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LEAD ARTICLE

Consistency and its converse: an introduction

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Abstract This essay is a didactic introduction to the literature on the “consistency principle” and its “converse”. An allocation rule is consistent if for each problem in its domain of definition and each alternative that it chooses for it, then for the “reduced problem” obtained by imagining the departure of an arbitrary subgroup of the agents with their “components of the alternative” and reassessing the options open to the remaining agents, it chooses the restriction of the alternative to that subgroup. Converse consistency pertains to the opposite operation. It allows us to deduce that a rule chooses an alternative for a problem from the knowledge that for each two-agent subgroup, it chooses its restriction to the subgroup for the associated reduced problem this subgroup faces. We present two lemmas that have played a critical role in helping understand the implications of these properties in a great variety of models, the Elevator Lemma and the Bracing Lemma. We describe several applications. Finally, we illustrate the versatility of consistency and of its converse by means of a sample of characterizations based on them.

Keywords Consistency · Converse consistency · Elevator Lemma · Bracing Lemma

JEL Classification C79 · D63 · D74

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1 Introduction

This essay is a didactic introduction to the literature on the "consistency principle" and its "converse". A "solution" associates with each decision problem in some domain one or several of its feasible alternatives, usually interpreted as recommendations for it. The principles pertain to the behavior of solutions whose domains of definition contain problems involving populations that may vary from one to the other, and for which it is therefore meaningful to compare the choices they make for different populations. (By contrast, most of the early axiomatic literature had been written for fixed populations.) Informally, the consistency principle states the following. Consider a problem and an alternative chosen by a solution for it. Then, imagine some agents "leaving with their components of the alternative", and examine the "reduced problem" that the remaining agents face, that is, reevaluate their opportunities at that point. The solution is consistent if for this problem, it chooses the restriction of the alternative to this subgroup. Thus, there is no need to reevaluate a chosen alternative as we start implementing it.

The principle of converse consistency pertains to the opposite operation. Consider a problem and an alternative in its feasible set. Suppose that for each two-agent subgroup of the agents this problem involves, a solution chooses the restriction of the alternative to the subgroup for the associated reduced problem this subgroup faces. The solution is conversely consistent if, under these conditions, it chooses the alternative for the initial problem. Thus, the desirability of an alternative can be deduced from the desirability, for each of its two-agent subgroups, of its restriction to the subgroup for the associated reduced problem the subgroup faces.

The principles, first investigated for abstract models of cooperative game theory, have now been examined in the context of a great variety of concrete problems of resource allocation, and for many models, their implications are quite well understood. We will survey some of these developments.

The essay is organized as follows. We first introduce the basic concepts of a "problem" and of a "solution", and our two principles. We also describe a number of variants of the principles. Then, we present two lemmas involving the principles that have been critical in understanding their implications in a wide range of situations. Finally, we sketch several characterizations based on the principles.

In order to illustrate various ideas and techniques, we specify a number of models in succession. Our objective however is not a comprehensive account of what is known of the principles for these models, but rather to show their usefulness in evaluating allocation rules, and in exposing some of the mechanics of proofs based on them. For a detailed survey of the vast literature devoted to their study, see Thomson (2011b).[1]

[1] Some of the recent studies in which consistency and its converse have played a central or important role concern airport problems (Potters and Sudhölter 1999), minimal cost spanning tree problems (Dutta and Kar 2004), probabilistic assignment (Chambers 2004), abstract social choice (Hwang et al. 2006; Yeh 2006), siting of public facilities (Ju 2008), queueing (Chun 2011), and roommate problems (Klaus and Nichifor 2010).

2 Basic concepts: domains and solutions

A **problem** is given by a set of **alternatives** and a set of **agents** whose preferences are defined over this set, or over "personal" components of it. A **domain of problems** satisfying some regularity conditions is specified, and the objective is to identify one or several feasible alternatives for each problem in the domain. Depending upon the context, such an alternative is interpreted as a recommendation that an arbitrator, (or a planner, a high-level manager, a judge ...) could make, or as a prediction of what the agents could choose if left to their own devices. A **solution** is a correspondence defined on the domain that associates with each problem in the domain a non-empty subset of its feasible set. For several of the models that we use as examples, the most interesting solutions are *single-valued*, but we will proceed under the assumption that this property may not hold. Indeed, for a number of others, it is very demanding, and the central solutions are multi-valued.

A number of tests can be devised to evaluate how satisfactory a solution is. Consistency involves testing each alternative it chooses for each group of agents and each problem involving this group, to the choices it makes for the "reduced" problems associated with subgroups and this alternative.

We use the following notation and language. There is an infinite set of "potential" agents indexed by the natural numbers, $\mathbb{N}$. To specify a problem, we first draw a finite number of them from this infinite population. Let $\mathcal{N}$ denote the family of nonempty finite subsets of $\mathbb{N}$. For each $N \in \mathcal{N}$, there is a class of problems $\boldsymbol{\mathcal{D}^N}$ that N could face. Solutions are defined over the union $\boldsymbol{\mathcal{D}} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{D}^N$ of these classes as N varies in $\mathcal{N}$. When an alternative is chosen by a solution S for a problem D, we say that it is ***S*-optimal for** D. If a solution $\bar{S}$ only chooses alternatives that are also chosen by some solution S, we say that $\bar{\boldsymbol{S}}$ **is a subsolution of** $\boldsymbol{S}$. For simplicity, we introduce each domain of problems, $\mathcal{D}^N$, for a fixed population $N \in \mathcal{N}$, but once again, solutions are understood to be defined over $\bigcup_{N \in \mathcal{N}} \mathcal{D}^N$.

To see the need for testing how solutions behave when the population of agents varies, we introduce our first domain, which pertains to fair division.

Domain 1 *A* **classical problem of fair division** *(see Thomson 2010a, for a survey of the literature on the subject) is a pair* (R, Ω) *where* $R \equiv (R_i)_{i \in N}$ *is a list of preference relations defined on the non-negative quadrant* $\mathbb{R}^\ell_+$ *of the* ℓ*-dimensional commodity space for some* $\ell \in \mathbb{N}$ *and* $\Omega \in \mathbb{R}^\ell_{++}$ *is a social endowment. Preferences are continuous, monotone increasing, and convex. The asymmetric part of* R_i *is denoted by* P_i *and the associated indifference relation is denoted by* I_i*. A feasible allocation for* (R, Ω) *is a list* $z \in \mathbb{R}^{\ell N}_+$ *such that* $\sum z_i = \Omega$.[2]

Several of the solutions defined for this model will also play a role in the analysis of models introduced later. We will not repeat the formal definitions then.

Examples of solutions for Domain 1 *The* **Pareto solution***,* $\boldsymbol{P}$*, chooses the feasible allocations* z *of* (R, Ω) *for which there is no other feasible allocation* z' *such that for*

[2] By $\mathbb{R}^N$ and $\mathbb{R}^{N\ell}$ we mean the cross-product of $|N|$ copies of $\mathbb{R}$ and $\mathbb{R}^\ell$ respectively, indexed by the members of N.

.	.	.	.	.	.
General n	W_{ed}	F	W_{ed}	P	P
.	.	.	.	.	.
.	.	.	.	.	.
$n=4$	W_{ed}	F	W_{ed}	P	W_{ed}
$n=3$	W_{ed}	F	FP	FP	P
$n=2$	W_{ed}	F	P	W_{ed}	F
	(a)	**(b)**	**(c)**	**(d)**	**(e)**

Fig. 1 Solutions defined on a domain of problems involving variable populations. Which one(s) of the solutions is (are) *consistent*? **a** This solution chooses the equal-division Walrasian allocations for each economy. **b** This solution chooses the envy-free allocations for each economy. **c** This hybrid solution is more and more restrictive as the number of agents increases: it chooses the efficient allocations for each two-agent economy, the envy-free and efficient allocations for each three-agent economy, and the equal-division Walrasian allocations for each economy with more than three agents. **d** By contrast, this solution is less and less restrictive as the number of agents increases. **e** This solution makes choices that do not seem to follow any particular pattern as the number of agents changes

each $i \in N$, $z'_i \; R_i \; z_i$, and for at least one $i \in N$, $z'_i \; P_i \; z_i$. The **no-envy solution***, $\boldsymbol{F}$, chooses the feasible allocations z such that for each $\{i, j\} \subseteq N$, $z_i \; R_i \; z_j$. The* **equal-division lower bound solution***, $\boldsymbol{B_{ed}}$, chooses the feasible allocations z such that for each $i \in N$, $z_i \; R_i \; \frac{\Omega}{|N|}$. The* **egalitarian-equivalence solution** *chooses the feasible allocations z such that for some "reference bundle" $z_0 \in \mathbb{R}^{\ell}_{+}$ and for each $i \in N$, $z_i \; I_i \; z_0$.* **The equal-division Walrasian solution***, $\boldsymbol{W_{ed}}$, chooses the feasible allocations z for which there is a price vector $p \in \Delta^{\ell}$ such that for each $i \in N$, z_i maximizes R_i in the budget set $\left\{z'_i \in \mathbb{R}^{\ell}_{+} : pz'_i \leq p\frac{\Omega}{|N|}\right\}$.*[3]

Figure 1 illustrates the great freedom available in defining solutions in a variable population framework. Column *a* represents the solution that chooses the equal-division Walrasian allocations for each economy, and Column *b* the solution that chooses the envy-free allocations for each economy. When a solution is defined over a domain of problems involving arbitrary sets of agents, it can in principle choose allocations in completely different ways as population varies. Giving free rein to our imagination, let us consider for instance the solution that chooses the efficient allocations for each two-agent economy, the envy-free and efficient allocations for each three-agent economy, and the equal-division Walrasian allocations for each economy involving more agents (Column *c*). Columns $d - e$ represent two other solutions, both of which also seem quite hard to justify. Note that in these examples as well as in several of the figures below, solutions are schematically shown as making recommendations in a manner that depends on the number of agents and not on their identify, but in general, a solution could take this data into consideration too. In any case, these examples make it clear that some tests are needed to relate the choices made by solutions for different sets of agents. In the next two sections we introduce two such tests.

[3] The notation Δ^{ℓ} designates the unit simplex in the ℓ-dimensional Euclidean space.

3 Consistency and its converse

In this section, we introduce the ideas of "consistency" "converse consistency", and illustrate them by means of several examples. We only give a few sample proofs that certain solutions pass or do not pass these tests. In most cases, these are simple exercises, which we suggest to the reader as a way of progressively strengthening his or her understanding of the subject, as well as gaining familiarity with the models, not all of which are standard, that we will discuss. The figures should be seen as an integral part of our exposition, as their detailed legends sometimes contain sketches of proofs.

3.1 Consistent allocation rules: the general definition

Very informally, we require of a solution that there should never be a need to revise an alternative it has chosen, once some agents "have received their components of it" and left. Here, the clause "have received their components of the alternative" is only meant to be suggestive, and we devote the next few pages to giving it substance.

To take a first step in this direction, let $N \in \mathcal{N}$ and $D \in \mathcal{D}^N$ be a problem that N could face. Let S be a solution and x be one of the S-optimal alternatives for D. Now, let us imagine some agents leaving with their components of x, and let us reevaluate the situation at this point, namely let us identify the options that are still available to the remaining agents. If N' is the subgroup they constitute, we refer to the set of alternatives at which the agents who leave (the subgroup $N \backslash N'$) receive their components of x, as the **reduced problem of D with respect to N' and x**. We denote it by $\boldsymbol{r^x_{N'}(D)}$. What has to be clarified for each model is what should be understood by this reduced problem. It is not always obvious how to do so, but we can already give our central definition. Let S be a solution.

Consistency: For each group $N \in \mathcal{N}$, each problem $D \in \mathcal{D}^N$, each subgroup $N' \subset N$, and each S-optimal alternative of D, x, if the reduced problem of D with respect to N' and x belongs to $\mathcal{D}^{N'}$, then the restriction of x to N' is S-optimal for it: $x_{N'} \in S\left(r^x_{N'}(D)\right)$.

We have presented the consistency principle as a robustness requirement on a solution: the solution should make coherent choices as population changes. The principle also has a fairness interpretation. The central idea of solidarity is that when the circumstances in which some group of agents find themselves change, and if none of them bears any particular responsibility for the change, or deserves any particular credit for it, then their welfare should be affected in the same direction. One expression of the idea involves imagining that some agents leave empty-handed, and it gives us the requirement of *population monotonicity* (Thomson 1983). Another expression is obtained by imagining instead that the members of the group $N \backslash N'$ leave *with their components of* x (as opposed to empty-handed). In many situations in which *efficiency* is imposed too, requiring that the welfare of all the members of N' should be affected in the same direction essentially leads to the invariance requirement expressed by *consistency*.

The following variants of *consistency* have been explored. (i) First is the property, which often turns out to be only slightly weaker, that is obtained by limiting attention to subgroups of *two* remaining agents, a variant called **bilateral consistency**. (ii) Apart from size, it is sometimes natural to impose other restrictions on the subgroups. The class they constitute may be endowed with some particular structure that reflects certain aspects of social organization, such as communication networks, trade groups, family relations, and so on. (iii) A third variant is obtained, for *single-valued* solutions, and in models in which feasible sets are equipped with a convex structure, as follows: instead of asking that the restriction of x to each subgroup be chosen for the associated reduced problem that this subgroup faces, we require that for each agent, his component of x coincide with the *average* of what he would receive in the reduced problems associated with x and all of the proper subgroups of N to which he belongs (Maschler and Owen 1989). This variant is called **average consistency**. A bilateral version of this variant can be defined in which the averages only involve reduced problems with two agents.

