)^{1/(\rho-1)} \right), \tag{4}$$

⁹ The gradient vector of a utility function should not be confused with the gradient vector of a preference that we defined in Sect. 2. Whereas $\frac{\partial U_{\alpha,\rho}}{\partial x}(x)$ is the gradient vector of the utility function $U_{\alpha,\rho}$, its normalization is the gradient vector of the preference $U_{\alpha,\rho}$ represented by the utility function.

¹⁰ In fact, α^i is obtained as the normalization of $\left(\frac{\bar{p}_1}{(\bar{g}_1^i)^{\rho-1}}, \dots, \frac{\bar{p}_L}{(\bar{g}_L^i)^{\rho-1}} \right)$.

by solving (3) with respect to x . Therefore, the consumption-direction vectors among agents are independent for any price $p \in S_{++}^{L-1}$ if and only if the vectors $\left((1/\alpha_1^i)^{1/(\rho-1)}, \dots, (1/\alpha_L^i)^{1/(\rho-1)} \right), i = 1, \dots, L$, are independent.

We modify the CES preferences if the independence does not hold. With a parameter $\delta^i \geq 0$ and a vector $z^i \in \mathbb{R}^L$, we define

$$\beta^i(\delta^i, z^i) = \left((1/\alpha_1^i)^{1/(\rho-1)}, \dots, (1/\alpha_L^i)^{1/(\rho-1)} \right) + \delta^i z^i,$$

for each $i = 1, \dots, N$. The next lemma shows that even if the vectors $\left((1/\alpha_1^i)^{1/(\rho-1)}, \dots, (1/\alpha_L^i)^{1/(\rho-1)} \right), i = 1, \dots, N$, are dependent, we can find the direction vectors $\bar{z}^i, i = 1, \dots, N$, with slight changes $\delta^i \leq \bar{\delta}$ to the directions in which the vectors $\beta^i(\delta^i, z^i), i = 1, \dots, N$, are independent.

Lemma 7 *There exist $\bar{\delta} > 0$ and $\bar{z}^i \in \mathbb{R}^L, i = 1, \dots, N$, such that $\beta^i(\delta^i, \bar{z}^i), i = 1, \dots, N$, are positive and independent with any $0 < \delta^i \leq \bar{\delta}, i = 1, \dots, N$.*

We fix $\bar{z}^i, i = 1, \dots, N$, and $\bar{\delta}$ that satisfy Lemma 7. For each $0 \leq \delta^i \leq \bar{\delta}$, we calculate $\hat{\gamma}^i = (\hat{\gamma}_1^i, \dots, \hat{\gamma}_L^i)$ by solving

$$\left((1/\hat{\gamma}_1^i)^{1/(\rho-1)}, \dots, (1/\hat{\gamma}_L^i)^{1/(\rho-1)} \right) = \beta^i(\delta^i, \bar{z}^i), \tag{5}$$

and define $\gamma^i = (\gamma_1^i, \dots, \gamma_L^i)$ as the normalization of $\hat{\gamma}^i: \gamma^i = \hat{\gamma}^i / \|\hat{\gamma}^i\|$. Note therefore that γ^i depends on δ^i .

We define the CES utility function U_{γ^i} with γ^i as the parameter by

$$U_{\gamma^i} = \left(\gamma_1^i (x_1)^\rho + \dots + \gamma_L^i (x_L)^\rho \right)^{1/\rho}$$

Then, for the preferences $U_{\gamma^i}, i = 1, \dots, N$, the consumption-direction vectors $g(U_{\gamma^i}, p), i = 1, \dots, N$, each of which is parallel to $\left((p_1/\gamma_1^i)^{1/(\rho-1)}, \dots, (p_L/\gamma_L^i)^{1/(\rho-1)} \right)$, are independent for any p as long as $0 < \delta^i \leq \bar{\delta}$. The independence of the consumption-direction vectors holds for preference profiles in a neighborhood of $(U_{\gamma^1}, \dots, U_{\gamma^N})$. Formally, this is stated in the next lemma.

Lemma 8 *Let $P \subset S_{++}^{L-1}$ be a compact price set. There exist functions $\epsilon^i : (0, \bar{\delta}] \rightarrow \mathbb{R}_{++}, i = 1, \dots, N$, such that for each $\delta^i \in (0, \bar{\delta}]$, $g(R^i, p), i = 1, \dots, N$, are independent for any $p \in P$ and any $R^i \in B_{\epsilon^i(\delta^i)}(U_{\gamma^i})$.*

(2) Preference construction: Although the CES utility functions satisfy the independence of the consumption-direction vectors for any price, they are not in a neighborhood of the original preferences $\bar{R}^i, i = 1, \dots, N$. We construct a preference that is close to \bar{R}^i and is represented by U_{γ^i} in a neighborhood of \bar{g}^i . See Momi (2017) for more details of the preference construction.

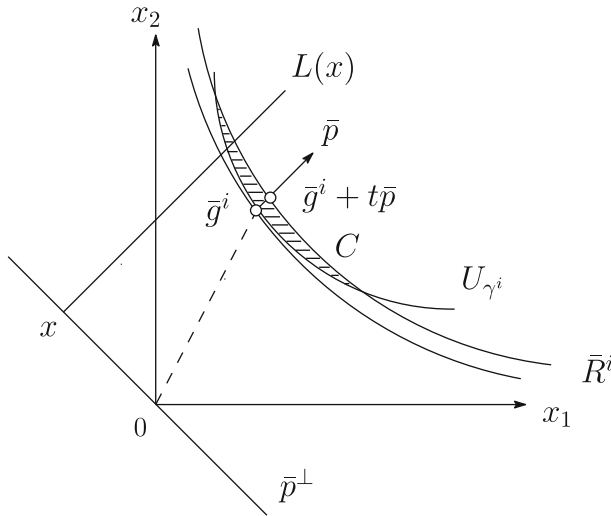


Fig. 2 The preference construction

Figure 2 depicts the preferences \bar{R}^i and U_{γ^i} . We let $t > 0$ be a sufficiently small parameter and define C as the set of consumption bundles x that are connected to \bar{g}^i in $UC(\bar{g}^i; U_{\gamma^i}) \cap LC(\bar{g}^i + t\bar{p}; \bar{R}^i)$. That is, C is the crescent-shaped set in Fig. 2 between the indifference sets $I(\bar{g}^i; U_{\gamma^i})$ and $I(\bar{g}^i + t\bar{p}; \bar{R}^i)$. The indifference set of the CES preference U_{γ^i} intersects the boundary of the consumption set, whereas that of \bar{R}^i is away from the boundary. Therefore, they intersect again, although this is not depicted in Fig. 2, and the crescent-shaped set C is not equal to $UC(\bar{g}^i; U_{\gamma^i}) \cap LC(\bar{g}^i + t\bar{p}; \bar{R}^i)$.

We define

$$A_{\delta^i}^i(t) = co\left(C \cup UC(\bar{g}^i + t\bar{p}; \bar{R}^i)\right)$$

where $co(Y)$ denotes the convex hull of a subset $Y \subset \mathbb{R}^L$. Therefore, this is the convex hull of the upper contour set of \bar{R}^i at $\bar{g}^i + t\bar{p}$ added to the crescent-shaped set C . The set C depends on the CES utility function U_{γ^i} , and U_{γ^i} depends on the parameter δ^i . Because this parameter plays a role, we write it explicitly as $A_{\delta^i}^i(t)$.

As the convex set $A_{\delta^i}^i(t)$ is not strictly convex, it cannot be an upper contour set of a preference in \mathcal{R} . To modify $A_{\delta^i}^i(t)$ into a strictly convex set, we prepare a strictly convex set as follows. Setting the parameter $t = 1$ in C , we let \bar{C} denote the set of consumption bundles x that are connected to \bar{g}^i in $UC(\bar{g}^i; U_{\gamma^i}) \cap LC(\bar{g}^i + \bar{p}; \bar{R}^i)$. That is, \bar{C} is the crescent-shaped set between the indifference sets $I(\bar{g}^i; U_{\gamma^i})$ and $I(\bar{g}^i + \bar{p}; \bar{R}^i)$ analogous to C . We define

$$\bar{D}^i = \bar{C} \cup \left(UC(\bar{g}^i; U_{\gamma^i}) \cap UC(\bar{g}^i + \bar{p}; \bar{R}^i)\right)$$

Note that \bar{D}^i is a strictly convex set; it does not intersect the boundary of the consumption set, but its surface is not smooth. To make its surface smooth, we let $D_{\delta^i}^i$ denote the union of closed balls with a sufficiently small radius ε included in \bar{D}^i : $D_{\delta^i}^i = \bigcup_{B_\varepsilon \subset \bar{D}^i} B_\varepsilon$. See Momi (2017, Lemma 6) for the proof that this makes a smooth surface. We explicitly write the index δ^i as in the case of $A_{\delta^i}^i(t)$.

Using $D_{\delta^i}^i$, we modify $A_{\delta^i}^i(t)$ into a strictly convex set. We let $s > 0$ be a sufficiently small parameter and define

$$B_{\delta^i}^i(t, s) = \bigcup_{x \in \bar{p}^\perp} \left\{ (1-s) \left(\partial A_{\delta^i}^i(t) \cap L(x) \right) + s \left(\left(\partial D_{\delta^i}^i \right) \cap L(x) \right) \right\},$$

where $L(x)$ is the half line starting from x and extending in the direction of the vector \bar{p} : $L(x) = \{y \in \mathbb{R}^L \mid y = x + t\bar{p}, t \geq 0\}$. Note that for $s > 0$, $B_{\delta^i}^i(t, s)$ is the boundary of a strictly convex set.¹¹ We let $R_{\delta^i, t, s}^i \in \mathcal{R}$ denote the preference that has $B_{\delta^i}^i(t, s)$ as its indifference set.

Note that, as long as $t > 0$, the boundaries of $A_{\delta^i}^i(t)$, $D_{\delta^i}^i(t)$, and $B_{\delta^i}^i(t, s)$ coincide in a neighborhood of \bar{g}^i , which is defined by the indifference set of the CES preference. That is, the indifference set of $R_{\delta^i, t, s}^i$ equals to that of U_{γ^i} in a neighborhood of \bar{g}^i . Hence, $R_{\delta^i, t, s}^i$ inherits the properties of U_{γ^i} . In particular, independence of the consumption-direction vectors holds for prices in a neighborhood of \bar{p} . In addition, note that, for $s > 0$ and $t > 0$, $R_{0, s, t}^i$, where $\delta^i = 0$, is an MMT of \bar{R}^i at \bar{g}^i .

Lemma 9 *For any $\epsilon > 0$, there exist values of $s > 0$, $t > 0$, $\bar{\delta}^i > 0$, $i = 1, \dots, N$, and $\varepsilon > 0$ satisfying the following properties.*

- (1) $R_{\delta^i, t, s}^i \in B_\epsilon(\bar{R}^i)$, $i = 1, \dots, N$, for any $0 \leq \delta^i \leq \bar{\delta}^i$.
- (2) $g\left(R_{\delta^i, t, s}^i, p\right)$, $i = 1, \dots, N$, are independent for any $0 < \delta^i \leq \bar{\delta}^i$ and any $p \in B_\epsilon(\bar{p})$.

The independence of the consumption-direction vectors holds for preferences in neighborhoods of $R_{\delta^i, t, s}^i$, $i = 1, \dots, N$.

Lemma 10 *Fix $s > 0$ and $t > 0$. There exist $\bar{\varepsilon} > 0$ and functions $\epsilon^i : (0, \bar{\delta}^i] \rightarrow \mathbb{R}_{++}$, $i = 1, \dots, N$, such that $g\left(R^i, p\right)$, $i = 1, \dots, N$, are independent for any $p \in B_{\bar{\varepsilon}}(\bar{p})$ and any $R^i \in B_{\epsilon^i(\delta^i)}\left(R_{\delta^i, t, s}^i\right)$, $i = 1, \dots, N$.*

Combining the preference construction with the results in the previous section, we prove the theorem. In the proof, we repeatedly find an agent who receives neither zero consumption nor the total consumption when increasing the number of agents whose consumption-direction vectors are independent. Then, finally, we make all agents have independent consumption-direction vectors, which contradicts Proposition 1.

