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One-loop leading logarithms in electroweak radiative corrections

I. Results

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Abstract. We present results for the complete one-loop electroweak logarithmic corrections for general processes at high energies and fixed angles. Our results are applicable to arbitrary matrix elements that are not mass suppressed. We give explicit results for 4-fermion processes and gauge-boson pair production in e^+e^- annihilation.

1 Introduction

In the LEP regime, at energies $s^{1/2} \sim M_Z$, electroweak radiative corrections are dominated by large electromagnetic effects from initial-state radiation, by the contributions of the running electromagnetic coupling, and by the corrections associated with the ρ parameter, typically of the order 10%. Future colliders, such as the LHC [1] or an e^+e^- linear collider (LC) [2], will explore a new energy range, $s^{1/2} \gg M_Z$. It is known since many years (see, for instance, [3,4]) that above the electroweak scale the structure of the leading electroweak corrections changes and double logarithms of Sudakov type [5] as well as single logarithms involving the ratio of the energy to the electroweak scale become dominating. These logarithms arise from virtual (or real) gauge bosons emitted by the initial- and final-state particles. They correspond to the well-known soft and collinear singularities observed in theories with massless gauge bosons.

In massless gauge theories such as QED and QCD, the soft and collinear logarithms in the virtual corrections are singular and have to be cancelled by adding the contribution of real gauge-boson radiation. In the electroweak theory, the masses of the weak gauge bosons, Z and W, provide a physical cutoff, and the massive gauge bosons can be detected as distinguished particles. Unlike for the photon, real Z and W bremsstrahlung need not be included, and the large logarithms originating from virtual corrections are of physical significance.

The typical size of double-logarithmic (DL) and single-logarithmic (SL) corrections is given by

$$\frac{\alpha}{4\pi s_{\mathrm{w}}^2}\log^2\frac{s}{M_W^2} = 6.6\%, \quad \frac{\alpha}{4\pi s_{\mathrm{w}}^2}\log\frac{s}{M_W^2} = 1.3\%, \quad (1.1)$$

at $s^{1/2} = 1\,\mathrm{TeV}$ and increases with the energy. If the experimental precision is at the few-percent level like at the LHC, both DL and SL contributions have to be included at the one-loop level. In view of the precision objectives of a LC, between the percent and the permil level, besides the complete one-loop corrections also two-loop DL effects have to be taken into account. The DL contributions represent a leading and negative correction, whereas the SL ones often have opposite sign, and are referred to as subleading. The compensation between DL and SL corrections can be quite important [6,7], and depending on the process and the energy, the SL contribution can be even larger than the DL one[1].

Owing to this phenomenological relevance, the infrared (IR) structure of the electroweak theory is receiving increasing interest recently. The one-loop structure and the origin of the DL corrections have been discussed for $e^+e^- \to f\bar{f}$ [8,9] and are by now well established. Recipes for the resummation of the DL corrections have been developed [10,9,7,11] and explicit calculations of the leading DL corrections for the processes $g \to f\bar{f}$ and $e^+e^- \to f\bar{f}$ have been performed [12–14]. On the other hand, for the SL corrections complete one-loop calculations are only available for 4-fermion neutral-current processes [6,7] and W-pair production [4]. The subleading two-loop logarithmic corrections have been evaluated for $e^+e^- \to f\bar{f}$ in [7]. A general recipe for a subclass of SL corrections to all orders has been proposed in [15], based on the infrared-evolution equation method.

In this paper, we present results for all DL and SL contributions to the electroweak one-loop virtual corrections. The results apply to exclusive processes with arbitrary external states, including transverse and longitudinal gauge bosons as well as Higgs fields. Above the electroweak scale, the photon, Z- and W-boson loops are most conveniently treated in a symmetric way, rather than split into elec-

[1] For instance in $e^+e^- \to \mu^+\mu^-$ [7] at $s^{1/2} = 1\,\mathrm{TeV}$ one has +13.8% for SL and −9.6% for DL corrections to the unpolarized cross section

tromagnetic and weak parts [10]; at the same time special care has to be taken for the gap between the photon mass and the weak scale M_W. To this end we split the logarithms originating from the electromagnetic and from the Z-boson loops into two parts: the contributions of a fictitious heavy photon and a Z-boson with mass M_W, which are added to the W-boson loops resulting in the "symmetric-electroweak" (sew) contribution, and the remaining part originating from the difference between the photon or Z-boson mass and the mass of the W-boson. The large logarithms originating in the photon loops owing to the gap between the electromagnetic and the weak scale are denoted as "pure electromagnetic" (em) contribution.

In contrast to predictions based on the unbroken phase, our results are obtained from the high-energy limit of the broken phase, i.e. with calculations in the physical fields. In this way all features of the electroweak theory are consistently implemented[2]. Especially, the mixing and the mass gap between photons and Z-bosons is well under control. Furthermore, the longitudinal components of massive gauge bosons and the scalar fields are included as external states.

On the method

We work within the 't Hooft–Feynman gauge and use dimensional regularization so that ultraviolet (UV) single logarithms depend on the regularization scale μ. Exploiting the μ independence of the S matrix, we choose $\mu^2 = s$ so that the logarithms $\log(\mu^2/s)$ related to the UV singularities are not enhanced, and only the mass-singular logarithms $\log(\mu^2/M^2)$ or $\log(s/M^2)$ are large. In order to be specific we fix the field-renormalization constants (FRCs) such that no extra wave-function renormalization constants are required [16]. For parameter renormalization we adopt the on-shell scheme for definiteness. This can easily be changed. In this setup large logarithms appear in the mass-singular loop diagrams as well as in the coupling and field-renormalization constants, and are distributed as follows:

(1) The DL contributions originate from those one-loop diagrams where soft–collinear gauge bosons are exchanged between pairs of external legs. These double logarithms are obtained with the eikonal approximation.
(2) The SL mass-singular contributions from loop diagrams originate from the emission of virtual collinear gauge bosons from external lines [17]. These SL contributions are extracted from the loop diagrams in the collinear limit by means of Ward identities, and are found to factorize into the Born amplitude times "collinear factors". These are the main results of this article, and a forthcoming publication [18] will be dedicated to a detailed description of their calculation.
(3) The remaining SL contributions originating from soft and collinear regions are contained in the FRCs.
(4) The parameter renormalization constants, i.e. the charge- and weak-mixing-angle renormalization constants as well as the renormalization constants for Yukawa and scalar self couplings, involve the SL contributions of UV origin. These are the leading logarithms that are controlled by the renormalization group.

The DL and SL mass-singular terms are extracted from loop diagrams by setting all masses to zero in the numerators of the loop integrals. This approach is applicable only if no inverse powers of gauge-boson masses are present in the Feynman rules. In the Feynman gauge this is true except for the polarization vectors of longitudinal gauge bosons. However, since we are only interested in the high-energy limit, we can use the Goldstone-boson equivalence theorem [19] for processes involving longitudinal gauge bosons taking into account the correction factors from higher-order contributions [20].

This paper is organized as follows: in Sect. 2 we introduce our basic definitions and conventions. The leading logarithms originating from the soft–collinear region, from the soft or collinear regions, and from parameter renormalization are considered in Sects. 3, 4, and 5, respectively. In Sect. 6 we discuss some applications of our general results to simple specific processes. Results for the electroweak logarithmic corrections to the production of an arbitrary number of transverse gauge bosons in fermion–antifermion annihilation are given in Appendix A. Finally, Appendix B summarizes explicit results for the various generic quantities appearing in our formulas.

2 Definitions and conventions

We consider electroweak processes involving n arbitrary external particles. As a convention, all these particles and their momenta p_k are assumed to be incoming, so that the process reads

$$\varphi_{i_1}(p_1)\dots\varphi_{i_n}(p_n) \to 0. \tag{2.1}$$

The particles (or antiparticles) φ_{i_k} correspond to the components of the various multiplets φ present in the standard model. Chiral fermions and antifermions are represented by f^κ_σ and $\bar{f}^\kappa_\sigma$, respectively, with the chirality $\kappa = \mathrm{R},\mathrm{L}$ and the isospin indices $\sigma = \pm$. The gauge bosons are denoted by $V_a = A, Z, W^\pm$, and can be transversally (T) or longitudinally (L) polarized. For neutral gauge bosons we use the symbol $N = A, Z$. The components Φ_i of the scalar doublet consist of the physical Higgs particle H and the unphysical Goldstone bosons $\chi, \phi^\pm$, which are used to describe the longitudinally polarized massive gauge bosons Z_L and $W^\pm_\mathrm{L}$ with help of the equivalence theorem.

The predictions for general processes,

$$\varphi_{i_1}(p_1^{\mathrm{in}})\dots\varphi_{i_m}(p_m^{\mathrm{in}}) \to \varphi_{j_1}(p_1^{\mathrm{out}})\dots\varphi_{j_{n-m}}(p_{n-m}^{\mathrm{out}}), \tag{2.2}$$

[2] As observed in [9], the Higgs mechanism is irrelevant for the IR structure at the DL level. This seems to be less clear at the SL level, where, through self-energy contributions, mixing effects between gauge bosons and Goldstone bosons enter

can be obtained by crossing symmetry from our predictions for the $n \to 0$ process

$$\varphi_{i_1}(p_1^{\mathrm{in}}) \dots \varphi_{i_m}(p_m^{\mathrm{in}}) \bar{\varphi}_{j_1}(-p_1^{\mathrm{out}}) \dots \bar{\varphi}_{j_{n-m}}(-p_{n-m}^{\mathrm{out}}) \to 0 \tag{2.3}$$

where $\bar{\varphi}_i$ represents the charge conjugate of φ_i. Thus, outgoing particles (antiparticles) are substituted by incoming antiparticles (particles) and the corresponding momenta are reversed. These substitutions can be directly applied to our results.

The couplings of the external fields φ_{i_k} to the gauge bosons V_a are denoted by $\mathrm{i}eI^{V_a}(\varphi)$, and correspond to the generators of infinitesimal global $\mathrm{SU}(2) \times \mathrm{U}(1)$ transformations of these fields,

$$\delta_{V_a}\varphi_i = \mathrm{i}eI^{V_a}_{\varphi_i\varphi_{i'}}(\varphi)\,\varphi_{i'}. \tag{2.4}$$

To be precise, $\mathrm{i}eI^{V_a}_{\varphi_i\varphi_{i'}}(\varphi)$ is the coupling corresponding to the $V_a\bar{\varphi}_i\varphi_{i'}$ vertex, where all fields are incoming. The indices of the matrix $I^{V_a}_{\varphi_i\varphi_{i'}}(\varphi)$ may be particles or antiparticles, and charge conjugation of the identity (2.4) gives

$$I^{\bar{V}_a}_{\bar{\varphi}_i\bar{\varphi}_{i'}}(\bar{\varphi}) = -\left(I^{V_a}_{\varphi_i\varphi_{i'}}(\varphi)\right)^*. \tag{2.5}$$

As a shorthand notation for those formulas where various fields labelled by $k = 1, \dots, n$ occur, the components φ_{i_k} are replaced by their indices i_k. For instance, the generators in (2.4) are denoted by $I^a_{i_k i'_k}(k)$. A detailed description of the generators and other group-theoretical operators is given in Appendix B, together with the explicit values for various representations.

We consider the process (2.1) with all external momenta on shell, $p_k^2 = m_k^2$, and in the limit where all invariants are much larger than the gauge-boson masses, in particular

$$r_{kl} = (p_k + p_l)^2 \sim 2p_k p_l \gg M_W^2. \tag{2.6}$$

Note that this condition is not fulfilled if the cross section is dominated by resonances. We restrict ourselves to Born matrix elements that are not mass suppressed in this limit, and we calculate the virtual one-loop corrections in leading and subleading logarithmic approximation (LA), i.e. we take into account only enhanced DL and SL terms and omit non-enhanced terms. The logarithmic contributions are written in terms of

$$L(|r_{kl}|, M^2) := \frac{\alpha}{4\pi}\log^2\frac{r_{kl}}{M^2}, \quad l(r_{kl}, M^2) := \frac{\alpha}{4\pi}\log\frac{r_{kl}}{M^2}, \tag{2.7}$$

and depend on different invariants r_{kl} and masses M, according to the Feynman diagrams they originate from. In order to render the results as symmetric as possible, we relate the energy-dependent part of all large logarithms to the scales M_W and s. To this end, we write all these logarithms in terms of

$$L(s) := L(s, M_W^2), \quad l(s) := l(s, M_W^2), \tag{2.8}$$

and logarithms of mass ratios and ratios of invariants. The DL contributions proportional to $L(s)$ and to $l(s)$ $\times \log(|r_{kl}|/s)$ as well as the SL contributions proportional to $l(s)$ are denoted as the symmetric-electroweak part of the corrections. The IR singularities are regularized by an infinitesimal photon mass λ, and owing to the mass hierarchy

$$M_H, m_t, M_W, M_Z \gg m_{f\neq t} \gg \lambda, \tag{2.9}$$

all logarithms of electromagnetic origin $l(M_W^2, \lambda^2)$ and $l(M_W^2, m_f^2)$ involving the photon mass or light charged fermion masses are large and have to be taken into account, whereas the logarithms $l(M_W^2, M_Z^2)$, $l(m_t^2, M_W^2)$, and $l(M_H^2, M_W^2)$ are neglected. Furthermore, in the limit (2.6), the pure angular-dependent contributions $\log(r_{kl}/s)$ and $\log^2(r_{kl}/s)$ can be neglected.

