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Ordinal Bayesian incentive compatibility in random assignment model

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Abstract

We explore the consequences of weakening the notion of incentive compatibility from strategy-proofness to ordinal Bayesian incentive compatibility (OBIC) in the random assignment model. If the common prior of the agents is the *uniform prior*, then a large class of random mechanisms are OBIC with respect to this prior—this includes the probabilistic serial mechanism. We then introduce a robust version of OBIC: a mechanism is *locally robust OBIC* if it is OBIC with respect all independent and identical priors in some neighborhood of a given independent and identical prior. We show that every locally robust OBIC mechanism satisfying a mild property called *elementary monotonicity* is strategy-proof. This leads to a strengthening of the impossibility result in Bogomolnaia and Moulin (J Econ Theory 100:295–328, 2001): if there are at least four agents, there is no locally robust OBIC and ordinally efficient mechanism satisfying equal treatment of equals.

Keywords Ordinal Bayesian incentive compatibility · Random assignment · Probabilistic serial mechanism

JEL Classification D47 · D82

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1 Introduction

This paper explores the consequences of weakening incentive compatibility from strategy-proofness to *ordinal Bayesian incentive compatibility* in the random assignment model (one-sided matching model). Ordinal Bayesian incentive compatibility (OBIC) requires that the truth-telling *expected share vector* of an agent first-order stochastically dominates the expected share vector from reporting any other preference. It is the natural analogue of Bayesian incentive compatibility in an ordinal mechanism. This weakening of strategy-proofness was proposed by d'Aspremont and Peleg ([1988]). We study OBIC by considering mechanisms that allow for randomization in the assignment model.

In the random assignment model, the set of mechanisms satisfying ex-post efficiency and strategy-proofness is quite rich.[1] Despite satisfying such strong incentive properties, all of them either fail to satisfy *equal treatment of equals*, a weak notion of fairness, or *ordinal efficiency*.[2] Indeed, Bogomolnaia and Moulin ([2001]) propose a new mechanism, called the *probabilistic serial mechanism*, which satisfies equal treatment of equals and ordinal efficiency. However, they show that it fails strategy-proofness, and no mechanism can satisfy all these three properties simultaneously if there are at least four agents. A primary motivation for weakening the notion of incentive compatibility to OBIC is to investigate if we can escape this impossibility result.

We show two types of results. First, if the (common) prior is a uniform probability distribution over the set of possible preferences, then every *neutral* mechanism satisfying a mild property called *elementary monotonicity* is OBIC.[3] An example of such a mechanism is the probabilistic serial mechanism. This is a positive result and provides a strategic foundation for the probabilistic serial mechanism. In particular, it shows that there exist ordinally efficient mechanisms satisfying equal treatment of equals which are OBIC with respect to the uniform prior.

Second, we explore the implications of strengthening OBIC as follows. A mechanism is *locally robust OBIC* (LROBIC) with respect to an independent and identical prior if it is OBIC with respect to every independent and identical prior in its "neighborhood". The motivation for such requirement of robustness in the mechanism design literature is now well-known, and referred to as the Wilson doctrine (Wilson [1987]). We show that every LROBIC mechanism satisfying elementary monotonicity is strategy-proof. An immediate corollary of this result is that the probabilistic serial mechanism is not LROBIC (though it is OBIC with respect to the uniform prior). As a corollary, we can show that when there are at least four agents, there is no LROBIC and *ordinally efficient* mechanism satisfying equal treatment of equals. This strengthens the

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[1] Pycia and Ünver ([2017]) characterize the set of deterministic, strategy-proof, Pareto efficient, and non-bossy mechanisms in this model. This includes generalizations of the top-trading-cycle mechanism.

[2] For instance, Bogomolnaia and Moulin ([2001]) show that the (uniform) random priority mechanism is ex-post efficient, strategy-proof and satisfies equal treatment of equals, but fails ordinal efficiency (which is stronger than ex-post efficiency).

[3] Neutrality is a standard axiom in social choice theory which requires that objects are treated symmetrically. Elementary monotonicity is a monotonicity requirement of a mechanism. We define it formally in Sect. 4.

seminal impossibility result of Bogomolnaia and Moulin (2001) by replacing strategy-proofness with LROBIC.

Both our results point to very different implications of OBIC in the presence of elementary monotonicity—if the prior is uniform, this notion of incentive compatibility is very permissive; but if we require OBIC with respect to a set of independent and identical priors in any neighborhood of a given prior, this notion of incentive compatibility is very restrictive.