¹¹ See Momi (2017, Lemma 5) for the proof.

Proof of Theorem 1 We let $L \geq N$ and the social choice function f be Pareto efficient and strategy-proof on $\mathbf{B} = \prod_{i=1}^N B^i$. We suppose that an agent j has neither zero nor the total consumption, $f^j(\bar{\mathbf{R}}) \notin \{0, \Omega\}$, at a preference profile $\bar{\mathbf{R}} = (\bar{R}^1, \dots, \bar{R}^N) \in \mathbf{B}$, and we show a contradiction. If $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \dots, N$, are independent, this immediately contradicts Proposition 1. We consider the case where these consumption-direction vectors are dependent.

We write \bar{p} and \bar{g}^i to denote the price vector and agent i 's consumption-direction vector at $\bar{\mathbf{R}}$, respectively: $\bar{p} = p(\bar{\mathbf{R}}, f)$ and $\bar{g}^i = g^i(\bar{\mathbf{R}}, f) = g(\bar{R}^i, p(\bar{\mathbf{R}}, f))$.

Here, we summarize our preparation for the proof. We fix a scalar $\bar{\epsilon} > 0$ such that $B_{\bar{\epsilon}}(\bar{R}^i) \subset B^i$, for any $i = 1, \dots, N$. For $\bar{R}^i, \bar{g}^i, i = 1, \dots, N$, and \bar{p} , we have prepared the preferences $R_{\delta^i, t, s}^i, i = 1, \dots, N$. By Lemma 9, we have $s, t, \bar{\delta}^i, i = 1, \dots, N$, and ϵ such that

$$R_{\delta^i, t, s}^i \in B_{\bar{\epsilon}}(\bar{R}^i), i = 1, \dots, N, \text{ for any } 0 \leq \delta^i \leq \bar{\delta}^i, \tag{6}$$

and

$$g(R_{\delta^i, t, s}^i, p), i = 1, \dots, N, \text{ are independent for any } 0 < \delta^i \leq \bar{\delta}^i \text{ and any } p \in B_{\bar{\epsilon}}(\bar{p}). \tag{7}$$

Furthermore, by Lemma 10, we have $\bar{\epsilon}$ and $\epsilon^i : (0, \bar{\delta}] \rightarrow \mathbb{R}_{++}, i = 1, \dots, N$, such that

$$g(R^i, p), i = 1, \dots, N, \text{ are independent for any } R^i \in B_{\epsilon^i(\delta^i)}(R_{\delta^i, t, s}^i) \text{ and } p \in B_{\bar{\epsilon}}(\bar{p}). \tag{8}$$

We set $\bar{\epsilon}$ sufficiently small such that $\bar{\epsilon} < \epsilon$. we also set the function ϵ^i such that its value is sufficiently small and $B_{\epsilon^i(\delta^i)}(R_{\delta^i, t, s}^i) \in B_{\bar{\epsilon}}(\bar{R}^i)$ for any $0 < \delta^i \leq \bar{\delta}^i$. From now on, parameters other than δ^i are fixed and we write $R_{\delta^i}^i = R_{\delta^i, t, s}^i$. Figure 3 depicts the neighborhoods of \bar{R}^i and \bar{p} . We will replace each preference \bar{R}^i with $R_{\delta^i}^i$ and then find a new preference in $B_{\epsilon^i(\delta^i)}(R_{\delta^i}^i)$ while keeping the price vector in $B_{\bar{\epsilon}}(\bar{p})$ so that the independence of the consumption vectors will be ensured by (8).

For the operation to keep the price vector in $B_{\bar{\epsilon}}(\bar{p})$, we prepare a function ϕ as follows. For each $i = 1, \dots, N$, R_0^i (that is $R_{\delta^i}^i$ where $\delta^i = 0$) is an MMT of \bar{R}^i at \bar{g}^i . Applying Lemma 3 to these preferences, we have a positive function $\bar{\phi}^i : \epsilon' \mapsto \bar{\phi}^i(\epsilon')$ for each $i = 1, \dots, N$, which maps ϵ' to ϕ' in Lemma 3:

$$\begin{aligned} &\text{If } f^i(\bar{R}^i, \mathbf{R}^{-i}) \neq 0 \text{ and } p((\bar{R}^i, \mathbf{R}^{-i}), f) \in B_{\bar{\phi}^i(\epsilon')}(\bar{p}), \\ &\text{then } p(R_0^i; G^i(\mathbf{R}^{-i})) \subset B_{\bar{\phi}^i(\epsilon')}(\bar{p}) \text{ for any } \mathbf{R}^{-i} \text{ and } \epsilon' > 0. \end{aligned} \tag{9}$$

We select a positive function ϕ such that $\phi(\epsilon') < \min\{\bar{\phi}^1(\epsilon'), \dots, \bar{\phi}^N(\epsilon'), \epsilon'\}$ for any ϵ' . We write $\phi^{(k)} = \phi \circ \dots \circ \phi$ to denote the k -times operation of ϕ .

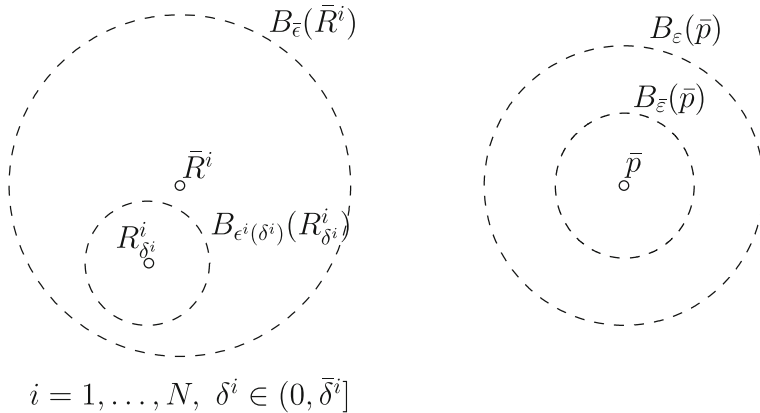


Fig. 3 Neighborhoods of \bar{R}^i , $R_{\delta^i}^i$, and \bar{p}

Without loss of generality, we assume that $f^1(\bar{\mathbf{R}}) \notin \{0, \Omega\}$. Starting from this agent, we will increase the number of agents whose consumption-direction vectors are independent in the following steps, and finally reach a situation contradicting Proposition 1.

Step 1: First, we modify agent 1’s preference for the later steps and show that there exists another agent who receives neither zero consumption nor the total consumption.

Replacing \bar{R}^1 with $R_{\delta^1}^1$ of any parameter $0 < \delta^1 < \bar{\delta}^1$, we have $f^1(R_{\delta^1}^1, \bar{\mathbf{R}}^{-i}) \notin \{0, \Omega\}$ because of the strategy-proofness of f , and there exists another agent receiving neither zero consumption nor the total consumption because of the Pareto efficiency of f .

Although the identity of the receiver might vary as δ^1 changes, there exists an agent $j \geq 2$ such that for any δ^1 , there exists $\delta^{1'} < \delta^1$ satisfying $f^j(R_{\delta^{1'}}^1, \bar{\mathbf{R}}^{-i}) \notin \{0, \Omega\}$. If there exists no such agent, then for each $j \geq 2$ there exists δ_j^1 such that for any $\delta^1 < \delta_j^1$, $f^j(R_{\delta^1}^1, \bar{\mathbf{R}}^{-i}) \in \{0, \Omega\}$. Then, for $\delta^1 < \min\{\delta_2^1, \dots, \delta_N^1\}$, we have $f^j(R_{\delta^1}^1, \bar{\mathbf{R}}^{-i}) \in \{0, \Omega\}$ for any $j \geq 2$. This contradicts that $f^1(R_{\delta^1}^1, \bar{\mathbf{R}}^{-i}) \notin \{0, \Omega\}$ for any parameter δ^1 .

Without loss of generality, we assume that agent 2 is such an agent. We let $\Delta^1 = \{\delta^1 \mid f^2(R_{\delta^1}^1, \bar{\mathbf{R}}^{-i}) \notin \{0, \Omega\}\}$ denote the set of δ^1 such that agent 2’s consumption is not zero at $(R_{\delta^1}^1, \bar{\mathbf{R}}^{-i})$. We consider δ^1 in Δ^1 .

If δ^1 in Δ^1 is sufficiently small, then $p(R_{\delta^1}^1; G^1(\bar{\mathbf{R}}^{-i}))$ can be arbitrarily close to \bar{p} because $f^1(\bar{\mathbf{R}}) = f^1(R_0^1, \bar{\mathbf{R}}^{-i})$ is the unique intersection between $UC(f^1(\bar{\mathbf{R}}); R_0^1)$ and $G^1(\bar{\mathbf{R}}^{-i})$. Hence, $F(R_{\delta^1}^1; G^1(\bar{\mathbf{R}}^{-i}))$ converges to $f^1(R_0^1, \bar{\mathbf{R}}^{-i})$ as $\delta^1 \rightarrow 0$ as shown in Lemma 2. We take a value of δ^1 that is sufficiently small so that $p(R_{\delta^1}^1; G^1(\bar{\mathbf{R}}^{-i})) \subset B_{\phi^{(N-1)}(\bar{\epsilon})}(\bar{p})$. \square

Step 2: Second, we modify agent 2’s preference, and then apply Corollary 2 to find another agent $j \geq 3$ who receives neither zero consumption nor the total consumption.

Keeping in mind that agent 2 now receives neither zero nor the total consumption at the preference profile $(R_{\delta^1}^1, \bar{\mathbf{R}}^{-1})$, we replace \bar{R}^2 with R_0^2 . Note that $p(R_{\delta^1}^1; G^1(\bar{\mathbf{R}}^{-1})) \subset B_{\phi^{(N-1)}(\bar{e})}(\bar{p})$ implies that the price vector at the preference profile is in $B_{\phi^{(N-1)}(\bar{e})}(\bar{p})$. Also note that $\phi^{(N-1)}(\bar{e}) \leq \bar{\phi}^2(\phi^{(N-2)}(\bar{e}))$ by the definitions of ϕ and $\bar{\phi}^i$. Therefore, we have $p((R_{\delta^1}^1, \bar{R}^2, \bar{\mathbf{R}}^{-\{1,2\}}), f) \in B_{\bar{\phi}^2(\phi^{(N-2)}(\bar{e}))}(\bar{p})$, and hence, $p(R_0^2; G^2(R_{\delta^1}^1, \bar{\mathbf{R}}^{-\{1,2\}})) \subset B_{\phi^{(N-2)}(\bar{e})}(\bar{p})$ by (9).

Then, we replace R_0^2 with $R_{\delta^2}^2$. As $\delta^2 \rightarrow 0$, any element in $F(R_{\delta^2}^2; G^2(R_{\delta^1}^1, \bar{\mathbf{R}}^{-\{1,2\}}))$ converges to a consumption bundle in $F(R_0^2; G^2(R_{\delta^1}^1, \bar{\mathbf{R}}^{-\{1,2\}}))$ as shown in Lemma 2. Then, any price vector in $p(R_{\delta^2}^2; G^2(R_{\delta^1}^1, \bar{\mathbf{R}}^{-\{1,2\}}))$ converges to a price vector in $p(R_0^2; G^2(R_{\delta^1}^1, \bar{\mathbf{R}}^{-\{1,2\}}))$. Therefore, we set δ^2 to be sufficiently small such that $p(R_{\delta^2}^2; G^2(R_{\delta^1}^1, \bar{\mathbf{R}}^{-\{1,2\}})) \subset B_{\phi^{(N-2)}(\bar{e})}(\bar{p})$. Of course, this implies that the price vector at the preference profile $(R_{\delta^1}^1, R_{\delta^2}^2, \bar{\mathbf{R}}^{-\{1,2\}})$ is in $B_{\phi^{(N-2)}(\bar{e})}(\bar{p})$.