The lowest-order matrix element for (2.1) is denoted by

$$\mathcal{M}_0^{i_1\dots i_n}(p_1, \dots, p_n). \tag{2.10}$$

In LA the corrections assume the form

$$\delta\mathcal{M}^{i_1\dots i_n}(p_1, \dots, p_n) = \mathcal{M}_0^{i'_1\dots i'_n}(p_1, \dots, p_n)\delta_{i'_1 i_1 \dots i'_n i_n}, \tag{2.11}$$

i.e. they factorize as a matrix, and are split into various contributions according to their origin:

$$\delta = \delta^{\mathrm{LSC}} + \delta^{\mathrm{SSC}} + \delta^{\mathrm{C}} + \delta^{\mathrm{PR}}. \tag{2.12}$$

The leading and subleading soft–collinear logarithms are denoted by δ^{LSC} and δ^{SSC}, respectively, the collinear logarithms by δ^{C}, and the logarithms resulting from parameter renormalization, which can be determined by the running of the couplings, by δ^{PR}.

$$\sum_{k=1}^{n}\sum_{l<k}\sum_{V_a=\mathrm{A,Z,W^\pm}}$$

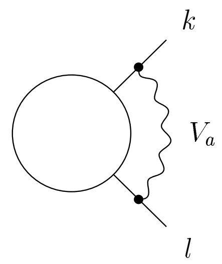


Fig. 1. Feynman diagrams leading to DL corrections

3 Soft–collinear contributions

The DL corrections originate from loop diagrams where virtual gauge bosons $V_a = A, Z, W^\pm$ are exchanged between pairs of external legs (Fig. 1). The double logarithms arise from the integration region where the gauge-boson momenta are soft and collinear to one of the external legs. As in QED, they can be evaluated using the eikonal approximation, where in the numerator of the loop integral the gauge-boson momentum is set to zero and all mass terms are neglected. In this approximation the one-loop corrections give

$$\delta\mathcal{M}^{i_1\dots i_n} = \sum_{k=1}^{n}\sum_{l<k}\sum_{V_a=A,Z,W^\pm}\int\frac{\mathrm{d}^4 q}{(2\pi)^4}$$

$$\times\,(-4\mathrm{i}e^2 p_k p_l I^{V_a}_{i'_k i_k}(k) I^{\bar{V}_a}_{i'_l i_l}(l)\mathcal{M}_0^{i_1\ldots i'_k\ldots i'_l\ldots i_n}) \,/\, \{(q^2-M^2_{V_a})[(p_k+q)^2-m^2_{k'}] \times [(p_l-q)^2-m^2_{l'}]\}, \quad (3.1)$$

and in LA, using the high-energy expansion of the scalar three-point function [21], one obtains

$$\delta\mathcal{M}^{i_1\ldots i_n} = \frac{1}{2}\sum_{k=1}^{n}\sum_{l\neq k}\sum_{V_a=A,Z,W^\pm} I^{V_a}_{i'_k i_k}(k) I^{\bar{V}_a}_{i'_l i_l}(l) \times\mathcal{M}_0^{i_1\ldots i'_k\ldots i'_l\ldots i_n} \times[L(|r_{kl}|,M^2_{V_a}) - \delta_{V_a A}L(m_k^2,\lambda^2)]. \quad (3.2)$$

The DL term containing the invariant r_{kl} depends on the angle between the momenta p_k and p_l. Writing

$$L(|r_{kl}|,M^2) = L(s,M^2) + 2l(s,M^2)\log\frac{|r_{kl}|}{s} + L(|r_{kl}|,s), \quad (3.3)$$

the angular-dependent part is isolated in logarithms of r_{kl}/s, and gives a subleading soft–collinear (SSC) contribution of order $l(s)\log(|r_{kl}|/s)$, whereas terms $L(|r_{kl}|,s)$ can be neglected in LA. The remaining part, together with the additional contributions from photon loops in (3.2), gives the leading soft–collinear (LSC) contribution and is angular-independent. The eikonal approximation (3.1) applies to chiral fermions, Higgs bosons, and transverse gauge bosons, and depends on their gauge couplings $I^{V_a}(k)$.

Owing to the longitudinal polarization vectors (4.24) which grow with energy, matrix elements involving longitudinal gauge bosons have to be treated with the equivalence theorem, i.e. they have to be expressed by matrix elements involving the corresponding Goldstone bosons. A detailed description of the equivalence theorem is given in Sect. 4. As explained there, the equivalence theorem for Born matrix elements (4.26) receives no DL one-loop corrections. Therefore, the soft–collinear corrections for external longitudinal gauge bosons can be obtained using the simple relations

$$\delta^{\mathrm{DL}}\mathcal{M}^{\ldots W^\pm_{\mathrm{L}}\ldots} = \delta^{\mathrm{DL}}\mathcal{M}^{\ldots\phi^\pm\ldots}, \qquad \delta^{\mathrm{DL}}\mathcal{M}^{\ldots Z_{\mathrm{L}}\ldots} = \mathrm{i}\delta^{\mathrm{DL}}\mathcal{M}^{\ldots\chi\ldots}, \quad (3.4)$$

from the corrections (3.2) for external Goldstone bosons.

Leading soft–collinear contributions

The invariance of the S matrix with respect to global $\mathrm{SU(2)}\times\mathrm{U(1)}$ transformations implies

$$0 = \delta_{V_a}\mathcal{M}^{i_1\ldots i_n} = \mathrm{i}e\sum_k I^{V_a}_{i'_k i_k}(k)\mathcal{M}^{i_1\ldots i'_k\ldots i_n}. \quad (3.5)$$

For external Goldstone fields extra contributions proportional to the Higgs vacuum expectation value appear, which are, however, irrelevant in the high-energy limit. Using (3.5), the LSC logarithms in (3.2) can be written as a single sum over external legs,

$$\delta^{\mathrm{LSC}}\mathcal{M}^{i_1\ldots i_n} = \sum_{k=1}^{n}\delta^{\mathrm{LSC}}_{i'_k i_k}(k)\mathcal{M}_0^{i_1\ldots i'_k\ldots i_n}. \quad (3.6)$$

After evaluating the sum over A, Z, and W, in (3.2), the correction factors read

$$\delta^{\mathrm{LSC}}_{i'_k i_k}(k) = -\frac{1}{2}\left[C^{\mathrm{ew}}_{i'_k i_k}(k)L(s) - 2(I^Z(k))^2_{i'_k i_k}\log\frac{M_Z^2}{M_W^2}l(s) + \delta_{i'_k i_k}Q_k^2 L^{\mathrm{em}}(s,\lambda^2,m_k^2)\right]. \quad (3.7)$$

The first term represents the DL symmetric-electroweak part and is proportional to the electroweak Casimir operator C^{ew} defined in (B.10). This is always diagonal in the SU(2) indices, except for external transverse neutral gauge bosons in the physical basis (B.14), where it gives rise to mixing between amplitudes involving photons and Z-bosons. The second term originates from Z-boson loops, owing to the difference between M_W and M_Z, and

$$L^{\mathrm{em}}(s,\lambda^2,m_k^2) := 2l(s)\log\left(\frac{M_W^2}{\lambda^2}\right) + L(M_W^2,\lambda^2) - L(m_k^2,\lambda^2) \quad (3.8)$$

contains all logarithms of pure electromagnetic origin. The LSC corrections for external longitudinal gauge bosons are directly obtained from (3.7) by using the quantum numbers of the corresponding Goldstone bosons. Formula (3.7) is in agreement with [9,11]. In [10] the logarithm $L(m_k^2,\lambda^2)$ that depends on the mass of the external state is missing.

Subleading soft–collinear contributions

The contribution of the second term of (3.3) to (3.2) remains a sum over pairs of external legs,

$$\delta^{\mathrm{SSC}}\mathcal{M}^{i_1\ldots i_n} = \sum_{k=1}^{n}\sum_{l<k}\sum_{V_a=A,Z,W^\pm} \times\delta^{V_a,\mathrm{SSC}}_{i'_k i_k i'_l i_l}(k,l)\mathcal{M}_0^{i_1\ldots i'_k\ldots i'_l\ldots i_n}, \quad (3.9)$$

with angular-dependent terms. The exchange of soft, neutral gauge bosons contributes with

$$\delta^{A,\mathrm{SSC}}_{i'_k i_k i'_l i_l}(k,l) = 2\left[l(s)+l(M_W^2,\lambda^2)\right]\log\frac{|r_{kl}|}{s}I^A_{i'_k i_k}(k)I^A_{i'_l i_l}(l),$$
$$\delta^{Z,\mathrm{SSC}}_{i'_k i_k i'_l i_l}(k,l) = 2l(s)\log\frac{|r_{kl}|}{s}I^Z_{i'_k i_k}(k)I^Z_{i'_l i_l}(l), \quad (3.10)$$

and, except for I^Z in the neutral scalar sector H,χ (see Appendix B), the couplings I^N are diagonal matrices. The exchange of charged gauge bosons yields

$$\delta^{W^\pm,\mathrm{SSC}}_{i'_k i_k i'_l i_l}(k,l) = 2l(s)\log\frac{|r_{kl}|}{s}I^\pm_{i'_k i_k}(k)I^\mp_{i'_l i_l}(l), \quad (3.11)$$

and owing to the non-diagonal matrices $I^{\pm}(k)$ [cf. (B.17), (B.22) and (B.26)], contributions of SU(2)-transformed Born matrix elements appear on the left-hand side of (3.9). In general, these transformed Born matrix elements are not related to the original Born matrix element and have to be evaluated explicitly.

The SSC corrections for external longitudinal gauge bosons are obtained from (3.9) with the equivalence theorem (3.4), i.e. the couplings and the Born matrix elements for Goldstone bosons[3] have to be used on the right-hand side of (3.9).

The application of the above formulas is illustrated in Sect. 6 for the case of 4-particle processes, where owing to $r_{12} = r_{34}$, $r_{13} = r_{24}$ and $r_{14} = r_{23}$, (3.9) reduces to

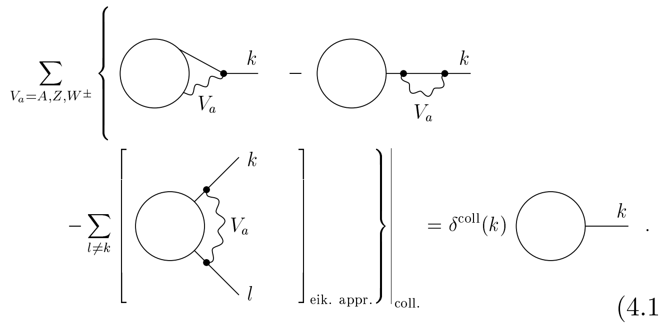
$$\begin{aligned}
\delta^{\mathrm{SSC}} \mathcal{M}^{i_1 i_2 i_3 i_4} = &\sum_{V_a = A, Z, W^{\pm}} 2\left[l(s) + l(M_W^2, M_{V_a}^2)\right] \\
&\times \Bigg\{ \log\frac{|r_{12}|}{s} \\
&\times \left[I^{V_a}_{i_1' i_1}(1) I^{\bar{V}_a}_{i_2' i_2}(2) \mathcal{M}_0^{i_1' i_2' i_3 i_4} + I^{V_a}_{i_3' i_3}(3) I^{\bar{V}_a}_{i_4' i_4}(4) \mathcal{M}_0^{i_1 i_2 i_3' i_4'} \right] \\
&+ \log\frac{|r_{13}|}{s} \\
&\times \left[I^{V_a}_{i_1' i_1}(1) I^{\bar{V}_a}_{i_3' i_3}(3) \mathcal{M}_0^{i_1' i_2 i_3' i_4} + I^{V_a}_{i_2' i_2}(2) I^{\bar{V}_a}_{i_4' i_4}(4) \mathcal{M}_0^{i_1 i_2' i_3 i_4'} \right] \\
&+ \log\frac{|r_{14}|}{s} \\
&\times \left[I^{V_a}_{i_1' i_1}(1) I^{\bar{V}_a}_{i_4' i_4}(4) \mathcal{M}_0^{i_1' i_2 i_3 i_4'} + I^{V_a}_{i_2' i_2}(2) I^{\bar{V}_a}_{i_3' i_3}(3) \mathcal{M}_0^{i_1 i_2' i_3' i_4} \right] \Bigg\},
\end{aligned} \tag{3.12}$$

and the logarithm with $r_{kl} = s$ vanishes. Note that this formula applies to $4 \to 0$ processes, where all particles or antiparticles and their momenta are incoming. Predictions for $2 \to 2$ processes are obtained by substituting outgoing particles (antiparticles) by the corresponding incoming antiparticles (particles), as explained in Sect. 2.