1.1 Related literature

There is a large literature on random assignment problems. We summarize them below.

The notion of incentive compatibility that we use, OBIC, has been used in voting models by Majumdar and Sen (2004); Bhargava et al. (2015); Mishra (2016); Hong and Kim (2018) to escape the dictatorship result in (Gibbard 1973; Satterthwaite 1975; Gibbard 1977). All these papers use deterministic mechanisms in voting models, whereas we apply OBIC to the random assignment model. Majumdar and Sen (2004) show that every deterministic neutral voting mechanism satisfying elementary monotonicity is OBIC with respect to uniform priors. Our Theorem 1 shows that this result generalizes to the random assignment model. Mishra (2016) shows that in the deterministic voting model, elementary monotonicity and OBIC with respect to "generic" prior is equivalent to strategy-proofness in a variety of restricted domains—see also Hong and Kim (2018) for a strengthening of this result. Though these results are similar to our Theorem 2, there are significant differences. First, we consider randomization while these results are only for deterministic mechanisms. Our notion of locally robust OBIC is stronger than OBIC with respect to generic priors used in these papers. Second, ours is a model of private good allocation (random assignment), while these papers deal with the voting model.

Bogomolnaia and Moulin (2001) introduce a family of mechanisms in the random assignment model. They call these the simultaneous eating algorithms. which generate *ordinally* efficient random assignments, a stronger notion of efficiency than ex-post efficiency. [4] The probabilistic serial mechanism belongs to this family and it is anonymous. However, it is not strategy-proof. In fact, Bogomolnaia and Moulin (2001) show that there is no ordinally efficient and strategy-proof mechanism satisfying equal treatment of equals when there are at least four agents.[5]

There is a large literature that provides strategic foundations to the probabilistic serial (PS) mechanism. Bogomolnaia and Moulin (2001) show that the PS mechanism satisfies *weak*-strategy-proofness. Their notion of weak strategy-proofness requires that the manipulation share vector *cannot* first-order-stochastic-dominate the truth-telling share vector. Bogomolnaia and Moulin (2002) study a problem where agents have an *outside option*. When agents have the same ordinal ranking over objects but the position of outside option in the ranking of objects is the only private information, they show that the PS mechanism is strategy-proof. Other contributions in this direction include Liu (2020); Liu and Zeng (2019), who identify domains where

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[4] Katta and Sethuraman (2006) extend the simultaneous eating algorithm to allow for ties in preferences.

[5] With three agent, the random priority mechanism satisfies these properties.

the probabilistic serial mechanism is strategy-proof. Che and Kojima (2010) show that the PS mechanism and the random priority mechanism (which is strategy-proof) are asymptotically equivalent. Similarly, Kojima and Manea (2010) show that when sufficiently many copies of an object are present, then the PS mechanism is strategy-proof. Thus, in large economies, the PS mechanism is strategy-proof. Balbuzanov (2016) introduce a notion of strategy-proofness which is stronger than weak strategy-proofness and show that the PS mechanism satisfies it. His notion of strategy-proofness is based on the "convex" domination of lotteries, and hence, called *convex strategy-proofness*. Mennle and Seuken (2021) define a notion called *partial strategy-proofness*, which is weaker than strategy-proofness and show that the PS mechanism satisfies it. They show that strategy-proofness is equivalent to upper invariant, lower invariant and elementary monotonicity (they call it swap monotonicity). Their notion of partial strategy-proofness is equivalent to upper invariance and elementary monotonicity, and hence, it is weaker than strategy-proofness.

The main difference between these weakenings of strategy-proofness and ours is that OBIC is a *prior-based* notion of incentive compatibility. It is the natural analogue of Bayesian incentive compatibility in an ordinal environment. Ehlers and Massó (2007) study OBIC in a *two-sided matching* problem. Their main focus is on OBIC mechanism that select a stable matching. They characterize the beliefs for which such a mechanism exists. There is a literature in computer science studying computational aspects of manipulation of the PS rule—see Aziz et al. (2014, 2015) and references therein.

2 Model

2.1 Assignments

There are n agents and n objects.[6] Let $N := \{1, \ldots, n\}$ be the set of agents and A be the set of objects. We define the notion of a feasible assignment first.