We can apply Corollary 2 to agents 1 and 2 because the price vector in $B_{\phi^{(N-2)}(\bar{e})}(\bar{p}) \subset B_{\bar{e}}(\bar{p})$ implies the independence of the consumption-direction vectors of $R_{\delta^1}^1$ and $R_{\delta^2}^2$ at the preference profile $(R_{\delta^1}^1, R_{\delta^2}^2, \bar{\mathbf{R}}^{-\{1,2\}})$ as in (8). Corollary 2 implies the existence of a preference subprofile $(R_{(1)}^1, R_{(1)}^2)$ arbitrarily close to $(R_{\delta^1}^1, R_{\delta^2}^2)$ such that the price vector is arbitrarily close to \bar{p} and another agent $j \geq 3$ receives neither zero consumption nor the total consumption at the preference profile $(R_{\delta^1}^1, R_{\delta^2}^2, \bar{\mathbf{R}}^{-\{1,2\}})$.

We set $\bar{\epsilon}^i, i = 1, 2$ to be sufficiently small such that $\bar{\epsilon}^i < \epsilon^i(\delta^i)$ and apply Corollary 2. Without loss of generality, we assume agent 3 is the receiver of positive consumption. As a result, we have $(R_{(1)}^1, R_{(1)}^2) \in B_{\bar{\epsilon}^1}(R_{\delta^1}^1) \times B_{\bar{\epsilon}^2}(R_{\delta^2}^2)$ such that the price vector is in $B_{\phi^{(N-2)}(\bar{e})}(\bar{p})$ and agent 3 receives neither zero nor the total consumption at the preference profile $(R_{(1)}^1, R_{(1)}^2, \bar{\mathbf{R}}^{-\{1,2\}})$. ◇

Step 3: In this step, we modify agent 3’s preference, and then apply Corollary 2 to find another agent $j \geq 4$ who receives neither zero nor the total consumption.

Because agent 3 now receives neither zero consumption nor the total consumption, we replace \bar{R}^3 with R_0^3 . By (9), we have $p(R_0^3; G^3(R_{(1)}^1, R_{(1)}^2, \bar{\mathbf{R}}^{-\{1,2,3\}})) \subset B_{\phi^{(N-3)}(\bar{e})}(\bar{p})$, as we observed for agent 2.

We replace R_0^3 with $R_{\delta^3}^3$, where δ^3 is sufficiently small such that $p(R_{\delta^3}^3; G^3(R_{(1)}^1, R_{(1)}^2, \bar{\mathbf{R}}^{-\{1,2,3\}})) \subset B_{\phi^{(N-3)}(\bar{e})}(\bar{p})$, as we observed for agent 2.

We can apply Corollary 2 to agents 1,2, and 3 because $R_{(1)}^1 \in B_{\bar{\epsilon}^1} \left(R_{\delta^1}^1 \right) \subset B_{\epsilon^1(\delta^1)} \left(R_{\delta^1}^1 \right)$, $R_{(1)}^2 \in B_{\bar{\epsilon}^2} \left(R_{\delta^2}^2 \right) \subset B_{\epsilon^2(\delta^2)} \left(R_{\delta^2}^2 \right)$, and $p \left(\left(R_{(1)}^1, R_{(1)}^2, R_{\delta^3}^3, \bar{\mathbf{R}}^{-\{1,2,3\}} \right), f \right) \in B_{\phi^{(N-3)}(\bar{\epsilon})}(\bar{p}) \subset B_{\bar{\epsilon}}(\bar{p})$ implies the independence of the consumption-direction vectors of $R_{(1)}^1$, $R_{(1)}^2$, and $R_{\delta^3}^3$ at the preference profile $\left(R_{(1)}^1, R_{(1)}^2, R_{\delta^3}^3, \bar{\mathbf{R}}^{-\{1,2,3\}} \right)$, as in (8).

Corollary 2 implies the existence of a preference subprofile $\left(R_{(2)}^1, R_{(2)}^2, R_{(2)}^3 \right)$ arbitrarily close to $\left(R_{(1)}^1, R_{(1)}^2, R_{\delta^3}^3 \right)$ such that the price vector is arbitrarily close to $p \left(\left(R_{(1)}^1, R_{(1)}^2, R_{\delta^3}^3, \bar{\mathbf{R}}^{-\{1,2,3\}} \right), f \right)$ and another agent $j \geq 4$ receives neither zero nor the total consumption at the preference profile $\left(R_{(2)}^1, R_{(2)}^2, R_{(2)}^3, \bar{\mathbf{R}}^{-\{1,2,3\}} \right)$.

We set $\bar{\epsilon}^3$ sufficiently small such that $\bar{\epsilon}^3 < \epsilon^3(\delta^3)$ and apply Corollary 2. Without loss of generality, we assume that agent 4 receives neither zero consumption nor the total consumption. As a result, we have $\left(R_{(2)}^1, R_{(2)}^2, R_{(2)}^3 \right) \in B_{\bar{\epsilon}^1} \left(R_{\delta^1}^1 \right) \times B_{\bar{\epsilon}^2} \left(R_{\delta^2}^2 \right) \times B_{\bar{\epsilon}^3} \left(R_{\delta^3}^3 \right)$ such that the price vector is in $B_{\phi^{(N-3)}(\bar{\epsilon})}(\bar{p})$ and agent 4 receives neither zero consumption nor the total consumption at the preference profile $\left(R_{(2)}^1, R_{(2)}^2, R_{(2)}^3, \bar{\mathbf{R}}^{-\{1,2,3\}} \right)$. □

Step 4: We modify agent 4's preference and then apply Corollary 2 to find another agent $j \geq 5$ who receives neither zero consumption nor the total consumption.

We replace \bar{R}^4 with $R_{\delta^4}^4$, where δ^4 is sufficiently small such that $p \left(R_{\delta^4}^4; G^4 \left(R_{(2)}^1, R_{(2)}^2, R_{(2)}^3, \bar{\mathbf{R}}^{-\{1,\dots,4\}} \right) \right) \in B_{\phi^{(N-4)}(\bar{\epsilon})}(\bar{p})$. Applying Corollary 2 to agents 1, ..., 4, we obtain a preference subprofile $\left(R_{(3)}^1, \dots, R_{(3)}^4 \right) \in B_{\bar{\epsilon}^1} \left(R_{\delta^1}^1 \right) \times \dots \times B_{\bar{\epsilon}^4} \left(R_{\delta^4}^4 \right)$ where $\bar{\epsilon}^4$ satisfies $\bar{\epsilon}^4 < \epsilon^4(\delta^4)$ such that the price vector is in $B_{\phi^{(N-4)}(\bar{\epsilon})}(\bar{p})$ and an agent $j \geq 5$ receives neither zero nor the total consumption at the preference profile $\left(R_{(3)}^1, \dots, R_{(3)}^4, \bar{\mathbf{R}}^{-\{1,\dots,4\}} \right)$. □

We repeat this process. Finally, in Step $N - 1$, we have a preference profile $\left(R_{(N-2)}^1, \dots, R_{(N-2)}^{N-1} \right)$ in $B_{\bar{\epsilon}^1} \left(R_{\delta^1}^1 \right) \times \dots \times B_{\bar{\epsilon}^{N-1}} \left(R_{\delta^{N-1}}^{N-1} \right)$, where $\bar{\epsilon}^i$, $i = 1, \dots, N - 1$, satisfies $\bar{\epsilon}^i < \epsilon^i(\delta^i)$ such that the price vector is in $B_{\phi(\bar{\epsilon})}(\bar{p})$ and agent N receives neither zero consumption nor the total consumption at the preference profile $\left(R_{(N-2)}^1, \dots, R_{(N-2)}^{N-1}, \bar{R}^N \right)$. Replacing \bar{R}^N with $R_{\delta^N}^N$ where δ^N is sufficiently small, we still have the price vector in $B_{\bar{\epsilon}}(\bar{p})$ and agent N receives neither zero consumption nor the total consumption at the preference profile $\left(R_{(N-2)}^1, \dots, R_{(N-2)}^{N-1}, R_{\delta^N}^N \right)$.

As in (8), the consumption-direction vectors $g^i \left(R_{(N-2)}^i, p \right)$, $i = 1, \dots, N - 1$, and $g \left(R_{\delta^N}^N, p \right)$ are independent at the price vector $p = p \left(R_{(N-2)}^1, \dots, R_{(N-2)}^{N-1}, R_{\delta^N}^N \right)$

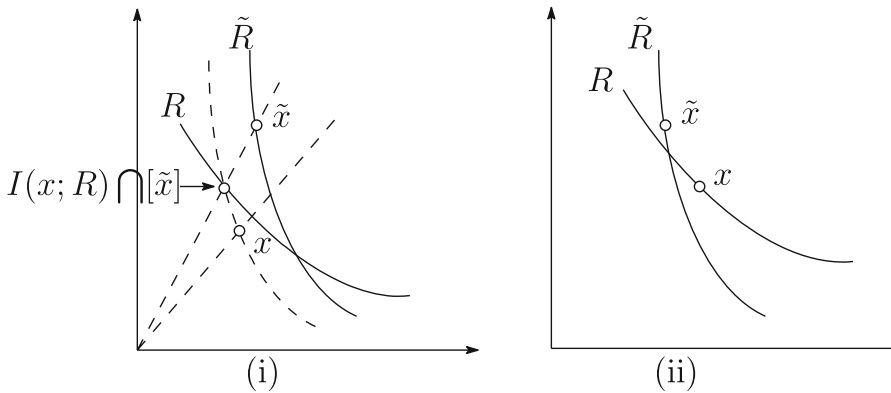


Fig. 4 Proof of Proposition 2

because $R^i_{(N-2)} \in B_{\bar{\varepsilon}^i}(R^i_{\delta^i})$, $i = 1, \dots, N - 1$, and $p \in B_{\bar{\varepsilon}}(\bar{p})$. This contradicts Proposition 1. □

6 Proof of Theorem 2

To prove the theorem, we reconstruct Momi’s (2013b) proof in a local preference set. Proposition 2 in Momi (2013b), which plays a key role in the proof, implies that if two preferences, R and \tilde{R} , and two consumption bundles, x and \tilde{x} , are as drawn in Fig. 3, then there exists a preference \bar{R} that is an MMT of R at x and of \tilde{R} at \tilde{x} . We reconstruct this proposition in a local preference set. We show that if R and \tilde{R} are sufficiently close, then \bar{R} can be selected to be close to R and \tilde{R} .

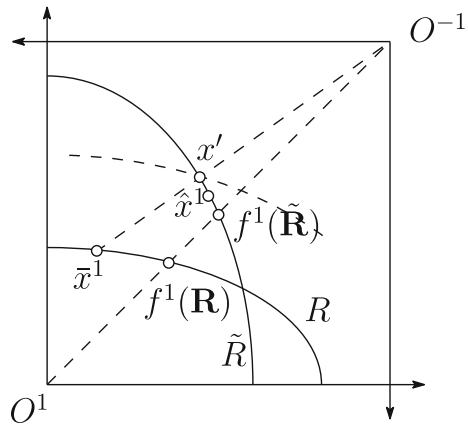
Proposition 2 *For any $R, \tilde{R} \in \mathcal{R}$, and any $x, \tilde{x} \in \mathbb{R}^L_{++}$, if $x \in P(I(x; R) \cap [\tilde{x}]; \tilde{R})$, then there exists a preference $\bar{R} \in \mathcal{R}$, that is an MMT of R at x and of \tilde{R} at \tilde{x} . Furthermore, we can select a \bar{R} that is arbitrarily close to R and \tilde{R} if R and \tilde{R} are sufficiently close.*

Proof The first statement is proved by Momi (2013b), so we need only show the second statement. Also note that as shown by Momi (2013b), the general case drawn in Fig. 4 (i) turns into Fig. 4 (ii), which is the case we have to consider.

We let \check{R} denote the homothetic, monotonic, and continuous preference whose indifference set is defined by $\partial(UC(x; R) \cap UC(\tilde{x}; \tilde{R}))$. It is clear that, following the construction of \bar{R} in Momi (2013b), we can construct a \bar{R} arbitrarily close to \check{R} with respect to the Kannai metric. It is clear from the definition of the Kannai metric that the distance between R and \check{R} or between \tilde{R} and \check{R} is less than the distance between R and \tilde{R} . Therefore, as R and \tilde{R} are closer, we can select \bar{R} closer to R and \tilde{R} . □

We can follow the proof by Momi (2013b) using preferences in the local set B to show the incompatibility of Pareto efficiency, strategy-proofness and the positive consumption on the local preference set.