We will illustrate various choices that we have in defining reduced problems by considering several applications.

3.1.1 The reduction operation for models formulated in commodity space

We start with classical problems of fair division (Domain 1). There, given the natural "separability" of allocation space, a very simple definition of a reduced economy suggests itself. When some agents leave, they take along the bundles intended for them. Thus, the options available to the remaining agents are the lists of bundles, one bundle for each of them, obtained by redividing the sum of the bundles that were intended for them in the first place.[4] Specifically, given $e \equiv (R, \Omega)$, $N' \subset N$, and $z \in S(e)$, the reduced economy of e with respect to N' and z is the pair $(R_{N'}, \Omega - \sum_{N \setminus N'} z_i)$, or equivalently $(R_{N'}, \sum_{N'} z_i)$.

It is easy to see that the Pareto solution is *consistent* since, if no Pareto-improving reallocation of the social endowment can be achieved by the group N, then of course no Pareto-improving reallocation of the resources they have received can be achieved by any subgroup $N' \subset N$. The no-envy solution is *consistent* too: if an agent would not want to exchange bundles with anyone in the initial group N, then *a fortiori*, he would not want to exchange bundles with anyone in any subgroup $N' \subset N$.

On the other hand, the equal-division lower bound solution is not *consistent* (Fig. 2b). It is tempting to say that this is because in a reduced economy, the point of equal division is typically not what it was in the initial economy. Thus, if initially, an agent finds his component of some allocation at least as desirable as equal division, there is no reason why this should still be true in a reduced economy associated with the allocation. But that is not the whole story because a solution such as the equal-division Walrasian solution, which obviously also depends on equal division, is *consistent* (Fig. 2a). The reason is that the points of equal division in the reduced economies associated with an equal-division Walrasian allocation z of some initial economy are related in a very special way, namely, they all have the same value at some prices supporting z. Thus, in each of these reduced economies, if these same

[4] This is because feasibility is defined with an equality sign. It does make a difference whether that is the case or not, as shown by Thomson (1988) and Ehlers and Klaus (2006).

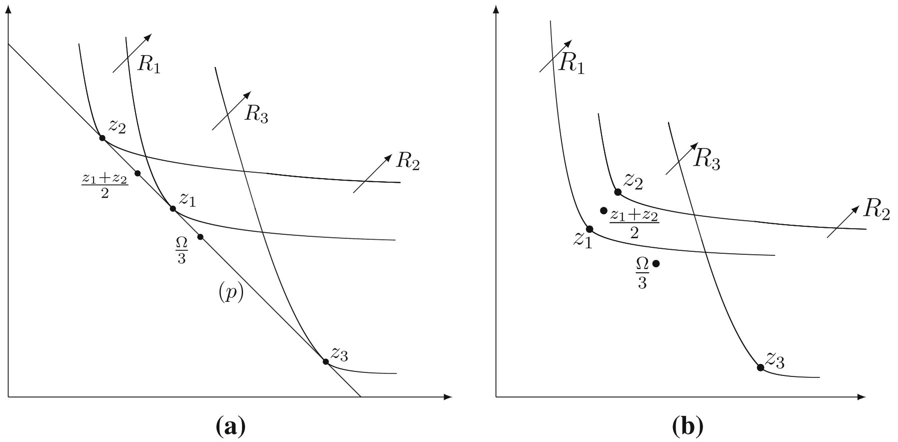

Fig. 2 Two solutions for the problem of fair division, one consistent and the other not. **a** The equal-division Walrasian solution is *consistent*: z is an equal-division Walrasian allocation in the economy with agent set $\{1, 2, 3\}$ depicted here; after agent 3 leaves with his bundle z_3, the resources that remain available to agents 1 and 2 are $z_1 + z_2$, and if each of them is endowed with $\frac{z_1+z_2}{2}$, equilibrium is indeed achieved by quoting the same prices: the pair (z_1, z_2) is a Walrasian allocation of the economy $(R_1, R_2, z_1 + z_2)$. **b** The equal-division lower bound and Pareto solution is not *consistent*: here, z meets the bound in the three-agent economy, but since $\frac{z_1+z_2}{2}\ P_1\ z_1$, the restriction (z_1, z_2) of z to the subgroup $\{1, 2\}$ does not meet the bound in $(R_1, R_2, z_1 + z_2)$

prices are quoted, the budget sets the remaining agents face are the same as they were initially; this guarantees that for each such agent, the same bundle maximizes his preferences, and in turn, equality of demand and supply.

We consider next the problem of rationing. Such a problem can be modeled as an economy with single-peaked preferences, as follows:

Domain 2 *A* **fair division problem with single-peaked preferences** (Sprumont 1991) *is a pair* (R, Ω) *where* $R \equiv (R_i)_{i\in N}$ *is a list of single-peaked preference relations defined on* $\mathbb{R}_+$*, and* $\Omega \in \mathbb{R}_+$ *is a social endowment of an infinitely divisible commodity. Single-peakedness of* R_i *means that* R_i *has a satiation amount, denoted by* $p(R_i)$*, and for each pair* $\{z_i, z'_i\}$ *such that* $z'_i < z_i \leq p(R_i)$ *or* $p(R_i) \leq z_i < z'_i$*, we have* $z_i\ P_i\ z'_i$*. A feasible allocation is a list* $z \in \mathbb{R}^N_+$ *such that* $\sum z_i = \Omega$*.*

Examples of solutions for Domain 2 *The* **Pareto solution***, the* **no-envy solution***, and the* **equal-division lower bound solution** *are defined as for classical problems of fair division. The* **uniform rule** *chooses the feasible allocation* z *of* (R, Ω) *such that for some* $\lambda \in \mathbb{R}_+$*, if* $\sum p(R_i) \geq \Omega$*, then for each* $i \in N$*,* $z_i = \min\{p(R_i), \lambda\}$*, and if* $\sum p(R_i) \leq \Omega$*, then for each* $i \in N$*,* $z_i = \max\{p(R_i), \lambda\}$*, (Fig. 13a illustrates the definition).*

Here too, when defining a reduction, we simply imagine some agents leaving with their assignments; the social endowment is whatever is left of the endowment. It is easy to see that the Pareto solution, the no-envy solution, and the uniform rule are *consistent*; the equal-division lower bound solution is not.

3.1.2 The reduction operation for models formulated in utility space

For the model of bargaining presented next, feasible sets are given in utility space.

Domain 3 *A* **bargaining problem** *is a compact, convex, and comprehensive*[5] *subset* T *of* $\mathbb{R}^N_+$, *containing at least one strictly positive point.*

Examples of solutions for Domain 3 *The* **Nash solution**, $\bar{\boldsymbol{N}}$, (Nash 1950) *chooses the point of* T *that maximizes the product of utilities. The* **egalitarian solution**, $\boldsymbol{E}$, (Kalai 1977) *chooses the maximal point of* T *of equal utilities. Now call the "ideal point of* T*" the point whose* i*-th coordinate is the maximal utility agent* i *can achieve in* T*. Then, the* **Kalai-Smorodinsky solution**, $\boldsymbol{K}$, (Kalai and Smorodinsky 1975) *chooses the maximal point of* T *proportional to its ideal point.*

Here, agents are assigned utility levels, and when some of them leave, it is most natural to define the set of options open to the remaining agents as the subset of the initial problem consisting of all the vectors at which the departing agents receive their promised payoffs. Given a problem T involving some group N, the reduced problem of T with respect to $N' \subset N$ and x is therefore $\{x' \in \mathbb{R}^{N'}:$ for some $y \in T$, $y_{N \setminus N'} = x_{N \setminus N'}$ and $y_{N'} = x'\}$. Geometrically, this is the section of T by the plane parallel to the N'-coordinate subspace through x, seen as a subset of $\mathbb{R}^{N'}$.

The Nash solution is *consistent* but the Kalai-Smorodinsky solution is not (Fig. 3). The egalitarian solution is *consistent* on the subdomain of problems on which it chooses Pareto-optimal outcomes (in general, this solution only selects "weakly" Pareto optimal points)[6].

To understand the difference between the definition considered here and the definition we gave above for classical problems of fair division, note that when a feasible set is defined as the image in utility space of a set of allocations obtained by distributing a social endowment, if the departing agents leave with physical amounts of goods giving them their agreed-upon utilities, the set of options available to the remaining agents is typically a subset of the feasible set of the reduced game as we just defined it for bargaining problems (except in the one-good case).

3.1.3 The reduction operation when the departing agents remain "available"

In some models, the phrase "an agent leaves with his payoff" should only be understood to mean that his payoff has been decided. Provided that he does get it in the end, the agents who stay may still be able to "cooperate" with him. This is illustrated by the next model.

Domain 4 *A (transferable utility)* **coalitional game** *is a vector* $v \in \mathbb{R}^{2^{|N|-1}}$, *with coordinates indexed by the nonempty subgroups of* N, *called coalitions, each coordinate of* v *being interpreted as what the corresponding coalition can achieve; it is the* **worth** *of the coalition.*

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[5] Comprehensiveness means that if $x \in \mathbb{R}^N$ is feasible, then so is any other vector y such that $0 \leq y \leq x$.

[6] This is a feasible point that is not dominated in every coordinate by some feasible point.

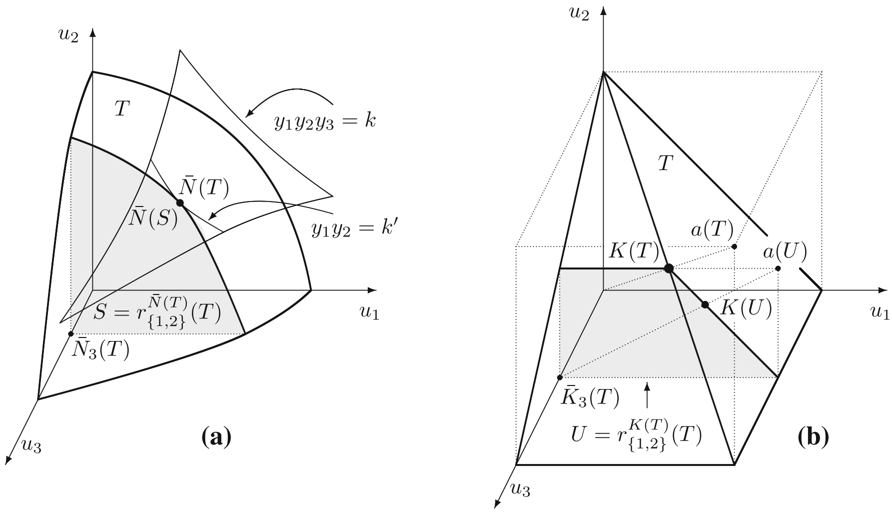

Fig. 3 Two bargaining solutions, one consistent, the other not. **a** The Nash solution is *consistent*: if x achieves maximal product of utilities in the problem T with agent set $\{1, 2, 3\}$, so that $x = \bar{N}(T)$, then (x_1, x_2) achieves maximal product of utilities in the section of T by the plane parallel to the $\{1, 2\}$-coordinate subspace passing through x (*the shaded area*). **b** The Kalai-Smorodinsky solution is not *consistent*: it chooses the point $K(T)$ for the problem T with agent set $\{1, 2, 3\}$ but not $(K_1(T), K_2(T))$ for the section of T by the plane parallel to the $\{1, 2\}$-coordinate subspace through $K(T)$ (*the shaded area*). The Kalai-Smorodinsky solution outcome of $U \equiv K^{K(T)}_{\{1,2\}}(T)$ differs from $K_{\{1,2\}}(T)$

Examples of solutions for Domain 4 *The* **core** (Gillies 1959) *chooses for v all payoff vectors $x \in \mathbb{R}^N$ such that $\sum_N x_i = v(N)$ and for each $C \subset N$, $\sum_C x_i \geq v(C)$. The* **Shapley value** (Shapley 1953) *chooses the payoff vector whose i-th coordinate is equal to $\sum_{C \subseteq N:\, i \in C} k_C\, (v(C) - v(C \backslash \{i\}))$ for certain combinatorial coefficient k_C. Next, define the "dissatisfaction of coalition C at the payoff vector x" to be the difference $v(C) - \sum_C x_i$; then, the* **prenucleolus** (Schmeidler 1969) *chooses the feasible payoff vector at which the dissatisfactions of coalitions are minimized in a lexicographic way, starting with the most dissatisfied coalition.*[7]

Here, in the definition of a reduced game, different scenarios can be imagined in which the commitment to give some agents the payoffs assigned to them by a solution is honored. Each results in a particular specification of the set of options available to the remaining agents. Given a coalition $C \subset N'$, a first possibility is to calculate, as for bargaining problems, what C can achieve by getting together with the departing agents—cooperating with them produces the worth $v(C \cup N \backslash N')$—and giving them their agreed-upon payoffs. What remains for C is the difference $v(C \cup N \backslash N') - \sum_{N \backslash N'} x_i$; it is its worth in the reduced game. For the grand coalition N', we propose the difference $v(N) - \sum_{N \backslash N'} x_i$. Since the complement of

[7] This means that it chooses the payoff vector at which the dissatisfaction of the most dissatisfied coalition is minimal if there is a unique such vector. Otherwise, among the minimizers, it chooses the vector at which the dissatisfaction of the second most dissatisfied coalition is minimal if there is a unique such vector; otherwise, and so on.

