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Two-sided unobservable investment, bargaining, and efficiency

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Abstract
Asymmetric information can lead to inefficient outcomes in many bargaining contexts. It is sometimes natural to think of asymmetric information as emerging from imperfect observation of previously taken actions (e.g., obtaining compliments or substitutes for the item being bargained over). How do such strategic investment choices prior to bargaining interact with the strategic problem of bargaining under private information? We focus on bilateral bargaining when players can make unobserved investments in the value of the item prior to their interaction. With two-sided hidden investment, strategic uncertainty induces a post-investment problem analogous to that in Myerson and Satterthwaite (J Econ Theory 29(2):265–281, 1983), and inefficiencies might be expected to arise. But, there are strong incentives to avoid investment levels that do not lead to trade and this must be anticipated by the other trader. This effect is shown to drive a form of unraveling; as a result in every equilibrium to the larger game the good ends up in the hands of the agent with the higher valuation.

Keywords Bargaining · Strategic uncertainty · Hold-up

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1 Introduction

One of the most important discoveries of information economics in the past half-century is that inefficiencies tend to be unavoidable in the presence of asymmetric information in interactions involving some conflicts of interests and limited external support. An important example of this phenomenon happens in bargaining over the sale of an item, where Myerson and Satterthwaite (1983) show that there are no voluntary mechanisms that can ensure trade happens whenever it is efficient if there is private information, and a positive probability that trade is inefficient (i.e., the buyer's and seller's distributions of possible values overlap).

The point of departure for this paper is the observation that while the sources of and interpretations for asymmetric information are varied, in some important bargaining settings asymetric information may emerge as strategic uncertainty induced by hidden actions. More concretely, it is natural to think that prior to negotiating over an exchange players may be able to take actions which influence their valuations and that these actions may be at best imperfectly observed. For example the value a buyer attaches to a particular item may be influenced by production technologies that compliment the item or connections that help marketing the item or output from the item. Alternatively, sellers may continue to market the item to other potential buyers or forego maintenance or search for substitutes for the item. All of these actions may be hidden. In settings where private information may arise in this way there is the potential for strategic interactions between expectations of how trading will occur and the pre-play investments that are made. The key insight here is that because investment decisions that are likely to lead to inefficiencies in bargaining do not provide the investor with high payoffs, equilibrium pressures will cut against the emergence of valuations that result in these inefficiencies. As a result of this incentive, equilibrium beliefs concentrate on types that can do well in bargaining and important inefficiencies are avoided.

Formally, we explore a setting where a buyer and seller can make unobserved investments in the value of an indivisible item and then interact in the kind of trading environment studied by Myerson and Satterthwaite. Thus, our treatment of private information as arising from hidden actions is parallel to the approach taken by Gul (2001) in the case of one-sided hidden actions. In our setting strategic uncertainty on the part of both players generates a post-investment bargaining problem with two-sided private information. As is well-known, such bargaining environments often result in conflicts between efficient trade and incentive-compatibility in trading mechanisms that also satisfy a participation constraint or budget balance. One might guess that, like in the case of the one shot game with one-sided incomplete information considered by Gul, equilibrium strategies result in underinvestment because the bargaining protocol cannot be efficient if both the buyer and seller possess (endogenously generated) private information. Alternatively, one might guess that the inefficiencies from Myerson and Satterthwaite are persistent in settings where players induce strategic uncertainty through unobserved investments. We show that both of these conjectures are wrong. In particular, there are no equilibria in which investment decisions induce post-investment bargaining inefficiencies. As a result, trade is always optimal in the sense that after bargaining the good is always possessed by the player who values it more. This conclusion stems from an unraveling effect that undermines any putative mixed investment equilibrium which involve distributions over valuations in which the optimal second stage bargaining mechanism is inefficient. To be sure, this finding is not a challenge to the relevance of the inefficiency result in Myerson and Satterthwaite. It does, however, represent a positive exhibition of how strategic investment in the value of trade may sometimes help avoid important strategic problems that emerge in equilibrium analysis of static models.

Our approach is to connect with extant work as much as possible. We augment, in a way that is familiar in the hold-up literature, Myerson and Satterthwaite's canonical description of bilateral trade by allowing players' valuation of the traded item to result from their unobserved investments. When mixed strategies are played this causes asymmetric information to emerge endogenously. Specifically, when players cannot observe each other's investment decisions, mixing in the investment stage induces strategic uncertainty and, therefore, asymmetric information at the bargaining stage.[1] As is the case in other work of this form, equilibrium conjectures will lead players to believe that they know the distribution from which unobserved choices emerge. We then proceed to analyze what is possible in bargaining using this approach and many results from Myerson and Satterthwaite. We do need to provide technical extensions to their characterization to cover the case of poorly-behaved distributions which might emerge when uncertainty is the result of strategic choices but we relegate these details to the Appendix.

In our framework there are three possible forms of inefficiency. First, at the *interim* bargaining stage, taking the investment decisions as fixed but unobserved, trade may exhibit the inefficiency documented in Myerson Satterthwaite. Second, the eventual winner of the item may not have made an investment decision that would be optimal if she knew she were guaranteed to obtain the item. Third, even if the previous two forms of inefficiency are absent, the winning agent might not be the player that can obtain the greatest value from owning and optimally investing in the good. We find that the possibility of the first two forms of inefficiency offset each other and in fact, under the assumption that players anticipate the use of a rule that is "optimal" given beliefs resulting from equilibrium mixing probabilities and incentive constraints, there are never equilibria with strategic uncertainty and inefficiencies from bargaining–so the first form of inefficiency is avoided. This is true because given equilibrium beliefs about valuations and participation constraints, if players anticipate the use of a second-best trading rule, then either the strategic uncertainty that emerges will not lead to allocation inefficiencies or strategic uncertainty will not emerge. This is our main result, Theorem 1. We end with an illustration that the third form of inefficiency is possible, although there are natural mechanism that can avoid it.

The intuition behind this result can be obtained by considering the optimal Myerson Satterthwaite mechanism for the case where both the buyer's and seller's valuations are independent draws from the same uniform distribution. In the second-best rule certain types of buyers do not trade with any seller; this is precisely the source of the famous wedge. But if valuations are the result of strategic decisions we would not expect buyers to be willing to expend resources in order to obtain these particular valuations. It is

[1] This is precisely the formulation used in Gul's case with one-sided hidden actions.

better to not invest than to pay to obtain a valuation that does not trade. Thus, some of the valuations in the conjectured support cannot be optimal investment choices given the anticipated mechanism. The emergence of mixed valuations given by this distribution function is then not possible given an expectation that bargaining is described by a second-best mechanism. Unraveling of this sort is not specific to the conjecture that equilibrium investment decisions induce uniform distributions over the valuations. It turns out to be pervasive regardless of the candidate equilibrium beliefs. Although it is possible to support lotteries over valuations which have overlapping supports, these distributions will not actually satisfy the conditions in Myerson Satterthwaite, and efficient allocations will be possible in the bargaining problem. Our conclusion is that knowledge that the bargaining mechanism is chosen optimally, given the relevant constraints and equilibrium beliefs about the investment strategies, implies that the form of allocation inefficiency that emerges in Myerson Satterthwaite is not consistent with equilibrium play in this model of valuation formation.[2] A corollary of this fact is that when there are bargaining inefficiencies due to asymmetric information, the source of asymmetric information is unlikely to be strategic investment in the value of trade.

Analytically, our focus on the interplay between the emergence of strategic uncertainty and expectations of how bargaining will unfold is most closely related to Gul's (2001) work on hold-up with one-sided unobservable investment. Our finding that inefficiencies are avoided when pre-play investment by both parties precedes bargaining represents a contrasting insight from the one-sided case studied by Gul. In the one-sided case the magnitude of the inefficiency resulting from hold-up is related to the efficiency of the equilibrium of the bargaining game. If the seller makes a one-shot offer to a buyer who makes a relationship-specific investment before trade, then the inefficiency resulting with unobservable investment is equivalent to that resulting with observable investment (Gibbons 1992). On the other end of the spectrum, when the seller makes repeated offers, and the time between offers vanishes, the investment decision of the buyer converges to the efficient level. Thus, Gul shows that if the equilibrium to the bargaining protocol is Coasian, extracting all the surplus, which is the case in the one-sided repeated offers game with one-sided incomplete information and vanishing time between periods (Gul and Sonnenschein 1988), then the underinvestment associated with the hold-up problem goes away. His result also demonstrates that when bargaining is itself not fully-efficient the presence of hidden investment decisions leads to additional distortions through the hold-up problem. We find that with two sided unobserved investments on the equilibrium path there are no distortions from the hold-up problem and bargaining is efficient even though the bargaining protocol would be inefficient in the presence of most forms of strategic uncertainty about valuations.