Definition 1 An $n \times n$ matrix L is an assignment if

$$L_{ia} \in [0, 1] \qquad \forall\, i \in N,\ \forall\, a \in A$$
$$\sum_{a \in A} L_{ia} = 1 \qquad \forall\, i \in N$$
$$\sum_{i \in N} L_{ia} = 1 \qquad \forall\, a \in A$$

Hence, an assignment is a bistochastic matrix. For any assignment L, we write L_i as the share vector of agent i.[7] Formally, a share vector is a probability distribution over the set of objects. For any $i \in N$ and any $a \in A$, L_{ia} denotes the "share" of agent i

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[6] All our results extend even if the number of objects is not the same as the number of agents. We assume this only to compare our results with the random assignment literature, where this assumption is common.

[7] Whenever we say an assignment, we mean a random assignment from now on.

of object a. The second constraint of the assignment definition requires that the total share of every agent is 1. The third constraint of the assignment requires that every object is completely assigned. Let $\mathcal{L}$ be the set of all assignments.

An assignment L is deterministic if $L_{ia} \in \{0, 1\}$ for all $i \in N$ and for all $a \in A$. Let $\mathcal{L}^d$ be the set of all deterministic assignments. By the Birkohff-von-Neumann theorem, for every $L \in \mathcal{L}$, there exists a set of deterministic assignments in $\mathcal{L}^d$ whose convex combination equals L.

2.2 Preferences

A preference is a strict ordering of A. The preference of an agent i will be denoted by P_i. The set of all preferences over A is denoted by $\mathcal{P}$. A preference profile is $\mathbf{P} \equiv (P_1, \ldots, P_n)$, and we will denote by P_{-i} the preference profile $\mathbf{P}$ excluding the preference P_i of agent i. We write $a P_i b$ to denote that a is strictly preferred over b in preference P_i.

2.3 Prior

We assume that the preference of each agent is independently and identically drawn using a common prior μ, which is a probability distribution over $\mathcal{P}$. From now on, whenever we say a prior, we refer to such an independent and identical prior. We will denote by $\mu(P_i)$ the probability with which agent i has preference P_i. We assume that $\mu(P_i) > 0$ for each P_i. With some abuse of notation, we will denote the probability with which agents in $N \setminus \{i\}$ have preference profile P_{-i} as $\mu(P_{-i})$. Note that by independence, $\mu(P_{-i}) = \times_{j \neq i} \mu(P_j)$.

3 Ordinal Bayesian incentive compatibility

Our solution concept is Bayes-Nash equilibrium but we restrict attention to ordinal mechanisms, i.e., mechanisms where we only elicit ranking over objects from each agent. Hence, whenever we say *mechanism*, we refer to such ordinal mechanisms.[8] Formally, a mechanism is a map $Q : \mathcal{P}^n \to \mathcal{L}$. A mechanism Q assigns a share vector $Q_i(\mathbf{P})$ to agent i at every preference profile $\mathbf{P}$.

Before discussing the notions of incentive compatibility, it is useful to think how agents compare share vectors in our model. Fix agent i with a preference P_i over the set of objects A. Denote the k-th ranked object in P_i as $P_i(k)$. Consider two share vectors π, π'. For every $a \in A$, we will denote by π_a and π'_a the share assigned to object a in π and π' respectively. We will say π first-order-stochastically-dominates (FOSD) π' according to P_i if

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[8] The restriction to not consider cardinal mechanisms is arguably arbitrary. It is usually done to simplify the process of elicitation. Such restriction is also consistent with the literature on random assignment models. The set of incentive compatible mechanisms expand if we consider cardinal mechanisms (Miralles 2012; Abebe et al. 2020).

$$\sum_{k=1}^{\ell} \pi_{P_i(k)} \geq \sum_{k=1}^{\ell} \pi'_{P_i(k)} \qquad \forall \, \ell \in \{1, \ldots, n\}.$$

In this case, we will write $\pi \succ_{P_i} \pi'$. Notice that $\succ_{P_i}$ is not a complete relation over the outcomes. An equivalent (and well known) definition of $\succ_{P_i}$ relation is that for *every* von-Neumann–Morgenstern utility representation of P_i, the expected utility from π is at least as much as π'.

The most standard notion of incentive compatibility is strategy-proofness (dominant strategy incentive compatibility), which uses the FOSD relation to compare share vectors.