Fig. 5 Proof of Theorem 2



Proof of Theorem 2 We suppose that f is a Pareto-efficient and strategy-proof social choice function on $\mathbf{B} = \prod_{i=1}^N B^i$ where $B^i = B$ satisfying $f^i(\mathbf{R}) > 0$ for any i and any $\mathbf{R} \in \mathbf{B}$, and show a contradiction.

When all agents have the same preference, all agents should be allocated positive portions of Ω : $f^i(\mathbf{R}) = \lambda^i \Omega$ with some $0 < \lambda^i < 1$ for $\mathbf{R} = (R, \dots, R)$. We pick two different preferences R and \tilde{R} in B that are sufficiently close, and consider the allocations given by f at $\mathbf{R} = (R, \dots, R)$ and $\tilde{\mathbf{R}} = (\tilde{R}, \dots, \tilde{R})$.

We let $A(x; R)$ denote the set of consumption bundles x' such that $\Omega - x'$ is indifferent to $\Omega - x$ with respect to R

$$A(x; R) = \left\{ x' \in \mathbb{R}_+^L \mid (\Omega - x') I_R (\Omega - x) \right\}$$

and we let $A^+(x; R) = \{x' \in \mathbb{R}_+^L \mid (\Omega - x) P_R (\Omega - x')\}$, which is the upper-right part of the consumption set partitioned by $A(x; R)$ and let $A^-(x; R) = \{x' \in \mathbb{R}_+^L \mid (\Omega - x') P_R (\Omega - x)\}$, which is the lower-left part.

Without loss of generality, we assume that $f^1(\mathbf{R}) \leq f^1(\tilde{\mathbf{R}})$. See the Edgeworth box described in Fig. 5, where the consumption of agent 1 is measured from the lower-left vertex and the sum of the consumptions of the other agents is measured from the upper-right vertex. We pick $\bar{x}^1 \in A(f^1(\mathbf{R}); R)$ in the neighborhood of $f^1(\mathbf{R})$ so that \bar{x}^1 is in $A(f^1(\tilde{\mathbf{R}}); \tilde{R})^-$ and \bar{x}^1 is not parallel to Ω . Next, let x' be the intersection of $A(f^1(\mathbf{R}); R)$ and the segment $[\bar{x}^1, \Omega]$ and pick $\hat{x}^1 \in A(f^1(\mathbf{R}); R)$ in the neighborhood of x' so that $\hat{x}^1 \in A(x'; \tilde{R})^-$.

As we observed in Proposition 1 of Momi (2013b), agent 1's consumption should be on $A(f^1(\mathbf{R}); R)$ (resp. $A(f^1(\tilde{\mathbf{R}}); \tilde{R})$) when other agents' preferences are R (resp. \tilde{R}) and agent 1's preference is changed. Let \bar{R} and \hat{R} be agent 1's preferences in B such that $f^1(\bar{R}, \mathbf{R}^{-1}) = \bar{x}^1$ and $f^1(\hat{R}, \tilde{\mathbf{R}}^{-1}) = \hat{x}^1$. When \bar{x}^1 is sufficiently close to $f^1(\mathbf{R})$, we can select such a \bar{R} in B . When \bar{x}^1 is sufficiently close to $f^1(\mathbf{R})$,

x' is sufficiently close to $f^1(\tilde{\mathbf{R}})$, and hence \hat{x}^1 is sufficiently close to $f^1(\tilde{\mathbf{R}})$. Then, we can select such a \hat{R} in B .

We let $\check{R} \in B$ be a preference that is an MMT of R at $\Omega - \hat{x}^1$ and of \tilde{R} at $\Omega - \bar{x}^1$. Observe that our choice of \bar{x}^1 and \hat{x}^1 ensures the condition in Proposition 2: $\Omega - \bar{x}^1 \in P(I(\Omega - \hat{x}^1; R) \cap [\Omega - \bar{x}^1]; \tilde{R})$. Furthermore, because R and \tilde{R} are sufficiently close, we can select such a \check{R} in B .

The following discussion is the same as that in Momi (2013b). We observe that the consumption allocated to agent 1 should not be changed when the preferences of agents other than agent 1 are changed to \check{R} from the profile $(\bar{R}, \mathbf{R}^{-1})$ or from $(\hat{R}, \tilde{\mathbf{R}}^{-1})$.

Because f is Pareto efficient and allocates positive consumption, at the profile $(\bar{R}, \mathbf{R}^{-1})$, agent 1 receives \bar{x}^1 and each of the other agents $i = 2, \dots, N$, receives a positive portion of $\Omega - \bar{x}^1$: $\bar{\lambda}^i (\Omega - \bar{x}^1)$, $i = 2, \dots, N$, where $0 < \bar{\lambda}^i < 1$ and $\sum_{i=2}^N \bar{\lambda}^i = 1$. Note that because we have chosen a \bar{x}^1 that is not parallel to Ω , the vectors \bar{x}^1 and $\Omega - \bar{x}^1$ are independent. Now, let us change agent 2's preference to \check{R} from R . We write the new profile as $(\bar{R}, \check{R}, \mathbf{R}^{-\{1,2\}})$ where agent 1's preference is \bar{R} , agent 2's is \check{R} and the other agents' preferences are R .

Given that \check{R} is an MMT of R at $\Omega - \bar{x}^1$, this is also the case at agent 2's consumption. Therefore agent 2's consumption should not be changed and nor should his/her gradient vector. Because of the Pareto efficiency, all agents' gradient vectors at their consumptions should be the same. Hence, all agents have the same gradient vector at both profiles $(\bar{R}, \mathbf{R}^{-1})$ and $(\bar{R}, \check{R}, \mathbf{R}^{-\{1,2\}})$. Because the preferences are homothetic, the equality of the gradient vectors implies that agents' consumptions at both profiles should be parallel. That is, at the new profile, agent 1's consumption is parallel to \bar{x}^1 and the other agents' consumptions are parallel to $\Omega - \bar{x}^1$. Because of the Pareto efficiency, the consumptions at the new profile should sum up to the total endowment Ω . Then, agent 1's consumption should remain \bar{x}^1 .

By applying the discussion repeatedly until all preferences except agent 1's are changed to \check{R} , we finally obtain that $f^1(\bar{R}, \check{\mathbf{R}}^{-1}) = \bar{x}^1$, where $\check{\mathbf{R}}^{-1} = (\check{R}, \dots, \check{R}) \in \mathcal{R}^{N-1}$. The discussions are the same for the profile $(\hat{R}, \tilde{\mathbf{R}}^{-1})$ and we obtain $f^1(\hat{R}, \check{\mathbf{R}}^{-1}) = \hat{x}^1$.

Remember our choice of \hat{x}^1 and \bar{x}^1 . From the construction, x' is strictly preferred to \bar{x}^1 with respect to any preference for agent 1 and \hat{x}^1 can be chosen arbitrarily close to x' . Therefore \hat{x}^1 could have been chosen to be preferred to \bar{x}^1 with respect to agent 1's preference \bar{R} . This violates the strategy-proofness of f . This ends the proof of Theorem 2. \square

7 Concluding remarks

In this paper, we focused on the local characterization of Pareto-efficient and strategy-proof social choice functions. We proved that Pareto-efficiency and strategy-proofness of a social choice function defined in a local preference set imply alternating dictator-

ship if the number of goods equals or exceeds the number of agents. In addition, we proved that Pareto efficiency and strategy-proofness of a social choice function defined in a local preference set are incompatible with the positive consumption condition if agents' individual preference sets are the same.

We comment on open questions left for future research. The most interesting question is whether local Pareto efficiency and strategy-proofness of a social choice function imply alternating dictatorship without the condition on the numbers of agents and goods. Although Momi (2020) proved the result on the whole preference domain, it seems difficult to extend the proof to the case of a local preference domain because Momi's proof changes all agents' preferences, except for those of two agents, to a common preference. Reflecting on the proof, the question may become slightly easier if all agents have the same local preference set. Without the condition on the numbers of agents and goods, it is also an open question whether Pareto-efficient and strategy-proof social choice functions defined on an arbitrarily given local preference set are incompatible with allocation conditions, such as the rationality, minimum consumption guarantee, and positive consumption conditions.

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A Appendix

A.1 An example

Consider an economy with two agents $i = 1, 2$, and two goods $l = 1, 2$. Let $U^i(x_1, x_2)$ be agent i 's utility function defined on the consumption set R_+^2 as

$$U^1(x_1, x_2) = (x_1 + 1)^{\frac{1}{2}}(x_2)^{\frac{1}{2}},$$

$$U^2(x_1, x_2) = (x_1)^{\frac{3}{4}}(x_2)^{\frac{1}{4}}.$$

We identify utility functions and the preferences represented by the utility functions. Let $\Omega = (2, 2)$ be the total endowment of the economy and define the allocation $\bar{x} = (\bar{x}^1, \bar{x}^2)$ where $\bar{x}^1 = (0, 1)$ and $\bar{x}^2 = (2, 1)$. Each agent's marginal rate of substitution (MRS) at the consumption bundle \bar{x}^i is calculated as

$$MRS^1 = \frac{\partial U^1}{\partial x_1} / \frac{\partial U^1}{\partial x_2} \Big|_{x=\bar{x}^1} = 1,$$

$$MRS^2 = \frac{\partial U^2}{\partial x_1} / \frac{\partial U^2}{\partial x_2} \Big|_{x=\bar{x}^2} = \frac{3}{2}.$$

Fig. 6 Edgeworth Box of the example

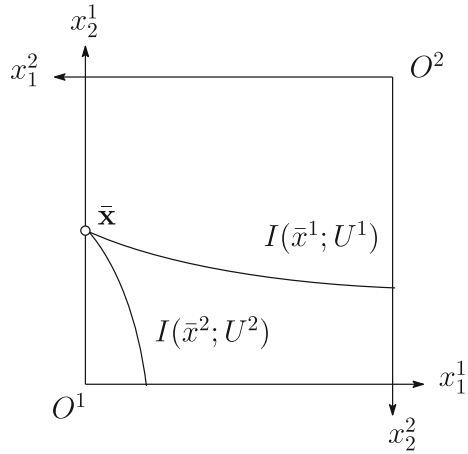


Figure 6 illustrates the Edgeworth box of this economy where the consumption of agent 1 is measured from the lower-left vertex and agent 2’s is measured from the upper-right vertex. The figure depicts the allocation \bar{x} and each agent’s indifference set $I(\bar{x}^i, U^i) = \{x \in \mathbb{R}_+^2 : U^i(x) = U^i(\bar{x}^i)\}$, which is the set of consumption indifferent to \bar{x}^i with respect to U^i on the possible allocation set X . Observe that \bar{x} , that is on the boundary of X , is Pareto efficient even though the MRSs are not equal between agents 1 and 2.

Consider a social choice function f that always allocates \bar{x} for any preferences. It is clear that f is strategy-proof and nondictatorial. It is also clear that for preference profiles in a neighborhood of (U^1, U^2) , the allocation \bar{x} remains Pareto efficient because the inequality $MRS^1 < MRS^2$ holds.

Although the agent 1’s preference represented by U^1 is not homothetic, it can be modified to be so without affecting the result. We pick up the indifference set of U^1 at \bar{x}^1 and construct a new homothetic preference by scaling this indifference set.

A.2 Proofs of lemmas and corollaries

Proof of Lemma 1 Let $R \in \mathcal{R}$ and $p \in S_{++}^{L-1}$ be given. We pick up a consumption bundle $\bar{x} \in \mathbb{R}_{++}^L$ and the corresponding indifference set $\bar{I} = I(\bar{x}, R)$. It is clear from the strict convexity of R that the consumption bundle $x' \in \bar{I}$ on the indifference set where the gradient vector is p : $p(R, x') = p$, is determined uniquely. Furthermore, by the homotheticity of R , the gradient vectors are the same on any consumption bundles on a ray starting from the origin. In particular, the gradient vector at a consumption bundle $x \in \mathbb{R}_{++}^L$ is p if and only if x is on the ray $[x']$. Therefore, the consumption-direction vector $g(R, p) \in S_{++}^{L-1}$, which is the normalization of these consumption bundles, is uniquely determined as $\frac{x'}{\|x'\|}$.