4 Collinear and soft single logarithms

In this section we consider the SL corrections originating from field-renormalization and from mass-singular loop diagrams. The PR contributions associated with the renormalization of the electric charge, the weak-mixing angle, and mass ratios are presented in Sect. 5. As explained in the introduction, in our approach to SL corrections we set the regularization scale $\mu^2 = s$ so that only mass-singular logarithms $\log(\mu^2/M^2)$ or $\log(s/M^2)$ are large.

On one hand the FRCs give the well-known factors $\delta Z_{\varphi}/2$ for each external leg, containing collinear as well as soft SL contributions. On the other hand, mass-singular logarithms arise from the collinear limit of loop diagrams where an external line splits into two internal lines [17], one of these internal lines being a virtual gauge boson A, Z or W. If the two internal lines involve only fermions and scalars no mass-singular terms emerge. The mass-singular diagrams are evaluated in the limit of collinear gauge-boson emission using Ward identities [18], and after subtraction of the contributions already contained in the FRCs and in the soft–collinear corrections, we find factorization into collinear factors δ^{coll} times Born matrix elements,



(4.1)

Then the complete SL contributions originating from soft or collinear regions can be written as a sum over the external legs,

$$\delta^{\mathrm{C}} \mathcal{M}^{i_1 \ldots i_n} = \sum_{k=1}^{n} \delta^{\mathrm{C}}_{i_k' i_k}(k) \mathcal{M}_0^{i_1 \ldots i_k' \ldots i_n}, \tag{4.2}$$

with

$$\delta^{\mathrm{C}}_{i_k' i_k}(k) = \delta^{\mathrm{coll}}_{i_k' i_k}(k) + \frac{1}{2} \delta Z^{\varphi}_{i_k' i_k}\Big|_{\mu^2 = s}. \tag{4.3}$$

The collinear factors $\delta^{\mathrm{coll}}(k)$ and the corrections $\delta^{\mathrm{C}}(k)$ depend on the quantum numbers of the external fields φ_{i_k}. In the following we give the results for chiral fermions, transverse charged gauge bosons W_{T}, transverse neutral gauge bosons $A_{\mathrm{T}}, Z_{\mathrm{T}}$, longitudinal gauge bosons $W_{\mathrm{L}}, Z_{\mathrm{L}}$, and Higgs bosons. We use the conventions of [16] for the Feynman rules, the self-energies, and the renormalization constants.

Chiral fermions

In LA the FRCs for fermions f^{κ}_{σ} with chirality $\kappa = \mathrm{R}, \mathrm{L}$ and isospin indices $\sigma = \pm$ are given by

$$\begin{aligned}
\delta Z^{\kappa}_{f_\sigma f_{\sigma'}} = \delta_{\sigma\sigma'} \Bigg\{ - \Bigg[& C^{\mathrm{ew}}_{f^\kappa} \\
& + \frac{1}{4 s_{\mathrm{w}}^2} \left((1 + \delta_{\kappa \mathrm{R}}) \frac{m_{f_\sigma}^2}{M_W^2} + \delta_{\kappa \mathrm{L}} \frac{m_{f_{-\sigma}}^2}{M_W^2} \right) \Bigg] l(\mu^2) \\
& + Q_{f_\sigma}^2 \left[2 l(M_W^2, \lambda^2) - 3 l(M_W^2, m_{f_\sigma}^2) \right] \Bigg\},
\end{aligned} \tag{4.4}$$

where the contribution of a non-trivial quark-mixing matrix is not considered. The FRCs depend on the chirality of the fermions, and contain Yukawa terms proportional to the masses of the fermion f_σ and of its isospin partner $f_{-\sigma}$. While these are negligible for leptons and light

[3] Note that for Goldstone bosons χ, the equivalence theorem as well as the couplings (B.23) and (B.21) contain the imaginary constant i

quarks, they give large contributions for $f^\kappa_\sigma = t^{\mathrm{R}}$, t^{L}, and b^{L}, where one of the masses is m_t.

From the mass-singular loop diagrams we obtain the factor [18]

$$\delta^{\mathrm{coll}}_{f_\sigma f_{\sigma'}}(f^\kappa) = \delta_{\sigma\sigma'}\left[2C^{\mathrm{ew}}_{f^\kappa} l(\mu^2) + 2Q^2_{f_\sigma} l(M_W^2, m^2_{f_\sigma})\right], \quad (4.5)$$

and the complete contribution (4.3) reads

$$\delta^{\mathrm{C}}_{f_\sigma f_{\sigma'}}(f^\kappa) = \delta_{\sigma\sigma'}\left\{\left[\frac{3}{2}C^{\mathrm{ew}}_{f^\kappa} - \frac{1}{8s_{\mathrm{w}}^2}\left((1+\delta_{\kappa\mathrm{R}})\frac{m^2_{f_\sigma}}{M_W^2} + \delta_{\kappa\mathrm{L}}\frac{m^2_{f_{-\sigma}}}{M_W^2}\right)\right] \times l(s) + Q^2_{f_\sigma} l^{\mathrm{em}}(m^2_{f_\sigma})\right\}, \quad (4.6)$$

where the pure electromagnetic logarithms

$$l^{\mathrm{em}}(m_f^2) := \frac{1}{2} l(M_W^2, m_f^2) + l(M_W^2, \lambda^2) \quad (4.7)$$

originate from the photonic loops as a result of the gap between the electromagnetic and weak scales. The symmetric-electroweak part of (4.6), i.e. the term proportional to $l(s)$, agrees with [7,15] up to the Yukawa contributions, and the electromagnetic part (4.7) agrees with [15].

Transverse charged gauge bosons W

The FRC of $W^\pm$ bosons in LA reads

$$\delta Z_{WW} = -\left.\frac{\partial \Sigma^{WW}_{\mathrm{T}}(k^2)}{\partial k^2}\right|_{k^2=M_W^2} = \left[b^{\mathrm{ew}}_W - 2C^{\mathrm{ew}}_W\right] l(\mu^2) + 2Q^2_W l(M_W^2, \lambda^2), \quad (4.8)$$

where b^{ew}_W is the coefficient of the β-function defined in (B.38) and contains the sum over gauge-boson, scalar, and fermion loops, whereas C^{ew}_W is the eigenvalue of the electroweak Casimir operator in the adjoint representation (B.24).

Combining the collinear factor [18]

$$\delta^{\mathrm{coll}}_{W^\sigma W^{\sigma'}}(V_{\mathrm{T}}) = \delta_{\sigma\sigma'} C^{\mathrm{ew}}_W l(\mu^2), \quad (4.9)$$

with the FRC $\delta Z_{W^\sigma W^{\sigma'}} = \delta_{\sigma\sigma'}\delta Z_{WW}$, results in

$$\delta^{\mathrm{C}}_{W^\sigma W^{\sigma'}}(V_{\mathrm{T}}) = \delta_{\sigma\sigma'}\left[\frac{1}{2} b^{\mathrm{ew}}_W l(s) + Q^2_W l^{\mathrm{em}}(M_W^2)\right]. \quad (4.10)$$

This result agrees with the revised version of [15].

Transverse neutral gauge bosons A,Z

The physical neutral gauge-boson fields $N = A, Z$ are renormalized by a non-symmetric matrix δZ, i.e.

$$N \to N + \delta N, \qquad \delta N = \frac{1}{2}\delta Z_{NN'} N' = \frac{1}{2}\left[\delta Z^{\mathrm{asymm}}_{NN'} + \delta Z^{\mathrm{symm}}_{NN'}\right] N'. \quad (4.11)$$

The matrix δZ has been split into antisymmetric and symmetric parts in order to facilitate the comparison with the corresponding FRC $\delta\tilde{Z}$ for the symmetric components $\tilde{N} = B, W^3$,

$$\delta\tilde{N} = \frac{1}{2}\delta\tilde{Z}_{\tilde{N}\tilde{N}'}\tilde{N}'. \quad (4.12)$$

In the following, we give $\delta Z_{NN'}$ as obtained in LA in the on-shell scheme [16] and compare it with $\delta\tilde{Z}_{\tilde{N}\tilde{N}'}$ using the matrix relation

$$\delta Z = 2\delta U(\theta_{\mathrm{w}})U^{-1}(\theta_{\mathrm{w}}) + U(\theta_{\mathrm{w}})\delta\tilde{Z}U^{-1}(\theta_{\mathrm{w}}), \quad (4.13)$$

resulting from the renormalization of the Weinberg rotation $U(\theta_{\mathrm{w}})$ defined in (B.4).

The results for symmetric and antisymmetric parts are expressed in terms of the coefficients of the β-function defined in (B.41):

(1) For the antisymmetric part the diagonal components vanish, whereas the non-diagonal ones are

$$\delta Z^{\mathrm{asymm}}_{AZ} = -\delta Z^{\mathrm{asymm}}_{ZA} = -\frac{\Sigma^{AZ}_{\mathrm{T}}(M_Z^2) + \Sigma^{AZ}_{\mathrm{T}}(0)}{M_Z^2}, \quad (4.14)$$

and in LA we find

$$\delta Z^{\mathrm{asymm}}_{AZ} = b^{\mathrm{ew}}_{AZ} l(\mu^2). \quad (4.15)$$

This part is related to the renormalization of the Weinberg angle (5.6), and in LA it corresponds to the first term in (4.13)

$$2\left[\delta U(\theta_{\mathrm{w}})U^{-1}(\theta_{\mathrm{w}})\right]_{NN'} = \frac{c_{\mathrm{w}}}{s_{\mathrm{w}}}\frac{\delta c_{\mathrm{w}}^2}{c_{\mathrm{w}}^2} E_{NN'} = b^{\mathrm{ew}}_{AZ} l(\mu^2) E_{NN'}, \qquad E := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.16)$$

(2) The symmetric part has components

$$\delta Z^{\mathrm{symm}}_{NN'} = -\left.\frac{\partial \Sigma^{NN'}_{\mathrm{T}}(k^2)}{\partial k^2}\right|_{k^2=M_N^2}, \qquad \delta Z^{\mathrm{symm}}_{AZ} = \delta Z^{\mathrm{symm}}_{ZA} = -\frac{\Sigma^{AZ}_{\mathrm{T}}(M_Z^2) - \Sigma^{AZ}_{\mathrm{T}}(0)}{M_Z^2}, \quad (4.17)$$

and in LA it reads

$$\delta Z^{\mathrm{symm}}_{NN'} = \left[b^{\mathrm{ew}}_{NN'} - 2C^{\mathrm{ew}}_{NN'}\right] l(\mu^2) + \delta_{NA}\delta_{N'A}\delta Z^{\mathrm{em}}_{AA}. \quad (4.18)$$

The AA component receives a pure electromagnetic contribution associated with the light-fermion loops,

$$\delta Z^{\mathrm{em}}_{AA} = -\frac{4}{3}\sum_{f,i,\sigma\neq t} N^f_{\mathrm{C}} Q^2_{f_\sigma} l(M_W^2, m^2_{f_{\sigma,i}}), \quad (4.19)$$

where the sum runs over the generations $i = 1,2,3$ of leptons and quarks $f = l, q$ with isospin σ, omitting the top-quark contribution.

Apart from these pure electromagnetic logarithms, (4.18) contains the same combination of b^{ew} and C^{ew} as (4.8). This part corresponds to the second term in (4.13) originating from the renormalization of the symmetric fields,

$$\delta\tilde{Z}_{\tilde{N}\tilde{N}'} = \delta_{\tilde{N}\tilde{N}'}\left[\tilde{b}^{\mathrm{ew}}_{\tilde{N}} - 2\tilde{C}^{\mathrm{ew}}_{\tilde{N}}\right] l(\mu^2), \tag{4.20}$$

which is diagonal, because the U(1) and SU(2) components do not mix in the unbroken theory.