Definition 2 A mechanism Q is strategy-proof if for every $i \in N$, every $P_{-i} \in \mathcal{P}^{n-1}$, and every $P_i, P'_i \in \mathcal{P}$, we have

$$Q_i(P_i, P_{-i}) \succ_{P_i} Q_i(P'_i, P_{-i}).$$

The interpretation of this definition is that fixing the preferences of other agents, the truth-telling share vector must FOSD other share vectors that can be obtained by deviation. This definition of strategy-proofness appeared in Gibbard (1977) for voting problems, and has been the standard notion in the literature on random voting and random assignment problems.

The ordinal Bayesian incentive compatibility notion is an adaptation of this by changing the solution concept to Bayes-Nash equilibrium. It was first introduced and studied in a voting committee model in d'Aspremont and Peleg (1988), and was later used in many voting models (Majumdar and Sen 2004). To define it formally, we introduce the notion of an interim share vector. Fix an agent i with preference P_i. Given a mechanism Q, the interim share of object a for agent i by reporting P'_i is:

$$q_{ia}(P'_i) = \sum_{P_{-i} \in \mathcal{P}^{n-1}} \mu(P_{-i}) Q_{ia}(P'_i, P_{-i}).$$

The interim share vector of agent i by reporting P'_i will be denoted as $q_i(P'_i)$.

Definition 3 A mechanism Q is ordinally Bayesian incentive compatible (OBIC) (with respect to prior μ) if for every $i \in N$ and every $P_i, P'_i \in \mathcal{P}$, we have

$$q_i(P_i) \succ_{P_i} q_i(P'_i).$$

It is immediate that if a mechanism Q is strategy-proof it is OBIC with respect to every (including correlated and non-identical) prior. Conversely, if a mechanism is OBIC with respect to *all* priors (including correlated and non-identical priors), then it is strategy-proof.

Table 1 Manipulation by agent 1

P_1	P_2	P_3	P_1'	P_2	P_3
$c; \frac{1}{2}$	$a; \frac{2}{3}$	$c; \frac{1}{2}$	$a; \frac{1}{2}$	$a; \frac{1}{2}$	$c; \frac{3}{4}$
$a; \frac{1}{6}$	$b; \frac{1}{3}$	$a; \frac{1}{6}$	$c; \frac{1}{4}$	$b; \frac{1}{2}$	$a; 0$
$b; \frac{1}{3}$	$c; 0$	$b; \frac{1}{3}$	$b; \frac{1}{4}$	$c; 0$	$b; \frac{1}{4}$

3.1 A motivating example

We investigate a simple example to understand the implications of strategy-proofness and OBIC for the probabilistic serial mechanism. Suppose $n = 3$ with three objects $\{a, b, c\}$. Consider the preference profiles (P_1, P_2, P_3) and (P_1', P_2, P_3) shown in Table 1—the table also shows the share vector of each agent in the probabilistic serial mechanism of Bogomolnaia and Moulin (2001). In the probabilistic serial mechanism, each agent starts "eating" her favorite object simultaneously till the object is finished. Then, she moves to the best available object according to her preference and so on. Each agent has the same eating speed. Table 1 shows the output of the probabilistic serial mechanism for preference profiles (P_1, P_2, P_3) and (P_1', P_2, P_3). Since $Q_{1a}(P_1', P_2, P_3) + Q_{1c}(P_1', P_2, P_3) > Q_{1a}(P_1, P_2, P_3) + Q_{1c}(P_1, P_2, P_3)$, we conclude that $Q_1(P_1, P_2, P_3) \nsucceq_{P_1'} Q_1(P_1', P_2, P_3)$. Hence, agent 1 can manipulate from P_1 to P_1', when agents 2 and 3 have preferences (P_2, P_3).

When can such a manipulation be prevented by OBIC? Note that P_1 is generated from P_1' by permuting a and c. Suppose we permute P_2 and P_3 also to get P_2' and P_3' respectively:

$$c \ P_2' \ b \ P_2' \ a \text{ and } a \ P_3' \ c \ P_3' \ b.$$

Since the probabilistic serial mechanism is *neutral* (with respect to objects), the share vector of agent 1 at (P_1, P_2, P_3) is a permutation of its share vector at (P_1', P_2', P_3'). Further, when all the preferences are equally likely, the probability of (P_2, P_3) is equal to the probability of (P_2', P_3'). So, the total expected probability of a and c for agent 1 at P_1 and P_1' is the same (where expectation is taken over (P_2, P_3) and (P_2', P_3')). As we show below, this argument generalizes and the expected share vector at P_1 first-order-stochastic-dominates the expected share vector at P_1' when the true preference is P_1 and prior is uniform.