To prove the continuity, we fix $\bar{R} \in \mathcal{R}$ and $p \in S_{++}^{L-1}$. We want to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that if R satisfies $d(R, \bar{R}) \leq \delta$, then

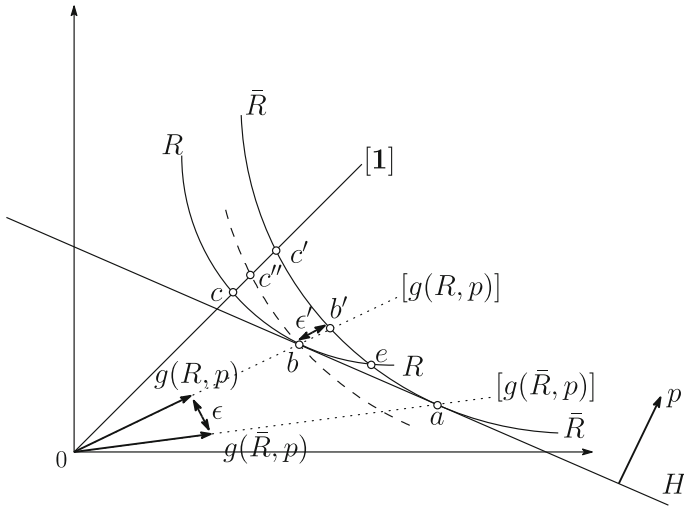


Fig. 7 Continuity of the consumption-direction vector

$\| g(R, p) - g(\bar{R}, p) \| < \epsilon$, where $d(\cdot, \cdot)$ is the Kannai metric and $\| \cdot \|$ is the Euclid norm in \mathbb{R}^L .

In contrast, we suppose that there exists an $\epsilon > 0$ such that for any $\delta > 0$ there exists some R satisfying $d(R, \bar{R}) < \delta$, and $\| g(R, p) - g(\bar{R}, p) \| \geq \epsilon$, and show a contradiction.

We suppose that a sufficiently small δ and R satisfy the condition. We fix a hyperplane H parallel to p^\perp as in Fig. 7. □

We define a as the intersection between $[g(\bar{R}, p)]$ and H , b as the intersection between $[g(R, p)]$ and H , b' as the intersection between $[g(R, p)]$ and $I(a; \bar{R})$, c as the intersection between $[1]$ and $I(b; R)$, and c' as the intersection between $[1]$ and $I(a; \bar{R})$. We define e as the intersection between $I(a; \bar{R})$ and $I(b; R)$, and c'' as the intersection between $I(b; \bar{R})$ and $[1]$.

Because $g(R, p)$ is away from $g(\bar{R}, p)$ by at least the distance ϵ , b' is away from b by some distance, which we write as ϵ' : $\| b' - b \| = \epsilon'$. Note that, because the preferences are homothetic, $\| c' - c'' \| : \| c' \| = \| b' - b \| : \| b' \|$. Therefore, $\| c' - c'' \| = \frac{\| c' \|}{\| b' \|} \epsilon'$.

We consider the Kannai metric between R and \bar{R} : $d(R, \bar{R}) = \max_{x \in \mathbb{R}_+^L} \frac{\| I(x; R) \cap [1] - I(x; \bar{R}) \cap [1] \|}{1 + \| x \|^2}$.

Letting e be the x in the definition of the Kannai metric, we have

$$d(R, \bar{R}) \geq \frac{\| I(e; R) \cap [1] - I(e; \bar{R}) \cap [1] \|}{1 + \| e \|^2} = \frac{\| c - c' \|}{1 + \| e \|^2}.$$

Letting b be the x in the definition of the Kannai metric, we have

$$d(R, \bar{R}) \geq \frac{\|I(b; R) \cap [1] - I(b; \bar{R}) \cap [1]\|}{1 + \|b\|^2} = \frac{\|c - c'\|}{1 + \|b\|^2}.$$

The condition $d(R, \bar{R}) < \delta$ with a sufficiently small δ implies that $\|c - c'\| < (1 + \|e\|^2)\delta$ and $\|c - c''\| < (1 + \|b\|^2)\delta$, that is, both c' and c'' are sufficiently close to c . However, this contradicts to that c' and c'' have the distance $\|c' - c''\| = \frac{\|c'\|}{\|b'\|} \epsilon'$ between them. □

Proof of Lemma 2 We fix δ arbitrarily. We let $\{\epsilon_n\}_{n=0}^\infty$ be a decreasing sequence of scalars converging to 0: $\epsilon_n < \epsilon_{n'}$ for $n > n'$ and $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. We assume that for any n , there exist some $R_{(n)}^i \in B_{\epsilon_n}(R^i)$ and $x_{(n)} \in F(R_{(n)}^i; G^i(\mathbf{R}^{-i}))$ satisfying $x_{(n)} \notin B_\delta(F(R^i; G^i(\mathbf{R}^{-i})))$, and we show a contradiction.

Because $x_{(n)} I_{R_{(n)}^i} f^i(R_{(n)}^i, \mathbf{R}^{-i}) R_{(n)}^i f^i(\mathbf{R})$, we have $x_{(n)} \in UC(f^i(\mathbf{R}); R_{(n)}^i)$. Because $x_{(n)} \in G^i(\mathbf{R}^{-i})$, it should not be preferred over $f^i(\mathbf{R})$ with respect to R^i , that is, $x_{(n)} \in LC(f^i(\mathbf{R}); R^i)$. Therefore, $x_{(n)}^i$ is in the intersection $UC(f^i(\mathbf{R}); R_{(n)}^i) \cap LC(f^i(\mathbf{R}); R^i)$. As $n \rightarrow \infty$, we have $R_{(n)}^i \rightarrow R^i$, and hence this intersection converges to some set in the indifference set $I(f^i(\mathbf{R}); R^i)$.

Note that $\{x_{(n)}\}_{n=0}^\infty$ has a convergent subsequence $\{x_{(n_k)}\}_{k=0}^\infty$ where $n_k < n_{k'}$ for $k > k'$ because the set of feasible allocations is compact. We let $\hat{x} = \lim_{k \rightarrow \infty} x_{(n_k)}$ denote the limit. Thus, we have $\hat{x} \in I(f^i(\mathbf{R}); R^i)$.

Because $x_{(n_k)} \in G^i(\mathbf{R}^{-i})$, we have $\hat{x} \in \overline{G^i(\mathbf{R}^{-i})}$, where $\overline{G^i(\mathbf{R}^{-i})}$ is the closure of $G^i(\mathbf{R}^{-i})$. This implies $\hat{x} \in G^i(\mathbf{R}^{-i})$, as proved by Momi (2017, Lemma 3).

Thus, we have $\hat{x} \in F(R^i; G^i(\mathbf{R}^{-i}))$ and this contradicts that $x_{(n_k)} \notin B_\delta(F(R^i; G^i(\mathbf{R}^{-i})))$. □

Proof of Lemma 3 We fix ϵ' arbitrarily. We suppose that for any ϕ' , there exists \mathbf{R}^{-i} such that $f^i(R^i, \mathbf{R}^{-i}) \neq 0$, $p((R^i, \mathbf{R}^{-i}), f) \in B_{\phi'}(\bar{p})$, and $p(\hat{R}^i; G^i(\mathbf{R}^{-i})) \notin B_{\epsilon'}(\bar{p})$, and we show a contradiction.

Because $p(\hat{R}^i; G^i(\mathbf{R}^{-i}))$ is not included in $B_{\epsilon'}(\bar{p})$, there exists $\hat{x}^i \in F(\hat{R}^i; G^i(\mathbf{R}^{-i}))$ such that $p(\hat{R}^i; \hat{x}^i) \notin B_{\epsilon'}(\bar{p})$. Because $p(\hat{R}^i; \bar{x}^i) = \bar{p}$, $p(\hat{R}^i; \hat{x}^i) \notin B_{\epsilon'}(\bar{p})$ implies that the ray $[\hat{x}^i]$ is away from the ray $[\bar{x}^i]$. Formally, we measure the distance between rays by the distance between their intersections with S_{++}^{L-1} , that is, we define the distance between two rays $[y]$ and $[z]$, where $y, z \in R_{++}^L$, by the distance between $y/\|y\|$ and $z/\|z\|$.

Note that, even if \hat{x}^i is different from $f^i(\hat{R}^i, \mathbf{R}^{-i})$, they are indifferent with respect to \hat{R}^i . Therefore, we have $\hat{x}^i \hat{R}^i f^i(R^i, \mathbf{R}^{-i})$ because of strategy-proofness. Conversely, because $\hat{x}^i \in G^i(\mathbf{R}^{-i})$, we have $f^i(R^i, \mathbf{R}^{-i}) R^i \hat{x}^i$. Thus, we have $\hat{x}^i \in LC(f^i(R^i, \mathbf{R}^{-i}); R^i) \cap UC(f^i(R^i, \mathbf{R}^{-i}); \hat{R}^i)$.

By taking a sufficiently small value of ϕ' , we have $p((R^i, \mathbf{R}^{-i}), f)$ arbitrarily close to \bar{p} . Then, the ray $[f^i(R^i, \mathbf{R}^{-i})]$ is arbitrarily close to $[\bar{x}^i]$. Then, for any $x \in LC(f^i(R^i, \mathbf{R}^{-i}); R^i) \cap UC(f^i(R^i, \mathbf{R}^{-i}); \hat{R}^i)$, the ray $[x]$ is arbitrarily close to the ray $[\bar{x}^i]$, and hence, the ray $[\hat{x}^i]$ should be arbitrarily close to $[\bar{x}^i]$. This is a contradiction. \square

Proof of Lemma 4 Note that $g^i(\bar{\mathbf{R}}, f) = g(R^i, p(\bar{\mathbf{R}}, f))$, $i = 1, \dots, K$, and they are independent. As $p \rightarrow p(\mathbf{R}, f)$, we have $g(\bar{R}^i, p) \rightarrow g(\bar{R}^i, p(\mathbf{R}, f))$. Therefore, there exists ϵ' such that $g(\bar{R}^i, p)$, $i = 1, \dots, K$, are independent for any $p \in B_{\epsilon'}(p(\bar{\mathbf{R}}, f))$, where $B_{\epsilon'}(p(\bar{\mathbf{R}}, f))$ denotes the closure of $B_{\epsilon'}(p(\bar{\mathbf{R}}, f))$.

For each $p \in B_{\epsilon'}(p(\bar{\mathbf{R}}, f))$, $g(R^i, p) \rightarrow g(\bar{R}^i, p)$ as $R^i \rightarrow \bar{R}^i$. Therefore, there exists ϵ_p such that $g(R^i, p)$, $i = 1, \dots, K$, are independent for any $R^i \in B_{\epsilon_p}(\bar{R}^i)$, $i = 1, \dots, K$. We define ϵ' as the minimum of ϵ_p as p moves over $B_{\epsilon'}(p(\bar{\mathbf{R}}, f))$: $\epsilon' = \min_{p \in B_{\epsilon'}(p(\bar{\mathbf{R}}, f))} \epsilon_p$. It is clear that ϵ' and ϵ' satisfy Lemma 4 (1).

We consider $R^i \in B_{\epsilon'}(\bar{R}^i)$, $i = 1, \dots, K$, and $p(\mathbf{R}, f) \in B_{\epsilon'}(p(\bar{\mathbf{R}}, f))$. Then, $g^i(\mathbf{R}, f)$, $i = 1, \dots, K$, are independent.

Note that $f^j(\mathbf{R}) = f^j(\bar{\mathbf{R}}) = 0$ for $j \geq K + 1$, means that $\sum_{i=1}^K f^i(\mathbf{R}) = \Omega$ and $\sum_{i=1}^K f^i(\bar{\mathbf{R}}) = \Omega$.

We define $\bar{\alpha}^i$, $i = 1, \dots, K$, as $f^i(\mathbf{R}) = \bar{\alpha}^i g^i(\mathbf{R}, f)$. Because $f^i(\bar{\mathbf{R}}) \in [g^i(\bar{\mathbf{R}}, f)]$, $i = 1, \dots, K$, and $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \dots, K$, are independent, these $\bar{\alpha}^i$'s are a unique solution of $\sum_{i=1}^K \bar{\alpha}^i g^i(\bar{\mathbf{R}}, f) = \Omega$.