The SL contributions (4.15) and (4.18) have to be combined with the collinear factor, for which we obtain [18]

$$\delta^{\mathrm{coll}}_{NN'}(V_{\mathrm{T}}) = C^{\mathrm{ew}}_{NN'} l(\mu^2). \tag{4.21}$$

Then, the complete correction is given by

$$\delta^{\mathrm{C}}_{N'N}(V_{\mathrm{T}}) = \frac{1}{2}\left[E_{N'N} b^{\mathrm{ew}}_{AZ} + b^{\mathrm{ew}}_{N'N}\right] l(s) + \frac{1}{2}\delta_{NA}\delta_{N'A}\delta Z^{\mathrm{em}}_{AA}. \tag{4.22}$$

Note that owing to the antisymmetric contribution ($E_{AZ} = -E_{ZA} = 1$) the non-diagonal components read

$$\delta^{\mathrm{C}}_{AZ}(V_{\mathrm{T}}) = b^{\mathrm{ew}}_{AZ} l(s), \quad \delta^{\mathrm{C}}_{ZA}(V_{\mathrm{T}}) = 0, \tag{4.23}$$

i.e. the correction factor for external photons does not involve mixing with Z-bosons. This is a consequence of the on-shell renormalization condition (4.14). The symmetric part of (4.22) agrees with the revised version of [15].

Longitudinally polarized gauge bosons

Our approach is not directly applicable to the calculation of the effective collinear factor (4.2) for longitudinal gauge bosons, because the amputated Green functions involving gauge bosons are contracted with longitudinal polarization vectors,

$$\epsilon^{\mu}_{\mathrm{L}}(p) = \frac{p^{\mu}}{M} + \mathcal{O}\left(\frac{M}{p^0}\right), \tag{4.24}$$

containing a mass term in the denominator so that in this case contributions of the order of the gauge-boson mass cannot be neglected. This problem can be circumvented by means of the Goldstone-boson equivalence theorem, expressing the Green functions involving longitudinal gauge bosons by Green functions with the corresponding Goldstone bosons. The equivalence theorem for bare amputated Green function reads (we denote bare quantities by an index 0)

$$\begin{aligned} p^{\mu}\langle W_{0,\mu}(p)\ldots\rangle &= M_{0,W}(1+\delta C_{W_0})\langle\phi_0(p)\ldots\rangle, \\ p^{\mu}\langle Z_{0,\mu}(p)\ldots\rangle &= \mathrm{i}M_{0,Z}(1+\delta C_{Z_0})\langle\chi_0(p)\ldots\rangle, \end{aligned} \tag{4.25}$$

where the dots represent arbitrary fields. In Born approximation, this gives the well-known relations

$$\begin{aligned} \mathcal{M}_0^{\ldots W^{\pm}_{\mathrm{L}}\ldots} &= \mathcal{M}_0^{\ldots\phi^{\pm}\ldots}, \\ \mathcal{M}_0^{\ldots Z_{\mathrm{L}}\ldots} &= \mathrm{i}\mathcal{M}_0^{\ldots\chi\ldots}, \end{aligned} \tag{4.26}$$

between matrix elements. Note however, that besides the lowest-order contribution, (4.25) contains non-trivial higher-order corrections δC_{W_0}, δC_{Z_0} owing to the mixing between gauge bosons and Goldstone bosons [20]. In one-loop approximation these corrections can be expressed in terms of bare self-energies involving Goldstone bosons and longitudinal gauge bosons evaluated at the mass of the gauge bosons,

$$\begin{aligned} \delta C_{W_0} &= -\frac{\Sigma^{WW}_{\mathrm{L}}(M_W^2) + M_W\Sigma^{W\phi}(M_W^2)}{M_W^2}, \\ \delta C_{Z_0} &= -\frac{\Sigma^{ZZ}_{\mathrm{L}}(M_Z^2) - \mathrm{i}M_Z\Sigma^{Z\chi}(M_Z^2)}{M_Z^2}. \end{aligned} \tag{4.27}$$

Since neither δC nor the counterterms involve double logarithms, the equivalence theorem can be applied to the DL corrections in the naive way, i.e. without higher-order corrections δC_{W_0}, δC_{Z_0}.

The renormalization of (4.25) leads to extra mass and field-renormalization counterterms. Especially, the renormalization in the neutral sector involves mixing effects, but as expected, the physical longitudinal Z-boson does not mix with the photon. Keeping the unphysical scalar fields unrenormalized, and absorbing correction factors and counterterms into new renormalized correction factors δC_{ϕ}, δC_{χ}, we can write

$$\begin{aligned} p^{\mu}\langle W_{\mu}(p)\ldots\rangle &= M_W(1+\delta C_{\phi})\langle\phi_0(p)\ldots\rangle, \\ p^{\mu}\langle Z_{\mu}(p)\ldots\rangle &= \mathrm{i}M_Z(1+\delta C_{\chi})\langle\chi_0(p)\ldots\rangle, \end{aligned} \tag{4.28}$$

with

$$\begin{aligned} \delta C_{\phi} &= \delta C_{W_0} + \frac{\delta M_W}{M_W} + \frac{1}{2}\delta Z_{WW}, \\ \delta C_{\chi} &= \delta C_{Z_0} + \frac{\delta M_Z}{M_Z} + \frac{1}{2}\delta Z_{ZZ}. \end{aligned} \tag{4.29}$$

In LA we find

$$\begin{aligned} \delta C_{\phi} &= \left[C^{\mathrm{ew}}_{\Phi} - \frac{N^t_{\mathrm{C}}}{4s^2_{\mathrm{w}}}\frac{m_t^2}{M_W^2}\right] l(\mu^2) + Q_W^2 l(M_W^2,\lambda^2), \\ \delta C_{\chi} &= \left[C^{\mathrm{ew}}_{\Phi} - \frac{N^t_{\mathrm{C}}}{4s^2_{\mathrm{w}}}\frac{m_t^2}{M_W^2}\right] l(\mu^2). \end{aligned} \tag{4.30}$$

The result is written in terms of the eigenvalue of C^{ew} for the scalar doublet Φ and contains large m_t-dependent contributions originating from the mass counterterms (5.4), which are proportional to the color factor $N^t_{\mathrm{C}} = 3$. With (4.28) and with the collinear factor for Goldstone bosons $S = \phi^{\pm}, \chi$ [18],

$$\delta^{\mathrm{coll}}_{SS'}(\Phi) = \delta_{SS'} C^{\mathrm{ew}}_{\Phi} l(\mu^2), \tag{4.31}$$

the complete collinear corrections (4.2) for longitudinal gauge bosons are obtained by means of amplitudes involving Goldstone bosons,

$$\begin{aligned} \delta^{\mathrm{C}}\mathcal{M}^{\ldots W^{\pm}_{\mathrm{L}}\ldots} &= \left[\delta^{\mathrm{coll}}_{\phi^{\pm}\phi^{\pm}}(\Phi) + \delta C_{\phi}\right]\mathcal{M}_0^{\ldots\phi^{\pm}\ldots} \\ &= \delta^{\mathrm{C}}_{\phi^{\pm}\phi^{\pm}}(\Phi)\,\mathcal{M}_0^{\ldots W^{\pm}_{\mathrm{L}}\ldots}, \\ \delta^{\mathrm{C}}\mathcal{M}^{\ldots Z_{\mathrm{L}}\ldots} &= \mathrm{i}\left[\delta^{\mathrm{coll}}_{\chi\chi}(\Phi) + \delta C_{\chi}\right]\mathcal{M}_0^{\ldots\chi\ldots} \\ &= \delta^{\mathrm{C}}_{\chi\chi}(\Phi)\,\mathcal{M}_0^{\ldots Z_{\mathrm{L}}\ldots}, \end{aligned} \tag{4.32}$$

with

$$\delta^{\mathrm{C}}_{\phi^\pm\phi^\pm}(\Phi) = \left[2C^{\mathrm{ew}}_{\Phi} - \frac{N^t_{\mathrm{C}}}{4s^2_{\mathrm{w}}}\frac{m_t^2}{M_W^2}\right] l(s) + Q_W^2 l^{\mathrm{em}}(M_W^2),$$
$$\delta^{\mathrm{C}}_{\chi\chi}(\Phi) = \left[2C^{\mathrm{ew}}_{\Phi} - \frac{N^t_{\mathrm{C}}}{4s^2_{\mathrm{w}}}\frac{m_t^2}{M_W^2}\right] l(s). \quad (4.33)$$

Note that, up to pure electromagnetic terms, the correction factors (4.29) correspond to FRCs for Goldstone bosons. In fact, in LA $\delta C_\chi = \delta Z_\chi/2$ and $\delta C_\phi = \delta Z_\phi/2 + Q_W^2 l(M_W^2, \lambda^2)$.

Higgs bosons

In LA the FRC for Higgs bosons reads

$$\delta Z_H = \left[2C^{\mathrm{ew}}_{\Phi} - \frac{N^t_{\mathrm{C}}}{2s^2_{\mathrm{w}}}\frac{m_t^2}{M_W^2}\right] l(\mu^2), \quad (4.34)$$

with a large Yukawa contribution coming from the top-quark loop. For the collinear factor we find [18]

$$\delta^{\mathrm{coll}}_{HH}(\Phi) = C^{\mathrm{ew}}_{\Phi} l(\mu^2), \quad (4.35)$$

and the complete correction is

$$\delta^{\mathrm{C}}_{HH}(\Phi) = \left[2C^{\mathrm{ew}}_{\Phi} - \frac{N^t_{\mathrm{C}}}{4s^2_{\mathrm{w}}}\frac{m_t^2}{M_W^2}\right] l(s). \quad (4.36)$$

Note that up to pure electromagnetic contributions, longitudinal gauge bosons and Higgs bosons receive the same collinear SL corrections.

5 Logarithms connected to parameter renormalization

Finally, there are logarithms related to UV divergences. These logarithms originate from the renormalization of the dimensionless parameters, i.e. the electric charge e, the weak-mixing angle c_{w}, and the mass ratios

$$h_t = \frac{m_t}{M_W}, \quad h_H = \frac{M_H^2}{M_W^2}, \quad (5.1)$$

and are obtained from the Born matrix element $\mathcal{M}_0 = \mathcal{M}_0(e, c_{\mathrm{w}}, h_t, h_H)$ in the high-energy limit by

$$\delta^{\mathrm{PR}}\mathcal{M} = \frac{\delta\mathcal{M}_0}{\delta e}\delta e + \frac{\delta\mathcal{M}_0}{\delta c_{\mathrm{w}}}\delta c_{\mathrm{w}} + \frac{\delta\mathcal{M}_0}{\delta h_t}\delta h_t + \frac{\delta\mathcal{M}_0}{\delta h_H}\delta h_H^{\mathrm{eff}}\bigg|_{\mu^2=s}. \quad (5.2)$$

The mass ratios h_t and h_H are related to the top-quark Yukawa coupling and to the scalar self-coupling, respectively. They appear only in processes where these couplings enter. The renormalization of the masses in the propagators or in couplings with mass dimension yields only mass-suppressed contributions which are irrelevant in the high-energy limit in amplitudes that are not mass suppressed.

The logarithms connected to parameter renormalization can simply be obtained by the replacements $e \to e + \delta e$, $c_{\mathrm{w}} \to c_{\mathrm{w}} + \delta c_{\mathrm{w}}$, $s_{\mathrm{w}} \to s_{\mathrm{w}} + \delta s_{\mathrm{w}}$, $h_t \to h_t + \delta h_t$ and $h_H \to h_H + \delta h_H^{\mathrm{eff}}$ in the lowest-order matrix elements in the high-energy limit. In the case of processes with longitudinal gauge bosons, these substitutions must be performed in the matrix elements resulting from the equivalence theorem.