N' is involved in the reduced game, we name the resulting condition **complement consistency** (Moulin 1988).

Alternatively, we can let C choose which ones of the departing agents to cooperate with. By getting together with $C' \subseteq N \backslash N'$, the worth $v(C \cup C')$ is generated, but since the members of C' have to be paid $\sum_{C'} x_i$ in total, what remains for C is the difference $v(C \cup C') - \sum_{C'} x_i$. Then, the worth of C in the reduced game is defined to be the *maximal* such difference when C' ranges over the subsets of $N \backslash N'$. For the grand coalition, we choose the same formula as for *complement consistency*. As this definition is based on a maximization exercise, we name it **max-consistency** (Davis and Maschler 1965).

The core satisfies both definitions and the Shapley value satisfies neither. The prenucleolus only satisfies the second one.

3.1.4 When the reduction operation suggests a reduction of consumption spaces

In any model formulated in commodity space, it is possible and sometimes natural to require that solutions only depend on the restrictions of preferences to the set of bundles that are actually feasible.[8] When some agents leave with certain resources, the set of bundles achievable by any one of the remaining agents gets smaller, which then calls for redefining their preferences. For some models, consumption spaces are "decomposable" in a way that makes requiring this kind of independence even more tempting. Then, we say that we "reduce" consumption spaces and preference relations. An illustration is provided by the next domain.

Domain 5 *An* **allocation problem with indivisible goods and one infinitely divisible good** (Svensson 1983) *is a list* (M, A, R) *where* $M \in \mathbb{R}$ *is some amount of an infinitely divisible good (often called "money"),* A *is a finite set of "objects" drawn from some infinite list* $\mathcal{A}$ *and* $R \equiv (R_i)_{i \in N}$ *is a list of preference relations defined over the product* $\mathbb{R} \times A$. *Preferences are continuous and strictly monotonic with respect to the divisible good. We assume* $|N| = |A|$. *A feasible allocation is a pair* (m, σ) *where* $m \in \mathbb{R}^N$ *is a list of amounts of the divisible good satisfying* $\sum_N m_i = M$ *and* σ *is a bijection from* N *to* A *indicating which object each agent receives.*

This model is illustrated in Fig. 4a for a three-agent example. With each object is associated an axis along which the amount of the divisible good that will go with it is measured, thereby defining a bundle assigned to one agent. We impose no sign constraint on the consumption of the divisible good. The broken lines connect bundles that are indifferent to each other for a particular agent.

Examples of solutions for Domain 5 *The* **Pareto**, **no-envy**, *and* **egalitarian-equivalence** *solutions are all still meaningful here, and there is no need to restate the definitions.*[9]

[8] A number of interesting solutions do not satisfy this requirement however, the equal-division Walrasian solution being an example.

[9] For this model, the no-envy solution is a subsolution of the Pareto solution (Svensson 1983).

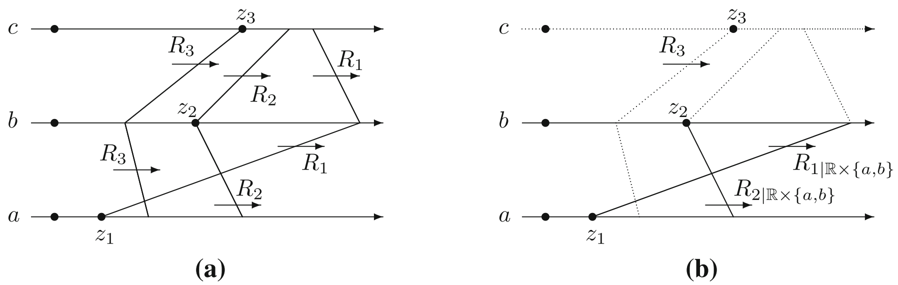

Fig. 4 Reducing a consumption space and a preference relation. **a** We start from an economy with agent set $\{1, 2, 3\}$ in which each of the three agents has preferences defined over the product of $\mathbb{R}$ with the set consisting of the three objects available, $\{a, b, c\}$. The chosen allocation is z. **b** Agent 3 leaves with his bundle z_3, which contains object c. In the reduced economy associated with the group $\{1, 2\}$ and z, the preferences of agents 1 and 2 are restricted to the product of $\mathbb{R}$ with the set consisting of the remaining objects, $\{a, b\}$

It is important to understand that whether or not we reduce consumption spaces affects which solutions are *consistent*. For instance, if we do, the egalitarian-equivalence solution violates the property; indeed, an agent could leave with the object appearing in the reference bundle associated with some egalitarian-equivalent allocation taken as point of departure. But if we do not, the solution is *consistent*, just as it is for classical problems of fair division (Domain 1). For the no-envy solution however, it does not matter which specification is adopted; it is *consistent* either way.

Another model for which it is natural to reduce consumption spaces pertains is the following:

Domain 6 *A* **matching problem** (Gale and Shapley 1962) *is defined by first partitioning the set of agents N, assumed to contain an even number of agents, into two groups of equal sizes, denoted by M and W, and called "men" and "women"; then, specifying a list R of strict preference relations for them; for each $i \in M$, R_i is defined over W, and for each $i \in W$, R_i is defined over M. A feasible allocation is a bijection, or "match", from the set of men to the set of women.*

Examples of solutions for Domain 6 *The* **Pareto solution** *is defined in the usual way. The* **stable solution** *chooses for each matching problem the set of matches such that there is no pair of a man and a woman who prefer each other to their assigned mates. (There always are such matches.)*[10] *The* **man-optimal solution** *chooses the stable match that is best for each man in the set of all stable matches.* (*There always is such a match.*) *The* **woman-optimal** *solution is defined in a symmetric way.*

Given a match chosen by a solution for a particular matching problem, we imagine agents leaving in matched pairs, and in a reduced economy, we define the preferences of each remaining agent to be the restriction of his or her preferences over his or her remaining possible partners. It is easy to see that as usual, the Pareto solution is

[10] This says that the match is not "blocked" by a pair of a man and a woman; requiring that the no-blocking conditions be met for each group, as in the usual definition of the core, is actually not more restrictive: the stable solution coincides with the core.

consistent. So is the stable solution. The man-optimal solution is not however, and of course neither is the woman-optimal solution.

3.1.5 Closedness of domains under the reduction operation

According to our definition of *consistency*, nothing is required of a solution if the reduced problem does not belong to its domain of definition. A stronger definition is obtained by *adding* the requirement that the reduced problem be in the domain.

How restrictive this additional requirement is depends on the situation. There are domains such that, for each feasible outcome, the natural way to define the reduction produces a problem that is automatically included. We then say that **the domain is closed under the reduction operation**. An example here are classical problems of fair division (Domain 1): if (R, Ω) is admissible and z is a feasible allocation for it, then $(R_{N'}, \sum_{N'} z_i)$ is admissible too. On the other hand, suppose that instead of thinking of the departing agents leaving with their components of z, we had imagined them leaving with the commitment that whatever allocation is eventually chosen should give them the welfare levels they experience at z. Then, the reduced problem of (R, Ω) with respect to N' and z would consists of all the lists $({z'}_i)_{i \in N'} \in \mathbb{R}_+^{\ell N'}$ such that for some list $({z'}_i)_{i \in N \setminus N'} \in \mathbb{R}_+^{\ell N \setminus N'}$, we have (i) $\sum_N {z'}_i = \Omega$ and (ii) for each $i \in N \setminus N'$, ${z'}_i \; I_i \; z_i$ (as discussed in Subsubsect. 3.1.3). Such a reduced economy could not be described as a pair $(R_{N'}, \Omega')$ for some $\Omega' \in \mathbb{R}_+^{\ell}$. Closedness would fail.

For another example, consider economies with production. There, what first comes to mind in defining the production set of a reduced economy is to translate that of the initial economy by the vector of goods taken with them by the departing agents. But it is unlikely that this resulting set will satisfy the same regularity conditions, such as “no free lunch”, “increasing returns to scale”, . . ., that may have been imposed on the initial production set.

In other cases, the reduced problem is admissible only if the outcome is chosen in a certain way. If this happens for the outcomes chosen by a particular solution, we say that the domain is closed under the reduction operation **for the solution**. This is illustrated by our next domain.

Domain 7 *A* **claims problem** (O’Neill 1982; Thomson 2003, *is a survey) is a list* $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$ *such that* $\sum c_i \geq E$. *The number* c_i *is the* **claim** *of agent* i *on the* **endowment** E *(for example, the liquidation value of a bankrupt firm). A feasible allocation, called an* **awards vector**, *for* (c, E) *is a list* $x \in \mathbb{R}_+^N$ *such that* $\sum x_i = E$.

Examples of solutions for Domain 7 *The* **proportional solution** *chooses the awards vector of* (c, E) *proportional to the claim vector. The* **constrained equal awards solution** *chooses the awards vector such that for some* $\lambda \in \mathbb{R}_+$, *each claimant* $i \in N$ *receives* $\min\{c_i, \lambda\}$. *The* **constrained equal losses solution** *chooses the awards vector such that for some* $\lambda \in \mathbb{R}_+$, *each claimant* $i \in N$ *receives* $\max\{c_i - \lambda, 0\}$. **Concede-and-divide**, *a solution defined for the two-claimant case, assigns to each claimant* $i \in N$ *the difference* $\max\{E - c_j, 0\}$ *between the endowment and the other agent’s claim (or 0 if this difference is negative)—this amount is conceded by that agent—and divides the remainder equally between them. For the* **random arrival solution**, *imagine agents arriving in random order, and calculate for each of them the minimum of* *his claim and whatever remains when he arrives. Assign to each claimant the average of these amounts under the assumption that all orders of arrival are equally likely.*

Here, the most natural way of defining the reduced problem of (c, E) with respect to $N' \subset N$ and an awards vector x is $(c_{N'}, E - \sum_{N \setminus N'} x_i)$, or equivalently, $(c_{N'}, \sum_{N'} x_i)$. It is easy to see that the proportional solution is *consistent*, and that so are the constrained equal awards and constrained equal losses solutions. On the other hand, simple examples can be constructed that reveal that the random arrival solution is not.

Now, note that in general, in a reduced problem associated with an arbitrary feasible allocation for (c, E), we may not have $\sum_{N'} c_i \geq \sum_{N'} x_i$. However it makes sense to require of a solution S that if $x = S(c, E)$, then for each $i \in N, \ x_i \leq c_i$. If this property of **claims boundedness** is satisfied, then for each $N' \subset N$, we have $\sum_{N'} c_i \geq \sum_{N'} x_i$. Thus, we can say that the domain of claims problems is closed under the reduction operation for any solution satisfying *claims boundedness*.

A surplus sharing problem (Moulin 1987) is defined like a claims problem except that the inequality $\sum c_i \leq E$ is imposed instead. The number c_i is interpreted as the investment made by agent $i \in N$ in a successful venture whose worth is E. A feasible allocation for (c, E) is a vector $x \in \mathbb{R}^N_+$ such that $\sum x_i = E$. For a solution that chooses vectors x satisfying the natural requirement that for each $i \in N, \ x_i \geq c_i$, closedness of the domain under the reduction operation holds.

Finally, consider the domain consisting of all claims problems *and* all surplus-sharing problems; for a problem in this enlarged domain, no relation is imposed between $\sum c_i$ and E. Starting from a surplus-sharing problem, and given a feasible allocation for it, an associated reduced problem may be a claims problem, and conversely. But all of these problems being admissible, the solution is applicable in each case.

3.2 Constructing consistent solutions

Here, we identify several operations preserving *consistency*. These operations will permit us to construct new *consistent* solutions using as building blocks solutions known to have the property.

We start with an informal observation that should provide some intuition for some of the assertions made in the remainder of this subsection: suppose that a solution is progressively "more and more restrictive" for groups of agents of increasing sizes. Then, starting from an alternative that it chooses for some problem, it has a better chance of choosing the restrictions of the alternative for the reduced problems associated with it. Still speaking informally, a *consistent* solution is more and more "tapered" for problems involving more and more agents (Fig. 5a).