Because $g^i(\bar{\mathbf{R}}, f) = g(\bar{R}^i, p(\mathbf{R}, f))$ and $g^i(\mathbf{R}, f) = g(R^i, p(\mathbf{R}, f))$, $g^i(\mathbf{R}, f)$ converges to $g^i(\bar{\mathbf{R}}, f)$ as R^i and $p(\mathbf{R}, f)$ converge to \bar{R}^i and $p(\bar{\mathbf{R}}, f)$, respectively. Therefore the scalars $\alpha^i(\mathbf{R}, f)$, $i = 1, \dots, K$ satisfying $\sum_{i=1}^K \alpha^i(\mathbf{R}, f) g^i(\mathbf{R}, f) = \Omega$, are determined uniquely and $\alpha^i(\mathbf{R}, f)$ converges to $\bar{\alpha}^i$, for any $i = 1, \dots, K$, as $g^i(\mathbf{R}, f)$ converges to $g^i(\bar{\mathbf{R}}, f)$, $i = 1, \dots, K$. Then, $f^i(\mathbf{R}) = \alpha^i(\mathbf{R}, f) g^i(\mathbf{R}, f)$ converges to $f^i(\bar{\mathbf{R}})$, $i = 1, \dots, K$. Thus $f^i(\bar{\mathbf{R}}) > 0$ implies that $f^i(\mathbf{R}) > 0$ \square

Proof of Lemma 5 We select ϵ and ϵ arbitrarily. If $f^j(\bar{\mathbf{R}}) \neq 0$ for some $j \geq K + 1$, the lemma holds. Therefore, we suppose that $f^j(\bar{\mathbf{R}}) = 0$ for any $j \geq K + 1$. We let $\bar{\epsilon} > 0$ and $\bar{\epsilon} > 0$ be scalars that support Lemma 4 with respect to $\bar{\mathbf{R}}$. We define $\bar{\epsilon} = \min\{\epsilon, \bar{\epsilon}\}$ and $\bar{\epsilon} = \min\{\epsilon, \bar{\epsilon}\}$. Note that, because of Lemma 4, at any \mathbf{R} such that $R^i \in B_{\bar{\epsilon}}(\bar{R}^i)$, $i = 1, \dots, K$, and $p(\mathbf{R}, f) \in B_{\bar{\epsilon}}(p(\bar{\mathbf{R}}, f))$, if $f^j(\mathbf{R}) = 0$ for $j \geq K + 1$, then $f^i(\mathbf{R})$, $i = 1, \dots, K$, are all positive and independent.

We let $\hat{R}^i \in B_{\bar{\epsilon}}(\bar{R}^i)$ be an MMT of \bar{R}^i at $f^i(\bar{\mathbf{R}})$ for each $i = 1, \dots, K$.

Case 1 We consider the case where some agent $j \geq K + 1$ receives positive consumption after replacements of \bar{R}^i with \hat{R}^i for some agents $i = 1, \dots, K$, that is, $f^j(R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N) \neq 0$ for some $j \geq K + 1$ and $R^i \in \{\bar{R}^i, \hat{R}^i\}$, $i = 1, \dots, K$.

Without loss of generality, by relabeling the consumer indexes if necessary, we assume that $f^j(\hat{R}^1, \dots, \hat{R}^{\bar{S}}, \bar{R}^{\bar{S}+1}, \dots, \bar{R}^N) \neq 0$ for some $j \geq K + 1$ where \bar{S} is

the smallest number of agents whose preference \bar{R}^i should be replaced with \hat{R}^i so that some agent $j \geq K + 1$ has positive consumption.

We show that the lemma holds at the preference profile $(\hat{R}^1, \dots, \hat{R}^{\bar{S}}, \bar{R}^{\bar{S}+1}, \dots, \bar{R}^N)$. It is clear that the preference subprofile $(\hat{R}^1, \dots, \hat{R}^{\bar{S}}, \bar{R}^{\bar{S}+1}, \dots, \bar{R}^K)$ is in $B_\epsilon(\bar{R}^1) \times \dots \times B_\epsilon(\bar{R}^K)$. All we have to show is that the price vector at the preference profile is in $B_\epsilon(p(\bar{\mathbf{R}}, f))$.

From the definition of \bar{S} , we have $f^j(R^1, \dots, R^{\bar{S}-1}, \bar{R}^{\bar{S}}, \dots, \bar{R}^N) = 0$ for any $j \geq K + 1$ and any $R^i \in \{\bar{R}^i, \hat{R}^i\}$, $i = 1, \dots, \bar{S} - 1$. Because the K agents, $i = 1, \dots, K$, share the total endowment among them with the independent consumption vectors, we have $p((R^1, \dots, R^{\bar{S}-1}, \bar{R}^{\bar{S}}, \dots, \bar{R}^N), f) = p(\bar{\mathbf{R}}, f)$ and $f^i(R^1, \dots, R^{\bar{S}-1}, \bar{R}^{\bar{S}}, \dots, \bar{R}^N) = f^i(\bar{\mathbf{R}})$ for any $i = 1, \dots, K$, and any $R^i \in \{\bar{R}^i, \hat{R}^i\}$, $i = 1, \dots, \bar{S} - 1$. In particular, we have $f^{\bar{S}}(\hat{R}^1, \dots, \hat{R}^{\bar{S}-1}, \bar{R}^{\bar{S}}, \dots, \bar{R}^N) = f^{\bar{S}}(\bar{\mathbf{R}})$ and $p((\hat{R}^1, \dots, \hat{R}^{\bar{S}-1}, \bar{R}^{\bar{S}}, \dots, \bar{R}^N), f) = p(\bar{\mathbf{R}}, f)$. Because $\hat{R}^{\bar{S}}$ is an MMT of $\bar{R}^{\bar{S}}$ at $f^{\bar{S}}(\bar{\mathbf{R}})$, we have $p((\hat{R}^1, \dots, \hat{R}^{\bar{S}}, \bar{R}^{\bar{S}+1}, \dots, \bar{R}^N), f) = p(\bar{\mathbf{R}}, f)$ as desired.

Case 2 We consider the case where agent $j \geq K + 1$ receives zero consumption after any replacement of \bar{R}^i with \hat{R}^i for any agents $i = 1, \dots, K$, that is, $f^j(R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N) = 0$ for any $j \geq K + 1$ and any $R^i \in \{\bar{R}^i, \hat{R}^i\}$, $i = 1, \dots, K$. Note that $f^i(R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N) = f^i(\bar{\mathbf{R}})$ for $i = 1, \dots, K$, and $p((R^1, \dots, R^K, \bar{R}^{K+1}, \dots, \bar{R}^N), f) = p(\bar{\mathbf{R}}, f)$ for any $R^i \in \{\bar{R}^i, \hat{R}^i\}$, $i = 1, \dots, K$, because these agents share the total endowment with independent consumption vectors.

Starting from the preference profile $\bar{\mathbf{R}}$, we replace \bar{R}^i with \hat{R}^i and then replace \hat{R}^i with a preference $R^{i'}$ in a neighborhood of \hat{R}^i for $i = 1, \dots, K$. If no agents $j \geq K + 1$ have positive consumption, then we reach a situation contradicting Proposition 1.

This replacement of preferences should be done so that the independence of the consumption vectors is maintained. For this purpose, we prepare a function ϕ^i as follows. For each $i = 1, \dots, K$, we consider a function $\bar{\phi}^i : \epsilon' \mapsto \bar{\phi}^i(\epsilon')$ that maps ϵ' to ϕ' in Lemma 3 with respect to \bar{R}^i and \hat{R}^i . Furthermore we define a function $\phi^i : \epsilon' \mapsto \phi^i(\epsilon')$ as $\phi^i(\epsilon') = \min\{\bar{\phi}^i(\epsilon'), \epsilon'\}$. Then, of course, for any ϵ and any \mathbf{R}^{-i} , if $f^i(\bar{R}^i, \mathbf{R}^{-i}) \neq 0$ and $p((\bar{R}^i, \mathbf{R}^{-i}), f) \in B_{\phi^i(\epsilon)}(p(\bar{\mathbf{R}}, f))$, then $p(\hat{R}^i; G^i(\mathbf{R}^{-i})) \in B_\epsilon(p(\bar{\mathbf{R}}, f))$ because of Lemma 3.

Step 1 We replace \bar{R}^1 with \hat{R}^1 . This replacement does not change the price vector. We let ϵ^1 be a sufficiently small scalar such that $B_{\epsilon^1}(\hat{R}^1) \subset B_{\epsilon}(\bar{R}^1)$ and

$$p(R^{1'}; G^1(\bar{\mathbf{R}}^{-1})) \subset B_{\phi^2 \circ \dots \circ \phi^K(\bar{\epsilon})}(p(\bar{\mathbf{R}}, f)) \tag{10}$$

for any $R^{1'} \in B_{\epsilon_1}(\hat{R}^1)$. Note that $f^1(\bar{\mathbf{R}})$ is the unique intersection of $UC(f^1(\bar{\mathbf{R}}); \hat{R}^1)$ and $G^1(\mathbf{R}^{-1})$. Therefore, if $R^{1'}$ is sufficiently close to \hat{R}^1 , then $f(R^{1'}, \bar{\mathbf{R}}^{-1})$ and $p(R^{1'}; G^1(\bar{\mathbf{R}}^{-1}))$ are sufficiently close to $f^1(\bar{\mathbf{R}})$ and $p(\bar{\mathbf{R}}, f)$, respectively, as discussed after Lemma 2. Thus, a value of ϵ^1 satisfying the condition exists.

The lemma holds if $f^j(R^{1'}, \bar{\mathbf{R}}^{-1}) \neq 0$ for some $j \geq K + 1$ with some $R^{1'} \in B_{\epsilon_1}(\hat{R}^1)$. Therefore, we suppose that $f^j(R^{1'}, \bar{\mathbf{R}}^{-1}) = 0$ for any $j \geq K + 1$ and any $R^{1'} \in B_{\epsilon_1}(\hat{R}^1)$. Then, $f^i(R^{1'}, \bar{\mathbf{R}}^{-1})$, $i = 1, \dots, K$, are all positive and independent because $R^{1'} \in B_{\epsilon_1}(\hat{R}^1) \subset B_{\bar{\epsilon}}(\bar{R}^1)$ and $p((R^{1'}, \bar{\mathbf{R}}^{-1}), f) \in B_{\phi^2 \circ \dots \circ \phi^K(\bar{\epsilon})}(p(\bar{\mathbf{R}}, f)) \subset B_{\bar{\epsilon}}(p(\bar{\mathbf{R}}, f))$. \square

Step 2 We replace R^2 with \hat{R}^2 . By (10) and Lemma 3, we have

$$p(\hat{R}^2; G^2(R^{1'}, \bar{\mathbf{R}}^{-\{1,2\}})) \subset B_{\phi^3 \circ \dots \circ \phi^K(\bar{\epsilon})}(p(\bar{\mathbf{R}}, f)) \tag{11}$$

for any $R^{1'} \in B_{\epsilon_1}(\hat{R}^1)$. We let ϵ^2 be sufficiently small such that $B_{\epsilon^2}(\hat{R}^2) \subset B_{\bar{\epsilon}}(\bar{R}^2)$ and

$$p(R^{2'}; G^2(R^{1'}, \bar{\mathbf{R}}^{-\{1,2\}})) \subset B_{\phi^3 \circ \dots \circ \phi^K(\bar{\epsilon})}(p(\bar{\mathbf{R}}, f)) \tag{12}$$

for any $R^{2'} \in B_{\epsilon^2}(\hat{R}^2)$ and any $R^{1'} \in B_{\epsilon_1}(\hat{R}^1)$. Note that a sufficiently small value of ϵ^2 ensures (12) because of (11).

