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ORIGINAL PAPER

Choice functions and weak Nash axioms

Susumu Cato[1]

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Abstract
The Nash axiom is a basic property of consistency in choice. This paper proposes weaker versions of the axiom and examines their logical implications. In particular, we demonstrate that weak Nash axioms are useful to understand the relationship between the Nash axiom and the path independence axiom. We provide an application of weak Nash axioms to the no-envy approach. We present a possibility result and an impossibility result.

Keywords Choice function · Rational behavior · Nash axiom · No-envy · Path-independence · Menu-dependent preference

JEL Classification D63 · D71 · D11

1 Introduction

Nash (1950) introduces the axiom of *independence of irrelevant alternatives* (IIA) in order to characterize his bargaining solution.[1] Collective decision problems are captured by feasible sets (bargaining set), A, B, C..., and a solution specifies a point over each feasible set.[2] Nash's axiom of IIA requires that if a solution for a bargaining set A is a member of a shrunk set B, then it must also be the solution of B. Since a solution can be regarded as a choice function, the Nash axiom can be reformulated as an axiom of choice consistency.

Several researchers have examined the Nash axiom in a general choice-function setting (Chernoff 1954; Suzumura 1983; Sakamoto 2013). As shown by them, the

[1] The Nash bargaining solution is characterized by IIA, Pareto optimality, the symmetry axiom, and a type of invariance. See Luce and Raiffa (1957). IIA is introduced by Chernoff (1954) (Postulate 5*) in another context.
[2] Standard bargaining problems contain thread points: each bargaining problem is specified by a pair of a feasible set and a thread point.

✉ Susumu Cato
susumu.cato@gmail.com

1 Institute of Social Science, The University of Tokyo, 7-3-1, Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

Nash axiom is implied by Plott's (1973) *path independence* as well as Arrow's (1959) axiom. A traditional approach to examine choice axioms is to make a decomposition of the axioms. Indeed, the Arrow axiom is decomposed into the Chernoff axiom (Sen's α) and dual Chernoff axiom (Sen's β); path independence is decomposed into weak path independence α and β (Ferejohn and Grether 1977). As pointed out by Sen (1977), choice axioms focus on contraction consistency or expansion consistency. The Chernoff axiom and weak path independence α (resp. dual Chernoff axiom and weak path independence α) are regarded as contraction consistency axioms (resp. expansion consistency axioms).

In this paper, we propose two weaker versions of the Nash axiom of choice consistency: the *weak Nash axiom* α and *weak Nash axiom* β. These two axioms constitute a decomposition of the Nash axiom: the two axioms are simultaneously satisfied if and only if the original Nash axiom is satisfied. As we will show, the weak Nash axiom α is intermediate in strength between weak path independence α and the stability axiom; the weak Nash axiom β is intermediate in strength between weak path independence β and the superset axiom. Therefore, the weak Nash axiom α is a contraction consistency axiom and the weak Nash axiom β is an expansion consistency axiom.

Employing the proposed axioms, we provide two characterizations of the Nash axiom. Nash axiom is satisfied if and only if weak Nash axiom α and superset axiom; Nash axiom is satisfied if and only if weak Nash axiom β and stability axiom. We compare our characterizations with a characterization of path independence by Blair et al. (1976).

We also discuss the implications of two weak Nash axioms in the context of the *revealed preference* approach. The weak Nash axiom α is necessary for an act of maximization with a menu-independent preference, while weak Nash axiom β is not. The weak Nash axiom β is a necessary condition for rationalization by a menu-dependent quasi-transitive preference. Thus, the weak Nash axiom β can be regarded as a natural weakening of the dual Chernoff axiom, which is necessary and sufficient for rationalization by a menu-dependent transitive preference (Tyson 2008).

Subsequently, we provide an application of our two axioms: we consider the no-envy approach, which is developed by Suzumura (1981a, b). This approach connects the problem of fairness with the framework of extended sympathy, which is introduced by Suppes (1966) and Sen (1970). A collective choice rule specifies desirable social states for each profile of "extended" preferences. We examine the existence of a collective choice rule that chooses efficient and equitable states and satisfies weak Nash axioms. The weak Nash axiom α yields a possibility result, while the weak Nash axiom β yields an impossibility result. Then, we discuss how our results can be related to a possibility or an impossibility of menu-dependent social choice with fairness concerns.

The rest of this paper is organized as follows. Section 2 introduces the basic axioms of choice. Section 3 presents our main results. Section 4 provides an application of our choice axioms to the no-envy approach. Section 5 concludes the paper.

2 Preliminaries

Let X be a finite set of alternatives. Let $\mathcal{X}$ be a set of subsets of X, excluding the empty set. A choice function $C : \mathcal{X} \to \mathcal{X}$ is a mapping such that $C(A) \subseteq A$ for all $A \in \mathcal{X}$.

The following axiom is introduced by Arrow (1959).

Arrow axiom (AA) For all $A, B \in \mathcal{X}$,

$$A \subseteq B \Rightarrow \left[A \cap C(B) = \emptyset \text{ or } A \cap C(B) = C(A)\right].$$

It is known that C is rationalizable by an ordering if and only if it satisfies AA (Arrow 1959; Sen 1971). Thus, the axiom is equivalent to the weak axiom of revealed preference under our formulation (Arrow 1959).

There are two axioms, the Chernoff axiom and dual Chernoff axiom, which together imply AA.[3]

Chernoff axiom (CA) For all $A, B \in \mathcal{X}$,

$$A \subseteq B \Rightarrow \left[A \cap C(B) = \emptyset \text{ or } A \cap C(B) \subseteq C(A)\right].$$

Dual Chernoff axiom (DCA) For all $A, B \in \mathcal{X}$,

$$A \subseteq B \Rightarrow \left[A \cap C(B) = \emptyset \text{ or } A \cap C(B) \supseteq C(A)\right].$$

A weak version of DCA is formulated as follows.[4]

Weak dual Chernoff axiom (WDCA) For all $K \in \mathbb{N}$ and $A_k \in \mathcal{X}$ $(k = 1, \ldots, K)$,

$$\bigcap_{k=1}^{K} C(A_k) \subseteq C\left(\bigcup_{k=1}^{K} A_k\right).$$

As shown by Sen (1971, 1997), C satisfies WDCA and CA if and only if C has a menu-independent rationalization.