4 Uniform prior and possibilities

In this section, we present our first result which shows that the set of OBIC mechanisms is much larger than the set of strategy-proof mechanisms if the prior is the uniform prior. A prior μ is the uniform prior if $\mu(P_i) = \frac{1}{|\mathcal{P}|} = \frac{1}{n!}$ for each $P_i \in \mathcal{P}$. Uniform prior puts equal probability on each of the possible preferences. We call a mechanism U-OBIC if it is OBIC with respect to the uniform prior.

We show that there is a large class of mechanisms which are U-OBIC - this will include some well-known mechanisms which are known to be not strategy-proof. This class is characterized by two axioms, neutrality and elementary monotonicity, which we define next. To define neutrality, consider any permutation $\sigma : A \to A$ of the set of objects. For every preference P_i, define P_i^σ as the preference that satisfies: aP_ib if and only if $\sigma(a)P_i^\sigma \sigma(b)$. Let $\mathbf{P}^\sigma$ be the preference profile generated by permuting each preference in the preference profile $\mathbf{P}$ by the permutation σ.

Definition 4 A mechanism Q is neutral if for every $\mathbf{P}$ and every permutation σ,

$$Q_{ia}(\mathbf{P}) = Q_{i\sigma(a)}(\mathbf{P}^\sigma) \qquad \forall i \in N, \ \forall a \in A.$$

Neutrality requires that objects be treated symmetrically by the mechanism. Any mechanism which does not use the "names" of the objects is neutral—this includes all priority mechanisms (including the random priority mechanism), the simultaneous eating algorithms (including the probabilistic serial mechanism) in Bogomolnaia and Moulin (2001).

Our next axiom is elementary monotonicity, an axiom which requires a mild form of monotonicity. This was introduced in Majumdar and Sen (2004). To define it, we need the notion of "adjacency" of preferences. We say preferences P_i and P_i' are adjacent if there exists a $k \in \{1, \ldots, n-1\}$ such that

$$P_i(k) = P_i'(k+1), \ P_i(k+1) = P_i'(k), \ \text{and} \ P_i(k') = P_i'(k') \ \forall k' \notin \{k, k+1\}.$$

In other words, P_i' is obtained by swapping consecutively ranked objects in P_i. Here, if $P_i(k) = a$ and $P_i(k+1) = b$, we say that P_i' is an (a,b)-swap of P_i.

Definition 5 A mechanism Q satisfies elementary monotonicity if for every $i \in N$, every $P_{-i} \in \mathcal{P}^{n-1}$, and every $P_i, P_i' \in \mathcal{P}$ such that P_i' is an (a,b)-swap of P_i for some a, b, we have

$$Q_{ib}(P_i', P_{-i}) \geq Q_{ib}(P_i, P_{-i}) \tag{1}$$
$$Q_{ia}(P_i', P_{-i}) \leq Q_{ia}(P_i, P_{-i}) \tag{2}$$

In other words, as agent i lifts alternative b in ranking by one position by swapping it with a (and keeping the ranking of every other object the same), elementary monotonicity requires that the share of object b should weakly increase for agent i, while share of object a should weakly decrease. A similar axiom called *swap monotonicity* is used in Mennle and Seuken (2021).[9]

It is not difficult to see that elementary monotonicity is a necessary condition for strategy-proofness—see Majumdar and Sen (2004). As we show later, elementary monotonicity is satisfied by a variety of mechanisms - including those which are not strategy-proof. However, every neutral mechanism satisfying elementary monotonicity is U-OBIC.

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[9] Swap monotonicity requires the following change in Definition 5: either $Q_i(P_i', P_{-i}) = Q_i(P_i, P_{-i})$ or Inequalities (1) and (2) hold with strict inequalities.

Theorem 1 *Every neutral mechanism satisfying elementary monotonicity is U-OBIC.*

Proof Fix a neutral mechanism Q satisfying elementary monotonicity. The proof goes in various steps.