The lemma holds if $f^j(R^{1'}, R^{2'}, \bar{\mathbf{R}}^{-\{1,2\}}) \neq 0$ for some $j \geq K + 1$ and some $(R^{1'}, R^{2'}) \in B_{\epsilon_1}(\hat{R}^1) \times B_{\epsilon^2}(\hat{R}^2)$. Therefore, we suppose that $f^j(R^{1'}, R^{2'}, \bar{\mathbf{R}}^{-\{1,2\}}) = 0$ for any $j \geq K + 1$ and any $(R^{1'}, R^{2'}) \in B_{\epsilon_1}(\hat{R}^1) \times B_{\epsilon^2}(\hat{R}^2)$. Then $f^i(R^{1'}, R^{2'}, \bar{\mathbf{R}}^{-\{1,2\}})$, $i = 1, \dots, K$, are all positive and independent because $(R^{1'}, R^{2'}) \in B_{\epsilon_1}(\hat{R}^1) \times B_{\epsilon^2}(\hat{R}^2) \subset B_{\bar{\epsilon}}(\bar{R}^1) \times B_{\bar{\epsilon}}(\bar{R}^2)$ and $p((R^{1'}, R^{2'}, \bar{\mathbf{R}}^{-\{1,2\}}), f) \in B_{\phi^3 \circ \dots \circ \phi^K(\bar{\epsilon})}(p(\bar{\mathbf{R}}, f)) \subset B_{\bar{\epsilon}}(p(\bar{\mathbf{R}}, f))$. \square

Step 3 We replace \bar{R}^3 with \hat{R}^3 . By (12) and Lemma 3, we have

$$p(\hat{R}^3; G^3(R^{1'}, R^{2'}, \bar{\mathbf{R}}^{-\{1,2,3\}})) \subset B_{\phi^4 \circ \dots \circ \phi^K(\bar{\epsilon})}(p(\bar{\mathbf{R}}, f))$$

for any $(R^{1'}, R^{2'}) \in B_{\epsilon_1}(\hat{R}^1) \times B_{\epsilon^2}(\hat{R}^2)$. We let ϵ^3 be a sufficiently small scalar such that $B_{\epsilon^3}(\hat{R}^3) \subset B_{\bar{\epsilon}}(\bar{R}^3)$ and

$$p(R^{3'}; G^3(R^{1'}, R^{2'}, \bar{\mathbf{R}}^{-\{1,2,3\}})) \subset B_{\phi^4 \circ \dots \circ \phi^K(\bar{\epsilon})}(p(\bar{\mathbf{R}}, f))$$

for any $R^{3'} \in B_{\epsilon^3}(\hat{R}^3)$ and any $(R^{1'}, R^{2'}) \in B_{\epsilon^1}(\hat{R}^1) \times B_{\epsilon^2}(\hat{R}^2)$.

The lemma holds if $f^j(R^{1'}, \dots, R^{3'}, \bar{\mathbf{R}}^{-\{1,2,3\}}) \neq 0$ for some $j \geq K + 1$ and some $(R^{1'}, \dots, R^{3'}) \in B_{\epsilon^1}(\hat{R}^1) \times \dots \times B_{\epsilon^3}(\hat{R}^3)$. Therefore, we suppose that $f^j(R^{1'}, \dots, R^{3'}, \bar{\mathbf{R}}^{-\{1,2,3\}}) = 0$ for any $j \geq K + 1$ and any $(R^{1'}, \dots, R^{3'}) \in B_{\epsilon^1}(\hat{R}^1) \times \dots \times B_{\epsilon^3}(\hat{R}^3)$. Then, $f^i(R^{1'}, \dots, R^{3'}, \bar{\mathbf{R}}^{-\{1,2,3\}})$, $i = 1, \dots, K$ all remain positive and independent because $(R^{1'}, \dots, R^{3'}) \in B_{\bar{\epsilon}}(\hat{R}^1) \times \dots \times B_{\bar{\epsilon}}(\hat{R}^3)$ and $p((R^{1'}, \dots, R^{3'}, \bar{\mathbf{R}}^{-\{1,2,3\}}), f) \in B_{\bar{\epsilon}}(p(\bar{\mathbf{R}}, f))$. \square

We repeat these steps. Finally, we replace \bar{R}^K with \hat{R}^K and obtain

$$p(\hat{R}^K; G^K(R^{1'}, \dots, R^{K-1'}, \bar{R}^{K+1}, \dots, \bar{R}^N)) \subset B_{\bar{\epsilon}}(p(\bar{\mathbf{R}}, f)),$$

for any $(R^{1'}, \dots, R^{K-1'}) \in B_{\epsilon^1}(\hat{R}^1) \times \dots \times B_{\epsilon^{K-1}}(\hat{R}^{K-1})$. We let ϵ^K be a sufficiently small scalar such that $B_{\epsilon^K}(\hat{R}^K) \subset B_{\bar{\epsilon}}(\bar{R}^K)$ and

$$p(R^{K'}; G^K(R^{1'}, \dots, R^{K-1'}, \bar{R}^{K+1}, \dots, \bar{R}^N)) \subset B_{\bar{\epsilon}}(p(\bar{\mathbf{R}}, f)),$$

for any $R^{K'} \in B_{\epsilon^K}(\hat{R}^K)$ and any $(R^{1'}, \dots, R^{K-1'}) \in B_{\epsilon^1}(\hat{R}^1) \times \dots \times B_{\epsilon^{K-1}}(\hat{R}^{K-1})$. There should exist some preference profile $(R^{1'}, \dots, R^{K'}) \in B_{\epsilon^1}(\hat{R}^1) \times \dots \times B_{\epsilon^K}(\hat{R}^K) \subset B_{\bar{\epsilon}}(\bar{R}^1) \times \dots \times B_{\bar{\epsilon}}(\bar{R}^K)$ such that $f^j(R^{1'}, \dots, R^{K'}, \bar{R}^{K+1}, \dots, \bar{R}^N) \neq 0$ for some $j \geq K + 1$. Otherwise, $f(\cdot, \dots, \cdot, \bar{R}^{K+1}, \dots, \bar{R}^N)$ becomes a social choice function in the economy with K agents $i = 1, \dots, K$, that is Pareto efficient, strategy-proof, and nonalternately dictatorial on $B_{\epsilon^1}(\hat{R}^1) \times \dots \times B_{\epsilon^K}(\hat{R}^K)$ with independent consumption directions. This contradicts Proposition 1. This ends the proof of Lemma 5. \square

Proof of Corollary 1 Contrary to the statement of the corollary, we suppose that for each $j \geq K + 1$ there exist ϵ^j and ε^j such that no $(\tilde{R}^1, \dots, \tilde{R}^K) \in B_{\epsilon^j}(\bar{R}^1) \times \dots \times B_{\epsilon^j}(\bar{R}^K)$ satisfies $p((\tilde{R}^1, \dots, \tilde{R}^K, \bar{R}^{K+1}, \dots, \bar{R}^N), f) \in B_{\varepsilon^j}(p(\bar{\mathbf{R}}, f))$ and $f^j(\tilde{R}^1, \dots, \tilde{R}^K, \bar{R}^{K+1}, \dots, \bar{R}^N) \notin \{0, \Omega\}$. We let $\bar{\epsilon} = \min\{\epsilon^{K+1}, \dots, \epsilon^N\}$ and $\bar{\varepsilon} = \min\{\varepsilon^{K+1}, \dots, \varepsilon^N\}$. Then, for these $\bar{\epsilon}$ and $\bar{\varepsilon}$, no $(\tilde{R}^1, \dots, \tilde{R}^K) \in B_{\bar{\epsilon}}(\bar{R}^1) \times \dots \times B_{\bar{\epsilon}}(\bar{R}^K)$ satisfies $p((\tilde{R}^1, \dots, \tilde{R}^K, \bar{R}^{K+1}, \dots, \bar{R}^N), f) \in B_{\bar{\varepsilon}}(p(\bar{\mathbf{R}}, f))$ and $f^j(\tilde{R}^1, \dots, \tilde{R}^K, \bar{R}^{K+1}, \dots, \bar{R}^N) \notin \{0, \Omega\}$ for any $j \geq K + 1$. This contradicts Lemma 5. \square

Proof of Lemma 6 We select ϵ and ε arbitrarily. If $f^j(\bar{\mathbf{R}}) \neq 0$ for some $j \geq K + 1$, the lemma holds. Therefore, we assume that $f^j(\bar{\mathbf{R}}) = 0$ for $j \geq K + 1$. When an agent

receives neither zero consumption nor the total consumption, there exists another agent who also receives neither zero nor the total consumption. Without loss of generality, by relabeling the consumer indexes if necessary, we assume that $f^i(\bar{\mathbf{R}}) \notin \{0, \Omega\}$ for $i = 1, \dots, M_1$ and $f^i(\bar{\mathbf{R}}) = 0$ for $i \geq M_1 + 1$, where $2 \leq M_1 \leq K$. Applying Corollary 1 to agents who receive positive consumption repeatedly, we can increase the number of agents who receive positive consumption. \square

Step 1 We start from the preference profile $\bar{\mathbf{R}}$. We let $\bar{\epsilon} > 0$ and $\bar{\varepsilon} > 0$ be scalars that support Lemma 4 with respect to $\bar{\mathbf{R}}$. We define that $\tilde{\epsilon} = \min\{\epsilon, \bar{\epsilon}\}$ and $\tilde{\varepsilon} = \min\{\varepsilon, \bar{\varepsilon}\}$.

We apply Corollary 1 to agents $i = 1, \dots, M_1$. There exists $j \geq M_1 + 1$ such that for any ϵ_1 and ε_1 , some $(R_{(1)}^1, \dots, R_{(1)}^{M_1}) \in B_{\epsilon_1}(\bar{R}^1) \times \dots \times B_{\epsilon_1}(\bar{R}^{M_1})$ satisfies

$$p\left(\left(R_{(1)}^1, \dots, R_{(1)}^{M_1}, \bar{R}^{M_1+1}, \dots, \bar{R}^N\right), f\right) \in B_{\epsilon_1}(p(\bar{\mathbf{R}}, f)), \tag{13}$$

and

$$f^j\left(R_{(1)}^1, \dots, R_{(1)}^{M_1}, \bar{R}^{M_1+1}, \dots, \bar{R}^N\right) \notin \{0, \Omega\}. \tag{14}$$

If (14) holds with some agent $j \geq K + 1$, then the lemma holds. Therefore, we assume that agent $j \geq K + 1$ has zero consumption at any preference profile $(R_{(1)}^1, \dots, R_{(1)}^{M_1}, \bar{R}^{M_1+1}, \dots, \bar{R}^N)$. We set sufficiently small $\epsilon_1 < \tilde{\epsilon}$ and $\varepsilon_1 < \tilde{\varepsilon}$, and we fix $(R_{(1)}^1, \dots, R_{(1)}^{M_1}) \in B_{\epsilon_1}(\bar{R}^1) \times \dots \times B_{\epsilon_1}(\bar{R}^{M_1})$ to satisfy (13) and (14). Without loss of generality, we assume that agents $j = M_1 + 1, \dots, M_2$, where $M_1 + 1 \leq M_2 \leq K$, satisfies (14).

We can prove that agents $i = 1, \dots, K$ have independent consumption vectors and agents $i = 1, \dots, M_1$ have positive consumption at the preference profile $(R_{(1)}^1, \dots, R_{(1)}^{M_1}, \bar{R}^{M_1+1}, \dots, \bar{R}^N)$. Remember Lemma 4 and our choice of the scalars ϵ_1 and ε_1 . First, $g^i\left(\left(R_{(1)}^1, \dots, R_{(1)}^{M_1}, \bar{R}^{M_1+1}, \dots, \bar{R}^N\right), f\right), i = 1, \dots, K$, are independent because $g^i(\bar{\mathbf{R}}, f), i = 1, \dots, K$, are independent, $R_{(1)}^i \in B_{\epsilon_1}(\bar{R}^i) \subset B_{\tilde{\epsilon}}(\bar{R}^i)$, $i = 1, \dots, M_1$, and $p\left(\left(R_{(1)}^1, \dots, R_{(1)}^{M_1}, \bar{R}^{M_1+1}, \dots, \bar{R}^N\right), f\right) \in B_{\epsilon_1}(p(\bar{\mathbf{R}}, f)) \subset B_{\tilde{\varepsilon}}(p(\bar{\mathbf{R}}, f))$. Furthermore $f^i(R_{(1)}^1, \dots, R_{(1)}^{M_1}, \bar{R}^{M_1+1}, \dots, \bar{R}^N) \neq 0$ for any $i = 1, \dots, M_1$, because $f^i(\bar{\mathbf{R}}) \neq 0$ for $i = 1, \dots, M_1$. Note that, as a result, agents $i = 1, \dots, M_2$ have positive consumption at the preference profile. \square

Step 2: Starting from the preference profile $(R_{(1)}^1, \dots, R_{(1)}^{M_1}, \bar{R}^{M_1+1}, \dots, \bar{R}^N)$, where agents $i = 1, \dots, M_2$ have positive consumption, we conduct the same operation as in Step 1. We let $\bar{\epsilon}_1$ and $\bar{\varepsilon}_1$ be scalars such that they support Lemma 4 with respect to $(R_{(1)}^1, \dots, R_{(1)}^{M_1}, \bar{R}^{M_1+1}, \dots, \bar{R}^N)$ and satisfy $B_{\bar{\epsilon}_1}(R_{(1)}^i) \subset B_{\bar{\epsilon}}(\bar{R}^i)$ for $i = 1, \dots, M_1$, $B_{\bar{\epsilon}_1}(\bar{R}^i) \subset B_{\bar{\epsilon}}(\bar{R}^i)$ for $i = M_1 + 1, \dots, K$, and $B_{\bar{\varepsilon}_1}(p(R_{(1)}^1, \dots, R_{(1)}^{M_1}, \bar{R}^{M_1+1}, \dots, \bar{R}^N), f) \subset B_{\bar{\varepsilon}}(p(\bar{\mathbf{R}}, f))$.