3.2.1 Constructing new consistent solutions by intersecting existing ones

Consistency is preserved under intersections: given two *consistent* solutions, if their intersection (the "inner envelope" of Fig. 5b) is well-defined, that is, if it is non-empty for each admissible problem, then it is *consistent*. To illustrate, for classical problems of fair division (Domain 1), both the Pareto solution and the no-envy solution are

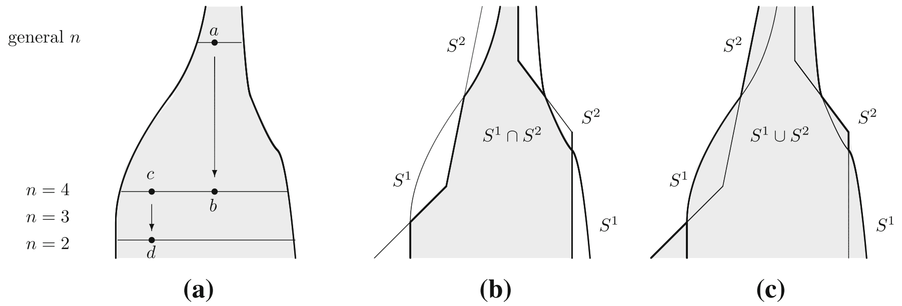

Fig. 5 The "shape" of consistent solutions. For each of these three panels, the range of possible alternatives is schematically indicated *horizontally*, and for each population, the subset of alternatives chosen by a solution is shown as a horizontal segment. The union of these segment as population varies is the "profile" of the solution. It is *the shaded area* of panel **a**. This panel shows that a *consistent* solution is more and more restrictive for groups of agents with larger and larger populations. This results in a "*tapered*" shape. Starting from some alternative a chosen for some problem with n agents, we symbolize *consistency* of the solution by indicating that the restriction of the alternative to a subgroup of agents, say to a subgroup of four agents—this restriction is symbolized by the *vertical downward arrow* to b—is an alternative that is chosen by the solution at that level for the associated reduced problem. We also show how some other alternative c chosen for a four-agent problem when the reduction is to a two-agent problem, is an alternative (denoted d in the figure) that would be chosen for it. **b** If two solutions S^1 and S^2 are *consistent*, then so is their intersection, if well defined: the "inner envelope" of the *two tapered* solutions, which represents this intersection, is also tapered. **c** If two solutions S^1 and S^2 are *consistent*, so is their union: the outer "envelope" of the two solutions is tapered

consistent. Thus, their intersection, which under standard assumptions on preferences is well-defined, is *consistent* too.

In fact, *consistency* is preserved under *arbitrary* intersections, and this permits us to define a *consistent* inner envelope to a solution that may not be *consistent*, as follows. Let S be a solution. In most situations of interest, the solution that associates with each problem its whole feasible set is *consistent*. Therefore, the family of *consistent* solutions containing S is non-empty. Let $\bar{S}$ denote the intersection of all of its members. Since they all contain S, so does $\bar{S}$. As we just argued, $\bar{S}$ is also *consistent*. Obviously then, it is the smallest *consistent* solution to contain S (Fig. 6a). Formally, the **minimal consistent enlargement**[11] of a solution S is defined as $\bigcap_{S' \in \mathcal{S}} S'$, where $\mathcal{S} \equiv \{S' : S' \supseteq S, S' \text{ is } \textit{consistent}\}$.

3.2.2 Constructing new consistent solutions by taking the union of existing ones

Similarly, *consistency* is preserved under arbitrary unions (the outer envelope of Fig. 5c), so that if a solution is not *consistent* but has at least one *consistent* subsolution, it has a maximal subsolution that is *consistent*, simply the union of all of its *consistent* subsolutions. Formally, the **maximal consistent subsolution** of a solu-

[11] This notion, and the notion of the *maximal consistent subsolution* of a given solution, presented next, are proposed and studied in Thomson (1994b). To say that the *minimal consistent enlargement* of a solution "approximates" it, as we did above, is not always justified however since a solution may differ considerably from its *minimal consistent enlargement*. Nevertheless, it represents the closest we can get to the solution so as to recover *consistency*.

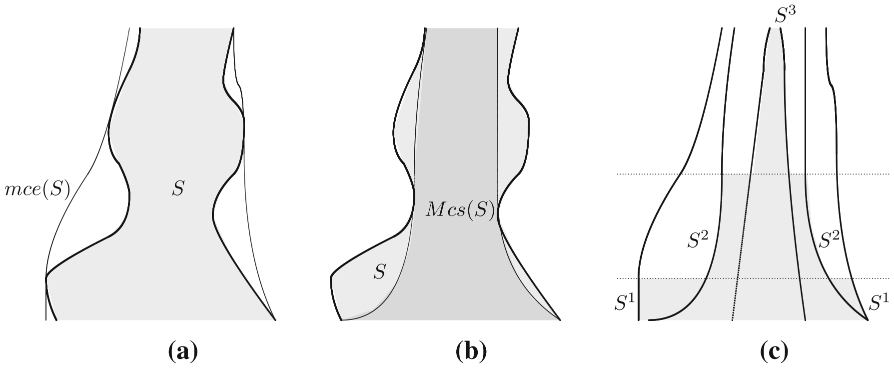

Fig. 6 Constructing consistent solutions. In each of the three panels, the solution that is constructed is indicated by *the thicker lines*. **a** *Minimal consistent enlargement of* S, $mce(S)$. **b** *Maximal consistent subsolution of S,* $Mcs(S)$. **c** If several solutions ordered by inclusion are *consistent*, any solution obtained by successively switching to less and less permissive ones as the number of agents increases is also *consistent*

tion S that contains at least one *consistent* solution is defined as $\bigcup_{S' \in \mathcal{S}'} S'$, where $\mathcal{S}' \equiv \{S' \colon S' \subseteq S,\ S' \text{ is } \textit{consistent}\}$.

3.2.3 Constructing new consistent solutions from consistent solutions ordered by inclusion

Let $(S^\ell)_{\ell \in \{1,\ldots,k\}}$ be a list of *consistent* solutions ordered by inclusion, $S^1 \subseteq \cdots \subseteq S^k$, and $(n^\ell)_{\ell \in \{1,\cdots,k-1\}}$ a list of natural numbers such that $n^1 < \cdots < n^{k-1}$. Now, consider the solution that coincides with S^1 for each problem involving no more than n^1 agents, with S^2 for each problem involving between $n^1 + 1$ and n^2 agents, ..., and with S^k for each problem involving more than n^{k-1} agents. This solution is clearly *consistent*. The operation is illustrated in Figs. 1c and 6c.

3.2.4 Constructing new consistent solutions by partitioning the domain into subdomains each of which is closed under the reduction operation for a particular consistent solution

Let $(S^\ell)_{\ell \in \{1,2,\ldots,k\}}$ be a list of *consistent* solutions having a common domain of definition $\mathcal{D}$ that can be partitioned into subdomains $(\mathcal{D}^\ell)_{\ell \in \{1,2,\ldots,k\}}$ such that (i) for each $\ell \in \{1, 2, \ldots, k\}$, $\mathcal{D}^\ell$ is closed under the reduction operation for the solution S^ℓ. Then, the solution that, for each $\ell \in \{1, 2, \ldots, k\}$, coincides with S^ℓ on $\mathcal{D}^\ell$, is *consistent*.

3.3 Conversely consistent solutions

Our second central property of a solution permits us to deduce that it chooses an alternative x for some problem involving some group of agents if it chooses the restriction of x to each two-agent subgroup for the reduced problem associated with the subgroup and x.

Converse consistency: For each group $N \in \mathcal{N}$, each problem $D \in \mathcal{D}^N$, and each feasible alternative x of D, if for each two-agent subgroup N' of N, the restriction $x_{N'}$ of x to N' is S-optimal for the reduced problem $r^x_{N'}(D)$ obtained from D by assigning to each agent in $N \backslash N'$ his component of x, then x is S-optimal for D.

This property does not appear as normatively compelling as *consistency* but it is of great computational interest, as it permits us to determine whether an alternative would be chosen for a problem possibly involving a large number of agents from the knowledge that its restrictions to subgroups of two agents, for which calculations are generally less complicated, are chosen for the associated reduced problems. Of course, if there are many agents initially, there are many reduced problems for which this simpler calculation has to be carried out.[12] Also, *converse consistency* does not help us *discover* S-optimal alternatives, but simply *check* whether a proposed alternative is S-optimal. Yet, the property suggests algorithms that sometimes converge to an S-optimal alternative. Starting from some alternative for some problem, visit pairs of agents in succession, and for each pair, adjust the alternative by replacing the components pertaining to the members of the pair by one of the two-agent alternatives that the solution would recommend for the reduced problem associated with the pair and the revised overall alternative of the previous stage. The possibility of convergence is explored by Thomson (2010b) who gives examples of models for which it occurs and others for which it does not.

A different formulation consists in writing the hypothesis for each reduced problem of cardinality up to $|N| - 1$, but it turns out that for many models, this amounts to the same thing.

For some models, the hypothesis for the two-agent case is actually no restriction on the solution. The property should then be rewritten with the hypothesis stated for the smallest number of agents for which it does constitute a meaningful restriction. An example of such a model is matching (Domain 6).

Converse consistency too is preserved under intersections and unions. Thus, the *minimal conversely consistent enlargement* and the *maximal conversely consistent subsolution* of a given solution can be defined analogously to the way we defined the notions of *minimal consistent enlargement* and *maximal consistent subsolution*. An example is calculated by Özkal-Sanver (2009) for matching (Domain 6).

Here are a few examples of *conversely consistent* solutions: for bargaining (Domain 3) the Nash solution is not in general *conversely consistent*, but it is on the subdomain of problems whose boundary is smooth; the egalitarian solution enjoys this property even if this assumption is not made.

For classical problems of fair division (Domain 1), the Pareto solution and the equal-division Walrasian solution are *conversely consistent* on the subdomain of economies with smooth preferences (Fig. 7a shows how the latter would violate the property without the smoothness assumption). The equal-division lower bound solution is not, and

[12] Sometimes, it is enough that the hypothesis be met for a (sufficient rich) class of pairs however.

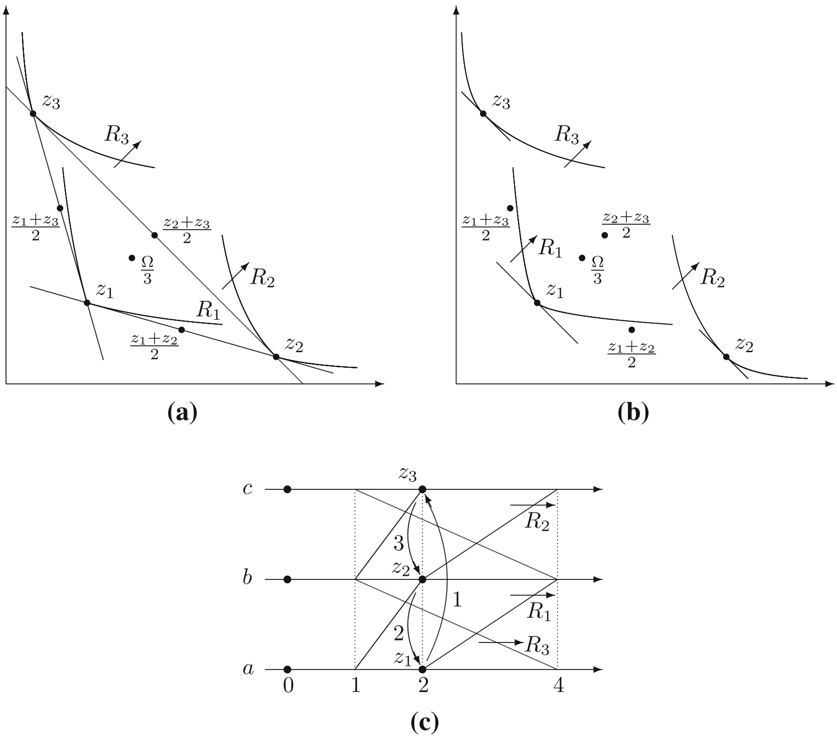

Fig. 7 Converse consistency. **a** For classical problems of fair division, the equal-division Walrasian solution is not *conversely consistent*. Consider the economy (R, Ω) with agent set $\{1, 2, 3\}$, and the allocation z. Note that for each two-agent group $\{i, j\}$, the restriction of z to the subgroup is an equal-division Walrasian for the reduced economy $(R_i, R_j, z_i + z_j)$, but z is not an equal-division Walrasian allocation for (R_1, R_2, R_3, Ω). (Under smoothness of preferences, the property holds however.) **b** The equal-division lower bound solution is not *conversely consistent* either. Again, we consider an economy (R, Ω) with agent set $N \equiv \{1, 2, 3\}$. For each $\{i, j\} \in N$, we have $(z_i, z_j) \in B_{ed}(R_i, R_j, z_i + z_j)$ but it is not true that $z \in B_{ed}(R, \Omega)$. **c** For the problem of allocating indivisible goods and one infinitely divisible good, the Pareto solution is not *conversely consistent*. Indeed, let (M, A, R) with agent set $N \equiv \{1, 2, 3\}$ be such that $M = 6$, $A \equiv \{a, b, c\}$, and that preferences R are such that $(2, a)\ I_1\ (4, b)\ I_1\ (1, c)$, and that the following statements, obtained by "rotation", hold for agents 2 and 3 : $(2, b)\ I_2\ (4, c)\ I_2\ (1, a)$ and $(2, c)\ I_3\ (4, a)\ I_3\ (1, b)$. It is easy to see that no Pareto improving trade can take place in any of the three two-agent reduced economies associated with the allocation z represented here, which is feasible. However, the allocation (z_2, z_3, z_1) Pareto-dominates z in the three-agent economy

neither is its intersection with the Pareto solution (Fig. 7b). Here, no natural restriction on preferences exists under which the property can be recovered. In any context where permutations of bundles across agents are meaningful, such as for various problems of fair allocation of privately appropriable goods, including classical problems of fair division (Domain 1), fair division problems with single-peaked preferences (Domain 2), or economies with indivisible goods and some infinitely divisible good (Domain 5), the no-envy solution is *conversely consistent*, precisely because it is based on two-agent tests. Whenever the notion of proportionality is well-defined, such as for fair division problems with single-peaked preferences (Domain 2), or for taxation

(Domain 7), the proportional solution is *conversely consistent*. For the allocation of indivisible goods and some infinitely divisible good (Domain 5), the Pareto solution is not *conversely consistent* (Fig. 7c).