Mixing-angle renormalization

In the on-shell scheme, the renormalization of the weak-mixing angle (B.5) is given by

$$\frac{\delta c_{\mathrm{w}}^2}{c_{\mathrm{w}}^2} = \frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2} = \frac{\Sigma^W_T(M_W^2)}{M_W^2} - \frac{\Sigma^Z_T(M_Z^2)}{M_Z^2}. \quad (5.3)$$

After tadpole renormalization, i.e. omitting the tadpole diagrams, the mass counterterms give

$$\frac{\delta M_W^2}{M_W^2} = -\left[b^{\mathrm{ew}}_W - 4C^{\mathrm{ew}}_{\Phi}\right] l(\mu^2) - \frac{N^t_{\mathrm{C}}}{2s^2_{\mathrm{w}}}\frac{m_t^2}{M_W^2} l(\mu^2),$$
$$\frac{\delta M_Z^2}{M_Z^2} = -\left[b^{\mathrm{ew}}_{ZZ} - 4C^{\mathrm{ew}}_{\Phi}\right] l(\mu^2) - \frac{N^t_{\mathrm{C}}}{2s^2_{\mathrm{w}}}\frac{m_t^2}{M_W^2} l(\mu^2), \quad (5.4)$$

and contain large $(m_t^2/M_W^2) l(\mu^2)$ terms. However, these terms cancel in (5.3), and using

$$b^{\mathrm{ew}}_{AZ} = \frac{c_{\mathrm{w}}}{s_{\mathrm{w}}}(b^{\mathrm{ew}}_{ZZ} - b^{\mathrm{ew}}_W), \quad (5.5)$$

which follows from (B.42), we can express the mixing-angle counterterm by the AZ component of the β-function:

$$\frac{\delta c_{\mathrm{w}}^2}{c_{\mathrm{w}}^2} = \frac{s_{\mathrm{w}}}{c_{\mathrm{w}}} b^{\mathrm{ew}}_{AZ} l(\mu^2). \quad (5.6)$$

Charge renormalization

In the on-shell scheme, the coupling-constant counterterms are related to the FRCs by Ward identities. For the electric charge counterterm we have

$$\delta Z_e = -\frac{1}{2}\left[\delta Z_{AA} + \frac{s_{\mathrm{w}}}{c_{\mathrm{w}}}\delta Z_{ZA}\right] = \frac{1}{2}\frac{\partial \Sigma^{AA}_T(k^2)}{\partial k^2}\bigg|_{k^2=0} - \frac{s_{\mathrm{w}}}{c_{\mathrm{w}}}\frac{\Sigma^{AZ}_T(0)}{M_Z^2} = -\frac{1}{2} b^{\mathrm{ew}}_{AA} l(\mu^2) + \delta Z_e^{\mathrm{em}}, \quad (5.7)$$

where the pure electromagnetic part

$$\delta Z_e^{\mathrm{em}} = -\frac{1}{2}\delta Z^{\mathrm{em}}_{AA} = \frac{2}{3}\sum_{f,i,\sigma\neq t} N^f_{\mathrm{C}} Q^2_{f_\sigma} l(M_W^2, m^2_{f_{\sigma,i}}) \quad (5.8)$$

is related to the running of the electromagnetic coupling constant from zero momentum transfer to the electroweak scale,

$$\Delta\alpha(M_W^2) = 2\delta Z_e^{\mathrm{em}}. \tag{5.9}$$

The counterterms to the U(1) and SU(2) gauge couplings,

$$g_1 = \frac{e}{c_\mathrm{w}}, \quad g_2 = \frac{e}{s_\mathrm{w}}, \tag{5.10}$$

can be written as

$$\begin{aligned}\frac{\delta g_1}{g_1} &= \delta Z_e - \frac{1}{2}\frac{\delta c_\mathrm{w}^2}{c_\mathrm{w}^2} = -\frac{1}{2}\tilde{b}_B^{\mathrm{ew}} l(\mu^2) + \delta Z_e^{\mathrm{em}},\\ \frac{\delta g_2}{g_2} &= \delta Z_e + \frac{1}{2}\frac{c_\mathrm{w}^2}{s_\mathrm{w}^2}\frac{\delta c_\mathrm{w}^2}{c_\mathrm{w}^2} = -\frac{1}{2}\tilde{b}_W^{\mathrm{ew}} l(\mu^2) + \delta Z_e^{\mathrm{em}},\end{aligned} \tag{5.11}$$

where we have used the relations (B.42).

Yukawa-coupling renormalization

In the on-shell scheme, the renormalization of the top-quark mass is given by

$$\delta m_t = \frac{m_t}{2}\left[\Sigma^{t,\mathrm{L}}(m_t^2) + \Sigma^{t,\mathrm{R}}(m_t^2) + 2\Sigma^{t,\mathrm{S}}(m_t^2)\right]. \tag{5.12}$$

This leads to the following logarithmic contributions

$$\frac{\delta m_t}{m_t} = \left[\frac{1}{4s_\mathrm{w}^2} + \frac{1}{8s_\mathrm{w}^2 c_\mathrm{w}^2} + \frac{3}{2c_\mathrm{w}^2}Q_t - \frac{3}{c_\mathrm{w}^2}Q_t^2 + \frac{3}{8s_\mathrm{w}^2}\frac{m_t^2}{M_W^2}\right] l(\mu^2). \tag{5.13}$$

Using (5.4), the counterterm for h_t reads

$$\begin{aligned}\frac{\delta h_t}{h_t} &= \frac{\delta m_t}{m_t} - \frac{1}{2}\frac{\delta M_W^2}{M_W^2}\\ &= \frac{1}{2}b_W^{\mathrm{ew}} l(\mu^2)\\ &+ \left[-\frac{3}{4s_\mathrm{w}^2} - \frac{3}{8s_\mathrm{w}^2 c_\mathrm{w}^2} + \frac{3}{2c_\mathrm{w}^2}Q_t - \frac{3}{c_\mathrm{w}^2}Q_t^2\right] l(\mu^2)\\ &+ \frac{3+2N_\mathrm{C}^t}{8s_\mathrm{w}^2}\frac{m_t^2}{M_W^2} l(\mu^2).\end{aligned} \tag{5.14}$$

The counterterm to the top-quark Yukawa coupling,

$$g_t = \frac{e}{\sqrt{2}s_\mathrm{w}} h_t, \tag{5.15}$$

is given by

$$\frac{\delta g_t}{g_t} = \frac{1}{2}\Delta\alpha(M_W^2) - \frac{1}{2}b_W^{\mathrm{ew}} l(\mu^2) + \frac{\delta h_t}{h_t}. \tag{5.16}$$

Scalar self-coupling renormalization

In the on-shell scheme, the renormalization of the Higgs mass is given by

$$\delta M_H^2 = \Sigma^H(M_H^2), \tag{5.17}$$

or in logarithmic accuracy

$$\begin{aligned}\frac{\delta M_H^2}{M_H^2} &= \frac{1}{2s_\mathrm{w}^2}\left[\frac{9M_W^2}{M_H^2}\left(1+\frac{1}{2c_\mathrm{w}^4}\right) - \frac{3}{2}\left(1+\frac{1}{2c_\mathrm{w}^2}\right)\right.\\ &\left.+ \frac{15}{4}\frac{M_H^2}{M_W^2}\right] l(\mu^2) + \frac{N_\mathrm{C}^t}{2s_\mathrm{w}^2}\frac{m_t^2}{M_W^2}\left(1 - 6\frac{m_t^2}{M_H^2}\right) l(\mu^2).\end{aligned} \tag{5.18}$$

The renormalization of the scalar self-couplings gets an extra contribution from the tadpole renormalization (cf. [22])

$$\begin{aligned}\delta t = -T &= \frac{1}{e s_\mathrm{w} M_W}\left[-\frac{3}{2}M_W^2\left(\frac{M_Z^2}{c_\mathrm{w}^2} + 2M_W^2\right)\right.\\ &\left.- \frac{M_H^2}{4}(2M_W^2 + M_Z^2 + 3M_H^2) + 2N_\mathrm{C}^t m_t^4\right] l(\mu^2).\end{aligned} \tag{5.19}$$

Including this in the renormalization of h_H and using (5.4), we find the effective counterterm

$$\begin{aligned}\frac{\delta h_H^{\mathrm{eff}}}{h_H} &= \frac{\delta M_H^2}{M_H^2} - \frac{\delta M_W^2}{M_W^2} + \frac{e}{2s_\mathrm{w}}\frac{\delta t}{M_W M_H^2}\\ &= b_W^{\mathrm{ew}} l(\mu^2) + \frac{3}{2s_\mathrm{w}^2}\left[\frac{M_W^2}{M_H^2}\left(2+\frac{1}{c_\mathrm{w}^4}\right)\right.\\ &\left.- \left(2+\frac{1}{c_\mathrm{w}^2}\right) + \frac{M_H^2}{M_W^2}\right] l(\mu^2)\\ &+ \frac{N_\mathrm{C}^t}{s_\mathrm{w}^2}\frac{m_t^2}{M_W^2}\left(1 - 2\frac{m_t^2}{M_H^2}\right) l(\mu^2).\end{aligned} \tag{5.20}$$

The counterterm to the scalar self-coupling

$$\lambda = \frac{e^2}{2s_\mathrm{w}^2} h_H \tag{5.21}$$

is given by

$$\frac{\delta\lambda}{\lambda} = \Delta\alpha(M_W^2) - b_W^{\mathrm{ew}} l(\mu^2) + \frac{\delta h_H^{\mathrm{eff}}}{h_H}. \tag{5.22}$$

The logarithms resulting from parameter renormalization are the ones that determine the running of the couplings.

6 Applications to simple processes

In this section, the above results for Sudakov DL, collinear or soft SL, and PR corrections are applied to simple processes. We discuss relative corrections to polarized Born amplitudes,

$$\delta_{A\to B} = \frac{\delta\mathcal{M}^{A\to B}}{\mathcal{M}_0^{A\to B}}. \tag{6.1}$$

Note that the corrections to the cross sections are twice as large. The complete logarithmic corrections are presented in analytic form. The numerical results are given for the coefficients of the genuine electroweak (ew) logarithms. These are obtained by omitting the pure electromagnetic contributions that result from the gap between the electromagnetic and the weak scale. Accordingly they include the symmetric-electroweak contributions and the $l(s)$ terms originating from Z-boson loops in (3.7). In order to keep track of the origin of the various $l(s)$ terms, we introduce different subscripts: collinear, Yukawa, PR contributions, and the Z-boson contributions from (3.7) are denoted by l_{C}, l_{Yuk}, l_{PR}, and l_Z respectively. The numerical results have been obtained using the following values for the physical parameters:

$$M_W = 80.35\,\mathrm{GeV},\ M_Z = 91.1867\,\mathrm{GeV},\ m_t = 175\,\mathrm{GeV},$$
$$\alpha = \frac{1}{137.036}, \quad s_{\mathrm{w}}^2 = 1 - \frac{M_W^2}{M_Z^2} \approx 0.22356. \tag{6.2}$$

6.1 Four-fermion neutral-current processes

The Sudakov DL corrections (3.7) and the collinear or soft SL corrections (4.6) depend only on the quantum numbers of the external legs, and can be applied to 4-fermion processes in a universal way. However, we are interested also in the SSC and PR corrections, which depend on the specific properties of the process. A general description of these corrections requires a decomposition of the Born matrix element into neutral-current (NC) and charged-current (CC) contributions. In order to simplify the discussion we restrict ourselves to pure NC transitions. To simplify notation, we consider processes involving a lepton–antilepton and a quark–antiquark pair. However, our analysis applies to the more general case of two fermion–antifermion pairs of different isospin doublets. The four external states and their momenta are chosen to be incoming, so that the process reads

$$\bar{l}^\kappa_\sigma l^\kappa_\sigma q^\lambda_\rho \bar{q}^\lambda_\rho \to 0, \tag{6.3}$$

where $\kappa, \lambda = \mathrm{R}, \mathrm{L}$ are the chiralities and $\sigma, \rho = \pm$ the isospin indices. All formulas for the $4 \to 0$ process (6.3) are expressed in terms of the particle eigenvalues $I^N_{l^\kappa_\sigma}$, $I^N_{q^\lambda_\rho}$.

In the high-energy limit, the Born amplitude is given by

$$\mathcal{M}_0^{\bar{l}^\kappa_\sigma l^\kappa_\sigma q^\lambda_\rho \bar{q}^\lambda_\rho} = e^2 R_{l^\kappa_\sigma q^\lambda_\rho} \frac{A_{12}}{r_{12}}, \tag{6.4}$$

where

$$R_{\phi_i \phi_k} := \sum_{N=A,Z} I^N_{\phi_i} I^N_{\phi_k} = \frac{1}{4c_{\mathrm{w}}^2} Y_{\phi_i} Y_{\phi_k} + \frac{1}{s_{\mathrm{w}}^2} T^3_{\phi_i} T^3_{\phi_k}, \tag{6.5}$$

and terms of order M_Z^2/r_{12}, originating from the difference between the photon and the Z-boson mass, are neglected. Note that (6.5) and the following formulas have an important chirality dependence, owing to the different values of the group-theoretical operators in the representations for right-handed and left-handed fermions.