The following axiom is introduced by Nash (1950) in order to characterize the Nash bargaining solution.

Nash axiom (NA) For all $A, B \in \mathcal{X}$,

$$[A \subseteq B \text{ and } C(B) \subseteq A] \Rightarrow C(A) = C(B).$$

According to NA, if each chosen alternative under menu B is still chosen under a shrunk menu A, then the choice under A is the same as that under B. Luce and Raiffa (1957) call this axiom *Independence of Irrelevant Alternatives*. AA implies NA, but not vice versa.

[3] The Chernoff axiom and dual Chernoff axiom are also called Sen's α and Sen's β. See Chernoff (1954, Postulate 4), Sen (1970), and Suzumura (1983).

[4] This axiom is also called Sen's γ (Sen 1971) or Sen's τ (Sen 1997).

Subsequently, we introduce various axioms of choice coherency, which are related to the two axioms.

Stability axiom (SA) For all $A \in \mathcal{X}$,

$$C(C(A)) = C(A).$$

Superset axiom (SUA) For all $A, B \in \mathcal{X}$,

$$[A \subseteq B \text{ and } C(B) \subseteq C(A)] \Rightarrow C(A) = C(B).$$

If NA is satisfied, then both the stability and superset axioms are satisfied (Suzumura 1983).

Following the argument of Arrow (1963), Plott (1973) introduces the following axiom.

Path independence (PI) For all $A, B \in \mathcal{X}$,

$$C(A \cup B) = C(C(A) \cup B).$$

Path independence implies NA, but not vice versa (Sakamoto 2013).

Ferejohn and Grether (1977) focus on two weak path independence conditions.

Weak path independence α ($WPI\alpha$) For all $A, B \in \mathcal{X}$,

$$C(A \cup B) \subseteq C(C(A) \cup B).$$

Weak path independence β ($WPI\beta$) For all $A, B \in \mathcal{X}$,

$$C(A \cup B) \supseteq C(C(A) \cup B).$$

It is clear that C satisfies path independence if and only if it satisfies both weak path independence α and β.

The logical implications of the above axioms are examined by researchers (Parks 1976; Ferejohn and Grether 1977; Suzumura 1983). We summarize their observations as follows.

Remark (i) Weak path independence α implies the stability axiom, but not vice versa.
(ii) Weak path independence β implies the superset axiom, but not vice versa.
(iii) Weak path independence α is equivalent to the Chernoff axiom.

3 Weak Nash axioms

3.1 The logical relationship with other choice axioms

Now, we introduce weak versions of NA.

Weak Nash axiom α ($WNA\alpha$) For all $A, B \in \mathcal{X}$,

$$[A \subseteq B \text{ and } C(B) \subseteq A] \Rightarrow C(A) \supseteq C(B).$$

The weak Nash axiom α requires that if each chosen alternative under menu B is still available under a shrunk menu A, then an alternative chosen under B is also chosen under A. More intuitively, if all world champions are Japanese, then they are champions in Japan, too.

Weak Nash axiom β (*WNA*β) For all $A, B \in \mathcal{X}$,

$$[A \subseteq B \text{ and } C(B) \subseteq A] \Rightarrow C(A) \subseteq C(B).$$

According to this, if each chosen alternative under menu B is still available under a shrunk menu A, then an alternative chosen under A is also chosen under B. More intuitively, if all world champions are Japanese, then all of champions in Japan are also world champions. The weak Nash axiom β is equivalent to the condition examined by Panda (1983) under the name of δ^*.[5] Panda (1983) introduces the axiom as a strong version of Sen's δ, defined as follows:[6]

Sen's δ: For all $A, B \in \mathcal{X}$,

$$[A \subseteq B \text{ and } \{x, y\} \subseteq C(A)] \Rightarrow \{x\} \neq C(B).$$

Note that Sen's δ is weaker than the superset axiom.

By definition, C satisfies NA if and only if it satisfies the weak Nash axioms α and β. Thus, if path independence is satisfied, then the weak Nash axioms α and β are also satisfied.

We first examine the logical relationship among the weak path independence axioms and the weak Nash axioms.

Proposition 1 (i) *If C satisfies weak path independence α, then it satisfies the weak Nash axiom α.*
(ii) *If C satisfies weak path independence β, then it satisfies the weak Nash axiom β.*

Proof (i) Let C be a choice function that satisfies weak path independence α. Suppose that $A \subseteq B$ and $C(B) \subseteq A$. From weak path independence α, it follows that

$$C(A \cup B) \subseteq C(A \cup C(B)).$$

Since $A \subseteq B$, we have $C(A \cup B) = C(B)$. Since $C(B) \subseteq A$, we have $C(A \cup C(B)) = C(A)$. Thus, $C(B) \subseteq C(A)$. Thus, the weak Nash axiom α is satisfied.

(ii) Let C be a choice function that satisfies weak path independence β. Suppose that $A \subseteq B$ and $C(B) \subseteq A$. From weak path independence β, it follows that

$$C(A \cup B) \supseteq C(A \cup C(B)).$$

Note that $A \cup B = B$ and $C(B) \cup A = A$. Thus, $C(B) \supseteq C(A)$. Then, the weak Nash axiom β is satisfied. □

[5] The formal definition of δ^* is as follows: $[A \subseteq B \text{ and } C(B) \subseteq A] \Rightarrow (A \setminus C(B)) \cap C(A) = \emptyset$. See Panda (1983, p.74).

[6] See Sen (1971).

The following example implies that the converse of Proposition 1(i) is not true.

Example 1 Assume that $X = \{x, y, z\}$. Let $S_1 = X$, $S_2 = \{x, y\}$, $S_3 = \{y, z\}$, $S_4 = \{x, z\}$, $S_5 = \{x\}$, $S_6 = \{y\}$, and $S_7 = \{z\}$. Consider the following choice function:

$$C(S_1) = \{x, y\}, C(S_2) = \{x, y\}, C(S_3) = \{y\},$$
$$C(S_4) = \{z\}, C(S_5) = \{x\}, C(S_6) = \{y\}, C(S_7) = \{z\}.$$

This choice function satisfies the weak Nash axiom α; weak path independence α is violated because $S_1 = S_4 \cup S_6$ but

$$\{x, y\} = C(S_1) \not\subseteq C(C(S_4) \cup S_6) = \{y\}.$$

□

The following example implies that the converse of Proposition 1(ii) is not true.