Rectifying these conclusions is instructive. A key effect that is present in our model as well as both the case of one-sided unobservable investment when the seller can extract all the rents (one shot) and a Coasian setting (for example repeated offers with vanishing time costs) is that in an equilibrium with strategic uncertainty investment

[2] This does not mean that strategic uncertainty cannot emerge. One can construct examples where the distribution of valuations have supports that overlap but in which Myerson and Satterthwaite's conclusion fails if gaps and atoms are present.

decisions are closely related to the bargaining strategies through the indifference condition(s). When the seller extracts all the rents a buyer cannot obtain value from her investment and the hold-up problem is severe. But, when the seller does not extract all the rents (as in the Coasian case), it is possible for buyers to invest and expect to be compensated for the investment. Similarly, in our two sided case, any type that does not trade cannot be supported and so equilibrium requires that all types in the equilibrium supports trade (sometimes). In the one-sided case when the bargaining protocol is inefficient and the seller extracts all the rents the buyer selects the lowest investment with probability one and the seller extracts all rents. As in our model, there is no inefficiency from asymmetric information because in equilibrium there is not asymmetric information. In the two sided case, investment is in pure strategies or strategic uncertainty persists (with small amounts of overlap) but in neither case is there inefficiency from the asymmetric information.

Now consider the Coasian environment in Gul. Again, the equilibrium lottery over valuations is dependent on the expected price obtained by different types through the equilibrium indifference condition. As one reduces the delay costs the equilibrium mixed strategy over investment changes. It changes at a rate to insure that the seller anticipates that delay will result in a quick future sale at a high price. This keeps the price from dropping too quickly. Thus equilibrium forces cause the distribution over types to change at a rate related to how the delay costs vanish and this allows for the delay in trade to vanish but the sale price to be near the buyer's privately known valuation. When the lottery over valuations is fixed the sale price becomes un-related to the true valuation. Because in Gul the sale price and valuation remain related the investment becomes optimal. In our treatment of the two-sided case, a similar phenomena occurs. The mapping over types that trade and the prices paid is closely related to the lotteries over valuations by a pair of indifference conditions. In Gul, this connection drives towards optimal investment in our case this connection drives towards efficient trading given investments.

2 Model

Our point of departure from existing theory is to model the buyer and seller's valuations of the indivisible good as a function of investment decisions. For example, suppose that the object in question is a computing technology such as a search algorithm or mapping software and the potential owners are two competing technology companies. Each potential owner could make investments in the ability to interface the new technology with its existing products. Each could also invest time or money in finding alternatives to the technology in question. These investments then influence the value of the trade to each player. If there is no trade, the seller can capitalize on his investment but investment returns are lost to him if the object is sold. The opposite is true for the buyer; her investment generates value only when she purchases the good.

Formally, consider a risk-neutral seller (player s) who owns an indivisible object and a buyer (player b) who may wish to acquire it. Before trade/bargaining takes place the seller and the buyer can make unobserved relationship-specific investments v_s and v_b. The value to player $i \in \{s, b\}$ of owning the item at the end of the buyer and seller's interactions is then $v_i - c_i(v_i)$. The cost function, $c_i(v_i)$, is strictly increasing (except possibly at the point 0), strictly convex and differentiable.[3] We assume that $c_i(0) = 0$ for both buyer and seller. We also assume that if either player knew she were going to own the item, her optimal investment would be finite, namely that for some finite level $\overline{\overline{v}}_i$ we have $c_i'(\overline{\overline{v}}_i) = 1$.

An investment strategy for a player is a choice of investment level. We allow the players to select mixed/behavioral strategies, so that the strategy for player i is a cumulative distribution function $F_i(\cdot)$ over valuations (non-negative reals). These investments are assumed to be unobservable hidden actions.[4]

After the investment stage, the players interact and ultimately the item ends up owned by the buyer or seller and a transfer is made. We follow the approach in Myerson and Saterthwaite and abstract away from the particulars of the bargaining protocol and equilibrium descriptions. Instead we rely on a direct bargaining mechanism and incentive and individual rationality constraints to describe outcomes that are consistent with equilibrium behavior under some bargaining protocol. Retaining the standard notation, we denote the result of such bargaining by way of a direct bargaining mechanism which has two pieces: a probability of trade p and transfer x from the buyer to the seller. Because the investments are hidden actions, this bargaining is similar to the problem of bilateral trade with private information albeit here the initiation of bargaining is at an interim stage in some larger game in which private information in bargaining possibly arises from hidden actions at an earlier stage of the game.

A direct bargaining mechanism is a game where the buyer and seller simultaneously report valuations, v_i to a broker or mediator who then determines whether the object will be transferred, p, and at what price, x. We let the message space be the set of all valuations that can result from investment. Formally, a direct bargaining mechanism is defined by two mappings. The first $p(m_s, m_b) : \mathbb{R}^2_+ \rightarrow [0, 1]$ determines the probability of trade and the second, $x(m_s, m_b) : \mathbb{R}^2_+ \rightarrow \mathbb{R}$ describes the transfer from the buyer to seller. The total payoffs for a profile of messages and valuations are

$$W_s(v_s, m_s, m_b) = v_s(1 - p(m_s, m_b)) + x(m_s, m_b) - c_s(v_s)$$
$$W_b(v_b, m_b, m_s) = v_b p(m_s, m_b) - x(m_s, m_b) - c_b(v_b).$$

We will employ standard techniques to restrict consideration to direct bargaining mechanism that are Bayesian incentive compatible, i.e truth telling is a mutual best response to the mechanism given the investment lotteries employed. Our focus is on settings in which the players first make simultaneous investment decisions and correctly anticipate the direct bargaining mechanism.[5] Treating bargaining as an interim stage requires augmenting the concept of Bayesian Nash equilibrium to ensure that in determining what messages are best responses players use beliefs that are consis-

[3] We sometimes refer to these cost functions as the exogenous investment technologies.

[4] To be clear, the investment choice of player i is unobservable to player j, but in equilibrium the players will correctly conjecture the other player's strategy (mixed or pure). Furthermore, in any equilibrium in which i employs a mixed strategy, she will be indifferent between all investment levels in the support of her mixture and weakly prefer these levels to investments not in the support of $F_i(\cdot)$.

[5] Perhaps a more appropriate term would be "interim direct bargaining mechanism," but since we do not have any other mechanism, we will drop the qualified "interim".

tent with an equilibrium conjecture of the other players investment strategy and that investment strategies are mutual best responses, given equilibrium conjectures about the reporting strategies. Given a direct bargaining mechanism, a strategy profile for the trading game is a pair of lotteries over investments and reports, where reports may depend on the realization of the possibly mixed investment actions. Thus, a strategy for player i is a lottery $F_i(\cdot)$ with support $V_i \subset \mathbb{R}_+$ and a reporting rule $\sigma_i(v_i)$ that defines for every realization of a player's valuation what message she will send to the mechanism.

Definition 1 An equilibrium is a direct bargaining mechanism $p(\cdot), x(\cdot)$, a pair of investment lotteries, (F_s, F_b) and messaging strategies $(\sigma_b(v_b), \sigma_s(v_s))$ s.t. given the lotteries (F_s, F_b), the messaging strategies constitute mutual best responses to direct bargaining mechanism $(p(\cdot), x(\cdot))$ for almost every. v_s and v_b in the respective supports of F_s and F_b.(i.e. they are Bayesian incentive compatible) and, given the valuation contingent payoffs associated with play of the bargaining mechanism and messaging strategies, the investment strategies (F_s, F_b) are simultaneous best responses. An equilibrium is truthful if the messaging strategies are the identity mapping, $\sigma_i(v_i) = v_i$.

Employing the logic in Myerson and Saterthwaite's proof of the revelation principle one can see that it is sufficient for us to focus on equilibria that are truthful. In the sequel we will focus only on truthful equilibria and for economy of exposition we suppress the adjective truthful, thus referring to equilibria to mean truthful equilibria.

Most interesting trading games will also satisfy the condition that participation is voluntary and that the trading game maximizes social welfare. In the bilateral trade setting with incomplete information Myerson and Satterthwaite (1983) restrict attention to games that satisfy an *interim* participation constraint where each type's expected net payoff from participating in the game is non-negative. In what follows, we will require that the equilibrium also satisfies this condition after investments are realized.

Definition 2 An equilibrium to a trading game satisfies the interim participation constraint (Condition IP) if each player's expected gain from trade is non-negative for almost every valuation, $v_i \geq 0$.

Second, we are interested in the relevance of time-consistency and pre-commitment to a bargaining mechanism or trading scheme that is optimal given rational expectations about investing behavior.