We apply Corollary 1 to agents $i = 1, \dots, M_2$. There exists $j \geq M_2 + 1$ such that for any ϵ_2 and ε_2 , some $(R_{(2)}^1, \dots, R_{(2)}^{M_2}) \in B_{\epsilon_2}(R_{(1)}^1) \times \dots \times B_{\epsilon_2}(R_{(1)}^{M_1}) \times B_{\epsilon_2}(\bar{R}^{M_1+1}) \times$

$\dots \times B_{\epsilon_2}(\bar{R}^{M_2})$ satisfies

$$\begin{aligned}
 & p((R_{(2)}^1, \dots, \tilde{R}_{(2)}^{M_2}, \bar{R}^{M_2+1}, \dots, \bar{R}^N), f) \in B_{\epsilon_2}(p((R_{(1)}^1, \dots, \\
 & \tilde{R}_{(1)}^{M_1}, \bar{R}^{M_1+1}, \dots, \bar{R}^N), f)),
 \end{aligned}
 \tag{15}$$

and

$$f^j \left(R_{(2)}^1, \dots, \tilde{R}_{(2)}^{M_2}, \bar{R}^{M_2+1}, \dots, \bar{R}^N \right) \notin \{0, \Omega\}.
 \tag{16}$$

If (15) holds with some agent $j \geq K + 1$, then the lemma holds. Therefore, we assume that agent $j \geq K + 1$ has zero consumption at any preference profile $(R_{(2)}^1, \dots, \tilde{R}_{(2)}^{M_2}, \bar{R}^{M_2+1}, \dots, \bar{R}^N)$. We set $\epsilon_2 < \bar{\epsilon}_1$ and $\epsilon_2 < \bar{\epsilon}_1$, and we fix $(R_{(2)}^1, \dots, \tilde{R}_{(2)}^{M_2}) \in B_{\epsilon_2}(R_{(1)}^1) \times \dots \times B_{\epsilon_2}(R_{(1)}^{M_1}) \times B_{\epsilon_2}(\bar{R}^{M_1+1}) \times \dots \times B_{\epsilon_2}(\bar{R}^{M_2})$ to satisfy (15) and (16). Without loss of generality, we assume that agents $j = M_2 + 1, \dots, M_3$, where $M_2 + 1 \leq M_3 \leq K$, satisfy (16).

We can prove that agents $i = 1, \dots, K$ have independent consumption vectors and agents $i = 1, \dots, M_2$ have positive consumption at the preference profile $(R_{(2)}^1, \dots, \tilde{R}_{(2)}^{M_2}, \bar{R}^{M_2+1}, \dots, \bar{R}^N)$ in a similar way to the previous step. As a result, agents $i = 1, \dots, M_3$ have positive consumption at the preference profile. \square

We repeat these steps. In each step, we have at least one additional agent who receives positive consumption at the preference profile in the ϵ -neighborhood of $\bar{\mathbf{R}}$, where the price vector is in the ϵ -neighborhood of $p(\bar{\mathbf{R}})$. Finally, some agent $j \geq K + 1$ should receive positive consumption. This ends the proof of Lemma 6. \square

Proof of Corollary 2 Contrary to the statement of the corollary, we suppose that for each $j \geq K + 1$, there exist values of ϵ^j and ϵ^j such that no $(\tilde{R}^1, \dots, \tilde{R}^K) \in B_{\epsilon^j}(\tilde{R}^1) \times \dots \times B_{\epsilon^j}(\tilde{R}^K)$ satisfies $p((\tilde{R}^1, \dots, \tilde{R}^K, \bar{R}^{K+1}, \dots, \bar{R}^N), f) \in B_{\epsilon^j}(p(\bar{\mathbf{R}}, f))$ and $f^j(\tilde{R}^1, \dots, \tilde{R}^K, \bar{R}^{K+1}, \dots, \bar{R}^N) \notin \{0, \Omega\}$. We let $\bar{\epsilon} = \min\{\epsilon^{K+1}, \dots, \epsilon^N\}$ and $\bar{\epsilon} = \min\{\epsilon^{K+1}, \dots, \epsilon^N\}$. Then, for these $\bar{\epsilon}$ and $\bar{\epsilon}$, no $(\tilde{R}^1, \dots, \tilde{R}^K) \in B_{\bar{\epsilon}}(\tilde{R}^1) \times \dots \times B_{\bar{\epsilon}}(\tilde{R}^K)$ satisfies $p((\tilde{R}^1, \dots, \tilde{R}^K, \bar{R}^{K+1}, \dots, \bar{R}^N), f) \in B_{\bar{\epsilon}}(p(\bar{\mathbf{R}}, f))$ and $f^j(\tilde{R}^1, \dots, \tilde{R}^K, \bar{R}^{K+1}, \dots, \bar{R}^N) \notin \{0, \Omega\}$ for any $j \geq K + 1$. This contradicts Lemma 6. \square

Proof of Lemma 7 We write $a^i = ((1/\alpha_i^j)^{1/(\rho-1)}, \dots, (1/\alpha_i^L)^{1/(\rho-1)})$. First, we show that the lemma holds when $a^i, i = 1, \dots, N$, are independent. If $a^i, i = 1, \dots, N$, are independent, $\beta^i(\delta, z^i), i = 1, \dots, N$, are positive and independent with any vectors $\bar{z}^i, i = 1, \dots, N$, when $\delta^i, i = 1, \dots, N$, are sufficiently small.

When they are dependent, $\bar{z}^i, i = 1, \dots, N$, can be obtained as follows, for example. Without loss of generality we assume that the vectors a^1, \dots, a^N span S -dimensional space, where $S < N$ and a^1, \dots, a^S are independent. We set $\bar{z}^i = a^i$ for $i = 1, \dots, S$. We let $\langle a^1, \dots, a^S \rangle$ denote the S -dimensional vector space spanned by a^1, \dots, a^S and let $\langle a^1, \dots, a^S \rangle^\perp$ denote the orthogonal complement space of $(L - S)$ -dimensions. We select $\bar{z}^i, i = S + 1, \dots, N$, so that they are independent vectors in $\langle a^1, \dots, a^S \rangle^\perp$.

When δ^i is sufficiently small $a^i + \delta^i \bar{z}^i$ is positive for any $i = 1, \dots, N$. Their independence is clear from the construction. \square

Proof of Lemma 8 In the proof of Lemma 7, we set \bar{z}^i so that $a^i + \delta^i \bar{z}^i, i = 1, \dots, N$, are independent. It is clear that any z^i that is sufficiently close to \bar{z}^i sustains the independence, and Lemma 7 holds with z^i instead of \bar{z}^i . That is, there exists a \bar{t} such that $\beta^i(\delta^i, z^i), i = 1, \dots, N$, are independent for any $0 < \delta^i \leq \bar{\delta}$ and any $z^i \in B_{\bar{t}}(\bar{z}^i), i = 1, \dots, N$.

For each δ^i and $z^i \in B_{\bar{t}^i}(\bar{z}^i)$, we calculate $\hat{\gamma}_{z^i}^i = (\gamma_{z^i 1}^i, \dots, \gamma_{z^i L}^i)$ by solving

$$((1/\hat{\gamma}_{z^i 1}^i)^{1/(\rho-1)}, \dots, (1/\hat{\gamma}_{z^i L}^i)^{1/(\rho-1)}) = \beta^i(\delta^i, z^i) \tag{17}$$

and define $\gamma_{z^i}^i = (\gamma_{z^i 1}^i, \dots, \gamma_{z^i L}^i)$ as the normalization of $\hat{\gamma}_{z^i}^i: \gamma_{z^i}^i = \hat{\gamma}_{z^i}^i / \|\hat{\gamma}_{z^i}^i\|$. We define the CES utility function $U_{\gamma_{z^i}^i}$ with $\gamma_{z^i}^i$ as the parameter. Then, as in the case of U_{γ^i} , for any price p , the consumption-direction vectors of $U_{\gamma_{z^i}^i}, i = 1, \dots, N$, are independent for any $0 < \delta^i < \bar{\delta}$ and any $z^i \in B_{\bar{t}^i}(\bar{z}^i)$ because of the independence of $\beta(\delta^i, z^i), i = 1, \dots, N$.

For each δ^i and $p \in P$, we define $G^i(\delta^i, \bar{t}^i, p) = \bigcup_{z^i \in B_{\bar{t}^i}(\bar{z}^i)} g(U_{\gamma_{z^i}^i}, p)$ as the set of the consumption-direction vectors of $U_{\gamma_{z^i}^i}$ for p while z^i moves over $B_{\bar{t}^i}(\bar{z}^i)$. Then, any vectors $b^i \in G^i(\delta^i, \bar{t}^i, p), i = 1, \dots, N$ are independent as observed in the previous paragraph.

As $R^i \rightarrow U_{\gamma^i}$, we have $g(R^i, p) \rightarrow g^i(U_{\gamma^i}, p)$. Therefore, for each $p \in P$ and $0 < \delta^i \leq \bar{\delta}$, there exists an $\epsilon^i(p, \delta^i)$ such that if $R^i \in B_{\epsilon^i(p, \delta^i)}(U_{\gamma^i})$, then $g^i(R^i, p) \in G^i(\delta^i, \bar{t}^i, p)$. Finally, we define $\epsilon^i(\delta^i) = \min_{p \in P} \epsilon^i(p, \delta^i)$. \square

Proof of Lemma 9 When $\delta^i = 0, R_{0,t,s}^i$ is an MMT of \bar{R}^i at \bar{g}^i and $R_{0,t,s}^i \rightarrow \bar{R}^i$ as $t \rightarrow 0$ and $s \rightarrow 0$. We fix s and t such that $R_{0,t,s}^i \in B_\epsilon(\bar{R}^i)$ for any $i = 1, \dots, N$.

Because $R_{\delta^i,t,s}^i \rightarrow R_{0,t,s}^i$ as $\delta^i \rightarrow 0$, we select $\bar{\delta}^i$ such that $R_{\delta^i,t,s}^i \in B_\epsilon(\bar{R}^i)$. Then we have $R_{\delta^i,t,s}^i \in B_\epsilon(\bar{R}^i)$ for any $0 \leq \delta^i \leq \bar{\delta}^i$.

For any $0 < \delta^i \leq \bar{\delta}^i$, the consumption-direction vectors of $R_{\delta^i,t,s}^i, i = 1, \dots, N$, are independent for prices in a neighborhood of \bar{p} . That is, there exists an ϵ such that $g(R_{\delta^i,t,s}^i, p), i = 1, \dots, N$, are independent for $p \in B_\epsilon(\bar{p})$. \square

Proof of Lemma 10 Let ϵ^i be the same as in Lemma 8. Because the preference $R_{\delta^i,t,s}^i$ coincides with the preference U_{γ^i} in a neighborhood of \bar{g}^i , we set $\bar{\epsilon}$ such that $g(R_{\delta^i,t,s}^i, p) = g(U_{\gamma^i}, p), i = 1, \dots, N$, for $p \in B_{\bar{\epsilon}}(\bar{p})$. Then, Lemma 10 follows Lemma 8. \square

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