Example 2 Assume that $X = \{x, y, z\}$. Let $S_1 = X$, $S_2 = \{x, y\}$, $S_3 = \{y, z\}$, $S_4 = \{x, z\}$, $S_5 = \{x\}$, $S_6 = \{y\}$, and $S_7 = \{z\}$. Consider the following choice function:

$$C(S_1) = \{x, y\}, C(S_2) = \{x\}, C(S_3) = \{y, z\},$$
$$C(S_4) = \{x, z\}, C(S_5) = \{x\}, C(S_6) = \{y\}, C(S_7) = \{z\}.$$

This choice function satisfies the weak Nash axiom β; weak path independence β is violated because $S_1 = S_2 \cup S_7$ but

$$\{x, y\} = C(S_1) \not\supseteq C(C(S_2) \cup S_7) = \{x, z\}.$$

□

Since weak path independence α is equivalent to the Chernoff axiom, we have the following proposition as a corollary to Proposition 1(i).

Corollary 1 *If C satisfies the Chernoff axiom, then it satisfies the weak Nash axiom α. The converse is not true.*

Next, we examine the logical relationship among the weak Nash axioms, stability axiom, and superset axiom.

Proposition 2 (i) *If C satisfies the weak Nash axiom α, then it satisfies the stability axiom.*
(ii) *If C satisfies the weak Nash axiom β, then it satisfies the superset axiom.*

Proof (i) Let C be a choice function that satisfies the weak Nash axiom α. Fix $A \subseteq X$. Let $B = C(A)$. Note that

$$B \subseteq A \text{ and } C(A) \subseteq B.$$

The weak Nash axiom α implies that $C(A) \subseteq C(B)$. Since $B = C(A)$, it follows that

$$C(A) \subseteq C(C(A)).$$

By the definition of C, $C(A) \supseteq C(C(A))$, and thus, $C(A) = C(C(A))$. Therefore, the stability axiom is satisfied.

(ii) Let C be a choice function that satisfies weak Nash axiom β. Suppose that $A \subseteq B$ and $C(B) \subseteq C(A)$. Since $C(B) \subseteq A$, the weak Nash axiom β implies that $C(A) \subseteq C(B)$. Thus, $C(B) = C(A)$. Therefore, the superset axiom is satisfied. □

The following example implies that the converse of Proposition 2(i) is not true.

Example 3 Assume that $X = \{x, y, z\}$. Let $S_1 = X$, $S_2 = \{x, y\}$, $S_3 = \{y, z\}$, $S_4 = \{x, z\}$, $S_5 = \{x\}$, $S_6 = \{y\}$, and $S_7 = \{z\}$. Consider the following choice function:

$$C(S_1) = \{z\}, C(S_2) = \{x\}, C(S_3) = \{y\},$$
$$C(S_4) = \{z\}, C(S_5) = \{x\}, C(S_6) = \{x\}, C(S_7) = \{y\}.$$

This choice function satisfies the stability axiom; the weak Nash axiom α is violated because

$$S_3 \subseteq S_1, C(S_1) \subseteq S_3, \text{ and } C(S_1) \nsubseteq C(S_3).$$

□

The following example implies that the converse of Proposition 2(ii) is not true.

Example 4 Assume that $X = \{x, y, z\}$. Let $S_1 = X$, $S_2 = \{x, y\}$, $S_3 = \{y, z\}$, $S_4 = \{x, z\}$, $S_5 = \{x\}$, $S_6 = \{y\}$, and $S_7 = \{z\}$. Consider the following choice function:

$$C(S_1) = \{x\}, C(S_2) = \{y\}, C(S_3) = \{y\},$$
$$C(S_4) = \{x\}, C(S_5) = \{x\}, C(S_6) = \{y\}, C(S_7) = \{z\}.$$

This choice function satisfies the superset axiom; the weak Nash axiom β is violated because

$$S_2 \subseteq S_1, C(S_1) \subseteq S_2, \text{ and } C(S_2) \nsubseteq C(S_1).$$

□

From Propositions 1 and 2, (i) the weak Nash axiom α is logically in between weak path independence α and the stability axiom; (ii) the weak Nash axiom β is logically in between weak path independence β and the superset axiom. Figure 1 summarizes Propositions 1 and 2 (we combine them with the results obtained by Plott 1973; Ferejohn and Grether 1977; Sen 1977; Suzumura 1983; Sakamoto 2013).

We investigate the logical relationship among Nash and weak Nash axioms.

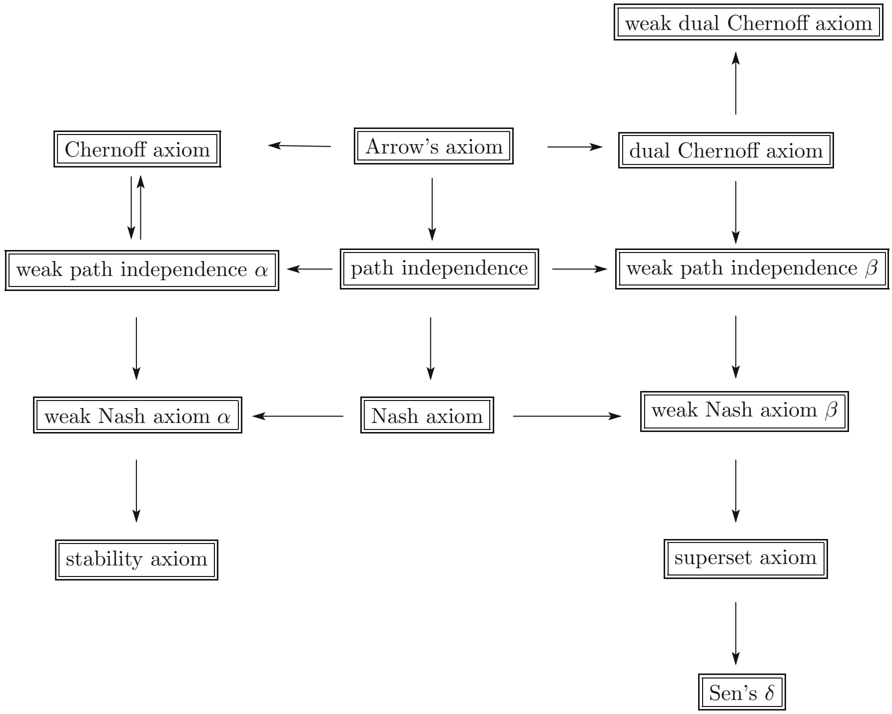

Fig. 1 "The logical relationship among choice axioms"

Proposition 3 *C satisfies the Nash axiom if and only if it satisfies the weak Nash axiom α and the superset axiom.*

Proof "If." Let C be a choice function that satisfies the weak Nash axiom α and the superset axiom. Suppose that $A \subseteq B$ and $C(B) \subseteq A$. The weak Nash axiom α implies

$$C(B) \subseteq C(A).$$

Thus, $A \subseteq B$ and $C(B) \subseteq C(A)$. From the superset axiom, $C(B) = C(A)$.