Definition 3 We say that an equilibrium is interim optimal (Condition O) if, given the investment lotteries (F_s, F_b), the bargaining mechanism $(p(\cdot), x(\cdot))$ maximizes the sum of players' payoffs within the class of mechanisms that are incentive compatible and which satisfy the interim participation constraint given the lotteries, $F_s(\cdot), F_b(\cdot)$.

This model and notion of equilibrium captures two ideas: (1) That when making investment decisions the traders have rational expectations about how bargaining will unfold and (2) trade will be conducted in a manner that is second-best given equilibrium conjectures about investing strategies. One way of motivating this definition of equilibrium is to think of a game with three players: buyer, seller, and a broker who selects a direct bargaining mechanism after investments and who seeks to maximize the total utility to the buyer and seller.[6] This broker would select the best of the direct bargaining protocols satisfying the relevant incentive constraints given correct beliefs about the mixed/behavioral investment strategies employed by the buyer and seller. Thus, every equilibrium satisfying condition O involves the choice of an optimal mechanism from the broker's perspective given equilibrium conjectures about the investing behavior and best responses by the buyer and seller given this bargaining mechanism, and would be supported as a Bayesian Nash equilibrium in this three player game. The converse is also true, any equilibrium to the 3 player game would also satisfy the conditions to be an equilibrium that also satisfies condition O. A second motivation would be to conceive of an evolutionary process where markets move toward efficient trading mechanisms. An equilibrium satisfying condition O can then be a steady-state to such a process.

3 Results

We begin by stating the main result.

Theorem 1 *Equilibria satisfying IP and O exist, and in every such equilibrium with probability one the good is allocated to a player with the highest realized valuation.*

In other words, there is no equilibrium satisfying conditions IP and O in which the player with the lower realized valuation obtains the good with positive probability. In the remainder of this section we prove this theorem by establishing several lemmas showing there are no equilibria that have inefficient trade in the bargaining stage after investments (i.e., the type of inefficiency in Myerson and Satterthwaite 1983). The argument involves showing that if a given mixed investment strategy only admits IC and IP mechanisms that are inefficient then that mixed investment strategy unravels and cannot be part of an equilibrium. We then construct an efficient equilibrium.

Let F_i be player i's equilibrium mixed-strategy distribution over the hidden action. Recall that our direct mechanism is a pair of functions $x(m_s, m_b)$ that describes the report-contingent transfer to the seller and $p(m_s, m_b)$ that determines the probability of trade. Expected gains from trade to the seller of reporting m_s in this direct mechanism, given investment v_s, can then be written as the integral[7]

$$U_s(m_s, v_s) = \int_{V_b} [x(m_s, v_b) - p(m_s, v_b)v_s] dF_b(v_b). \tag{1}$$

Similarly, for the buyer we have:

$$U_b(m_b, v_b) = \int_{V_s} [p(v_s, m_b)v_b - x(v_s, m_b)] dF_s(v_s). \tag{2}$$

[6] It is worth noting that it only makes sense to require Condition O to hold on the path as it is a condition on the mechanism and a distribution of valuations. Investments are hidden actions and thus if a player deviates from equilibrium the broker will not know this and cannot adjust and select the second-best given the distribution induced by the deviation.

[7] Throughout we denote Lebesgue–Stieltjes integrals with $dF_i(v_i)$ and Riemann integrals by $f_i(v_i)dv_i$- using the latter on intervals in which a density exists.

In a slight abuse of notation, let $U_i(v_i) = U_i(v_i, v_i)$.

We note a convenient feature of the supports of investment strategies. Since $c_i(0) = 0$, if

$$c_i(\hat{v}_i) > \hat{v}_i,$$

then the investment $\hat{v}_i$ is strictly dominated by $v_i = 0$. Recall that $\overline{\overline{v}}_i$ is the investment that makes $c_i'(v_i) = 1$ and so given strict convexity of the cost function any investment higher than this level is strictly dominated by $\overline{\overline{v}}_i$. In equilibrium investments must have support contained in the interval $[0, \overline{\overline{v}}_i]$. Because the support of any random variable is closed by definition, we can then conclude that equilibrium investment strategies always have compact support.

We now turn to the study of what types of investment strategies are possible in an equilibrium. We find that the equilibrium conditions from strategic investment pin down a number of characteristics of the bargaining problem.

Lemma 1 (MIXING) *In any equilibrium, if v_i is an accumulation point of the support of i's mixed strategy, then*

$$1 + U_s'(v_s) = c_s'(v_s), \tag{3}$$

$$U_b'(v_b) = c_b'(v_b). \tag{4}$$

The proof is given in the Appendix. With investment in mixed strategies, it must be the case that for every point in the support of the investment actions either the derivative of the cost function and the utility are equal (if player 2) or differ by exactly 1 (if player 1). The derivative of the utility for the trading game is pinned down by incentive compatibility so there must be a close connection between investment strategies, their implied trading probabilities, and the marginal cost of investment for the traders. Below, we show that this connection precludes equilibria with investment decisions that lead to Myerson–Satterthwaite inefficiencies.

We start by considering the classical bilateral trading case investigated by Myerson and Satterthwaite, where both the buyer's and seller's valuations are distributed continuously over a connected domain. Myerson and Satterthwaite's classical result is that as long as the distributions of the players overlap and have full support on an interval, no efficient mechanism exists that is both incentive compatible and individually rational. The theorem below, on the other hand, shows that such distributions cannot emerge from a mixed-strategy investment equilibrium, if the mechanism designer is choosing second-best mechanisms that maximize aggregate gains from trade.

Lemma 2 (NO CONNECTED SUPPORTS WITH IC, O, IP) *Assume the cost function is strictly increasing. When the designer chooses an optimal IC and IP mechanism that maximizes aggregate gains from trade given the investment strategies (condition O) there is no mixed-strategy equilibrium with connected and overlapping supports containing no atoms.*

Proof To see this, suppose the seller and the buyer are following mixed strategies with positive probability densities over $[a_s, b_s]$ and $[a_b, b_b]$, respectively, and that the interiors of the supports have a non-empty intersection. With nice densities like this, Myerson and Satterthwaite show that no efficient mechanism exists and satisfies incentive compatibility and the participation constraint. So, the aggregate gains from trade will be maximized by a second-best mechanism characterized by Theorem 2 of Myerson and Satterthwaite. This result states that the mechanism that ensures $U_s(b_s) = U_b(a_b) = 0$ is a optimal second-best mechanism. Further, it is easy to see that this is the only optimal second-best mechanism (see Appendix). This means that the lowest type buyer will not gain any benefit from their investment. It immediately follows that any $a_b > 0$ is strictly dominated by investing 0 and not paying a cost; hence $a_b = 0$. Now, from Theorem 1 and the envelope theorem of Myerson and Satterthwaite we have for any incentive compatible mechanism:

$$c_b'(v_b) = \overline{p}_b(v_b) \tag{5}$$

Now, we need to distinguish between "regular" and "irregular" distributions. The regular case is covered by Theorem 2 of Myerson and Satterthwaite shows that when $c_s(v_s) = v_s + \frac{F_s(v_s)}{f_s(v_s)}$ and $c_b(v_b) = v_b - \frac{1-F_b(v_b)}{f_b(v_b)}$ are increasing in v_s and v_b, respectively, that the optimal second-best mechanism prescribes trade when

$$v_b - v_s \geq \alpha \left(\frac{F_s(v_s)}{f_s(v_s)} + \frac{1 - F_b(v_b)}{f_b(v_b)} \right), \tag{6}$$

where $\alpha \in [0, 1]$ and $F_i(\cdot)$ and $f_i(\cdot)$ are the cumulative and probability density functions for the players. Note that $\frac{1-F_b(0)}{f_b(0)} > 0$; hence, the right-hand side of (6) is strictly positive. This means that there exists an $\epsilon > 0$, for which $\frac{1-F_b(\epsilon)}{f_b(\epsilon)} > \epsilon$; and consequently, $\overline{p}_b(\epsilon) = 0$. In other words, even if the lower bound of the seller's mixed strategy is at 0, the ϵ-type buyer will not be able to trade with any seller because the IC and interim participation (IP) constraints mean that the ϵ type will not trade even with the 0-type seller. However, since $c_b'(\epsilon) > 0$ by assumption, it follows that the buyer strictly prefers a lower investment to the ϵ-investment. This means that ϵ cannot be part of the equilibrium support, a contradiction.

For the irregular case, where the functions $c_s(v_s)$ or $c_b(v_b)$ are decreasing for some range, the monotonicity of p becomes the binding constraint. In such cases, the optimal mechanism is found by "ironing" (Myerson 1981; Fudenberg and Tirole 1991, pp. 303–306), which involves pooling a range of types together and treating them the same, meaning players in the ironed range will have an expected gain from trade that is linear in their valuations. But since $c(\cdot)$ is strictly increasing and convex, the mixing condition (Eq. 5) cannot be satisfied, meaning that investments in the ironed range cannot be part of the equilibrium support.[8] □

The above result shows that "nice" lotteries over valuations and the inefficiencies from Myerson–Satterthwaite cannot arise in equilibria when valuations emerge from hidden investments. One reason is the wedge introduced by the IP conditions in the second-best bargaining mechanism, which ensure that the lowest type buyers cannot trade with anyone. But then these types cannot be supported by equilibrium investment decisions.