3.4 Logical relations between consistency and its converse

Consistency and its *converse* are not logically related in general. Indeed, for bargaining (Domain 3) we already noted the following facts: the Nash solution is *consistent* (Fig. 3a) but not *conversely consistent*, and the reverse holds for the egalitarian solution. Both are *single-valued*, so these solutions can serve to answer, in the negative, a question that is often raised, namely whether this property helps relate *consistency* and its *converse*. However, for some models, interesting logical relations do hold, or hold under minor additional conditions (Domain 7) (Chun 1999).

A property that bears an interesting conceptual relation to *consistency* and its *converse* is **flexibility**: starting from an S-optimal allocation for some economy, suppose that a subgroup redistributes between its members what they have jointly received. Then, provided it performs these redistributions according to S, the conjunction of the list of resulting bundles for its members together with the bundles initially assigned to the complementary group, defines an allocation for the initial economy that is also S-optimal (Balinski and Young 1982).

It is easy to see that a *single-valued* solution is *consistent* if and only if it is *flexible*.

3.5 Lifting properties across cardinalities by means of consistency

It is often the case that if a certain property is imposed on a *bilaterally consistent* solution for the two-agent case, then the property is transferred, we say **lifted**, to the other cardinalities.

For instance, consider the requirement of **equal treatment of equals**, which says that two agents with the same characteristics should be treated in the same way; for resource allocation problems, this means that they should receive bundles that are indifferent to each other according to their common preferences.[13] Let S be a *consistent* solution. Let e be an economy in which two agents i and $j \in N$ have the same characteristics. Then, we claim that at an allocation x that is S-optimal for e, the two agents should receive indifferent bundles. Indeed, by *consistency* of S, in the reduced economy of e with respect to $\{i, j\}$ and x, the two agents still receive their components of x; since they have the same characteristics and S satisfies *equal treatment of equals* in the two-agent case, these components of x are indifferent bundles for them.

Other properties are lifted by *consistency*. As a result, theorems involving *consistency* can often be stated with these properties being only required for the two-agent case. Lifting results for claims problems (Domain 7), are established in Hokari and Thomson (2008). Examples of properties that are lifted are certain continuity and invariance properties. However, lifting does not occur as generally as one might

[13] A stronger requirement can be expressed in physical terms.

expect, as illustrated there. Occasionally, properties are lifted provided a solution satisfies some other basic properties. Still in the context of claims problems, a property that is particularly helpful in that regard is *endowment monotonicity*, the requirement that, if the endowment increases, each claimant should receive at least as much as he received initially (again, see Hokari and Thomson).

Similarly, one can define the lifting of an order defined on a space of solutions. An order is lifted by *consistency* if given two solutions that are related in that order in the two-agent case, if both are *consistent*, then they are related in the same way for any number of agents. Here too, lifting can be assisted by certain properties of the solutions. An application to the Lorenz order on the space of solutions to claims problems (Domain 7) is proposed and studied by Thomson (2011a). Indeed, the Lorenz order can be used to compare the awards vectors of a problem, but it can also be used to compare solutions: **a rule Lorenz dominates another one** if for each problem, the awards vector it chooses Lorenz dominates the awards vector the other one chooses. If two solutions are *consistent*, *endowment monotonic*, *order awards as claims are* (given two claimants, both always assign at least as much to the larger claimant as they do to the smaller claimant), and one Lorenz dominates the other in the two-agent case, this domination extends to all cardinalities. Thus, the Lorenz order is lifted with the assistance of these two basic properties.

4 The Elevator Lemma

Consistency and its *converse* are versatile principles and they have been studied in models exhibiting great diversity in their mathematical structures. An unfortunate consequence of this diversity is that few theorems are available that apply across all, or most models. However, we can offer two extremely useful lemmas that are "model-free". We illustrate them in several contexts. In Sect. 6, where we present a number of characterizations, we will see that much of the work in proving them often consists in showing that the Lemmas are applicable.

4.1 Statement of the Elevator Lemma

The first lemma identifies conditions on two solutions guaranteeing that if an inclusion relation holds between them for two agents, then this relation holds for any number of agents: if $S \subseteq \bar{S}$ for the two-agent case, S is *consistent*, and $\bar{S}$ is *conversely consistent*, then $S \subseteq \bar{S}$ for each cardinality. Given a problem involving an arbitrary number of agents, and an alternative that is S-optimal for it, its proof consists in applying the *consistency* of S to deduce the S-optimality of its restrictions to each of the two-agent associated reduced problems, invoking the inclusion relation that holds in the two-agent case to derive a parallel list of statements for $\bar{S}$, and deriving the $\bar{S}$-optimality of the alternative by means of the *converse consistency* of $\bar{S}$. Using the image of a building whose floors are indexed by the cardinalities of problems (Fig. 8), we refer to this lemma as the "Elevator Lemma": *consistency* is the "Down" button and *converse consistency* the "Up" button. Fig. 8 shows no first floor because for most models,

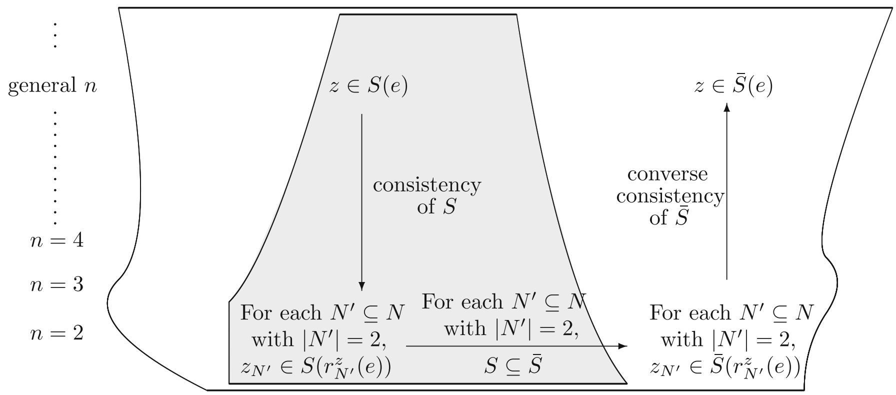

Fig. 8 The Elevator Lemma. Classes of problems involving increasing numbers of agents are stacked up like the floors of a building. The Elevator Lemma states that if a *consistent* solution S is a subsolution of a *conversely consistent* solution S' in the two-agent case, then this inclusion holds in general

nothing is learned from how a solution behaves on the class of one-agent problems. Therefore, no generality is lost in excluding these problems from the domain.[14]

Lemma 1 (The "Elevator Lemma") *Let S and $\bar{S}$ be two solutions defined on a domain $\mathcal{D}$ that is closed under the reduction operation for S. If (i) on the subdomain of two-agent problems, S is a subsolution of $\bar{S}$, (ii) S is consistent, and (iii) $\bar{S}$ is conversely consistent, then S is a subsolution of $\bar{S}$ on the entire domain $\mathcal{D}$.*

Proof Let $N \in \mathcal{N}$, $D \in \mathcal{D}^N$, and x be S-optimal for D. We need to show that x is $\bar{S}$-optimal for D. Since S is *consistent*, then for each subgroup N' of N, the restriction $x_{N'}$ of x to N' is S-optimal for the associated reduced problem $r^x_{N'}(D)$. This is true in particular for each $N' \subset N$ such that $|N'| = 2$. Since in the two-agent case, S is a subsolution of $\bar{S}$, then for each subgroup N' of N such that $|N'| = 2$, the restriction $x_{N'}$ of x to N' is $\bar{S}$-optimal for $r^x_{N'}(D)$. Thus, x satisfies the hypotheses of *converse consistency* for $\bar{S}$. Since $\bar{S}$ is *conversely consistent*, x is $\bar{S}$-optimal for D. □

Note that *bilateral consistency* would suffice in the Elevator Lemma. We stated earlier that *converse consistency* may not be as compelling as *consistency*, but as we have seen, many solutions do satisfy the former property and the Elevator Lemma shows how this fact can be profitably exploited.[15]

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[14] An important exception is the domain of strategic games (Peleg and Tijs 1996) where the objective is precisely to relate the way multi-agent interactions are resolved from the knowledge of how one-agent decision problems are solved. We will not discuss strategic games here, but only note that *consistency* has played an important role in linking the study of such games to the study of cooperative games. See for example, Serrano (1995).

[15] In Subsect. 3.5, we explained how *consistency* helps extend properties from the two-agent case to the general case. At the risk of causing the cable holding that elevator snapping under the weight of our metaphor, we will say that *consistency* "lifts" the property from the second floor to the other floors.

4.2 Applications of the Elevator Lemma

The importance of the Elevator Lemma stems from the fact that there are many models for which an inclusion relation, or even coincidence, holds between certain solutions in the two-agent case that do not hold for more agents. We give three examples:

1. For classical problems of fair division (Domain 1) or fair division problems with single-peaked preferences (Domain 2), in the two-agent case, any allocation that meets the equal-division lower bound is envy-free.
2. For coalitional games (Domain 4), requiring that a solution be a subsolution of the core is very mild in the two-agent case; it simply means that the chosen payoff vectors are "imputations", namely that they meet the individual rationality conditions and are efficient. (There are no coalitions of intermediate size then.)
3. For claims problems (Domain 7), a number of different ways of thinking about the problem give us concede-and-divide in the two-claimant case.

5 The Bracing Lemma

We now turn to the second lemma, which identifies conditions on two solutions only assumed to be related by inclusion, guaranteeing that in fact they coincide.

5.1 Statement of the Bracing Lemma

Let S be a *consistent* solution and suppose that it is a subsolution of some solution $\bar{S}$. Given a problem D and an alternative x that is $\bar{S}$-optimal for D, there will in general be some freedom to move away from x without leaving the $\bar{S}$-optimal set. However, suppose that additional agents can be introduced and D augmented so as to include the additional agents in such a way that (i) only one alternative is $\bar{S}$-optimal for the augmented problem, (ii) the restriction of that alternative to the initial group is precisely x—we will say that this alternative is **an augmentation of** x—, and (iii) the reduction of the augmented problem with respect to the initial group of agents and that augmented alternative is D. Now, since $S \subseteq \bar{S}$ and (i) holds, the augmented alternative is the only S-optimal alternative for the augmented problem. Then, since S is *consistent*, and since (ii) and (iii) hold, we conclude that x is S-optimal for D. If this can be done for each $N \in \mathcal{N}$, each $D \in \mathcal{D}^N$ and each $x \in \bar{S}(D)$, we conclude that $S = \bar{S}$.

We will illustrate the Bracing Lemma with another building metaphor. Consider the "house" of Fig. 9a, constructed by nailing boards together. This structure will not be very stable because the nails will serve as axes of rotation for the boards. Two of their infinitely many possible configurations are indicated. However, it is possible to eliminate the unwanted degrees of freedom by adding "braces", as shown in Fig. 9b. In the example, we have several choices of where to place braces but note that two of them are needed. If the structure to be stabilized were more complex, we could need more. These braces are the additional agents of the lemma.

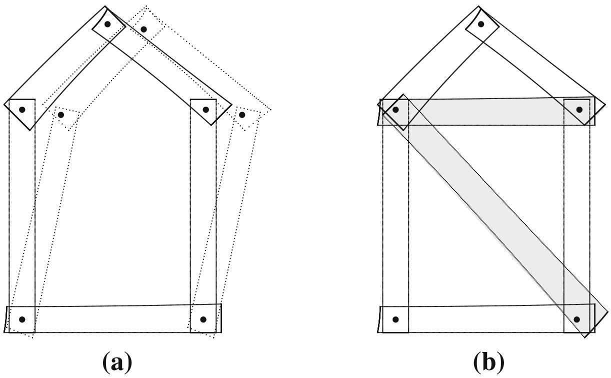

Fig. 9 The Bracing Lemma. **a** The house on the *left* is not stable because the boards it is made of have several degrees of freedom. One of the possible configurations is indicated by *the solid lines* and another one by *the dotted lines*. **b** To stabilize it, we add two boards connecting two pairs of corners. Note that if only one of the two boards were added, the structure would not be completely stabilized. In fact, one board would not be sufficient no matter where it would be placed. On the other hand, there are other ways to position two new boards so as to obtain a stable structure. Also, a third board would be redundant

Lemma 2 (The "Bracing Lemma") *Let S be a consistent subsolution of some solution $\bar{S}$. Suppose that $\bar{S}$ is such that for each $N \in \mathcal{N}$, each $D \in \mathcal{D}^N$, and each $x \in \bar{S}(D)$, there are $N' \supset N$, $D' \in \mathcal{D}^{N'}$, and x' in the feasible set of D', such that (i) x' is the only $\bar{S}$-optimal alternative for D', (ii) the restriction of x' to N' is x, and (iii) the reduced problem of D' with respect to N and x' is D. Then, $S = \bar{S}$.*

The proof follows directly from the definitions and in fact we have essentially given it in the paragraph preceding the lemma. Perhaps, it is not so much a bracing "lemma" as a bracing "construction".