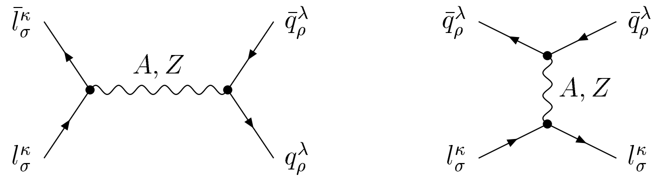


Fig. 2. Lowest-order diagrams for $\bar{l}^\kappa_\sigma l^\kappa_\sigma \to \bar{q}^\lambda_\rho q^\lambda_\rho$ and $\bar{q}^\lambda_\rho l^\kappa_\sigma \to \bar{q}^\lambda_\rho l^\kappa_\sigma$

The Sudakov soft–collinear corrections give according to (3.7) the leading contribution

$$\delta^{\mathrm{LSC}}_{\bar{l}^\kappa_\sigma l^\kappa_\sigma q^\lambda_\rho \bar{q}^\lambda_\rho} = - \sum_{f^\mu_\tau = l^\kappa_\sigma, q^\lambda_\rho} \left[C^{\mathrm{ew}}_{f^\mu} L(s) - 2(I^Z_{f^\mu_\tau})^2 \log \frac{M_Z^2}{M_W^2}\, l_Z \right.$$
$$\left. + Q^2_{f_\tau} L^{\mathrm{em}}(s, \lambda^2, m^2_{f_\tau}) \right]. \tag{6.6}$$

The angular-dependent SSC corrections are obtained from (3.12). The contribution of the neutral gauge bosons $N = A, Z$ is diagonal in the SU(2) indices, and factorizes into the Born matrix element (6.4) times the relative correction

$$\sum_{N=A,Z} \delta^{N,\mathrm{SSC}}_{\bar{l}^\kappa_\sigma l^\kappa_\sigma q^\lambda_\rho \bar{q}^\lambda_\rho} = -2l(s)$$
$$\times \left\{ (R_{l^\kappa_\sigma l^\kappa_\sigma} + R_{q^\lambda_\rho q^\lambda_\rho}) \log \frac{|r_{12}|}{s} + 2R_{l^\kappa_\sigma q^\lambda_\rho} \log \frac{|r_{13}|}{|r_{14}|} \right\}$$
$$-2l(M_W^2, \lambda^2)$$
$$\times \left[(Q^2_{l_\sigma} + Q^2_{q_\rho}) \log \frac{|r_{12}|}{s} + 2Q_{l_\sigma} Q_{q_\rho} \log \frac{|r_{13}|}{|r_{14}|} \right], \tag{6.7}$$

where $I^N_{\bar{f}\bar{f}} = -I^N_{ff}$ has been used and terms involving $l(M_W^2, M_Z^2)$ have been omitted. The contribution of the charged gauge bosons to (3.12) gives

$$\sum_{V_a = W^\pm} \delta^{V_a,\mathrm{SSC}} \mathcal{M}^{\bar{l}^\kappa_\sigma l^\kappa_\sigma q^\lambda_\rho \bar{q}^\lambda_\rho} =$$
$$-\frac{1}{s_{\mathrm{w}}^2} l(s) \left\{ \left(\delta_{\kappa\mathrm{L}} \mathcal{M}_0^{\bar{l}^\kappa_{-\sigma} l^\kappa_{-\sigma} q^\lambda_\rho \bar{q}^\lambda_\rho} + \delta_{\lambda\mathrm{L}} \mathcal{M}_0^{\bar{l}^\kappa_\sigma l^\kappa_\sigma q^\lambda_{-\rho} \bar{q}^\lambda_{-\rho}} \right) \log \frac{|r_{12}|}{s} \right.$$
$$+ \delta_{\kappa\mathrm{L}} \delta_{\lambda\mathrm{L}} \left[\delta_{\sigma\rho} \left(\mathcal{M}_0^{\bar{l}^\kappa_{-\sigma} l^\kappa_\sigma q^\lambda_{-\rho} \bar{q}^\lambda_\rho} + \mathcal{M}_0^{\bar{l}^\kappa_\sigma l^\kappa_{-\sigma} q^\lambda_\rho \bar{q}^\lambda_{-\rho}} \right) \log \frac{|r_{13}|}{s} \right.$$
$$\left.\left. - \delta_{-\sigma\rho} \left(\mathcal{M}_0^{\bar{l}^\kappa_{-\sigma} l^\kappa_\sigma q^\lambda_\rho \bar{q}^\lambda_{-\rho}} + \mathcal{M}_0^{\bar{l}^\kappa_\sigma l^\kappa_{-\sigma} q^\lambda_{-\rho} \bar{q}^\lambda_\rho} \right) \log \frac{|r_{14}|}{s} \right] \right\}, \tag{6.8}$$

where the non-diagonal couplings (B.17) have been used. On the left-hand side, the SU(2)-transformed Born matrix elements involving the isospin partners $l^\kappa_{-\sigma}$, $q^\lambda_{-\rho}$, have to be evaluated explicitly. The NC matrix elements (first line) are obtained from (6.4), and for the CC amplitudes we find up to mass-suppressed terms using the non-diagonal couplings (B.17),

$$\mathcal{M}_0^{\bar{l}^\kappa_{\sigma'} l^\kappa_{-\sigma'} q^\lambda_{\rho'} \bar{q}^\lambda_{-\rho'}} = \frac{e^2}{2s_{\mathrm{w}}^2} \frac{A_{12}}{r_{12}}. \tag{6.9}$$

Then, dividing (6.8) by the Born matrix element, we obtain the relative correction

$$\sum_{V_a=W^\pm} \delta^{V_a,\mathrm{SSC}}_{\bar{l}^\kappa_\sigma l^\kappa_\sigma q^\lambda_\rho \bar{q}^\lambda_\rho} = -\frac{1}{s_\mathrm{w}^2 R_{l^\kappa_\sigma q^\lambda_\rho}} l(s) \left\{ \left(\delta_{\kappa\mathrm{L}} R_{l^\kappa_{-\sigma} q^\lambda_\rho} + \delta_{\lambda\mathrm{L}} R_{l^\kappa_\sigma q^\lambda_{-\rho}}\right) \log\frac{|r_{12}|}{s} + \frac{\delta_{\kappa\mathrm{L}}\delta_{\lambda\mathrm{L}}}{s_\mathrm{w}^2}\left[\delta_{\sigma\rho}\log\frac{|r_{13}|}{s} - \delta_{-\sigma\rho}\log\frac{|r_{14}|}{s}\right]\right\}. \tag{6.10}$$

The angular-dependent corrections for $2 \to 2$ processes, like those depicted in Fig. 2, are directly obtained from (6.7) and (6.10) by substituting the invariants r_{kl} by the corresponding Mandelstam variables s, t, u. For the s-channel processes $\bar{l}^\kappa_\sigma l^\kappa_\sigma \to \bar{q}^\lambda_\rho q^\lambda_\rho$, we have to substitute $r_{12} = s, r_{13} = t, r_{14} = u$, and the SSC corrections simplify to

$$\delta^{\mathrm{SSC}}_{\bar{l}^\kappa_\sigma l^\kappa_\sigma \to \bar{q}^\lambda_\rho q^\lambda_\rho} = -l(s)\left[4R_{l^\kappa_\sigma q^\lambda_\rho}\log\frac{t}{u} + \frac{\delta_{\kappa\mathrm{L}}\delta_{\lambda\mathrm{L}}}{s_\mathrm{w}^4 R_{l^\kappa_\sigma q^\lambda_\rho}} \times \left(\delta_{\sigma\rho}\log\frac{|t|}{s} - \delta_{-\sigma\rho}\log\frac{|u|}{s}\right)\right] - 4Q_{l_\sigma}Q_{q_\rho} l(M_W^2,\lambda^2)\log\frac{t}{u}. \tag{6.11}$$

If one subtracts the photonic contributions from (6.11) one finds agreement with (50) of [7]. For the t-channel processes $\bar{q}^\lambda_\rho l^\kappa_\sigma \to \bar{q}^\lambda_\rho l^\kappa_\sigma$, the substitution reads $r_{12} = t, r_{13} = s, r_{14} = u$, whereas for $q^\lambda_\rho l^\kappa_\sigma \to q^\lambda_\rho l^\kappa_\sigma$ one has to choose $r_{12} = t, r_{13} = u, r_{14} = s$.

The collinear or soft SL contributions (4.6) give

$$\delta^{\mathrm{C}}_{\bar{l}^\kappa_\sigma l^\kappa_\sigma q^\lambda_\rho \bar{q}^\lambda_\rho} = \sum_{f^\mu_\tau = l^\kappa_\sigma, q^\lambda_\rho} \Big[3C^{\mathrm{ew}}_{f^\mu} l_\mathrm{C} - \frac{1}{4s_\mathrm{w}^2}\left((1+\delta_{\mu R})\frac{m^2_{f_\tau}}{M_W^2} + \delta_{\mu L}\frac{m^2_{f_{-\tau}}}{M_W^2}\right) l_\mathrm{Yuk} + 2Q^2_{f_\tau} l^\mathrm{em}(m^2_{f_\tau})\Big], \tag{6.12}$$

and the Yukawa contribution depends on the chiralities μ and on the masses of the fermions f^μ_τ and their isospin partners $f^\mu_{-\tau}$.

The PR logarithms for NC processes are obtained from the renormalization of the electric charge and the weak-mixing angle in the Born amplitude (6.4). Using (5.6) and (5.7) this gives the relative correction

$$\delta^{\mathrm{PR}}_{\bar{l}^\kappa_\sigma l^\kappa_\sigma q^\lambda_\rho \bar{q}^\lambda_\rho} = \left[\frac{s_\mathrm{w}}{c_\mathrm{w}} b^\mathrm{ew}_{AZ} \Delta_{l^\kappa_\sigma q^\lambda_\rho} - b^\mathrm{ew}_{AA}\right] l_\mathrm{PR} + 2\delta Z^\mathrm{em}_e, \tag{6.13}$$

where

$$\Delta_{\phi_i\phi_k} := \frac{-\frac{1}{4c_\mathrm{w}^2} Y_{\phi_i} Y_{\phi_k} + \frac{c_\mathrm{w}^2}{s_\mathrm{w}^4} T^3_{\phi_i} T^3_{\phi_k}}{R_{\phi_i\phi_k}} \tag{6.14}$$

gives a chirality-dependent contribution owing to mixing-angle renormalization of (6.5), and b^ew_{AA} represents the universal contribution of electric charge renormalization.

In order to give an impression of the size of the genuine electroweak part of the corrections, we consider the relative corrections $\delta^{\kappa_e\kappa_f,\mathrm{ew}}_{e^+e^-\to\bar{f}f}$ to NC processes $e^+e^- \to \bar{f}f$ with chiralities $\kappa_e, \kappa_f = $ R or L, and give the numerical coefficients of the electroweak logarithms for the cases $f = \mu, t, b$. For muon-pair production we have

$$\begin{aligned}
\delta^{\mathrm{RR,ew}}_{e^+e^-\to\mu^+\mu^-} &= -2.58L(s) - 5.15\left(\log\frac{t}{u}\right)l(s) + 0.29l_Z \\
&\quad + 7.73l_\mathrm{C} + 8.80l_\mathrm{PR}, \\
\delta^{\mathrm{RL,ew}}_{e^+e^-\to\mu^+\mu^-} &= -4.96L(s) - 2.58\left(\log\frac{t}{u}\right)l(s) + 0.37l_Z \\
&\quad + 14.9l_\mathrm{C} + 8.80l_\mathrm{PR}, \\
\delta^{\mathrm{LL,ew}}_{e^+e^-\to\mu^+\mu^-} &= -7.35L(s) \\
&\quad - \left(5.76\log\frac{t}{u} + 13.9\log\frac{|t|}{s}\right)l(s) + 0.45l_Z \\
&\quad + 22.1l_\mathrm{C} - 9.03l_\mathrm{PR},
\end{aligned} \tag{6.15}$$

and $\delta^{\mathrm{LR,ew}}_{e^+e^-\to\mu^+\mu^-} = \delta^{\mathrm{RL,ew}}_{e^+e^-\to\mu^+\mu^-}$. For top-quark pair production we find