Step 1 Pick an agent i and two preferences P_i and P_i'. Pick any $k \in \{1, \ldots, n\}$ and suppose $P_i(k) = a$ and $P_i'(k) = b$. We show that the interim shares of a and b are same for agent i in preferences P_i and P_i': $q_{ia}(P_i) = q_{ib}(P_i')$. This is a consequence of uniform prior and neutrality. To see this, let $P_i' = P_i^\sigma$ for some permutation σ of objects in A. Then, $b = \sigma(a)$ and hence, for every P_{-i}, we have

$$Q_{ia}(P_i, P_{-i}) = Q_{i\sigma(a)}(P_i^\sigma, P_{-i}^\sigma) = Q_{ib}(P_i', P_{-i}^\sigma).$$

Due to uniform prior and using the above expression,

$$
\begin{aligned}
q_{ia}(P_i) &= \frac{1}{(n!)^{n-1}} \sum_{P_{-i}} Q_{ia}(P_i, P_{-i}) = \frac{1}{(n!)^{n-1}} \sum_{P_{-i}} Q_{ib}(P_i', P_{-i}^\sigma) \\
&= \frac{1}{(n!)^{n-1}} \sum_{P_{-i}} Q_{ib}(P_i', P_{-i}) = q_{ib}(P_i'),
\end{aligned}
$$

where the third equality follows from the fact that $\{P_{-i} : P_{-i} \in \mathcal{P}^{n-1}\} = \{P_{-i}^\sigma : P_{-i} \in \mathcal{P}^{n-1}\}$.

In view of step 1, with some abuse of notation, we write q_{ik} to denote the interim share of the object at rank k in the preference. We call q_i the interim rank vector of agent i.

Step 2 Pick an agent i and a preference P_i. We show that interim shares are non-decreasing with rank: $q_{ik} \geq q_{i(k+1)}$ for all $k \in \{1, \ldots, n-1\}$. Fix a number k and let $P_i(k) = a$ and $P_i(k+1) = b$. Then, consider the preference P_i', which is an (a, b)-swap of P_i. For every P_{-i}, elementary monotonicity implies $Q_{ia}(P_i, P_{-i}) \geq Q_{ia}(P_i', P_{-i})$. Due to uniform prior, $q_{ia}(P_i) \geq q_{ia}(P_i')$. But by Step 1,

$$q_{ik} = q_{ia}(P_i) \geq q_{ia}(P_i') = q_{i(k+1)}.$$

Step 3 We show that Q is OBIC with respect to the uniform prior. Suppose agent i has preference P_i. By Steps 1 and 2, she gets interim rank vector $(q_{i1}, \ldots, q_{in})$ by reporting P_i with $q_{ij} \geq q_{ij+1}$ for all $j \in \{1, \ldots, n-1\}$. Suppose she reports $P_i' = P_i^\sigma$, where σ is some permutation of set of objects. By Steps 1 and 2, the interim share vector is a permutation of interim rank vector q_i. Using non-decreasingness of this interim share vector with respect to ranks, we get $q_i(P_i) \succ_{P_i} q_i(P_i')$. Hence, Q is OBIC with respect to uniform prior. □

Theorem 1 generalizes, an analogous result in Majumdar and Sen (2004), who consider the voting problem and *only* deterministic mechanisms. They arrive at the same conclusion as Theorem 1 in their model. Theorem 1 shows that their result holds even in the *random* assignment problem.

4.1 Probabilistic serial mechanism and U-OBIC

Bogomolnaia and Moulin (2001) define a family of mechanisms, which they call the simultaneous eating algorithms (SEA). Though the SEAs are not strategy-proof, they satisfy compelling efficiency and fairness properties, which we discuss in Sect. 5. We informally introduce the SEAs—for a formal discussion, see Bogomolnaia and Moulin (2001).

Each SEA is defined by a (possibly time-varying) *eating speed* function for each agent. At every preference profile, agents simultaneously start "eating" their favorite objects at a rate equal to their eating speed. Once an object is completely eaten (i.e., the entire share of 1 is consumed), the amount eaten by each agent is the share of that agent of that object. Once an object completely eaten, agents go to their next preferred object and so on.

If the eating speed of each agent is the same, then the simultaneous eating algorithm is *anonymous*. Bogomolnaia and Moulin (2001) call the unique anonymous SEA, the probabilistic serial mechanism. [10]

Corollary 1 *Every simultaneous eating algorithm is U-OBIC.*

Proof Clearly, the SEAs are neutral since eating speeds do not depend on the objects. The SEAs also satisfy elementary monotonicity: Theorem 3 in Cho (2018) and Theorem 1 in Mennle and Seuken (2021). Hence, by Theorem 1, we are done. ☐

5 Locally robust OBIC

While the uniform prior is an important prior in decision theory, it is natural to ask if Theorem 1 extends to other "generic" priors. Though we do not have a full answer to this question, we have been able to answer this question in negative under a natural robustness requirement. Our robustness requirement is *local*. Take any (independent and identical) prior μ, and let μ' be any (independent and identical) prior in the ϵ-radius ball around μ (where $\epsilon > 0$), i.e., $||\mu(P) - \mu'(P)|| < \epsilon$ for all $P \in \mathcal{P}$. In this case, we write $\mu' \in B_\epsilon(\mu)$. Our local robustness requirement is the following.