Sometimes bracing requires only one additional agent—we will see an illustration in the context of allocation of indivisible goods and an infinitely divisible good (Domain 5)— sometimes two are required, as for the allocation of *identical* indivisible goods and an infinitely divisible good (a special case of Domain 5) and sometimes almost as many agents as are present initially are needed; an example here can be found in the context of fair division problems with single-peaked preferences (Domain 2). In some situations, instead of working with an infinite set of potential agents, it is natural to impose an upper bound on the size of the set; then, the Bracing Lemma only applies to a restricted class of situations.

5.2 Applications and variants of the Bracing Lemma

In applications, the question is when the "augmentation to uniqueness" of the Bracing Lemma is possible, and this depends on the richness of the domain of problems over which the solution is defined. To return to our architectural metaphor, bracing is possible there only if we have available a board that is long enough to be nailed diagonally. Of course, it will also depend on the solution under consideration.

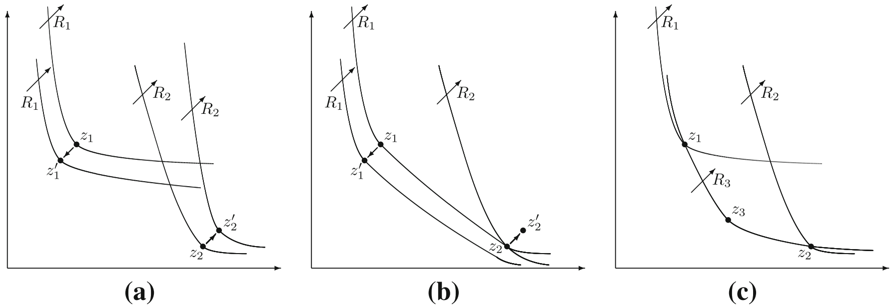

Fig. 10 Illustrating the Bracing Lemma. **a** If both envy constraints are met strictly (as they are at z), reallocations are possible *in any direction* without envy being violated, provided they are small enough (z' is just one example of an allocation that can be reached). **b** Here, one of the envy constraints is met at indifference and redistributions in certain directions would lead to violations of envy (a move to z' would create envy). **c** This figure illustrates a natural attempt at augmenting an economy with agent set $\{1, 2\}$ that admits many envy-free allocations, by introducing agent 3 and additional resources z_3 so as to obtain a *unique* such allocation. This attempt will be unsuccessful however. Indeed the equal-division Walrasian solution is a *consistent* subsolution of the no-envy solution

5.2.1 When bracing is not possible

The following example will make it obvious that an augmentation to uniqueness is not always possible. For classical problems of fair division with strictly monotonic and strictly convex preferences, consider the Pareto solution. Starting from some economy and an arbitrary Pareto-optimal allocation for it, there is in general no way to introduce additional agents and additional resources—these resources being intended to provide bundles for them, thereby defining an augmented allocation, the allocation obtained by concatenating the allocation that is the point of departure with the list of bundles created for the new agents—so that in the augmented economy, this augmented allocation is the only one to be Pareto-optimal.[16] If this augmentation were possible, then by the Bracing Lemma, there would be no *consistent* subsolution of the Pareto solution, but we know this not to be true: the equal-division Walrasian solution is one (of course, we have other information about the structure of the set of Pareto-optimal allocations that confirms this).

Here is an example, in the same context of classical problems of fair division, for which the answer is a little less clear. It involves the no-envy solution (Fig. 10). Let us say that the "envy constraints are met at an allocation for an agent" if he finds his bundle at least as desirable as each of the bundles assigned to the other agents. If the envy constraints are all met strictly for each agent, then reallocations will be possible within the envy-free set in each direction (Fig. 10a) (except when the boundary of the consumption spaces would get in the way). If some of them are met as indifferences, such reallocations may still be feasible but we will have to be more careful

[16] It is of course easy to perform the augmentation in such a way that there is a Pareto-optimal allocation whose restriction to the initial set of agents is z. For instance, add an arbitrary agent and no new resources; assign nothing to the new agent.

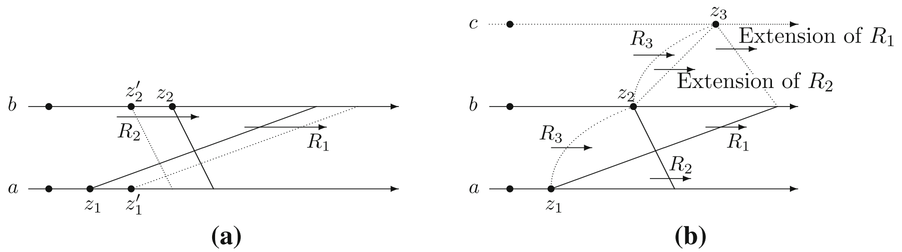

Fig. 11 The Bracing Lemma for the allocation of indivisible goods. **a** An economy with agent set $\{1, 2\}$ and an envy-free allocation z for it at which the no-envy constraints are met strictly: by redistributing the divisible good (to z' for example), we do not violate the no-envy constraints. **b** This freedom to move in the envy-free set is eliminated up to a neutral exchange by adding one object (object c in the figure) and some arbitrary amount of the divisible good (given by the abscissa of z_3), introducing a new agent (agent 3) and specifying his preferences so that he is indifferent between the bundle of additional resources (z_3)—this bundle is intended for him—and the components z_1 and z_2 of z; finally, extending the preferences of agents 1 and 2 so that each of them is indifferent between his old bundle and z_3

(Fig. 10b), and it may be that if sufficiently many of the no-envy constraints are met as indifferences, there will be no freedom to move at all. Figure 10c illustrates then a natural way to go about augmenting a two-agent economy with agent set $\{1, 2\}$ in an attempt to obtain a unique envy-free allocation in the augmented economy that results. Starting from an envy-free allocation $z \equiv (z_1, z_2)$, introduce agent 3, and let z_3 designate a bundle of additional resources. These resources are intended for him, which is the reason for designating them as z_3. As we just noted, the augmented allocation (z, z_3) will have a chance to be the only envy-free allocation in the augmented economy only if sufficiently many of the envy constraints are met as indifferences. Since we can choose z_3 as well as agent 3′s preferences, we should probably make these choices in such a way that he is indifferent between z_3 and as many as possible of the bundles received by the agents initially present, and this is what we have done in the figure. If that does not work, the option of introducing more than one agent gives us the opportunity of increasing the proportion of the envy constraints that are met as indifferences, thereby getting us closer to our objective of creating a structure in which all degrees of freedom are eliminated.

However, once again, this construction will not work because the equal-division Walrasian solution is a *consistent* subsolution of the no-envy solution too. Nevertheless, we proceeded in essentially the right way to brace an allocation, and our approach will be successful for other models: for the allocation of indivisible goods and an infinitely divisible good for example (Domain 5), it does allow bracing, as explained in Subsubsect. 5.2.4 (Figs. 11 and 12).

5.2.2 A model in which the Bracing Lemma is directly applicable

An important model illustrating the usefulness of the Bracing Lemma is the domain of coalitional games, but we will not go into details as it is a more difficult domain with which to work.

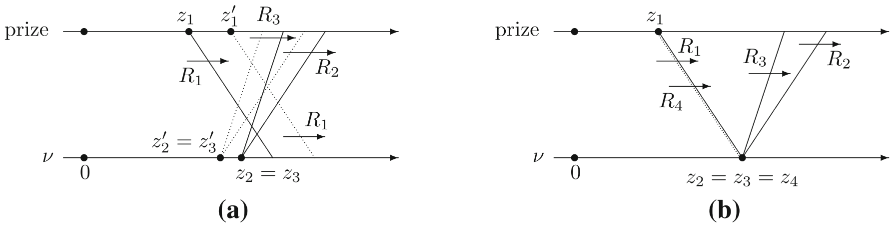

Fig. 12 Lemma 3 applied to the allocation of a prize. **a** We start with an economy with agent set $\{1, 2, 3\}$. At z, agent 1 is the winner of the prize. By no-envy, agents 2 and $3'$s bundles are the same (they consist of the same amount of the divisible good and the null object), and their indifference curves through this common bundle pass to the right of z_1; also, agent $1'$s indifference curve through z_1 passes to the right of z_2. The allocation z' is another envy-free allocation, obtained from z by transferring the divisible good from the losers to the winner. The allocation z cannot be braced. **b** The allocation at which the winner is indifferent between his bundle and the common bundle of the losers can be braced. By introducing a new agent, agent 4 and specifying his preferences so that he is indifferent between z_1 and z_2, adding $m_4 \equiv m_2$ units of the divisible good, and giving him the bundle $z_4 \equiv (m_4, \nu)$, we obtain an augmented allocation (z, z_4) that in the augmented economy, is the only envy-free allocation up to a neutral exchange (between agents 1 and 4)

5.2.3 Bracing requiring an augmentation of the consumption spaces

We saw earlier that for some models, the reduction operation is most naturally accompanied by a reduction of consumption spaces and a restriction to the reduced spaces of the preferences of the agents who stay. This means that conversely, an augmentation of a problem will have to involve an augmentation of the consumption spaces of the agents initially present and an extension of their preferences to the augmented spaces. Examples of domains illustrating this operation are the allocation of indivisible goods and an infinitely divisible good (Domain 5) and matching (Domain 6).

5.2.4 Bracing "up to neutral exchanges" or "up to indifferent exchanges"

For a number of economic domains and for certain solutions $\bar{S}$ of interest, the $\bar{S}$-optimal set is often not a singleton but the allocations it contains are Pareto-indifferent. A useful variant of the Bracing Lemma in such situations involves the requirement that the solution S should also satisfy **Pareto-indifference**: if x and x' are feasible alternatives of D such that x is S-optimal for D and x' is Pareto-indifferent to x, then x' should also be S-optimal for D. This requirement, which seems innocuous enough, is not always met however.[17] Nevertheless, if imposed on S and if satisfied by $\bar{S}$, the conclusion $S = \bar{S}$ of the Bracing Lemma is obtained by a slight modification of the proof given above.

For the allocation of indivisible goods and an infinitely divisible good (Domain 5), each envy-free allocation can be braced but only up to a "neutral exchange": the allocation z' is related to the allocation z by such an exchange if its components are obtained by reshuffling the components of z but each agent is indifferent between his

[17] For instance, for classical problems of fair division, the no-envy solution violates it.

old and new bundles (in Fig. 11b, (z_1, z_3, z_2) is obtained from (z_1, z_2, z_3) by a neutral exchange).[18]

5.2.5 When bracing is possible only for distinguished alternatives

In some cases, not all $\bar{S}$-optimal alternatives can be braced but only distinguished ones. If these distinguished alternatives exist for each problem, they provide the basis for defining a solution—let us call it S^*—and our conclusion will be a containment of S^*, as formally stated in the following lemma:

Lemma 3 (Variant of the Bracing Lemma). *Let S be a consistent subsolution of some solution $\bar{S}$. Let S^* be a subsolution of $\bar{S}$. If the "augmentation to uniqueness" described in the Bracing Lemma is possible for each $x \in S^*(D)$, then $S \supseteq S^*$. Therefore, if S^* is consistent, it is the minimal consistent subsolution of S.*

An illustration of Lemma 3 is provided by the allocation of a single indivisible good (a prize, say) and an infinitely divisible good, the equality between the numbers of objects and agents being reestablished by introducing "null objects". Getting a null object simply means not getting the prize (Fig. 12).

Domain 8 *A problem of* **allocating a single indivisible good and an infinitely divisible good** *is the simple version of Domain 5 when there is only one real object. The domain also includes economies where only some of the divisible good is to be allocated.*

Examples of solutions for Domain 8 *The* **winner's curse solution** (Tadenuma and Thomson 1993) *chooses the envy-free allocation(s) at which the* **winner***, the agent who is assigned the real object, is indifferent between his bundle and the common bundle of the losers.*

See the legend of Fig. 12 for an explanation of how Lemma 3 applies. There, we use the notation ν for the null object.

5.2.6 When bracing is achieved approximately

In some situations, bracing is achievable only approximately but with a "tolerance" that can be chosen arbitrarily small. Then, another useful variant of the Bracing Lemma is obtained by imposing some form of continuity on the solution. This possibility is illustrated by fair division problems with single-peaked preferences (Domain 2).

We refer to Fig. 13 for a sketch of the analysis of this domain. It pertains to a solution assumed to be a *consistent* subsolution of the no-envy and Pareto solution. It shows that for each economy, allocations that are arbitrarily close to its uniform allocation has to be chosen. It represents a two-agent economy for which $\sum p(R_i) \geq \Omega$ and whose uniform allocation is denoted by z (there, $\lambda = z_2$). The proof uses the obvious

[18] Note that this is a special case of a Pareto-indifferent exchange. For this model the no-envy solution in invariant under such exchanges.