"Only if." By definition, the Nash axiom implies the weak Nash axiom α. It also implies the superset axiom (Suzumura 1983, Theorem 2.12. (b)). The proof is complete. □

Proposition 4 *C satisfies Nash axiom if and only if it satisfies weak Nash axiom β and stability axiom.*

Proof "If." Let C be a choice function that satisfies the weak Nash axiom β and the stability axiom. From Proposition 2, C satisfies the superset axiom. Suppose that $A \subseteq B$ and $C(B) \subseteq A$. Weak Nash axiom β implies

$$C(A) \subseteq C(B).$$

From stability axiom, $C(C(B)) = C(B)$. From superset axiom, we have the following:

$$[C(B) \subseteq A \text{ and } C(A) \subseteq C(C(B))] \Rightarrow C(A) = C(C(B)).$$

Thus, $C(A) = C(B)$.

"Only if." By definition, Nash axiom implies weak Nash axiom β. It also implies stability axiom (Suzumura 1983, Theorem 2.12. (c)). The proof is complete. □

Propositions 3 and 4 characterize Nash axioms.

Blair et al. (1976) show that C satisfies path independence if and only if it satisfies Chernoff axiom and superset axiom. Since Chernoff axiom is equivalent to weak path independence α, we paraphrase as follows.

Proposition 5 (Blair et al. 1976) *C satisfies path independence if and only if it satisfies weak path independence α and superset axiom.*

The following corollary immediately follows from Propositions 2 and 5.

Corollary 2 *C satisfies path independence if and only if it satisfies weak path independence α and weak Nash axiom β.*

Proposition 5 and our result lead us to the following question: do weak path independence β and weak Nash axiom α (or stability axiom) together imply path independence? The following example demonstrates that the answer is negative.

Example 5 Assume that $X = \{x, y, z\}$. Let $S_1 = X$, $S_2 = \{x, y\}$, $S_3 = \{y, z\}$, $S_4 = \{x, z\}$, $S_5 = \{x\}$, $S_6 = \{y\}$, and $S_7 = \{z\}$. Consider the following choice function:

$$C(S_1) = X, C(S_2) = \{x\}, C(S_3) = \{y\},$$
$$C(S_4) = \{z\}, C(S_5) = \{x\}, C(S_6) = \{y\}, C(S_7) = \{z\}.$$

This choice function satisfies weak Nash axiom α and weak path independence β; path independence is violated. □

3.2 Revealed preference and choice behavior

In this section, we consider the implication of our axioms in the revealed preference theory. A choice function represents observable data of behavior. If a decision maker is "rational," there must be some reasoning underlying the choice function. A common assumption in economics is the act of maximization: a choice function is a consequence of maximization with respect to some preference. A preference is not directly observable, but it can be revealed by the choice function. A standard approach is as follows: if x is chosen in the presence of y under some opportunity, then x is at least as good as y.

The analysis of revealed preferences in our framework has a long history. A path-breaking work is provided by Arrow (1959) and Sen (1971). The basic notion of revealed preference is *rationalizability*: C is said to be rationalizable if there exists a preference $\succsim$ on X such that for all $A \in \mathcal{X}$, $x \in C(A)$ if and only if $x \succsim y$ for $y \in A$. A rationalizable choice is regarded as if the decision maker engages in maximization given a preference over the set of alternatives.

A fundamental assumption of rationalizability is the universality of preferences: the decision maker has a preference in advance of choice, and thus, he/she does not change the preference depending on the choice circumstance. This implies that the decision maker has the same preference whatever the circumstance around him/her. C satisfies weak Nash axiom α if C is rationalized by some preference $\succsim$. This implies that weak Nash axiom α is a necessary condition for the standard single-preference rationalization.[7] On the other hand, weak Nash axiom β is not necessary for rationalization. Consider $X = \{x, y, z\}$ with the following preference:

$$\succsim^* = \{(x, y), (y, z), (x, z), (z, x), (x, x), (y, y), (z, z)\}.$$

It can generate a choice function. However, the choice function rationalized by $\succsim^*$ does not satisfy weak Nash axiom β. To check this, we need only to consider the following:

$$\{x, z\} = C(\{x, z\}) \text{ and } \{x\} = C(\{x, y, z\}).$$

A possibility shown by this counterexample is that weak Nash axiom β cannot be captured by a single-preference framework.

Now, we consider *menu-dependent preferences*. As described above, a universal preference is invariant with respect to menus or opportunities which are faced by the individual. In an actual situation, preferences sometimes are affected by menus. Such preferences are called menu-dependent: a preference is menu-dependent if it is dependent on menus A. A menu-dependent preference can be expressed by a preference over menu A, $\succsim_A$. Then, it is easy to see that a preference is menu-independent if, for all $A, B \subseteq X$

$$x, y \in A, B \Rightarrow x \succsim_A y \Leftrightarrow x \succsim_B y.$$

The importance of menu-dependent preferences has been widely recognized because many choice behaviors violates maximization with a single preference (Sen 1993, 1997). A menu (an opportunity set) describes a circumstance faced by a decision maker. If the decision maker cares about contexts, social norms, information from a menu, or the cost of choice, then the preference depends on the menu. Such concerns are important in working places or social lives (Sen 1994).

Tyson (2008) introduces the concept of a preference system to examine "satisficing." The concept is a fundamental tool to analyze menu-dependent preferences. Let $\succsim_A$ denote a binary relation on $A \subseteq X$. A *system of menu-dependent preferences*, say, $\langle \succsim_A \rangle_{A \in \mathcal{X}}$ on X, is a collection of menu-dependent preferences. $\langle \succsim_A \rangle_{A \in \mathcal{X}}$ is called *nested* if $\forall A, B \in \mathcal{X}$,

$$A \subseteq B \Rightarrow \succsim_A \subseteq \succsim_B .$$

7 Necessary and sufficient conditions are found in Sen (1993, 1997).

If x is preferred to y in menu A, then x is preferred to y in a menu with more options.