Definition 6 A mechanism Q is locally robust OBIC (LROBIC) with respect to a prior μ if there exists an $\epsilon > 0$ such that for every prior $\mu' \in B_\epsilon(\mu)$, Q is OBIC with respect to μ'.

It is well known that Bayesian incentive compatibility with respect to all priors lead to strategy-proofness (Ledyard 1978). Here, we require OBIC with respect to all independent and identical priors in the ϵ-neighborhood of an independent and identical prior. Bhargava et al. (2015) study a version of robust OBIC with respect to *uniform* prior but their robustness also allows the mechanism to be OBIC with respect to correlated priors. They show that a large class of voting rules satisfy their notion of LROBIC. We show that in the random assignment model, LROBIC with respect to *any* independent and identical prior has a very different implication.

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[10] For axiomatic characterization of the PS mechanism, see Bogomolnaia and Heo (2012) and Hashimoto et al. (2014).

Theorem 2 *A mechanism is LROBIC with respect to a prior and satisfies elementary monotonicity if and only if it is strategy-proof.*

Remark Though relatively strong, the notion of LROBIC is in the spirit of robust mechanism design (Bergemann and Morris 2005). It remains to be seen if Theorem 2 continues to hold under weaker notions of OBIC than LROBIC—for instance, OBIC with respect to *generic* priors as in Majumdar and Sen (2004).[11] It may be possible to construct a OBIC mechanism satisfying elementary monotonicity which is not strategy-proof if the prior is *approriately* chosen. Theorem 1 shows this possibility for uniform priors.

The proof of Theorem 2 builds on some earlier results. Before giving the proof, we define some notions and preliminary results. We first decompose OBIC into three conditions. This decomposition is similar to the decomposition of strategy-proofness in Mennle and Seuken (2021)—there are some minor differences in axioms and we look at interim share vectors whereas they look at ex-post share vectors.

Our decomposition of OBIC uses the following three axioms.

Definition 7 A mechanism Q satisfies interim elementary monotonicity if for every $i \in N$ and every P_i, P_i' such that P_i' is an (a, b)-swap of P_i, we have

$$q_{ib}(P_i') \geq q_{ib}(P_i)$$
$$q_{ia}(P_i') \leq q_{ia}(P_i).$$

Give a preference P_i of agent i and an object $a \in A$, define $U(a, P_i) := \{x \in A : x\, P_i\, a\}$ and $L(a, P_i) := \{x \in A : a\, P_i\, x\}$.

Definition 8 A mechanism Q satisfies interim upper invariance if for every $i \in N$ and every P_i, P_i' such that P_i' is an (a, b)-swap of P_i, and for every $x \in U(a, P_i)$, we have

$$q_{ix}(P_i') = q_{ix}(P_i).$$

Definition 9 A mechanism Q satisfies interim lower invariance if for every $i \in N$ and every P_i, P_i' such that P_i' is an (a, b)-swap of P_i, and for every $x \in L(b, P_i)$, we have

$$q_{ix}(P_i') = q_{ix}(P_i).$$

The following proposition characterizes OBIC using these axioms.

Proposition 1 *A mechanism Q is OBIC with respect to a prior if and only if it satisfies interim elementary monotonicity, interim upper invariance, and interim lower invariance.*

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[11] For three objects and three agents, we can show that the probabilistic serial mechanism cannot be OBIC with respect to a generic prior.

Since the proof of Proposition 1 is similar to the characterization of strategy-proofness in Mennle and Seuken (2021), we skip its proof.[12] The proof of Theorem 2 is based on Proposition 1 and the following lemma.