$$\begin{aligned}
\delta^{\mathrm{RR,ew}}_{e^+e^-\to\bar{t}t} &= -1.86L(s) + 3.43\left(\log\frac{t}{u}\right)l(s) + 0.21l_Z \\
&\quad + 5.58l_\mathrm{C} - 10.6l_\mathrm{Yuk} + 8.80l_\mathrm{PR}, \\
\delta^{\mathrm{RL,ew}}_{e^+e^-\to\bar{t}t} &= -4.68L(s) + 0.86\left(\log\frac{t}{u}\right)l(s) + 0.50l_Z \\
&\quad + 14.0l_\mathrm{C} - 5.30l_\mathrm{Yuk} + 8.80l_\mathrm{PR}, \\
\delta^{\mathrm{LR,ew}}_{e^+e^-\to\bar{t}t} &= -4.25L(s) + 1.72\left(\log\frac{t}{u}\right)l(s) + 0.29l_Z \\
&\quad + 12.7l_\mathrm{C} - 10.6l_\mathrm{Yuk} + 8.80l_\mathrm{PR}, \\
\delta^{\mathrm{LL,ew}}_{e^+e^-\to\bar{t}t} &= -7.07L(s) \\
&\quad + \left(4.90\log\frac{t}{u} - 16.3\log\frac{|u|}{s}\right)l(s) + 0.58l_Z \\
&\quad + 21.2l_\mathrm{C} - 5.30l_\mathrm{Yuk} - 12.2l_\mathrm{PR},
\end{aligned} \tag{6.16}$$

and for bottom-quark pair production we obtain

$$\begin{aligned}
\delta^{\mathrm{RR,ew}}_{e^+e^-\to\bar{b}b} &= -1.43L(s) - 1.72\left(\log\frac{t}{u}\right)l(s) + 0.16l_Z \\
&\quad + 4.29l_\mathrm{C} + 8.80l_\mathrm{PR}, \\
\delta^{\mathrm{RL,ew}}_{e^+e^-\to\bar{b}b} &= -4.68L(s) + 0.86\left(\log\frac{t}{u}\right)l(s) + 0.67l_Z \\
&\quad + 14.0l_\mathrm{C} - 5.30l_\mathrm{Yuk} + 8.80l_\mathrm{PR}, \\
\delta^{\mathrm{LR,ew}}_{e^+e^-\to\bar{b}b} &= -3.82L(s) - 0.86\left(\log\frac{t}{u}\right)l(s) + 0.24l_Z \\
&\quad + 11.5l_\mathrm{C} + 8.80l_\mathrm{PR}, \\
\delta^{\mathrm{LL,ew}}_{e^+e^-\to\bar{b}b} &= -7.07L(s) \\
&\quad - \left(4.04\log\frac{t}{u} + 19.8\log\frac{|t|}{s}\right)l(s) + 0.75l_Z \\
&\quad + 21.2l_\mathrm{C} - 5.30l_\mathrm{Yuk} - 16.6l_\mathrm{PR}.
\end{aligned} \tag{6.17}$$

The Mandelstam variables are defined as usual, i.e. $s = (p_{e^+} + p_{e^-})^2$, $t = (p_{e^+} - p_{\bar{f}})^2$ and $u = (p_{e^+} - p_f)^2$.

Note that the corrections to light quark-pair production $f = u, c(d, s)$ are obtained from the results for heavy quarks $f = t(b)$ by omitting the Yukawa contributions. Independently of the process and of the chirality, the DL and SL terms appear in the combination $(-L(s)+3l_{\mathrm{C}})$, so that the negative DL contribution becomes dominating only above 400 GeV, and at $s^{1/2} = 1\,\mathrm{TeV}$ the cancellation between SL and DL corrections is still important. The SU(2) interaction, which is stronger than the U(1) interaction, generates large corrections for left-handed fermions. Also the PR logarithms show a strong chirality dependence: the RR and RL transitions receive positive corrections from the running of the abelian U(1) coupling, whereas the LL transition is dominated by the non-abelian SU(2) interaction and receives negative PR corrections.

6.2 Production of W-boson pairs in e^+e^- annihilation

We consider the polarized scattering process[4] $e^+_\kappa e^-_\kappa \to W^+_{\lambda_+} W^-_{\lambda_-}$, where $\kappa = \mathrm{R}, \mathrm{L}$ is the electron chirality, and $\lambda_\pm = 0, \pm$ represent the gauge-boson helicities. In the high-energy limit only the following helicity combinations are non-suppressed [4,23]: the purely longitudinal final state $(\lambda_+, \lambda_-) = (0, 0)$, which we denote by $(\lambda_+, \lambda_-) = (\mathrm{L}, \mathrm{L})$, and the purely transverse and opposite final state $(\lambda_+, \lambda_-) = (\pm, \mp)$, which we denote by $(\lambda_+, \lambda_-) = (\mathrm{T}, \mathrm{T})$. All these final states can be written as $(\lambda_+, \lambda_-) = (\lambda, -\lambda)$. The Mandelstam variables are $s = (p_{e^+} + p_{e^-})^2$, $t = (p_{e^+} - p_{W^+})^2 \sim -s(1 - \cos\theta)/2$, and $u = (p_{e^+} - p_{W^-})^2 \sim -s(1 + \cos\theta)/2$, where θ is the angle between e^+ and W^+. The Born amplitude gets contributions of the s- and t-channel diagrams in Fig. 3 and reads

$$\mathcal{M}_0^{e^+_\kappa e^-_\kappa \to W^+_{\mathrm{L}} W^-_{\mathrm{L}}} = e^2 R_{e^-_\kappa \phi^-} \frac{A_s}{s},$$
$$\mathcal{M}_0^{e^+_{\mathrm{L}} e^-_{\mathrm{L}} \to W^+_{\mathrm{T}} W^-_{\mathrm{T}}} = \frac{e^2}{2 s_{\mathrm{w}}^2} \frac{A_t}{t}, \qquad (6.18)$$

up to terms of order M_W^2/s, where R is defined in (6.5). The amplitude involving longitudinal gauge bosons W_{L} is expressed by the amplitude involving Goldstone bosons $\phi^\pm$ and is dominated by the s-channel exchange of neutral gauge bosons. The amplitude for transverse gauge-boson production is dominated by the t-channel contribution, which involves only the SU(2) interaction. Therefore, it is non-vanishing only for left-handed electrons in the initial state.

The DL corrections read

$$\delta^{\mathrm{LSC}}_{e^+_\kappa e^-_\kappa \to W^+_\lambda W^-_{-\lambda}} = - \sum_{\varphi = e^-_\kappa, W^-_\lambda} \left[C^{\mathrm{ew}}_\varphi L(s) - 2 (I^Z_\varphi)^2 \log \frac{M_Z^2}{M_W^2} l_Z + Q_\varphi^2 L^{\mathrm{em}}(s, \lambda^2, m_\varphi^2) \right]. \qquad (6.19)$$

[4] The momenta and fields of the initial states are incoming, and those of the final states are outgoing

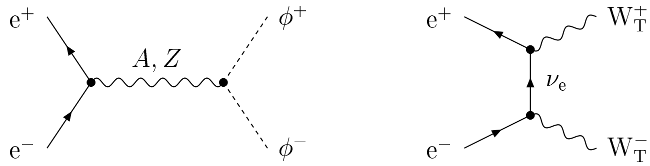


Fig. 3. Dominant lowest-order diagrams for $e^+e^- \to \phi^+\phi^-$ and $e^+e^- \to W^+_{\mathrm{T}} W^-_{\mathrm{T}}$

Here and in the following formulas, for longitudinally polarized gauge bosons $W^\pm_{\mathrm{L}}$ the quantum numbers of the Goldstone bosons $\phi^\pm$ have to be used.

The SSC corrections are obtained by applying (3.12) to the crossing symmetric process $e^+_\kappa e^-_\kappa W^-_\lambda W^+_{-\lambda} \to 0$. The contribution of the neutral gauge bosons $N = A, Z$ gives

$$\sum_{N=A,Z} \delta^{N,\mathrm{SSC}}_{e^+_\kappa e^-_\kappa \to W^+_\lambda W^-_{-\lambda}} = -4 R_{e^-_\kappa W^-_\lambda} l(s) \log \frac{t}{u} - 4 Q_{e^-} Q_{W^-} l(M_W^2, \lambda^2) \log \frac{t}{u}, \qquad (6.20)$$

and corresponds to the result (6.7) for 4-fermion s-channel NC processes. The contribution of soft $W^\pm$-bosons to (3.12) yields

$$\sum_{V_a = W^\pm} \delta^{V_a,\mathrm{SSC}} \mathcal{M}^{e^+_\kappa e^-_\kappa \phi^- \phi^+} = - \frac{2 l(s) \delta_{\kappa \mathrm{L}}}{\sqrt{2} s_{\mathrm{w}}} \times \sum_{S=H,\chi} \left[I^+_S \mathcal{M}_0^{\bar{\nu}_\kappa e^-_\kappa S \phi^+} - I^-_S \mathcal{M}_0^{e^+_\kappa \nu_\kappa \phi^- S} \right] \log \frac{|t|}{s},$$
$$\sum_{V_a = W^\pm} \delta^{V_a,\mathrm{SSC}} \mathcal{M}^{e^+_{\mathrm{L}} e^-_{\mathrm{L}} W^-_{\mathrm{T}} W^+_{\mathrm{T}}} = - \frac{2 l(s)}{\sqrt{2} s_{\mathrm{w}}} \times \sum_{N=A,Z} \left[I^+_N \mathcal{M}_0^{\bar{\nu}_{\mathrm{L}} e^-_{\mathrm{L}} N_{\mathrm{T}} W^+_{\mathrm{T}}} - I^-_N \mathcal{M}_0^{e^+_{\mathrm{L}} \nu_{\mathrm{L}} W^-_{\mathrm{T}} N_{\mathrm{T}}} \right] \log \frac{|t|}{s}, \qquad (6.21)$$

where, depending on the polarization of the final states, one has to use the non-diagonal $W^\pm$ couplings to Goldstone bosons ($I^\pm_S$) defined in (B.22) or the $W^\pm$ couplings to gauge bosons ($I^\pm_N$) defined in (B.26). The SU(2)-transformed Born matrix elements on the left-hand side of (6.21) have to be evaluated explicitly. For Goldstone bosons, we have s-channel CC amplitudes

$$\mathcal{M}_0^{\bar{\nu}_\kappa e^-_\kappa S \phi^+} = -e^2 I^-_S \frac{\delta_{\kappa \mathrm{L}}}{\sqrt{2} s_{\mathrm{w}}} \frac{A_s}{s},$$
$$\mathcal{M}_0^{e^+_\kappa \nu_\kappa \phi^- S} = e^2 I^+_S \frac{\delta_{\kappa \mathrm{L}}}{\sqrt{2} s_{\mathrm{w}}} \frac{A_s}{s}, \qquad (6.22)$$

similar to the NC Born amplitude in (6.18), whereas for transverse gauge bosons we have

$$\mathcal{M}_0^{\bar{\nu}_\kappa e^-_\kappa N_{\mathrm{T}} W^+_{\mathrm{T}}} = \mathcal{M}_0^{e^+_\kappa \nu_\kappa W^-_{\mathrm{T}} N_{\mathrm{T}}} = e^2 \frac{\delta_{\kappa \mathrm{L}}}{\sqrt{2} s_{\mathrm{w}}} \left(I^N_{\nu_\kappa} \frac{A_t}{t} + I^N_{e^-_\kappa} \frac{A_u}{u} \right), \qquad (6.23)$$

where $A_t = A_u$ up to mass-suppressed contributions. In contrast to (6.18), the transformed amplitude (6.23) receives contributions from both t- and u-channels. Expressing (6.21) as relative corrections to the Born matrix elements we obtain

$$\sum_{V_a=W^\pm} \delta^{V_a,\mathrm{SSC}}_{e^+_\kappa e^-_\kappa \to W^+_\mathrm{L} W^-_\mathrm{L}} = -l(s)\frac{\delta_{\kappa \mathrm{L}}}{s_\mathrm{w}^4 R_{e^-_\mathrm{L}\phi^-}} \log\frac{|t|}{s},$$
$$\sum_{V_a=W^\pm} \delta^{V_a,\mathrm{SSC}}_{e^+_\mathrm{L} e^-_\mathrm{L} \to W^+_\mathrm{T} W^-_\mathrm{T}} = -\frac{2}{s_\mathrm{w}^2}\left(1-\frac{t}{u}\right) l(s)\log\frac{|t|}{s}. \qquad (6.24)$$

The SL corrections can be read off from (4.6), (4.10), and (4.33),

$$\delta^{\mathrm{C}}_{e^+_\kappa e^-_\kappa \to W^+_\mathrm{L} W^-_\mathrm{L}} = \left[3C^{\mathrm{ew}}_{e_\kappa} + 4C^{\mathrm{ew}}_{\Phi}\right] l_\mathrm{C} - \frac{3}{2s_\mathrm{w}^2}\frac{m_t^2}{M_W^2} l_\mathrm{Yuk} + \sum_{\varphi=e,W} 2l^{\mathrm{em}}(m_\varphi^2),$$
$$\delta^{\mathrm{C}}_{e^+_\mathrm{L} e^-_\mathrm{L} \to W^+_\mathrm{T} W^-_\mathrm{T}} = \left[3C^{\mathrm{ew}}_{e_\mathrm{L}} + b^{\mathrm{ew}}_{W}\right] l_\mathrm{C} + \sum_{\varphi=e,W} 2l^{\mathrm{em}}(m_\varphi^2). \qquad (6.25)$$

Despite of their different origin, the l_C contributions for longitudinal and transverse gauge bosons have similar numerical values $4C^{\mathrm{ew}}_\Phi = 14.707$ and $b^{\mathrm{ew}}_W = 14.165$. The strong W-polarization dependence of δ^C is due to the large Yukawa contributions occurring only for longitudinal gauge bosons.