Tyson (2008) provides the following characterization result on menu-dependent preferences.

Proposition 6 (Tyson 2008)

(i) *C is rationalized by a nested system of orderings if and only if it satisfies dual Chernoff axiom.*
(ii) *C is rationalized by a nested system of complete and quasi-transitive binary relations if and only if it satisfies weak Nash axiom β and weak dual Chernoff axiom.*
(iii) *C is rationalized by a nested system of complete and acyclic binary relations if and only if it satisfies weak dual Chernoff axiom.*

For our analysis, the second result is especially important. It implies that weak Nash axiom β is a necessary condition for the existence of a quasi-transitive rationalization with a nested system of menu-dependent preferences. A comparison with a single-preference framework is useful for our purpose. In the single-preference framework, the Chernoff axiom and weak dual Chernoff axiom are necessary and sufficient for rationalizability with a preference ordering (Arrow 1959; Sen 1971). On the other hand, the dual Chernoff axiom is necessary and sufficient for rationalizability with menu-dependent preferences.

In the single-preference framework, path independence is a necessary condition for rationalization with a quasi-transitive preference. Indeed, path independence and the weak dual Chernoff axiom are satisfied if and only if there is a quasi-transitive rationalization (Plott 1973; Suzumura 1983). Since the superset axiom is implied by path independence, it is also a necessary condition for quasi-transitive rationalization (Weymark 1983). To obtain menu-dependent rationalizations, we can weaken path independence to the weak Nash axiom β.

For the single-preference case, the Chernoff axiom and weak dual Chernoff axiom are necessary and sufficient for rationalizability with an acyclic preference. If we allow menu-dependence with a nested system, we can get a rationalization when the Chernoff axiom is dropped: weak dual Chernoff axiom is necessary and sufficient for rationalizability with acyclic menu-dependent preferences.

4 An application: no-envy approach and extended sympathy

In this section, we provide an application of the weak Nash axioms to collective decision making. We consider the no-envy approach, developed by Suzumura (1981a, b, 1983). The concept of no-envy (envy-freeness) is introduced by Foley (1967) and is examined by many researchers in models of exchange economies (Varian 1974, 1976; Thomson 1982, 1988, 1999). An efficient and envy-free allocation is said to be *fair* (see Varian 1974, 1976). Suzumura (1981a, b, 1983) incorporates the concept of no-envy into social choice theory by employing the framework of extended sympathy of Suppes (1966) and Sen (1970).

Suzumura (1981a, b, 1983) and Sakamoto (2013) clarify the fundamental trade-off between fairness and social rationality. They prove that collective decision making regarding “fairness” cannot be rational in the sense that a social choice function might violate the axioms of choice consistency, such as superset axiom and Chernoff axiom.

Let N be a finite set of individuals. Each individual $i \in N$ has an *extended preference* $\succsim_i$ on $X \times N$. Then, $\succsim_i$ can be interpreted as the subjective preference of individual $i \in N$. That is, $(x, j) \succsim_i (y, k)$ means that “the position of individual j at state x is at least good as that of k at y for individual i”. A preference profile $\succsim_N = (\succsim_i)_{i \in N}$ is a list of individual preferences. The profile $\succsim_N$ satisfies the *axiom of identity* if and only if, for all $x, y \in X$, and for all $i, j \in N$,

$$(x, j) \succsim_i (y, j) \Leftrightarrow (x, j) \succsim_j (y, j).$$

Let Ψ be the set of preference profiles that satisfies the axiom of identity.[8]

An *extended collective choice rule (ECCR)* is a mapping $C_e : \mathcal{X} \times \Psi \to \mathcal{X}$ such that $C_e(A, \succsim_N) \subseteq A$ for all $A \in \mathcal{X}$ and for all $\succsim_N \in \Psi$. Then, given $\succsim_N \in \Psi$, $C_e(\cdot, \succsim_N)$ must be a choice function.

The set of *envy-free* states in A is defined as follows:

$$\mathcal{E}(A, \succsim_N) = \{x \in A : (x, i) \succsim_i (x, j) \text{ for all } i, j \in N\}.$$

The *Pareto* set in A is defined as follows:

$$\mathcal{P}(A, \succsim_N) = \{x \in A : \nexists y \in A \text{ such that } (y, i) \succ_i (x, i) \text{ for all } i \in N\}.$$

The *fair* set in A is defined as the intersection of $\mathcal{E}(A, \succsim_N)$ and $\mathcal{P}(A, \succsim_N)$:

$$\mathcal{F}(A, \succsim_N) = \mathcal{E}(A, \succsim_N) \cap \mathcal{P}(A, \succsim_N).$$

Then, a social state is fair if and only if it is Pareto efficient and envy-free.

Before introducing conditions for C_e, we present some properties of $\mathcal{E}$ and $\mathcal{P}$.

Remark 1 (i) $\mathcal{E}(A, \succsim_N) = A \cap \mathcal{E}(X, \succsim_N)$; (ii) If $A \subseteq B$ and $\mathcal{P}(B, \succsim_N) \subseteq A$, then $\mathcal{P}(B, \succsim_N) = \mathcal{P}(A, \succsim_N)$.

Proof (i) Note that

$$\begin{aligned}\mathcal{E}(A, \succsim_N) &= \{x \in A : (x, i) \succsim_i (x, j) \text{ for all } i, j \in N\} \\ &= \{x \in X : (x, i) \succsim_i (x, j) \text{ for all } i, j \in N\} \cap A.\end{aligned}$$

Claim (i) follows.

(ii) Suppose that $A \subseteq B$ and $\mathcal{P}(B, \succsim_N) \subseteq A$. Let $x \in \mathcal{P}(B, \succsim_N)$. There exists no $y \in B$ such that $(y, i) \succ_i (x, i)$ for all $i \in N$. Since $A \subseteq B$, there exists no $y \in A$ such that $(y, i) \succ_i (x, i)$ for all $i \in N$. Since $\mathcal{P}(B, \succsim_N) \subseteq A$, we have $x \in A$. Thus, $x \in \mathcal{P}(A, \succsim_N)$.