Lemma 1 *Suppose Q mechanism is LROBIC with respect to a prior. Then, for every i, for every P_{-i}, for every P_i and P_i' such that P_i' is an (a, b) swap of P_i, we have*

$$Q_{ic}(P_i, P_{-i}) = Q_{ic}(P_i', P_{-i}) \qquad \forall\, c \notin \{a, b\}.$$

Proof Pick an agent $i \in N$ and $P_i, P_i' \in \mathcal{P}$ such that P_i' is an (a, b)-swap of P_i. Fix some P_{-i}. By Proposition 1, Q satisfies interim upper invariance and interim lower invariance. Hence, we know that for all $c \notin \{a, b\}$, we get

$$\sum_{P_{-i}} \mu(P_{-i}) \Big[Q_{ic}(P_i, P_{-i}) - Q_{ic}(P_i', P_{-i}) \Big] = 0. \qquad (3)$$

Since μ is a probability distribution over $\mathcal{P}$, we can treat it as a vector in $\mathbb{R}^{n!-1}$. Using $\mu(P_{-i}) \equiv \times_{j \neq i} \mu(P_j)$, we note that the LHS of the Equation (3) is a polynomial function of $\{\mu(P)\}_{P \in \mathcal{P}}$. The equation describes the zero set of this polynomial function. For non-zero polynomials, the set of zeros has measure zero (Caron and Traynor 2005), i.e., the set of μ satisfying Equation (3) has measure zero.[13] Hence, given any prior μ^* and $\epsilon > 0$, if Equation 3 has to hold for *all* $\mu \in B_\epsilon(\mu^*)$ (which has non-zero measure), then $Q_{ic}(P_i, P_{-i}) = Q_{ic}(P_i', P_{-i})$ for all $c \notin \{a, b\}$.   □

We now complete the proof of Theorem 2.

Proof of Theorem 2

Proof: Every strategy-proof mechanism is OBIC with respect to any prior. A strategy-proof mechanism satisfies elementary monotonicity. So, we now focus on the other direction of the proof. Let Q be an LROBIC mechanism with respect to a prior μ. Suppose Q satisfies elementary monotonicity.

By Lemma 1, any LROBIC mechanism Q, Q satisfies *ex-post* versions of interim lower invariance and interim upper invariance. Mennle and Seuken (2021) refer to these properties as *upper invariance* and *lower invariance* (see also Cho 2018). They show that upper invariance, lower invariance, and elementary monotonicity are equivalent to strategy-proofness. By the assumption of the theorem, Q satisfies elementary monotonicity. Hence, it is strategy-proof.   □

We now explore the compatibility of LROBIC and ordinal efficiency.

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[12] As we discussed, swap monotonicity discussed in Mennle and Seuken (2021) is slightly different than elementary monotonicity (neither conditions imply the other), but in the presence of the other two axioms in Mennle and Seuken (2021) (upper invariance and lower invariance), they are equivalent.

[13] Note that we are not characterizing the set of μ for which (3) has a solution. Our claim is only about the measure of the set of solutions.

Definition 10 A mechansim Q is ordinally efficient if at every preference profile **P** there exists no assignment L such that

$$L_i \succ_{P_i} Q_i(\mathbf{P}) \qquad \forall\, i \in N,$$

with $L_i \neq Q_i(\mathbf{P})$ for some i.

Bogomolnaia and Moulin (2001) show that every ordinally efficient mechanism is ex-post efficient but the converse is not true if $n \geq 4$. In fact, for $n \geq 4$, strategy-proofness is incompatible with ordinally efficiency along with the following weak fairness criterion.

Definition 11 A mechanism Q satisfies equal treatment of equals if at every preference profile **P** and for every $i, j \in N$, we have

$$\left[P_i = P_j \right] \Rightarrow \left[Q_i(\mathbf{P}) = Q_j(\mathbf{P}) \right]$$

Due to Theorem 2, we can strengthen the impossibility results in Bogomolnaia and Moulin (2001) and Mennle and Seuken (2017) as follows.

Corollary 2 *Suppose $n \geq 4$. Then, there is no locally robust OBIC and ordinally efficient mechanism satisfying equal treatment of equals.*

Proof By Lemma 1, a locally robust OBIC mechanism satisfies *ex-post* versions of upper invariance and lower invariance. Mennle and Seuken (2017) show that the proof in Bogomolnaia and Moulin (2001) can be adapted by replacing strategy-proofness with ex-post versions of upper invariance and lower invariance. Hence, these two properties are incompatible with ordinal efficiency and equal treatment of equals for $n \geq 4$, and we are done. □

Note that Corollary 2 does not use elementary monotonicity, and hence, cannot be directly inferred from Theorem 2.

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