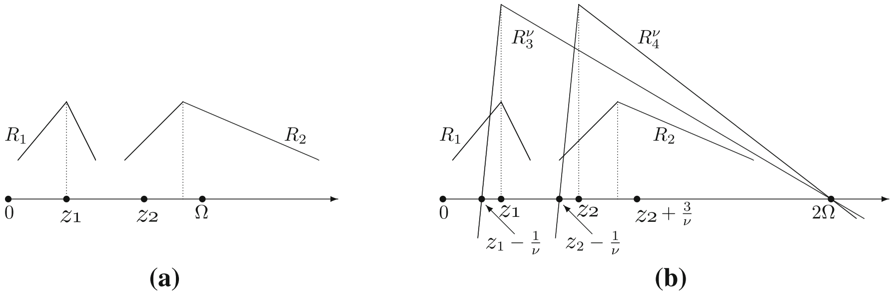

Fig. 13 Approximate bracing. **a** We start with an economy $e \equiv (R, \Omega)$ with agent set $\{1, 2\}$ for which $\sum p(R_i) > \Omega$, and we identify its uniform allocation, (z_1, z_2). **b** We introduce two new agents, agents 3 and 4. Given $\nu \in \mathbb{N}$, we specify their preferences R_3^ν and R_4^ν so that $p(R_3^\nu) = p(R_1)$, $p(R_4^\nu) = z_2$, $(z_1 - \frac{1}{\nu})\ I_3^\nu\ 2\Omega$, and $(z_2 - \frac{1}{\nu})\ I_4^\nu\ 2\Omega$. We double the social endowment. We consider a solution that is a selection from the no-envy and Pareto solution. Let y^ν be an allocation that it chooses for the augmented economy $(R_1, R_2, R_3^\nu, R_4^\nu, 2\Omega)$. By efficiency, for each $i \in \{1, \ldots, 4\}$, $y_i^\nu \le p(R_i)$. Then by no-envy, $y_1^\nu = y_3^\nu$ and $y_2^\nu = y_4^\nu$. This implies that at least one of agents 2 and 4 consumes more than $p(R_3^\nu)$, and for agent 3 not to envy him, he should consume at least $p(R_3^\nu) - \frac{1}{\nu} = p(R_1^\nu) - \frac{1}{\nu} = z_1 - \frac{1}{\nu}$. So, agent 1 should consume at least that amount. Similarly, agent 4 should consume at least $p(R_4) - \frac{1}{\nu} = x_2 - \frac{1}{\nu}$, and therefore agent 2 should consume at least that amount. We then deduce that agent 2 should not consume more than $z_2 + \frac{3}{\nu}$. Altogether, y_1^ν belongs to the interval $[z_1 - \frac{1}{\nu}, z_1]$ and y_2^ν to the interval $[z_2 - \frac{1}{\nu}, z_2 + \frac{3}{\nu}]$, so that (y_1^ν, y_2^ν) converges to (z_1, z_2) as $\nu \to \infty$

fact that at an efficient allocation, if $\sum p(R_i) \ge \Omega$, then each agent receives at most his satiation amount, and if $\sum p(R_i) \le \Omega$, each agent receives at least his satiation amount.

6 Characterizations: a sampler

In this section, we state a few results involving *consistency* and its *converse*. They constitute but a small fraction of the literature, but we have selected them so as to give a flavor of the range of existing applications of the principles and whet the reader's appetite. Several of them use one or the other of the two Lemmas and variants. In some cases, very little work is required beyond showing that the hypotheses of the Lemmas are met.

6.1 Bargaining

Our first result, which pertains to bargaining (Domain 3), involves two basic properties. One is *Pareto-optimality*, whose definition we will not repeat. The other is **anonymity**, which says that the solution should be invariant under renamings of agents. We also impose **scale invariance**, according to which a linear rescaling, independent agent by agent, of their utilities, should be accompanied by a similar rescaling of the outcome.

Theorem 1 (Lensberg 1988) *The Nash solution is the only solution satisfying single-valuedness, Pareto-optimality, anonymity, scale invariance, and consistency.*

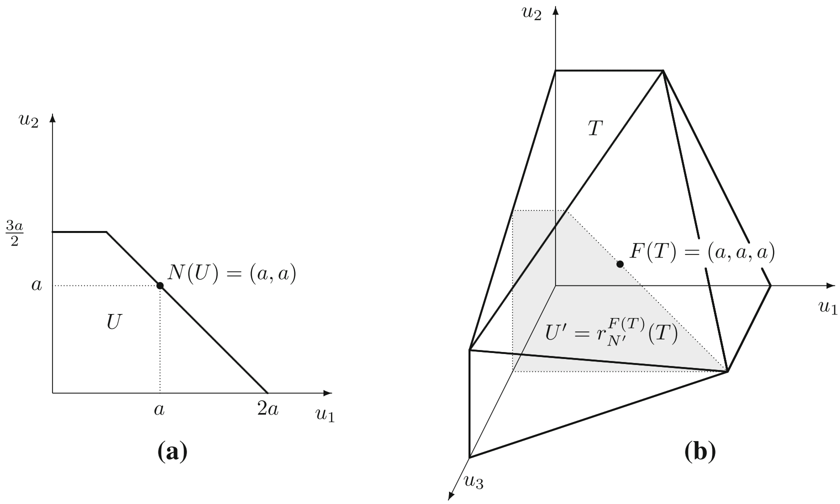

Fig. 14 Characterization of the Nash solution (Theorem 1). **a** We start with a problem U with agent set $\{1, 2\}$. By *scale invariance*, we can assume that its Nash outcome has equal coordinates, (a, a). **b** We introduce a third agent, agent 3, and construct a three-agent problem as follows. First, we translate U along the third axis by the amount a. Let U' denote the result (*shaded area*). Next, we replicate U' twice by having the roles played by agents 1 and 2 in U' be played by agents 2 and 3 respectively, and then by agents 3 and 1 respectively. Finally, we construct the smallest convex and comprehensive problem, T, containing U' and its two replicas. Now, by *Pareto-optimality* and *anonymity*, the point chosen for T is (a, a, a). The reduced problem of T with respect to $\{1, 2\}$ and (a, a, a) happens to be U'. By *consistency*, the solution outcome of U' is (a, a). Since $U = U'$, we are done

The proof, illustrated in Fig. 14, and sketched in its caption, involves an operation that can be described as an "augmentation to anonymity". Starting from an arbitrary problem S that may not have any particular symmetry, we augment it so as to obtain a problem T that has enough symmetries so that we can deduce by *Pareto-optimality* and *anonymity* the point that that it should choose; moreover, the reduced problem of T with respect to the initial group of agents and that point is S.

The construction works for any S whose boundary contains a segment centered at its Nash outcome that "extends sufficiently" on both sides. (By *scale invariance*, this outcome can be given equal coordinates, (a, a) in Fig. 14a). Otherwise, the section of T through (a, a, a)—which is the point the solution has to choose for the three-agent problem T of Fig. 14b—contains $S' = S$ as a *strict* subset, and we cannot derive what we want about S. However, the desired conclusion can be obtained then by introducing more than one new agent and extending the replication. The number of new agents that are needed is all the greater, the shorter the segment centered at the Nash outcome of S in relation to a. A continuity argument is required for a problem whose boundary is strictly convex at its Nash outcome.

A very general result that involves neither *symmetry* nor *scale invariance* is given by Lensberg (1987). He obtains a characterization of a class of solutions defined by maximizing a sum of functions of the agents' utilities having certain properties. A study of *converse consistency* for this model is Chun (2002).

6.2 Coalitional games with transferable utility

The literature on *consistency* for coalitional games (Domain 4) is extensive, partly because on this domain, as we have already seen, the reduction operation can be defined in more than one way. We list three basic results involving one or the other of the two notions of *consistency* introduced in Subsubsect. 3.1.3. The hypotheses of Theorems 2 and 3 include **individual rationality**, the requirement that at each of the payoff vectors chosen by a solution, each agent's payoff should be at least as large as his individual worth. The proofs of both theorems rely on the Bracing Lemma.

Theorem 2 (Tadenuma 1992) *On the domain of TU coalitional games whose core is non-empty, the core is the only solution satisfying individual rationality and complement consistency.*

The next result includes the condition of **super-additivity**, which says that if x is chosen for some game v and y is chosen for some game w, then $x + y$ should be chosen for the game $v + w$.

Theorem 3 (Peleg 1985) *On the domain of TU coalitional games whose core is non-empty, the core is the only solution satisfying individual rationality, super-additivity and max consistency.*

Zero independence and invariance under common rescaling of utilities which appears in the next theorem is the requirement that if arbitrary constants are added to all utilities and utilities are scaled by the same positive number, the chosen payoff vector should be subjected to the same transformation. The proof of the theorem is by means of an augmentation to anonymity analogous, although considerably more complex, to that carried out in the proof of Theorem 1.[19]

Theorem 4 (Sobolev 1975) *The prenucleolus is the only solution satisfying single-valuedness, anonymity, zero independence and invariance under common rescaling of utilities, and max consistency.*

A third notion of *consistency* in which the solution itself appears, is proposed by Hart and Mas-Colell (1988), and it essentially leads to a characterization of the Shapley value.

Counterparts of Theorems 2 and 3 for the more general class of coalitional games without transferable utility are available (Tadenuma 1992; Peleg 1985). Numerous other contributions have been made to the analysis of *consistency* for coalitional games. Examples are Maschler and Owen (1989), Dutta (1990), Hokari and Kıbrıs (2003) and Hokari (2005).

6.3 Fair division

To present the results of this section, which pertain to classical problems of fair division (Domain 1), we need two additional properties. Given a natural number k, in a

[19] It is a weak form of *scale invariance* encountered earlier.

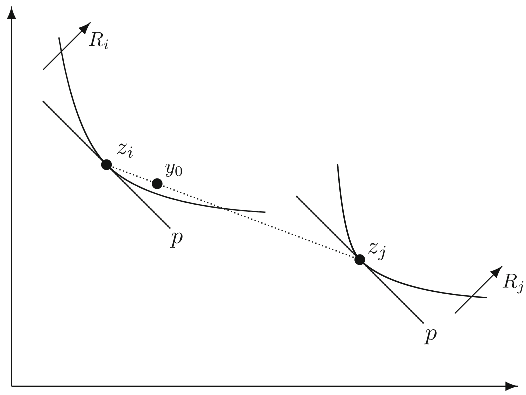

Fig. 15 Characterization of the equal-division Walrasian solution (Theorem 5). Given a solution satisfying the assumptions of the theorem, suppose that there are an economy and an efficient allocation chosen for it by the solution, z, at which the values of the bundles assigned to two agents, say agents i and j, calculated at the prices supporting z (these prices exist and they are *unique*), are not equal. By smoothness of preferences, there is $k \in \mathbb{N}$ such that $y_0 \; P_j \; z_i$, where $y_0 \equiv \frac{kz_i+z_j}{k_i+1}$. Then, we replicate the economy k times. Let $k * z$ be the corresponding k-replica of z. Since z is chosen for the initial economy, then, by *replication-invariance*, $k * z$ is chosen for $(k * R, k\Omega)$. Let N' be a subgroup of the augmented set of agents consisting of the k agents of type i and agent j. By *consistency*, $(k * z_i, z_j)$ is chosen for $((R_\ell)_{\ell \in N'}, kz_i + z_j)$. Since the solution is a subsolution of the equal-division lower bound solution, we should have $z_i \; R_i \; y_0$, but this is in contradiction with the way we specified y_0

k-replica of an economy, each of the preference relations is represented k times and the social endowment is multiplied by k. **Replication-invariance** says that if an allocation is chosen for some economy, then for each $k \in \mathbb{N}$, and each economy obtained by replicating the economy k times, the corresponding k-replica of the allocation should be chosen.

Theorem 5 (Thomson 1988) *Suppose that preferences are smooth. If a subsolution of the equal-division lower bound and Pareto solution satisfies replication invariance and consistency, then it is a subsolution of the equal-division Walrasian solution.*

The proof is sketched in the legend of Fig. 15a, where the replication operation is denoted with a star ($k * z$ being obtained by replicating z k times and $(k * R, k\Omega)$ by replicating (R, Ω) k times). It involves a variant of the Elevator Lemma, in which the role of *converse consistency* is played by *replication invariance*, which is a (very) weak form of it.

We also have the following result, in which **anonymity** appears: this is the requirement that the chosen allocations should be independent of the names of agents:

Theorem 6 (Thomson 1994a) *Suppose that preferences are smooth. If a subsolution of the equal-division lower bound and Pareto solution satisfies anonymity and converse consistency, then in the two-agent case, it is a subsolution of the equal-division Walrasian solution. If for the two-agent case, equality holds, then it is a subsolution of the equal-division Walrasian solution for each cardinality.*

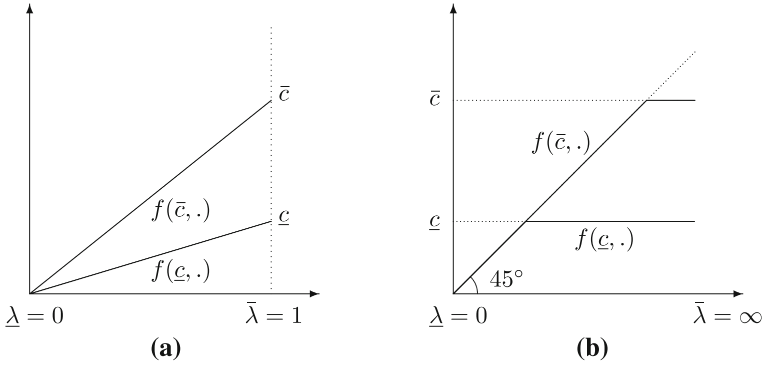

Fig. 16 Parametric representations of two solutions. **a** The proportional solution. **b** The constrained equal awards solution

Other results for this domain are due to Maniquet (1996) and Fleurbaey and Maniquet (1996). Roemer (1988) formulates and studies a notion of *consistency* where it is the number of commodities that varies.