Let $x \in \mathcal{P}(A, \succsim_N)$. There exists no $y \in A$ such that $(y, i) \succ_i (x, i)$ for all $i \in N$. Suppose that there exists $y_0 \in B$ such that $(y_0, i) \succ_i (x, i)$ for all $i \in N$. Then,

[8] The axiom of identity is introduced by Sen (1970).

$y_0 \in B \backslash A$. Since $\mathcal{P}(B, \succsim_N) \subseteq A$, $y_0 \notin \mathcal{P}(B, \succsim_N)$, and thus, there exists $y_1 \in B$ such that $(y_1, i) \succ_i (y_0, i)$ for all $i \in N$. Then, $(y_1, i) \succ_i (x, i)$ for all $i \in N$. If $y_1 \in B \setminus A$, there exists $y_2 \in B$ such that $(y_2, i) \succ_i (x, i)$ for all $i \in N$. The repeated application of the above argument implies that there exists $y^* \in A$ such that $(y^*, i) \succ_i (x, i)$ for all $i \in N$. This is a contradiction. □

Remark 1(i) implies that $\mathcal{E}(A, \succsim_N) = \bigcup_{x \in A} \mathcal{E}(\{x\}, \succsim_N)$. Thus, an envy-free state is not affected by what states are available. Such a property does not hold for $\mathcal{P}$. Whether or not a state is Pareto efficient is crucially dependent on opportunity sets. In most cases, it is true that $\mathcal{P}(A, \succsim_N) \neq A \cap \mathcal{P}(X, \succsim_N)$. Remark 1(ii) implies that $\mathcal{P}$ satisfies Nash axiom.

The following condition is introduced by Suzumura (1981a, b, 1983).

Fairness extension $C_e(A, \succsim_N) = \mathcal{F}(A, \succsim_N)$ if $\mathcal{F}(A, \succsim_N) \neq \emptyset$.

Suzumura (1981a) shows that fairness extension is incompatible with superset axiom.

Denicolo (1999) introduces a weak version of fairness extension.

Weak fairness extension $C_e(A, \succsim_N) \subseteq \mathcal{F}(A, \succsim_N)$ if $\mathcal{F}(A, \succsim_N) \neq \emptyset$.

Since Denicolo (1999) considers a decision problem with a fixed agenda, he does not examine coherency of social choice. Sakamoto (2013) closely examines what kind of rationality axioms are compatible with the weak fairness extension.

Now, we present our results. A possibility result is obtained under weak Nash axiom α, while an impossibility result is obtained under weak Nash axiom β.

Proposition 7 *Suppose that X contains at least four alternatives.*

(i) *There exists an ECCR C_e that satisfies fairness extension and weak Nash axiom α.*
(ii) *There exists no ECCR C_e that satisfies weak fairness extension and weak Nash axiom β.*

Proof (i) Define C_e^* as follows:

$$C_e^*(A, \succsim_N) = \begin{cases} \mathcal{F}(A, \succsim_N) \text{ if } \mathcal{F}(A, \succsim_N) \neq \emptyset, \\ A \text{ otherwise.} \end{cases}$$

It is obvious that C_e^* satisfies fairness extension. Thus, it suffices to show that C_e^* satisfies weak Nash axiom α. Let $A, B \in \mathcal{X}$ and $\succsim_N \in \Psi$ such that

$$A \subseteq B \text{ and } C_e^*(B, \succsim_N) \subseteq A.$$

We focus on the case where $A \neq B$. By the definition of C_e^*, either $C_e^*(B, \succsim_N) = \mathcal{F}(B, \succsim_N)$ or $C_e^*(B, \succsim_N) = B$ must be true. In the latter case, we have $C_e^*(B, \succsim_N) = B \subseteq A$, which contradicts our assumption that $A \neq B$. In the former case, $C_e^*(B, \succsim_N) = \mathcal{F}(B, \succsim_N) \subseteq A$. Note that $C_e^*(A, \succsim_N) = \mathcal{F}(A, \succsim_N)$ or $C_e^*(A, \succsim_N) = A$ must be true. If $C_e^*(A, \succsim_N) = A$, then $C_e^*(B, \succsim_N) \subseteq C_e^*(A, \succsim_N)$. In this case, weak Nash axiom α is satisfied. Consider the case where $C_e^*(A, \succsim_N) = \mathcal{F}(A, \succsim_N)$. Since $C_e^*(B, \succsim_N) \subseteq A$, it follows that

$$\mathcal{E}(B, \succsim_N) \cap \mathcal{P}(B, \succsim_N) \subseteq A.$$

Let $x \in \mathcal{E}(B, \succsim_N) \cap \mathcal{P}(B, \succsim_N)$. By the definition of $\mathcal{E}$, we have

$$(x, i) \succsim_i (x, j) \text{ for all } i, j \in N.$$

Since $x \in A$, we have $x \in \mathcal{E}(A, \succsim_N)$. Since $x \in \mathcal{P}(B, \succsim_N)$, there exists no $y \in B$ such that $(y, i) \succ_i (x, i)$ for all $i \in N$. Since $A \subseteq B$, there exists no $y \in A$ such that $(y, i) \succ_i (x, i)$ for all $i \in N$. We have $x \in \mathcal{P}(A, \succsim_N)$. Then, $x \in \mathcal{E}(A, \succsim_N) \cap \mathcal{P}(A, \succsim_N)$. We obtain $\mathcal{F}(B, \succsim_N) \subseteq \mathcal{F}(A, \succsim_N)$. Thus, $C_e^*(B, \succsim_N) \subseteq C_e^*(A, \succsim_N)$. Then, weak Nash axiom α is satisfied.