[8] See Toika (2011) for a more recent treatment of ironing with discussion of applications to bargaining.

However, Lemma 2 above does not rule out potential investment strategies that involve atoms or gaps. For example, one might think that placing a probability mass of sellers at zero investment, and a gap between the zero-type buyers and the next highest type in the mixed strategy's support might resolve the issue identified in the previous section. Therefore, we next consider this possibility by extending the Myerson–Satterthwaite analysis to distributions with gaps and/or atoms. We show that this extension might allow efficient mechanisms satisfying IC and IP in some cases. However, we also show that distributions that do not admit first-best efficiency in the trading stage cannot be equilibrium mixed strategies. As a result, the conclusion that Myerson–Satterthwaite inefficiencies cannot occur with endogenously determined investments extends beyond the case of lotteries described in Lemma 2. This result holds generally when the valuations are equilibrium choices as modeled here. We treat the case where each player's distribution has atoms and gaps, but the special case of each result when only gaps or only atoms or only pathologies for one player apply is covered by these results.

As probabilities sum to 1 there can be at most a countable number of atoms (discontinuities in the distribution functions).[9] For this reason the supports of these mixed strategies must be the closure of the union of intervals and a countable number of single points. Thus each support is the union of a countable number of intervals and isolated points.[10] Consider a distribution whose support consists of an arbitrary (but countable) number of atoms, gaps, and compact intervals. Figure 1 illustrates a simple example of such a distribution where the atoms are at the upper and lower limits of the distribution.[11] Denote by Θ_i the set of atoms and by $\mathcal{K}_i$ the union of all compact intervals in the support of player i's mixed strategy distribution. We use $\mathcal{I}_i^j$ to denote the jth such interval (counted in increasing order). Likewise, for notational convenience, we define the sets $\underline{V}_i$ and $\overline{V}_i$ to be the sets of infima and suprema of the compact intervals in the distribution of the seller and the buyer, respectively, with $\underline{v}_i^j = \inf(\mathcal{I}_i^j)$ and $\overline{v}_i^j = \sup(\mathcal{I}_i^j)$.

To focus on efficient mechanisms, we make the technical assumption that $p(v_s, v_b)$ is left-continuous in v_s and right-continuous on v_b, which is satisfied for both efficient mechanisms and second-best mechanisms because any discontinuity in such mechanisms will involve $v_b \geq v_s$, and the mechanism designer will weakly prefer trade to non-trade. Given a mechanism with allocation function $p(v_s, v_b)$ and transfer function $x(v_s, v_b)$, we define the expected probability of trading for a seller that reports her type as v_s, $\overline{p}_s(v_s)$, with the Lebesque–Stieltjes integral:

$$\overline{p}_s(v_s) = \int_{a_b}^{b_b} p(v_s, t_b) dF_b, \tag{7}$$

[9] See for example Billingsley (1995), p. 256.

[10] Note that taking the closure is definitional as the support is the smallest closed set that has measure 1. The number of intervals is countable because otherwise the total probability of the intervals would be unbounded.

[11] The following result shows that unraveling occurs if there are gaps in combination with atoms. We can rule out the possibility of atoms in the interior of either player's support with standard arguments on all-pay auctions (e.g. Baye et al. 1996) and the fact that an optimal M-S mechanism induces a probability of trade equalling 0 or 1. This argument is made in the beginning of the proof of lemma 6.

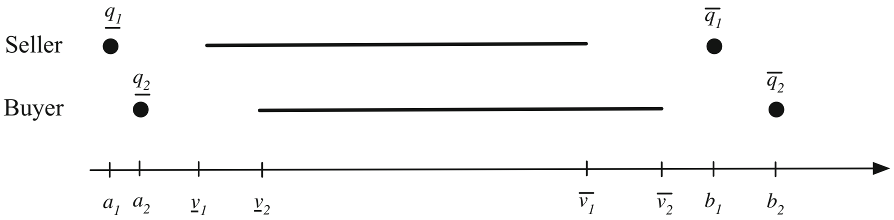

Fig. 1 Diagram of the buyer and seller distributions. Solid lines signify continuous supports of the players' investment strategies

where a_b and b_b are again the lower and upper bounds of the buyer's distribution. Under our assumptions $\overline{p}_s$ is left-continuous (and $\overline{p}_b$ right-continuous). Likewise, the expected payment to the seller reporting v_s is defined as

$$\overline{x}_s(v_s) = \int_{a_b}^{b_b} x(v_s, t_b) dF_b. \tag{8}$$

The expected gain (relative to non-participation) for the seller from declaring v_s' when his real type is v_s, is:

$$U_s(v_s', v_s) = \int_{\underline{v}_b}^{\overline{v}_b} [x(v_s', v_b) - p(v_s', v_b)v_s] dF_b. \tag{9}$$

Similar definitions apply to the buyer. These integrals exist because $p(\cdot, \cdot)$ and $x(\cdot, \cdot)$ are non-negative and F_i are monotone and right-continuous. First, we show that the envelope theorem applies in the connected parts of the seller and buyer's distribution, and put bounds on the difference between the expected payoffs for types bordering the gaps. For brevity, the proofs of the following results are given in the "Appendix".

Lemma 3 (Envelope theorem with atoms and/or gaps) *Consider an IC mechanism. For any $v_s \in \mathcal{K}_s$, the expected payoff satisfies $U_s'(v_s) = -\overline{p}_s(v_s)$, and if $v_s \in \mathcal{I}_s^j$ then the following holds:*

$$U_s(v_s, v_s) = U_s(\overline{v}_s^j, \overline{v}_s^j) + \int_{v_s}^{\overline{v}_s^j} \overline{p}_s(t_s) dt_s. \tag{10}$$

Likewise, for any $v_b \in \mathcal{K}_b$, the expected payoff satisfies $U_b'(v_b) = \overline{p}_b(v_b)$ and for $v_b \in \mathcal{I}_b^j$:

$$U_b(v_b, v_b) = U_b(\underline{v}_b^j, \underline{v}_b^j) + \int_{\underline{v}_b^j}^{v_b} \overline{p}_b(t_b) dt_b. \tag{11}$$

For two values v_s and v_s' with $v_s < v_s'$ that border a gap in the seller's distribution, we have:

$$-\overline{p}_s(v_s') \geq \frac{U_s(v_s) - U_s(v_s')}{v_s' - v_s} \geq -\overline{p}_s(v_s), \tag{12}$$

and likewise, for two values v_b and v_b' with $v_b < v_b'$ that border a gap in the buyer's distribution, we have:

$$\overline{p}_b(v_b') \geq \frac{U_b(v_b') - U_b(v_b)}{v_b' - v_b} \geq \overline{p}_b(v_b). \tag{13}$$

For the next lemma which gives the necessary and sufficient condition for an IP and IC mechanism to satisfy, let us denote by $\text{supp}(F_i)$ the support of the distribution of player i, and the functions $\overline{\pi}_s(v_s)$ and $\overline{\pi}_b(v_b)$, as follows:

$$\overline{\pi}_s(v_s) = \begin{cases} \overline{p}_s(v_s) & \text{if } v_s \in \text{supp}(F_s) \\ \overline{p}_s(\hat{v}_s) & \text{s.t. } \hat{v}_s = \inf\{x \in \text{supp}(F_s)\,|x \geq v_s\} \text{ otherwise,} \end{cases} \tag{14}$$

$$\overline{\pi}_b(v_b) = \begin{cases} \overline{p}_b(v_b) & \text{if } v_b \in \text{supp}(F_b) \\ \overline{p}_b(\hat{v}_b) & \text{s.t. } \hat{v}_b = \sup\{x \in \text{supp}(F_b)\,|x \leq v_b\} \text{ otherwise.} \end{cases} \tag{15}$$

In other words, $\overline{\pi}_s(v_s)$ is equal to $\overline{p}_s(v_s)$ whenever v_s is in the support of the seller's distribution, and equal to $\overline{p}_s$ for the next higher point in the seller's distribution, if v_s is not in the support. Likewise for the buyer, except if v_b is not in the support of the buyer's distribution, $\overline{\pi}_b(v_b)$ is equal to the expected probability of trade for the next lower point in the distribution.