6.4 Claims problems

To state our next result, which pertains to claims problems (Domain 7), we need the concept of a **parametric solution**. Consider a family of real-valued, continuous, and nowhere decreasing functions $f \equiv \{f(c_0, .)\}_{c_0 \in \mathbb{R}_+}$ defined on some interval $[\underline{\lambda}, \bar{\lambda}]$ of the extended reals and such that for each $c_0 \in \mathbb{R}_+$, $f(c_0, \underline{\lambda}) = 0$ and $f(c_0, \bar{\lambda}) = c_0$. Now, for each $N \in \mathcal{N}$ and each claims problem (c, E) with agent set N, let $x \in \mathbb{R}^N_+$ be such that $\sum_N x_i = E$ and for some $\lambda \in [\underline{\lambda}, \bar{\lambda}]$ and each $i \in N$, $f(c_i, \lambda) = x_i$: this is the choice made by the parametric solution associated with f. It is easy to see that the proportional, constrained equal awards, constrained equal losses, and Talmud solutions (defined next), are parametric solutions. Figure 16 gives parametric representations for two of them. To define the **Talmud solution**, we distinguish two cases: if $\sum \frac{c_i}{2} \geq E$, each claimant $i \in N$ receives what he would be assigned by the constrained equal awards solution applied to the problem $(\frac{c}{2}, E)$; if $\sum \frac{c_i}{2} \leq E$, each claimant $i \in N$ receives $\frac{c_i}{2}$ plus what he would be assigned by the constrained equal losses solution applied to the problem $(\frac{c}{2}, E - \sum \frac{c_i}{2})$.

The following is a simple application of the Elevator Lemma. Many solutions coincide with concede-and-divide in the two-claimant case, but only one of them is *consistent*. The proof uses the fact that the parametric solutions are *conversely consistent*.

Theorem 7 (Aumann and Maschler 1985) *The Talmud solution is the only consistent solution to coincide with concede-and-divide in the two-claimant case.*

The following theorem describes the structure of the class of *consistent* solutions very completely. **Equal treatment of equals** is the requirement that two agents with equal claims should be assigned equal amounts.

Theorem 8 (Young 1987) *A solution satisfies continuity, equal treatment of equals, and consistency if and only if it is a parametric solution.*

By imposing additional properties on solutions, interesting subfamilies of the parametric family can be identified (Young 1988), including several solutions that have played a prominent role in the public finance literature (as the model can also be seen as a cost sharing model). Another relevant contribution, in which the notion of *consistency on average* is developed (Sect. 3.1), is due to Dagan and Volij (1997).

6.5 Allocation of indivisible goods and some amount of an infinitely divisible good

The two theorems below, which pertain to the allocation of indivisible goods and an infinitely divisible good (Domains 5 and 8), involve bracing up to neutral exchanges (operations illustrated in Figs. 10 and 11).

Theorem 9 (Tadenuma and Thomson 1993) *In the one-object case, there is a minimal (in terms of inclusion) subsolution of the no-envy solution satisfying neutrality and consistency. It is the winner's curse solution.*

Theorem 10 (Tadenuma and Thomson 1991) *In the multiple-object case, if a subsolution of the no-envy solution satisfies neutrality and consistency, then in fact, it is the no-envy solution.*

If several identical objects have to be allocated, any envy-free allocation can be braced by introducing two agents, specifying their preferences in such a way that they are indifferent between the two bundles initially received by the losers and the winners (all members of each group have to receive the same bundle, by no-envy), and adding resources so that one of these new agents can be given the winners' initial bundle and the other can be given the losers' initial bundle (that is, adding one object and an amount of the divisible good equal to the sum of the amounts of the good contained initially in the losers' common bundle and the winners' common bundle).

Bevia (1996) studies *consistency* in situations where each agent may receive more than one object. The case of economies with only indivisible goods is treated by Ergin (2000).

6.6 Allocation with single-peaked preferences

For fair division problems with single-peaked preferences (Domain 2), we have the following characterization:

Theorem 11 (Thomson 1994a) *There is a minimal subsolution of the no-envy and Pareto solution satisfying upper semi-continuity with respect to the social endowment and consistency. It is the uniform rule. The same conclusion holds when no-envy is replaced by the equal-division lower bound.*

The proof for a solution S required to be a subsolution of the no-envy solution relies on the approximate bracing illustrated in Fig. 13. Using the notation introduced there, it concludes as follows. Let $\Omega^\nu \equiv \sum_{\{1,2\}} y_i^\nu$ and $e^\nu \equiv (R_1, R_2, \Omega^\nu)$. By *consistency*, $y^\nu_{\{1,2\}} \in S(e^\nu)$. Since $y^\nu_{\{1,2\}} \to z$ as $\nu \to \infty$, it follows that $\Omega^\nu \to \sum_{\{1,2\}} z_i = \Omega$. By *upper semi-continuity with respect to the social endowment*, $z \in S(e)$.

When no-envy is replaced by the equal-division lower bound solution, the conclusion is obtained as a direct consequence of the Elevator Lemma and of the fact that for the two-agent case, the no-envy solution is a subsolution of the equal-division lower bound solution.

A counterpart of Theorem 6 holds for this model, and its form is even a little simpler since the uniform rule is *single-valued*. A result related to Theorem 11 is due to Dagan (1996).

6.7 Matching

For matching problems (Domain 6), we will impose both *consistency* and *converse consistency*, but note that here, for this condition to make sense, we need its hypotheses to hold for each problem involving two men and two women. The following theorem involves the Bracing Lemma.

Theorem 12 (Sasaki and Toda 1992) *The stable solution is the only subsolution of the Pareto solution satisfying anonymity, consistency, and converse consistency.*

Additional results are obtained by Toda (2006) for variants of this model, (remaining single may be an option; the reduction operation can be defined differently.) Results for the model in which the formation of each pair creates a monetary value, and the sum of the values created at a match has to be shared are developed by Sasaki (1995) and Toda (2003).

References

Aumann R, Maschler M (1985) Game theoretic analysis of a bankruptcy problem from the Talmud. J Econ Theo 36:195–213

Balinski M, Young P (1982) Fair representation. Yale University Press, New Haven

Bevia C (1996) Identical preferences lower bound solution and consistency in economies with indivisible goods. Soc Choice Welfare 13:113–126

Chambers C (2004) Consistency in the probabilistic assignment model. J Math Econ 40:953–962

Chun Y (1999) Equivalence of axioms for bankruptcy problems. Int J Game Theory 28:511–520

Chun Y (2002) The converse consistency principle in bargaining. Games Econ Behav 40:25–43

Chun Y (2011) Consistency and monotonicity in sequencing problems. Int J Game Theory 40:29–41

Dagan N (1996) A note on Thomson's characterizations of the uniform rule. J Econ Theory 96:255–261

Dagan N, Volij O (1997) Bilateral comparisons and consistent fair division rules in the context of bankruptcy problems. Int J Game Theory 26:11–25

Davis M, Maschler M (1965) The Kernel of a cooperative game. Naval Res Logis Q 12:223–259

Dutta B (1990) The egalitarian solution and reduced game properties in convex games. Int J Game Theory 19:153–169

Dutta B, Kar A (2004) Cost monotonicity, consistency, and minimum cost spanning tree games. Games Econ Behav 48:223–248

Ehlers L, Klaus B (2006) Efficient priority rules. Games Econ Behav 55:372–384

Ergin H (2000) Consistency in house allocation problems. J Math Econ 34:77–97

Fleurbaey M, Maniquet F (1996) Fair allocation with unequal production skills: the solidarity approach to compensation. Math Soc Sci 32:71–93

Gale D, Shapley LS (1962) College admissions and the stability of marriage. Am Math Mon 69:9–15
Gillies DB (1959) Solutions to general non-zero sum games. In: Tucker AW, Luce RD (eds) Contributions to the theory of games IV, annals of mathematical studies 40. Princeton University Press, Princeton pp 595–614
Hart S, Mas-Colell A (1988) The potential of the Shapley-value, Chapter 9 in the shapley-value: essays in honor of Lloyd S. Shapley Roth AE (ed), Cambridge University Press, pp 127–137
Hokari T (2005) Consistency implies equal treatment in TU games. Games Econ Behav 51:63–82
Hokari T, Kıbrıs O (2003) Consistency, converse consistency, and aspirations in coalitional games. Math Soc Sci 45:249–297
Hokari T, Thomson W (2008) On properties of division rules lifted by bilateral consistency. J Math Econ 44:1057–1071
Hwang Y-A, Yeh C-H, Ju B-G (2006) Reduction-consistency and the Condorcet principle in collective choice problems, mimeo
Ju B-G (2008) Efficiency and consistency for locating multiple public facilities. J Econ Theory 138:165–183
Kalai E (1977) Proportional solution to bargaining problems: interpersonal utility comparisons. Econometrica 45:1023–1030
Kalai E, Smorodinsky M (1975) Other solutions to Nash's bargaining problem. Econometrica 43:513–518
Klaus B, Nichifor A (2010) Consistency in one-sided assignment problems. Soc Choice Welfare 35: 415–433
Lensberg T (1987) Stability and collective rationality. Econometrica 55:935–961
Lensberg T (1988) Stability and the Nash solution. J Econ Theory 45:330–341
Maniquet F (1996) Horizontal equity and stability when the number of agents is variable in the fair division problem. Econ Lett 50:85–90
Maschler M, Owen G (1989) The consistent shapley-value for hyperplane games. Int J Game Theory 18:390–407
Moulin H (1987) Equal or proportional division of a surplus, and other methods. Int J Game Theory 16: 161–186
Moulin H (1988) Axioms of cooperative decision making. Cambridge University Press, Cambridge
Nash J (1950) The bargaining problem. Econometrica 18:155–162
O'Neill B (1982) A problem of rights arbitration from the Talmud. Math Soc Sci 2:345–371
Özkal-Sanver I (2009) Minimal converse consistent extension of the men-optimal solution, mimeo
Peleg B (1985) An axiomatization of the core of cooperative games without side-payments. J Math Econ 14:203–214
Peleg B (1986) On the reduced game property and its converse. Int J Game Theory 15 , 187–200. A correction. International Journal of Game Theory 16 (1987)
Peleg B, Tijs S (1996) The consistency principle for games in strategic form. Int J Game Theory 25:13–34
Potters JAM, Sudhölter P (1999) Airport problems and consistent solution rules. Math Soc Sci 38:83–102
Roemer J (1988) Axiomatic bargaining on economic environments. J Econ Theory 45:1–31
Sasaki H (1995) Consistency and monotonicity in assignment problems. Int J Game Theory 24:373–397
Sasaki H, Toda M (1992) Consistency and characterization of the core of two-sided matching problems. J Econ Theory 56:218–227
Schmeidler D (1969) The nucleolus of a characteristic function game. SIAM J Appl Math 17:1163–1170
Serrano R (1995) Strategic bargaining, surplus sharing and the nucleolus. J Math Econ 24:319–329
Shapley L (1953) A value for n-person games. In: Kuhn HW, Tucker AW (eds) Contributions to the theory of games II (annals of matheLLLmatics studies 28). Princeton University Press, Princeton pp 307–317
Shapley L, Shubik M (1972) The assignment game I: the core. Int J Game Theory 1:111–130
Sobolev AI (1975) The characterization of optimality principles in cooperative games by functional equations. Math Methods Soc Sci 6:150–165 (in Russian)
Sprumont Y (1991) The division problem with single-peaked preferences. Econometrica 59:509–519
Svensson L-G (1983) Large indivisibilities: an analysis with respect to price equilibrium and fairness. Econometrica 51:939–954
Tadenuma K (1992) Reduced games, consistency, and the core. Int J Game Theory 20:325–334
Tadenuma K, Thomson W (1991) No-envy and consistency in economies with indivisible goods. Econometrica 59:1755–1767
Tadenuma K, Thomson W (1993) The fair allocation of an indivisible good when monetary compensations are possible. Math Soc Sci 25:117–132

Thomson W (1983) The fair division of a fixed supply among a growing population. Math Oper Res 8: 319–326
Thomson W (1988) A study of choice correspondences in economies with a variable number of agents. J Econ Theory 46:247–259
Thomson W (1994a) Consistent solutions to the problem of fair division when preferences are single-peaked. J Econ Theory 63:219–245
Thomson W (1994b) Consistent extensions. Math Soc Sci 28:35–49
Thomson W (2003) Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Math Soc Sci 45:249–297
Thomson W (2010a) Fair allocation rules. In: Arrow K, Sen A, Suzumura K (eds) Handbook of social choice and welfare. North-Holland, Amsterdam pp 393–506
Thomson W (2010b) On the computational implications of converse consistency, University of Rochester mimeo
Thomson W (2011a) Lorenz rankings of rules for the adjudication of conflicting claims. Economic Theory (forthcoming)
Thomson W (2011b) Consistent allocation rules, monograph
Toda M (2003) Consistency and its converse in assignment games. Int J Math Game Theory Algebra 13: 1–14
Toda M (2006) Monotonicity and consistency in matching markets. Int J Game Theory 34:13–31
Yeh C-H (2006) Reduction-consistency in collective choice problems. J Math Econ 42:637–652
Young P (1987) On dividing an amount according to individual claims or liabilities. Math Oper Res 12: 398–414
Young P (1988) Distributive justice in taxation. J Econ Theory 44:321–335