(ii) Suppose that C_e satisfies weak fairness extension and weak Nash axiom β.[9] For simplicity, we assume that $N = \{1, 2\}$ and $X = \{x_1, x_2, x_3, x_4\}$. Let $\succsim_N \in \Psi$ be such that

$$(x_4, 2) \succ_1 (x_3, 1) \succ_1 (x_1, 1) \succ_1 (x_4, 1) \succ_1 (x_2, 1) \succ_1 (x_2, 2) \succ_1 (x_3, 2) \succ_1 (x_1, 2),$$
$$(x_4, 2) \succ_2 (x_2, 2) \succ_2 (x_3, 2) \succ_2 (x_3, 1) \succ_2 (x_1, 2) \succ_2 (x_1, 1) \succ_2 (x_4, 1) \succ_2 (x_2, 1).$$

It is easy to check that $\mathcal{E}(X, \succsim_N) = \{x_1, x_2\}$, $\mathcal{P}(\{x_1, x_2\}, \succsim_N) = \{x_1, x_2\}$, $\mathcal{P}(\{x_1, x_2, x_3\}, \succsim_N) = \{x_2\}$, $\mathcal{P}(\{x_1, x_2, x_4\}, \succsim_N) = \{x_1, x_4\}$. Thus, weak fairness extension implies that

$$C_e((\{x_1, x_2\}, \succsim_N) \subseteq \{x_1, x_2\}, C_e(\{x_1, x_2, x_3\}, \succsim_N) = \{x_2\}, C_e(\{x_1, x_2, x_4\}, \succsim_N) = \{x_1\}.$$

From weak Nash axiom β,

$$C_e(\{x_1, x_2, x_3\}, \succsim_N) \supseteq C_e((\{x_1, x_2\}, \succsim_N) \subseteq C_e(\{x_1, x_2, x_4\}, \succsim_N).$$

This is a contradiction. □

Suzumura (1981b) shows that there exists an ECCR that satisfies fairness extension and stability axiom. Proposition 7 (i) implies that a stronger property can be satisfied. Proposition 7 (ii) in combination with Proposition 2 (i) is a strong version of Proposition 2 of Sakamoto (2013), which states that there exists no ECCR C_e satisfying weak fairness extension and Nash axiom. Our result implies that the impossibility is robust under weak Nash axiom β.

We have the following possibility result.

Proposition 8 *Suppose that X contains at least three alternatives. There exists an ECCR C_e that satisfies fairness extension and weak dual Chernoff axiom.*

[9] The strategy of our proof is the same as that of Sakamoto (2013, Proposition 2).

Proof Define C_e^P as follows:

$$C_e^P(A, \succsim) = \begin{cases} \mathcal{F}(A, \succsim_N) \text{ if } \mathcal{F}(A, \succsim_N) \neq \emptyset, \\ \mathcal{P}(A, \succsim_N) \text{ otherwise.} \end{cases}$$

It is obvious that C_e^P satisfies fairness extension. Thus, it suffices to show that C_e^P satisfies weak dual Chernoff axiom. Take $A_k \in \mathcal{X}$ $(k = 1, \ldots, K)$ and $\succsim_N \in \Psi$. Suppose that

$$x \in C_e^P(A_k, \succsim_N) \text{ for all } k \in \{1, \ldots, K\}.$$

Since $\mathcal{F}(A_k, \succsim_N) \subseteq \mathcal{P}(A_k, \succsim_N)$ $(k = 1, \ldots, K)$, $C_e^P(A_k, \succsim_N) \subseteq \mathcal{P}(A_k, \succsim_N)$ for all $k \in \{1, \ldots, K\}$. Thus,

$$x \in \mathcal{P}(A_k, \succsim_N) \text{ for all } k \in \{1, \ldots, K\}.$$

This implies that there is no $y \in \bigcup_{k=1}^K A_k$ such that $(y, i) \succ_i (x, i)$ for all $i \in N$. Thus,

$$x \in \mathcal{P}\left(\bigcup_{k=1}^K A_k, \succsim_N\right).$$

Either $x \in \mathcal{E}(X, \bigcup_{k=1}^K A_k)$ or not. If $x \in \mathcal{E}(\bigcup_{k=1}^K A_k, \succsim_N)$, then $x \in \mathcal{F}(\bigcup_{k=1}^K A_k, \succsim_N$) and, thus,

$$C_e^P\left(\bigcup_{k=1}^K A_k, \succsim_N\right) = \mathcal{F}\left(\bigcup_{k=1}^K A_k \succsim_N\right).$$

Suppose that $x \notin \mathcal{E}(\bigcup_{k=1}^K A_k, \succsim_N)$. If there exists $y \in \bigcup_{k=1}^K A_k$ such that $y \in \mathcal{P}(\bigcup_{k=1}^K A_k, \succsim_N)$ and $y \in \mathcal{E}(\bigcup_{k=1}^K A_k, \succsim_N)$, then there exists A_ℓ such that

$$y \in \mathcal{P}(A_\ell, \succsim_N) \text{ and } y \in \mathcal{E}(A_\ell, \succsim_N).$$

This implies that $x \notin C_e^P(A_\ell, \succsim_N)$. A contradiction. Thus, $\mathcal{F}(\bigcup_{k=1}^K A_k, \succsim_N) = \emptyset$ and, thus,

$$C_e^P\left(\bigcup_{k=1}^K A_k, \succsim_N\right) = \mathcal{P}\left(\bigcup_{k=1}^K A_k \succsim_N\right).$$

As a result, $x \in C_e^P(\bigcup_{k=1}^K A_k, \succsim_N)$. □

Now, we explain the implication of our results in the relationship with menu-dependent preferences. Suzumura (1981a) shows no ECCR that satisfies fairness extension and superset axiom. Since superset axiom is a necessary condition for a quasi-transitive rationalization (with a single preference), this result implies that fairness extension is not compatible with quasi-transitive rationalizability. Sakamoto (2013) shows that no ECCR satisfies weak fairness extension and the Chernoff axiom. Thus, weak fairness extension is not compatible with rationalizability because the Chernoff axiom is a necessary condition for the existence of a rationalization. In this line, Proposition 7(ii) means that weak fairness extension is not compatible with quasi-transitive menu-dependent rationalizability (Proposition 6(ii)). Remind that the weak dual Chernoff axiom is a necessary and sufficient condition for the existence of a nested system of complete and acyclic binary relation, which rationalizes C. Proposition 8 implies that fairness extension is compatible with menu-dependent rationalizability. We have an impossibility result for quasi-transitive menu-dependent rationalizability, while we can obtain a possibility result for acyclic menu-dependent rationalizability.

As mentioned earlier, Sen's δ is weaker than weak Nash axiom β. We present the possibility and impossibility results for Sen's δ.