Lemma 4 (Myerson and Satterthwaite with atoms and/or gaps) *Given buyer and seller distributions F_s and F_b, consisting of a countable number of atoms and compact supports given by the union of intervals, for any IC and IP mechanism (x, p) it must hold that:*

$$\begin{aligned} &\int_{a_b}^{b_b} v_b\overline{p}(v_b)dF_b - \int_{a_s}^{b_s} v_s\overline{p}(v_s)dF_s \\ &\quad - \int_{a_s}^{b_s} F_s(t_s)\overline{\pi}_s(t_s)dt_s - \int_{a_b}^{b_b} (1 - F_b(t_b))\overline{\pi}_b(t_b)dt_b \\ &\geq U_s(b_s) + U_b(a_b) \geq 0. \end{aligned} \tag{16}$$

Furthermore, for any function $p(v_s, v_b)$ that maps from $\text{supp}(F_s) \times \text{supp}(\text{F}_\text{b})$ *to* $[0, 1]$, *a payment function $x(v_s, v_b)$ exists such that (x, p) is IC and IP if and only if* (16) *holds and $\overline{p}_s(v_s)$ and $\overline{p}_b(v_b)$ are weakly decreasing and decreasing, respectively.*

When it is impossible to implement efficient incentive compatible individually rational mechanism, we have the following results about the second-best mechanisms:

Lemma 5 *Consider any gap in the seller's distribution, and denote the lower and upper boundary of the gap as $\underline{v}_s$ and $\overline{v}_s$, respectively (i.e., $\underline{v}_s, \overline{v}_s \in \text{supp}(F_s)$ but for $\underline{v}_s < t_s < \overline{v}_s$, $t_s \notin \text{supp}(F_s)$). When an efficient mechanism does not satisfy both IC and IP, the second-best mechanism maximizing aggregate gains from trade has:*

$$U_s(\underline{v}_s) = U_s(\overline{v}_s) + (\overline{v}_s - \underline{v}_s)\overline{p}_s(\overline{v}_s)$$

Similarly, consider any gap in the buyer's distribution, bounded by $\underline{v}_b$ and $\overline{v}_b$. A second-best mechanism has

$$U_b(\overline{v}_b) = U_b(\underline{v}_b) + (\overline{v}_b - \underline{v}_b)\overline{p}_b(\underline{v}_b)$$

Using Lemma 5, we can now prove that a second-best mechanism when efficiency is not possible rules out a mixing investment equilibrium.

Lemma 6 (Mixed investment unravels) *Suppose that given F_s, F_b with atoms and or gaps there is no IC and IP mechanism that is efficient and the designer chooses a second-best mechanism (p, x) to maximize the aggregate gains from trade given these lotteries (condition O). Then it is not possible to support F_s, F_b as equilibrium mixed investment strategies with (p, x) for any strictly convex and continuous cost functions.*

One direction of Theorem 1 follows directly from Lemma (6). The other direction follows from our last lemma

Lemma 7 (Existence) *There is an equilibrium satisfying conditions IP and O.*

In order to exhibit an equilibrium (and ensure the result is not vacuous) we borrow on some of the logic behind second-price auctions. Consider the following construction. Let $e \in \{b, s\}$ denote the more efficient owner, i.e. the player for whom ownership and optimal investment is maximal, and $-e$ denote the other player (and s if there is a tie). Let $\overline{\overline{v}}_i$ denote the investment i makes if she knows she will obtain the item. Thus,

$$\overline{\overline{v}}_e - c_e(\overline{\overline{v}}_e) \geq \overline{\overline{v}}_{-e} - c_{-e}(\overline{\overline{v}}_{-e})$$

Define $p^*(m_s, m_b)$ to be 0 if $e = s$ and 1 otherwise. Define the transfer $x^*(m_s, m_b) = p(m_s, m_b)^*(\overline{\overline{v}}_{-e} - c_{-e}(\overline{\overline{v}}_{-e}))$. Further consider the investment strategies: F^*_e is concentrated at $\overline{\overline{v}}_e$ and F^*_{-e} is concentrated at 0.

Observe that that the transfer and allocation do not depend on reports and so the mechanism is IC.[12] Given the investment strategies, the mechanism allocates the item to the player with the highest valuation and thus satisfies O. Given the investment strategies both players obtain non-negative rents and thus IP is satisfied. Moreover given this rule, the less efficient owner, $-e$ has no incentive to invest whereas e maximizes her payoff by selecting $\overline{\overline{v}}_e$ and so the investment strategies are mutual best responses.

[12] An alternative mechanism that allocates the item to the player that makes the highest report and maintains this transfer would satisfy IC for valuations in the supports of F_s, F_b.

Thus the proof the the theorem is complete. Do all equilibria need to obtain the efficiency captured by the one detailed above? The equilibrium described in the proof of the last lemma results in an efficient allocation, under some conditions there are equilibria that allocate the good to a player that has optimally invested given that she anticipates obtaining the good but there is still an inefficiency because the other player would obtain higher value from optimally investing and owning the item. When $\overline{\overline{v}}_{-e} > \overline{\overline{v}}_e$ miscoordination can result in stable allocations where $-e$ owns the item and invests while e does not invest because following a deviation to investing $\overline{\overline{v}}_e$, e is not willing to pay a sufficient price to induce $-e$ to sell. In practice this problem could be avoided if, prior to the investment stage, the buyer (here e) can commit to buy at a price greater than $\overline{\overline{v}}_{-e} - c_{-e}(\overline{\overline{v}}_{-e})$. In this case, the seller (here $-e$) can always ensure a higher payoff by not investing and selling at that price instead of investing optimally and not trading - regardless of the buyer's actions. Accordingly, in equilibrium, the seller will invest 0 and sell the item. Given this commitment the buyer's best response is to invest optimally. In fact, for any $V \in [\overline{\overline{v}}_{-e}, \overline{\overline{v}}_e]$ we can construct an equilibrium that obtains expected surplus of V. For $V = \lambda\overline{\overline{v}}_{-e}, +(1-\lambda)\overline{\overline{v}}_e$ we let the designer randomize between making an announcement that $-e$ will be required to own the item and the announcement that e will be required to own the item (with probabilities λ and $1-\lambda$ respectively). Following the announcements the players are free to make any investment decisions.

4 Discussion

As noted above Gul (2001) shows that if the buyer's investment is a hidden action, then, even when the seller has all the bargaining power, the underinvestment problem can be resolved if repeated offers are allowed and the time between offers vanishes. Gul also considers the case of two-sided investments but assumes that the seller's investment is observed prior to bargaining.[13] Gul's model only allow asymmetric information to emerge from strategic uncertainty caused by mixed strategies and hidden actions. We share this feature but allow for hidden actions by both players.

In the other papers on pre-bargaining investment four central distinctions appear. In some of this scholarship the relevant fundamentals, like investment in value, are assumed to be observable at the time of bargaining (Grossman and Hart 1986; Hart and Moore 1988). More recently, Schmitz (2002), Gonzlez (2004) and Lau (2008) also consider investments as hidden actions, however, these papers consider the case where only one player can invest and Lau (2008) allows for partial observability of investments. Lau (2011) considers the case of one-sided hidden investments and exogenous asymmetric information, capturing some of the relevant tradeoffs but in her paper the asymmetric information is not directly attributed to a choice by the players.

Perhaps closer to our paper is Rogerson (1992) who provides a quite general treatment of the case where multiple players can invest before trade and where there are no externalities. His case of completely private information is closest in spirit to the

[13] Incidentally, Gul finds that the seller will have an incentive to underinvest, and points out the challenges to applying his arguments to the case of a continuum of types.

environments we consider. The key distinction is that Rogerson assumes that there is a random component connecting each player's investment to its type. In particular, by assuming that investment decisions always admit unique optima, he excludes the case where investments completely determine a player's type [as in Gul (2001) and our paper]. Rogerson also does not impose the individual rationality constraint imposed by Myerson–Satterthwaite and thus, in principle, is free to work with a larger set of mechanisms (he does require budget balance and incentive compatibility). Finally, Rogerson assumes that the mechanism is committed to prior to investment decisions and shows that d'Aspremont and Grard-Varet (1979) and Cremer and Riordan (1985) mechanisms also create incentives for optimal investment.[14] We are interested in the same participation constraints as Myerson and Satterthwaite, and focus only on mechanisms that are optimal given equilibrium beliefs about investments. Thus, we do not analyze the full-mechanism design problem in which a designer commits to a mechanism (either before or after learning something from the traders) that is not the constrained optimal.

As Gul (2001) notes, a common feature of his result and the literature on moral hazard and renegotiation (Che and Chung 1999; Che and Hausch 1999; Fudenberg and Tirole 1990; Hermalin and Katz 1991; Ma 1991, 1994; Matthews 1995) is that pure strategies by the agent generate strong reactions from the principal and thus, in equilibrium, the agents' randomization generates asymmetric information. Our analysis offers a counter-point to this result. The presence of randomization by both traders is typically hard to support, and impossible to support when they admit no first-best trading mechanisms, and if the traders anticipate that a designer is using a second-best mechanism. One possible exception to this result occurs if the investment cost functions support an equilibrium in which the buyer and seller mix over a small interval and disconnected atoms, with most of the probability being allocated to the atoms. In cases like this first best trading rules exist, even though there is overlap in the supports. Therefore, this form of strategic uncertainty is not consistent with the inefficiency that emerges in Myerson and Satterthwaite, as first-best given investment choices becomes possible with this information environment.

Sometimes the value of a trade between two economic agents is determined by choices that the traders make prior to the transaction. In these circumstances a rational expectation about how the trading game might be played can be seen to have important effects on the incentives to invest and, as a consequence, generate bilateral trade games where the information environment looks very different from those well-studied in the economics literature. When valuations are the product of hidden pre-trade investment, the standard case of overlapping connected sets of possible types cannot emerge as the result of equilibrium mixing. Furthermore, in every equilibrium of the trading game, given investments, trade occurs in every instance where the net gains are positive. We interpret this as good news; in models that endogenize a larger set of the key economic variables there are sometimes strong incentives that help avoid the deep source of inefficiency that drives the result in Myerson–Satterthwaite.

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[14] See also Hart and Moore (1988) for a similar observation in case of two-players and an indivisible item–as in our model.

Compliance with ethical standards

Conflict of interest The authors declare they have no conflict of interests.

Appendix

Myerson–Sattertwaite second-best mechanism with $U_s(b_s) = U_b(a_b) = 0$ maximizes gains from trade

First, define constraint on the optimal mechanism $G(\alpha)$ in the same way as Myerson and Satterthwaite as:

$$G(\alpha) = \int_{a_b}^{b_b} \int_{a_s}^{b_s} (c_b(v_b, 1) - c_s(v_s, 1)) p^{\alpha}(v_s, v_b) f_s(v_s) f_b(v_b) dv_s dv_b \\ = U_b(a_b) + U_s(b_s), \tag{17}$$

where

$$c_s(v_s, \alpha) = v_s + \alpha \frac{F_s(v_s)}{f_s(v_s)} \quad c_b(v_b, \alpha) = v_b - \alpha \frac{1 - F_b(v_b)}{f_b(v_b)}.$$

Furthermore, define $p^{\alpha}(v_s, v_b)$ is 1 if $c_b(v_b, \alpha) \geq c_s(v_s, \alpha)$ and zero otherwise, M-S show that $G(\alpha)$ is increasing in α, and continuous, with $G(0) < 0$ and $G(1) \geq 0$, thus ensuring there is a positive α^* that satisfied the constraint from their Theorem 1 with equality, i.e., $G(\alpha^*) = 0$.

The second-best mechanism maximizes the expected gain from trade, given by:

$$\int_{a_b}^{b_b} \int_{a_s}^{b_s} (v_b - v_s) p(v_s, v_b) f_s(v_s) f_b(v_b) dv_s dv_b. \tag{18}$$

M-S show that α^* is a second-best mechanism. We can show that it is the only one by noting any function $p'(v_s, v_b)$ that differs from $p^{\alpha^*}(v_s, v_b)$ can only do so by being less than 1 outside the "wedge" of types that donot efficiently trade as determined by α^*, where $p^{\alpha^*} = 1$. This is because within the wedge, we have $p^{\alpha^*} = 0$ and $c_b(v_b, \alpha^*) - c_s(v_s, \alpha^*) < 0$. Thus, having non-zero probability of trade inside the wedge would violate the constraint in Theorem 1 of M-S. But outside the wedge, $v_b > v_s$, hence, any such function $p'(\cdot, \cdot)$ that doesn't prescribe trade with probability one where p^{α^*} does will lead to a strictly lower expected gains from trade than p^{α^*}.

Proof of Lemma 1 Consider an equilibrium involving the direct mechanism (x, p). To begin, consider the case of the seller. Take any two investments v_s, v_s' in the support of F_s. Then, because the seller is mixing over these values

$$\int_{V_b} (1 - p(v_s, v_b)) v_s + x(v_s, v_b) dF_b(v_b) - c_s(v_s) \\ = \int_{V_b} (1 - p(v_s', v_b)) v_s' + x(v_s', v_b) dF_b(v_b) - c_s(v_s').$$

The left-hand side equals

$$U_s(v_s) - c_s(v_s) + v_s,$$

and the right-hand side equals

$$U_s(v_s') - c_s(v_s') + v_s',$$

and we can rewrite the equation above as

$$U_s(v_s) - c_s(v_s) + v_s = U_s(v_s') - c_s(v_s') + v_s' \tag{19}$$

$$1 + \frac{U_s(v_s) - U_s(v_s')}{v_s - v_s'} = \frac{c_s(v_s) - c_s(v_s')}{v_s - v_s'}. \tag{20}$$

At an accumulation point of the support of F_s, we can take the limits as $v_i' \to v_i$ and

$$1 + U_s'(v_s) = c_s'(v_s). \tag{21}$$

This is the first equation in the theorem. Similar calculations give the identity for the seller. □

Proof of Lemma 3 The proof follows the familiar argument of Myerson and Satterthwaite. Incentive compatibility means that for all v_s, v_s' in the support of the seller's distribution:

$$U_s(v_s, v_s) \geq U_s(v_s', v_s) \tag{22}$$

$$U_s(v_s', v_s') \geq U_s(v_s, v_s'). \tag{23}$$

By subtracting the RHS of the second inequality from the LHS of the first and the RHS of the first from the second and canceling the payment terms, we get:

$$-\overline{p}_s(v_s)[v_s - v_s'] \geq U_s(v_s, v_s) - U_s(v_s', v_s') \geq -\overline{p}_s(v_s')[v_s - v_s'].$$

For either v_s or v_s' in Θ_s, we can stop here. For $v_s \in \mathcal{K}_s$ and $v_s \notin \underline{V}_s$, we assume $v_s > v_s'$, divide by $v_s - v_s'$ and take the limit as $v_s' \to v_s$ to obtain:

$$U_s'(v_s) = -\overline{p}_s(v_s). \tag{24}$$

For $v_s \in \underline{V}_s$ we simply take the limit from the right, with $v_s' > v_s$ to obtain the same result. Integrating Eq. (24) within an interval $\mathcal{I}_s^j$, we obtain (10) The same method applies to the buyers. □

Proof of Lemma 4 The proof proceeds analogously to the canonical case (Theorem 1 of Myerson and Satterthwaite), except that we need to make use of Lebesgue–Stieltjes integrals to account for the fact that we integrate over distributions that have gaps and atoms. First, observe that Lemma 3 implies that $U_s(b_s) \leq U_s(v_s)$ for all v_s in the seller's support, and $U_b(a_b) \leq U_b(v_b)$ for all v_b in the buyer's support. Next, consider the expected gains from trade under a direct mechanism (x, p).

$$\begin{aligned}\int_{a_s}^{b_s}\int_{a_b}^{b_b}(v_b - v_s)p(v_s, v_b)dF_b dF_s &= \int_{a_s}^{b_s}\int_{a_b}^{b_b} v_b p(v_s, v_b)dF_b dF_s \\ &- \int_{a_s}^{b_s}\int_{a_b}^{b_b} -v_s p(v_s, v_b)dF_b dF_s \\ &= \int_{a_b}^{b_b} v_b \overline{p}(v_b)dF_b - \int_{a_s}^{b_s} v_s \overline{p}(v_s)dF_s,\end{aligned} \tag{25}$$

where the last line follows from integrating the two integrals in different orders, permissible by Tolleni's theorem.