Proposition 9 *Suppose that X contains at least three alternatives.*

(i) *There exists an ECCR C_e that satisfies weak fairness extension and Sen's δ.*
(ii) *There exists no ECCR C_e that satisfies fairness extension and Sen's δ.*

Proof (i) Sakamoto (2013, Proposition 5) provides an ECCR that satisfies weak fairness extension and superset axiom. Since superset axiom implies Sen's δ, it satisfies Sen's δ. (ii) Suppose that C_e satisfies fairness extension and Sen's δ.[10] For simplicity, we assume that $N = \{1, 2\}$ and $X = \{x_1, x_2, x_3\}$. Let $\succsim_N \in \Psi$ be such that

$$(x_3, 1) \succ_1 (x_1, 1) \succ_1 (x_2, 1) \succ_1 (x_2, 2) \succ_1 (x_3, 2) \succ_1 (x_1, 2),$$
$$(x_2, 2) \succ_2 (x_3, 1) \succ_2 (x_3, 2) \succ_2 (x_1, 2) \succ_2 (x_1, 1) \succ_2 (x_2, 1).$$

It is easy to check that $\mathcal{E}(X, \succsim_N) = \{x_1, x_2\}$, $\mathcal{P}(\{x_1, x_2\}, \succsim_N) = \{x_1, x_2\}$, and $\mathcal{P}(\{x_1, x_2, x_3\}, \succsim_N) = \{x_1, x_3\}$. Thus, fairness extension implies that

$$C_e(\{x_1, x_2\}, \succsim_N) = \{x_1, x_2\} \text{ and } C_e(\{x_1, x_2, x_3\}, \succsim_N) = \{x_1\}.$$

This contradicts Sen's δ. □

Since Sen's δ is weaker than superset axiom as well as weak Nash axiom β, Proposition 9 (ii) is a strong version of Theorem 2 of Suzumura (1981a).

5 Concluding remarks

In this paper, we introduced weak versions of the Nash axiom. First, we examine the relationship among the axioms of choice coherency. Second, we examine the implications of our weak Nash axioms for the analysis of collective decision making.

10 Our proof follows the approach of Suzumura Suzumura (1981a).

Focusing on the no-envy approach, we provide possibility and impossibility results. Weak Nash axioms serve as new insights for the approach. Weak Nash axiom α is the strongest axiom that yields a possibility result, and weak Nash axiom β is one of the weakest axioms that yield an impossibility result.

The Arrovian social choice problem is not addressed in this paper. That is, we do not consider whether or not there exists reasonable collective decision making that satisfies the Pareto principle and Arrow's Independence of Irrelevant Alternatives. Few works demonstrate impossibility results under Nash-type axioms.[11] We can emphasize the importance of menu-dependence in the context of social choice as well as individual choice. For example, consider the transitive-closure majority rule, which generates social preferences by taking the smallest transitive binary relation of majority-rule preferences.[12] The transitive-closure majority rule generates a nested system of menu-dependent preferences. It preserves many attractive properties of majority rule. This type of rule is not allowed under the assumption of menu-independent social preference. Thus, menu-dependent social preferences might provide a meaningful and useful basis of collective decision-making. Weak Nash axioms, especially weak Nash axiom β, are potentially applicable in this direction because it can serve as a basis of menu-dependent preferences. Investigating the problem remains a task for future research.

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References

Arrow KJ (1959) Rational choice functions and orderings. Economica 26:121–127

Arrow KJ (1963) Social choice and individual values, 2nd edn. Wiley, New York

Blair DH, Bordes G, Kelly JS, Suzumura K (1976) Impossibility theorems without collective rationality. J Econ Theory 13:361–379

Bordes G (1976) Consistency, rationality and collective choice. Rev Econ Stud 43(3):451–457

Chernoff H (1954) Rational selection of decision functions. Econometrica 22:422–443

Denicolo V (1999) No-envy and Arrow's conditions. Soc Choice Welfare 16:585–597

Ferejohn JA, Grether DM (1977) Weak path independence. J Econ Theory 14:19–31

Foley D (1967) Resource allocation and the public sector. Yale Econ Essays 7:45–98

Luce RD, Raiffa H (1957) Games and decisions: introduction and critical survey. Wiley, New York

Nash JF (1950) The bargaining problem. Econometrica 18:155–162

Panda SC (1983) On non-binary social choice. Math Soc Sci 4:73–78

Parks RP (1976) Further results on path independence, quasitransitivity, and social choice. Public Choice 26:75–87

Plott CR (1973) Path independence, rationality, and social choice. Econometrica 41:1075–1091

Sakamoto N (2013) No-envy, efficiency, and collective rationality. Soc Choice Welf 40:1033–1045

Sen AK (1970) Collective choice and social welfare. Holden-Day, San Francisco

Sen AK (1971) Choice functions and revealed preference. Rev Econ Stud 38:307–317

Sen AK (1977) Social choice theory: a re-examination. Econometrica 45:53–89

Sen AK (1993) Internal consistency of choice. Econometrica 61:495–521

[11] Panda (1983) examines the existence of collective choice rules satisfying weak Nash axiom β. He imposes "rejection," which implies base acyclicity, as well as weak Nash axiom β.

[12] For detailed analysis of the transitive-closure majority rule, see Bordes (1976) and Suzumura (1983).

Sen AK (1997) Maximization and the act of choice. Econometrica 65:745–779
Suppes P (1966) Some formal models of grading principles. Synthese 16:284–306
Suzumura K (1981a) On Pareto-efficiency and the no-envy concept of equity. J Econ Theory 25:367–379
Suzumura K (1981b) On the possibility of 'fair' collective choice rule. Int Econ Rev 22:351–364
Suzumura K (1983) Rational choice, collective decisions and social welfare. Cambridge University Press, Cambridge
Thomson W (1982) An informationally efficient equity criterion. J Pub Econ 18(2):243–263
Thomson W (1988) A study of choice correspondences in economies with a variable number of agents. J Econ Theory 46(2):237–254
Thomson W (1999) Welfare-domination under preference-replacement: a survey and open questions. Soc Choice Welf 16(3):373–394
Tyson CJ (2008) Cognitive constraints, contraction consistency, and the satisficing criterion. J Econ Theory 138(1):51–70
Varian HR (1974) Equity, envy, and efficiency. J Econ Theory 9:63–91
Varian HR (1976) Two problems in the fairness. J Pub Econ 5:249–260
Weymark JA (1983) Quasitransitive rationalization and the superset property. Math Soc Sci 6(1):105–108