At the same time, since the payments are zero sum, the expected gains from trade is equal to the sum of the average gains of the buyers and sellers:

$$\int_{a_s}^{b_s}\int_{a_b}^{b_b}(v_b - v_s)p(v_s, v_b)dF_b dF_s = \int_{a_s}^{b_s} U_s(v_s)dF_s + \int_{a_b}^{b_b} U_b(v_b)dF_b \tag{26}$$

Take the seller's term, the first integral. Using the envelope theorem (Lemma 3) and using the definition of the function $\overline{\pi}_s(v_s)$ above, we can write

$$U_s(v_s) \geq U_s(b_s) + \int_{v_s}^{b_s} \overline{\pi}_s(t_s)dt_s \tag{27}$$

So, we have:

$$\begin{aligned}\int_{a_s}^{b_s} U_s(v_s)dF_s &\geq \int_{a_s}^{b_s}\left[U_s(b_s) + \int_{v_s}^{b_s}\overline{\pi}_s(t_s)dt_s\right]dF_s \\ &= U_s(b_s) + \int_{a_s}^{b_s}\int_{v_s}^{b_s}\overline{\pi}_s(t_s)dt_s dF_s = U_s(b_s) + \int_{a_s}^{b_s} F_s(t_s)\overline{\pi}_s(t_s)dt_s,\end{aligned}$$

where the change in the order of integration again is permissible by Tolleni's theorem. Similarly for the buyer, we have:

$$\begin{aligned}\int_{a_b}^{b_b} U_b(v_b)dF_b &\geq \int_{a_b}^{b_b}\left[U_b(a_b) + \int_{a_b}^{t_b}\overline{\pi}_b(t_b)dt_b\right]dF_b \\ &= U_b(a_b) + \int_{a_b}^{b_b}(1 - F_b(t_b))\overline{\pi}_b(t_b)dt_b.\end{aligned}$$

Putting these together, we have:

$$\int_{a_b}^{b_b} v_b \overline{p}(v_b) dF_b - \int_{a_s}^{b_s} v_s \overline{p}(v_s) dF_s \geq U_s(b_s) + U_b(a_b) + \int_{a_s}^{b_s} F_s(t_s)\overline{\pi}_s(t_s) dt_s + \int_{a_b}^{b_b} (1 - F_b(t_b))\overline{\pi}_b(t_b) dt_b, \tag{28}$$

or,

$$\int_{a_b}^{b_b} v_b \overline{p}(v_b) dF_b - \int_{a_s}^{b_s} v_s \overline{p}(v_s) dF_s - \int_{a_s}^{b_s} F_s(t_s)\overline{\pi}_s(t_s) dt_s - \int_{a_b}^{b_b} (1 - F_b(t_b))\overline{\pi}_b(t_b) dt_b \geq U_s(b_s) + U_b(a_b) \geq 0. \tag{29}$$

This proves the "only if" part of Lemma 4. To prove the "if" part, we need to show that for a function $p(\cdot, \cdot)$ satisfying (16), and when $\overline{p}_s(\cdot)$ and $\overline{p}_b(\cdot)$ are weakly decreasing and increasing, respectively, a payment function exists that makes the mechanism satisfy IC and IP. First, we observe that for $\overline{p}_s(\cdot)$ and $\overline{p}_b(\cdot)$ are weakly decreasing and increasing, respectively, $\overline{\pi}_s(\cdot)$ and $\overline{\pi}_b(\cdot)$, defined in (14) and (15) are also weakly decreasing and increasing, respectively.

Next, consider the following payment function:

$$x(v_s, v_b) = \chi_b(v_b) - \chi_s(v_s) + K, \tag{30}$$

where $\chi_s(\cdot)$ and $\chi_b(\cdot)$ are given by the Lebesgue–Stieltjes integrals:

$$\chi_b(v_b) = \int_{t_b=a_b}^{v_b} t_b d[\overline{\pi}_b(t_b)] \tag{31}$$

$$\chi_s(v_s) = \int_{t_s=a_s}^{v_s} t_s d[-\overline{\pi}_s(t_s)] \tag{32}$$

and K is a constant. To see that this payment function satisfies incentive compatibility, consider for any pair v_s, v_s' in the seller's support:

$$U_s(v_s, v_s) - U_s(v_s', v_s) = -v_s(\overline{p}_s(v_s) - \overline{p}_s(v_s')) - \chi_s(v_s) + \chi_s(v_s')$$

Since $\overline{\pi}_s(v_s) = \overline{p}_s(v_s)$ whenever v_s is in the support of the seller, we have $\overline{p}_s(v_s) - \overline{p}_s(v_s') = -\int_{t_s=v_s'}^{v_s} d[-\overline{\pi}_s(t_s)]$, and $-\chi_s(v_s) + \chi_s(v_s') = -\int_{t_s=v_s'}^{v_s} t_s d[-\overline{\pi}_s(t_s)]$ thus we have:

$$U_s(v_s, v_s) - U_s(v_s', v_s) = v_s \int_{t_s=v_s'}^{v_s} d[-\overline{\pi}_s(t_s)] - \int_{t_s=v_s'}^{v_s} t_s d[-\overline{\pi}_s(t_s)]$$
$$= \int_{t_s=v_s'}^{v_s} (v_s - t_s) d[-\overline{\pi}_s(t_s)] \geq 0, \tag{33}$$

since $\overline{\pi}_s(\cdot)$ is a weakly decreasing function. The proof for the buyer proceeds analogously.

Now, consider the difference $U_s(v_s') - U_s(v_s)$ for some $v_s' \leq v_s$ in the seller's support:

$$U_s(v_s') - U_s(v_s) = - v_s'\overline{p}_s(v_s') + v_s\overline{p}_s(v_s) - \chi_s(v_s') + \chi_s(v_s)$$
$$= - v_s'\overline{p}_s(v_s') + v_s\overline{p}_s(v_s) + \int_{t_s=v_s'}^{v_s} t_s d[-\overline{\pi}_s(t_s)]$$
$$= - v_s'\overline{p}_s(v_s') + v_s\overline{p}_s(v_s) + \int_{t_s=v_s'}^{v_s} \overline{\pi}_s(t_s) dt_s - \Big[t_s\overline{\pi}_s(t_s)\Big]_{t_s=v_s'}^{v_s}$$
$$= \int_{t_s=v_s'}^{v_s} \overline{\pi}_s(t_s) dt_s, \tag{34}$$

where the second to last step follows from integration by parts (we note that $\overline{p}_s$ is left-continuous and non-increasing under our assumptions), and the last step is due to the fact that $\overline{\pi}_s(v_s) = \overline{p}_s(v_s)$ by definition whenever v_s is in the support of the seller. Thus, the payment function (30) yields for any v_s in the seller's support:

$$U_s(v_s) = U_s(b_s) + \int_{t_s=v_s}^{b_s} \overline{\pi}_s(t_s) dt_s \tag{35}$$

A similar calculation shows that for any v_b in the buyer's support, we have:

$$U_b(v_b) = U_b(a_b) + \int_{t_b=a_b}^{v_b} \overline{\pi}_b(t_b) dt_b \tag{36}$$

These two relations imply that under this payment function, the inequality in (27) (and the corresponding one for the buyer) is satisfied with equality, and through the steps that follow, the first inequality in (16) must also be satisfied with equality, and that if the LHS of it is non-negative, $U_s(b_s) + U_b(a_b)$ must also be non-negative.

Now consider $U_s(b_s)$. We have

$$U_s(b_s) = \int_{a_b}^{b_b} (x(b_s, v_b) - b_s p(b_s, v_b)) dF_b$$
$$= \int_{a_b}^{b_b} \int_{t_b=a_b}^{v_b} t_b d[\overline{\pi}_b(t_b)] dF_b - \int_{t_s=a_s}^{b_s} t_s d[-\overline{\pi}_s(t_s)] + K - b_s\overline{p}_s(b_s)$$
$$= \int_{t_b=a_b}^{b_b} (1 - F_b(t_b)) t_b d[\overline{\pi}_b(t_b)] - \int_{t_s=a_s}^{b_s} t_s d[-\overline{\pi}_s(t_s)] - b_s\overline{p}_s(b_s) + K \tag{37}$$

Setting

$$K = -\int_{t_b=a_b}^{b_b} (1 - F_b(t_b))t_b d[\overline{\pi}_b(t_b)] + \int_{t_s=a_s}^{b_s} t_s d[-\overline{\pi}_s(t_s)] + b_s \overline{p}_s(b_s) \qquad (38)$$

ensures that $U_s(b_s) = 0$. Since, in addition, we assume that the LHS of (16) is non-negative, and have shown that it is equal to $U_s(b_s) + U_b(a_b)$, it must follow that $U_b(a_b)$ is also non-negative. This implies, by the envelope theorem, that the mechanism is IP for all buyer and seller types. □

Proof of lemma 5 We will prove the lemma for the case of the seller; the proof works exactly the same way for the buyer.

For a certain trading probability function $p(v_s, v_b)$, that yields non-increasing and non-decreasing $\overline{p}_s(\cdot)$ and $\overline{p}_b(\cdot)$, respectively, consider a mechanism resulting in the following relationship at the focal gap in the seller's distribution:

$$U_s(\underline{v}_s) = U_s(\overline{v}_s) + (\overline{v}_s - \underline{v}_s)(\overline{p}_s(\overline{v}_s) + \gamma)$$

where $0 \leq \gamma \leq (\overline{p}_s(\overline{v}_s) - \overline{p}_s(\underline{v}_s))$, such that incentive compatibility is satisfied for the sellers of type $\overline{v}_s$ and $\underline{v}_s$. Under this mechanism, for all $v_s \leq \underline{v}_s$, the envelope theorem will have an additional payoff increment $\gamma(\overline{v}_s - \underline{v}_s)$ for all $v_s \leq \underline{v}_s$, so we need to modify inequality (27) to read:

$$U_s(v_s) \geq \begin{cases} U_s(b_s) + \int_{v_s}^{b_s} \overline{\pi}_s(t_s)dt_s + \gamma(\overline{v}_s - \underline{v}_s) & \text{for } v_s \leq \underline{v}_s \\ U_s(b_s) + \int_{v_s}^{b_s} \overline{\pi}_s(t_s)dt_s & \text{for } v_s > \underline{v}_s \end{cases} \qquad (39)$$

Furthermore, by choosing the following payment function, we can make sure that (39) is satisfied with equality

$$x(v_s, v_b) = \begin{cases} \chi_b(v_b) - \chi_s(v_s) + K + \gamma(\overline{v}_s - \underline{v}_s) & \text{for } v_s \leq \underline{v}_s \\ \chi_b(v_b) - \chi_s(v_s) + K & \text{for } v_s > \underline{v}_s \end{cases}, \qquad (40)$$

This statement follows straightforwardly from the same calculations as in the proof of the *if* part of lemma 4. To see that the payment function remains IC, note that for the buyer and $v_s, v_s' \leq \underline{v}_s$ or $v_s, v_s' > \underline{v}_s$ the addition of a constant on to payment function makes no difference for incentive compatibility. For $v_s' \leq \underline{v}_s < \overline{v}_s \leq v_s$, we have

$$\begin{aligned} &U_s(v_s, v_s) - U_s(v_s', v_s) \\ &\quad = -v_s(\overline{p}(v_s) - \overline{p}_s(v_s')) - \chi_s(v_s) + \chi_s(v_s') - \gamma(\overline{v}_s - \underline{v}_s) \\ &\quad \geq \int_{t_s=v_s'}^{v_s} (v_s - t_s)d[-\overline{\pi}_s(t_s)] - (\overline{p}_s(\overline{v}_s) - \overline{p}_s(\underline{v}_s))(\overline{v}_s - \underline{v}_s) \geq 0, \end{aligned} \qquad (41)$$

where the first inequality follows from the upper limit we imposed on γ, and the last one from the fact that $\overline{p}_s(\cdot)$ is a non-increasing function. It is easy to verify this payment function results in (39) being satisfied with equality.

Retracing the steps that lead up to (29) in the proof of lemma 4, we can then arrive at a modified condition:

$$G(\gamma) \equiv \int_{a_b}^{b_b} v_b \overline{p}(v_b) dF_b - \int_{a_s}^{b_s} v_s \overline{p}(v_s) dF_s - \gamma(\overline{v}_s - \underline{v}_s) F_s(\underline{v}_s)$$
$$- \int_{a_s}^{b_s} F_s(t_s) \overline{\pi}_s(t_s) dt_s - \int_{a_b}^{b_b} (1 - F_b(t_b)) \overline{\pi}_b(t_b) dt_b$$
$$= U_s(b_s) + U_b(a_b) \geq 0, \tag{42}$$

where we have defined the left hand side as $G(\gamma)$.

Now, the second-best mechanism is given by maximizing the aggregate welfare subject to (42), i.e., maximizing the Lagrangian through the choice of $p(\cdot, \cdot)$ and γ:

$$L = \int_{a_b}^{b_b} \int_{a_s}^{b_s} (v_b - v_s) p(v_s, v_b) dF_s dF_b + \lambda G(\gamma), \tag{43}$$

where $\lambda \geq 0$ is a Lagrange multiplier.[15] But since $G(\gamma)$ is decreasing in γ and γ is bounded by zero from below (by the envelope theorem), the maximum of the Lagrangian requires γ to be zero, which finishes the proof. □

Proof of Lemma 6 We first show that in an equilibrium investment strategy with atoms there must also be a gap; this allows the remainder of the proof to focus on unraveling caused by a gap. Suppose the seller's mixture has an atom of probability mass q_s at some value $v_s^* > 0$, and that there is a v_b^* where the optimal mechanism prescribes that $p(v_s^*, v_b) = 1$ for $v_b \geq v_b^*$ and $p(v_s^*, v_b) = 0$ for $v_b < v_b^*$.[16] Since there is a jump discontinuity in $\overline{p}$ (of magnitude q_s) at v_b^* and we assume $c_b(\cdot)$ is differentiable, the mixing condition cannot be satisfied at v_b just below v_b^*, and therefore some interval below v_b^* cannot be in the support of the buyer's mixed strategy. Furthermore, this means that for some $v_s^* > v_s > v_s^* - \epsilon$, with $\epsilon > 0$, the probability of trade $\overline{p}(v_s)$ will be constant, since the gap in the buyers' distribution means sellers with valuations slightly less than v_s^* cannot trade with additional buyers relative to a seller at v_s^*. This again is inconsistent with the mixing condition for the sellers that states $\overline{p}(v_s) = 1 - c_s'(v_s)$, where $c_s'(v_s)$ is strictly increasing. Similar arguments can be made with regard to an atom in the buyer's distribution, implying that atoms cannot be in the interior of an interval part of either player's support. Given this, to prove Lemma 6, we will focus on a gap of the seller's candidate mixed investment strategy in an equilibrium; similar arguments apply for the buyer. First, note that the mixing condition for the seller is given by:

$$v_s + U_s(v_s) - c_s(v_s) = v_s' + U_s(v_s') - c_s(v_s'), \tag{44}$$

15 See for example Gelfand and Fomin (1963) for the use of the method of Lagrange multipliers in the calculus of variations.

16 We can restrict our attention to such deterministic mechanisms, since the gains from trade is linear in $p(\cdot, \cdot)$ and the differences in valuation. Therefore, the designer will always be willing to trade an intermediate probability of trade with lower (higher) valuation buyer (seller) with the same probability of trade with higher (lower) valuation buyer (seller).

for any v_s, v'_s in the support of the seller's mixed strategy. Hence, for a pair of values $\underline{v}_s$, $\overline{v}_s$ bordering a gap, we must have $U_s(\underline{v}_s) - U_s(\overline{v}_s) = \overline{v}_s - \underline{v}_s + c_s(\underline{v}_s) - c_s(\overline{v}_s)$. Dividing by $\overline{v}_s - \underline{v}_s$, we get:

$$1 + \frac{c_s(\underline{v}_s) - c_s(\overline{v}_s)}{\overline{v}_s - \underline{v}_s} = \frac{U_s(\underline{v}_s) - U_s(\overline{v}_s)}{\overline{v}_s - \underline{v}_s} = \overline{p}_s(\overline{v}_s), \tag{45}$$

where the last equality follows from Lemma 5 for a second-best mechanism. From the convexity of the cost function, we have:

$$1 + \frac{c_s(\underline{v}_s) - c_s(\overline{v}_s)}{\overline{v}_s - \underline{v}_s} = \overline{p}_s(\overline{v}_s) < 1 + c'_s(\overline{v}_s), \tag{46}$$

Now, consider a deviation from a candidate mixed-strategy equilibrium where a seller invests at $\overline{v}_s - \epsilon$, but reports $\overline{v}_s$. For small ϵ, the expected change in payoff from this deviation is given by:

$$(1 - \overline{p}_s(\overline{v}_s) + c'_s(\overline{v}_s))\epsilon > 0, \tag{47}$$

meaning that such a deviation will be profitable. Hence, the candidate equilibrium is not an equilibrium strategy. This implies there cannot be any gaps in the seller's equilibrium investment strategy.

The above argument applies when $v_s^* > 0$, since it shows unraveling below v_s^*. The unraveling in the case with an atom at $v_s^* = 0$ can be seen by showing the buyer's support cannot have a gap. As noted on page 45, the jump in the trading probability $\overline{p}$ at some v_b^* means that some interval below v_b^* cannot be in the buyer's support. This fact is true for $v_s^* = 0$ as well. Since this v_b^* is the lowest valuation buyer that trades with $v_s = 0$, it must be that $v_b^* = a_b$, unless $a_b = 0$ (since buyers that do not trade by incur a non-zero cost of investment would be better off not investing at all). If $a_b > 0$, this mixing strategy is not stable (provided that first-best is not attainable), since the second-best mechanism gives the lowest valuation buyer zero expected payoff from trade,[17] and therefore, this buyer is better off investing nothing. If $a_b = 0$, there must be a gap between a_b and v_b^*: since v_b^* is the lowest buyer that can trade with the lowest seller, no positive investment lower than v_b^* can be part of the equilibrium support. Having established a gap in the buyer's distribution, we can use Lemma 5 and the same argument as above for the seller to show that the following deviation from the equilibrium is profitable: invest $\underline{v}_b^* + \epsilon$ with $\epsilon > 0$ small and report $\underline{v}_b^*$. □

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[17] This can be seen by noting that our Lemma 5 implies that the inequalities in Lemma 4 are satisfied with equality for a second-best mechanism.

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