The PR logarithms are obtained from the renormalization of (6.18) and read according to (5.6)–(5.11)

$$\delta^{\mathrm{PR}}_{e^+_\kappa e^-_\kappa \to W^+_\mathrm{L} W^-_\mathrm{L}} = \left[\frac{s_\mathrm{w}}{c_\mathrm{w}} b^{\mathrm{ew}}_{AZ}\Delta_{e^-_\kappa\phi^-} - b^{\mathrm{ew}}_{AA}\right] l_\mathrm{PR} + 2\delta Z^{\mathrm{em}}_e,$$
$$\delta^{\mathrm{PR}}_{e^+_\mathrm{L} e^-_\mathrm{L} \to W^+_\mathrm{T} W^-_\mathrm{T}} = -b^{\mathrm{ew}}_W l_\mathrm{PR} + 2\delta Z^{\mathrm{em}}_e, \qquad (6.26)$$

where Δ is defined in (6.14). Note that for transverse polarizations, the symmetric-electroweak parts of the PR corrections ($-b^{\mathrm{ew}}_W l_\mathrm{PR}$) and the collinear SL corrections originating from external gauge bosons ($b^{\mathrm{ew}}_W l_\mathrm{C}$) cancel. As illustrated in Appendix A, this kind of cancellation takes place for all processes with production of arbitrary many charged or neutral transverse gauge bosons in fermion–antifermion annihilation.

The results (6.19)–(6.26) can be compared with those of [4]. After subtracting the real soft-photonic corrections from the results of [4] we find complete agreement for the logarithmic corrections. The coefficients for the various electroweak logarithmic contributions to the relative corrections $\delta^{\kappa\lambda}_{e^+e^-\to W^+W^-}$ read

$$\delta^{\mathrm{LL,ew}}_{e^+e^-\to W^+W^-} = -7.35L(s) - \left(5.76\log\frac{t}{u} + 13.9\log\frac{|t|}{s}\right) l(s) + 0.45 l_Z + 25.7 l_\mathrm{C} - 31.8 l_\mathrm{Yuk} - 9.03 l_\mathrm{PR},$$

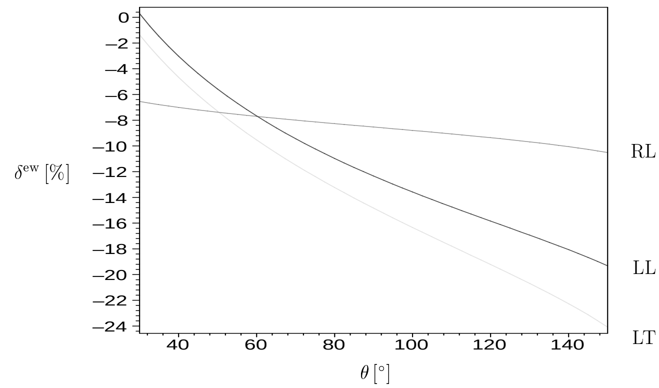


Fig. 4. Dependence of the electroweak correction factor $\delta^{\mathrm{ew}}_{e^+_\kappa e^-_\kappa \to W^+_\lambda W^-_{-\lambda}}$ on the scattering angle θ at $s^{1/2} = 1\,\mathrm{TeV}$ for polarizations RL, LL, and LT

$$\delta^{\mathrm{RL,ew}}_{e^+e^-\to W^+W^-} = -4.96L(s) - 2.58\left(\log\frac{t}{u}\right) l(s) + 0.37 l_Z + 18.6 l_\mathrm{C} - 31.8 l_\mathrm{Yuk} + 8.80 l_\mathrm{PR},$$
$$\delta^{\mathrm{LT,ew}}_{e^+e^-\to W^+W^-} = -12.6L(s) - 8.95\left[\log\frac{t}{u} + \left(1-\frac{t}{u}\right)\log\frac{|t|}{s}\right] l(s) + 1.98 l_Z + 25.2 l_\mathrm{C} - 14.2 l_\mathrm{PR}. \qquad (6.27)$$

Recall that the pure electromagnetic contributions have been omitted. These correction factors are shown in Figs. 4 and 5 as a function of the scattering angle and the energy, respectively. If the electrons are left-handed, large negative DL and PR corrections originate from the SU(2) interaction. Instead, for right-handed electrons the DL corrections are smaller, and the PR contribution is positive. For transverse W-bosons, there are no Yukawa contributions and the other contributions are in general larger than for longitudinal W-bosons. Nevertheless, for energies around 1 TeV, the corrections are similar. Finally, note that the angular-dependent contributions are very important for the LL and LT corrections: at $s^{1/2} \approx 1\,\mathrm{TeV}$ they vary from +15% to −5% for scattering angles $30^\circ < \theta < 150^\circ$, whereas the angular-dependent part of the RL corrections remains between ±2%.

6.3 Production of neutral gauge-boson pairs in e^+e^- annihilation

We consider the polarized scattering process $e^+_\kappa e^-_\kappa \to N^1_\mathrm{T} N^2_\mathrm{T}$ with incoming electrons of chirality $\kappa = \mathrm{R,L}$ and outgoing gauge bosons $N^k = A, Z$. The amplitude is non-suppressed only for transverse and opposite gauge-boson polarizations $(\lambda_1, \lambda_2) = (\pm, \mp)$ [24]. In lowest order the t- and u-channel diagrams (Fig. 6) yield

$$\mathcal{M}_0^{e^+_\kappa e^-_\kappa \to N^1_\mathrm{T} N^2_\mathrm{T}} = e^2 I^{N^1}_{e^-_\kappa} I^{N^2}_{e^-_\kappa}\left[\frac{A_t}{t} + \frac{A_u}{u}\right] \qquad (6.28)$$

up to terms of order M_W^2/s, where the Mandelstam variables are defined as in Sect. 6.2. In the ultra-relativistic

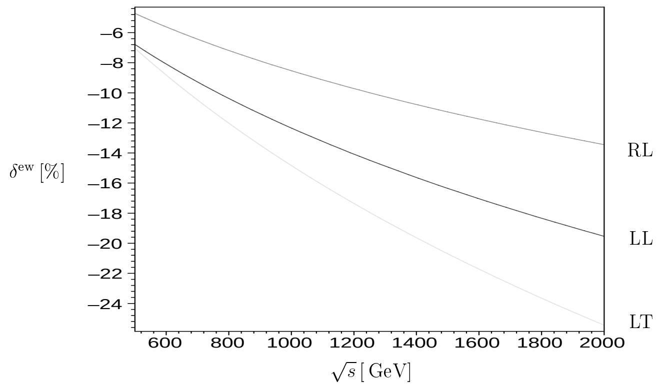


Fig. 5. Dependence of the electroweak correction factor $\delta^{\mathrm{ew}}_{e^+_\kappa e^-_\kappa \to W^+_\lambda W^-_{-\lambda}}$ on the energy $s^{1/2}$ at $\theta = 90°$ for polarizations RL, LL, and LT

Fig. 6. Lowest-order diagrams for $e^+e^- \to N^1N^2$

limit the amplitude is symmetric with respect to exchange of the gauge bosons, and up to mass-suppressed contributions we have

$$A_t = A_u. \tag{6.29}$$

The DL corrections read [cf. (3.7)]

$$\begin{aligned}
&\delta^{\mathrm{LSC}}\mathcal{M}^{e^+_\kappa e^-_\kappa \to N^1_{\mathrm{T}} N^2_{\mathrm{T}}} \\
&= -\left[C^{\mathrm{ew}}_{e_\kappa} L(s) - 2(I^Z_{e_\kappa})^2 \log\frac{M_Z^2}{M_W^2} l_Z + L^{\mathrm{em}}(s,\lambda^2,m_e^2)\right] \\
&\quad \times \mathcal{M}_0^{e^+_\kappa e^-_\kappa \to N^1_{\mathrm{T}} N^2_{\mathrm{T}}} - \frac{1}{2}\left[C^{\mathrm{ew}}_{N' N^1}\mathcal{M}_0^{e^+_\kappa e^-_\kappa \to N'_{\mathrm{T}} N^2_{\mathrm{T}}}\right. \\
&\quad \left. + C^{\mathrm{ew}}_{N' N^2}\mathcal{M}_0^{e^+_\kappa e^-_\kappa \to N^1_{\mathrm{T}} N'_{\mathrm{T}}}\right] L(s),
\end{aligned} \tag{6.30}$$

with a non-diagonal contribution associated with the external neutral gauge bosons. Using

$$C^{\mathrm{ew}}_{N'N} I^{N'} = \frac{2}{s^2_{\mathrm{w}}} U_{NW^3}(\theta_{\mathrm{w}}) \tilde{I}^{W^3} = \frac{2}{s^2_{\mathrm{w}}} U_{NW^3}(\theta_{\mathrm{w}}) \frac{T^3}{s_{\mathrm{w}}}, \tag{6.31}$$

where $U_{N\tilde{N}}(\theta_{\mathrm{w}})$ is the Weinberg rotation defined in (B.4), we can derive a correction relative to the Born matrix element,

$$\begin{aligned}
&\delta^{\mathrm{LSC}}_{e^+_\kappa e^-_\kappa \to N^1_{\mathrm{T}} N^2_{\mathrm{T}}} = \\
&-\left[C^{\mathrm{ew}}_{e_\kappa} L(s) - 2(I^Z_{e_\kappa})^2 \log\frac{M_Z^2}{M_W^2} l_Z + L^{\mathrm{em}}(s,\lambda^2,m_e^2)\right] \\
&-\frac{T^3_{e^-_\kappa}}{s^3_{\mathrm{w}}} \sum_{k=1,2} \frac{U_{N^k W^3}(\theta_{\mathrm{w}})}{I^{N^k}_{e^-_\kappa}} L(s).
\end{aligned} \tag{6.32}$$

Note that only the SU(2) component of the neutral gauge bosons is self-interacting and can exchange soft gauge bosons. For this reason, only left-handed electrons ($T^3 \neq 0$) yield a contribution to (6.31) and to the corresponding term in (6.32).

Angular-dependent logarithmic corrections (3.12) arise only from the exchange of soft $W^\pm$-bosons between initial and final states, and with the non-diagonal couplings (B.26)

$$\begin{aligned}
\delta^{\mathrm{SSC}}\mathcal{M}^{e^+_\kappa e^-_\kappa N^1_{\mathrm{T}} N^2_{\mathrm{T}}} &= \frac{2l(s)\delta_{\kappa\mathrm{L}}}{\sqrt{2}s_{\mathrm{w}}} \\
&\times \left\{\left[I^+_{N^1}\mathcal{M}_0^{\bar{\nu}_\kappa e^-_\kappa W^+_{\mathrm{T}} N^2_{\mathrm{T}}} - I^-_{N^2}\mathcal{M}_0^{e^+_\kappa \nu_\kappa N^1_{\mathrm{T}} W^-_{\mathrm{T}}}\right] \log\frac{|t|}{s}\right. \\
&\left. + \left[I^+_{N^2}\mathcal{M}_0^{\bar{\nu}_\kappa e^-_\kappa N^1_{\mathrm{T}} W^+_{\mathrm{T}}} - I^-_{N^1}\mathcal{M}_0^{e^+_\kappa \nu_\kappa W^-_{\mathrm{T}} N^2_{\mathrm{T}}}\right] \log\frac{|u|}{s}\right\}.
\end{aligned} \tag{6.33}$$

The SU(2)-transformed Born matrix elements on the left-hand side are given by

$$\begin{aligned}
\mathcal{M}_0^{\bar{\nu}_\kappa e^-_\kappa W^+_{\mathrm{T}} N_{\mathrm{T}}} &= \mathcal{M}_0^{e^+_\kappa \nu_\kappa N_{\mathrm{T}} W^-_{\mathrm{T}}} \\
&= e^2 \frac{\delta_{\kappa\mathrm{L}}}{\sqrt{2}s_{\mathrm{w}}} \left(I^N_{e^-_\kappa}\frac{A_t}{t} + I^N_{\nu_\kappa}\frac{A_u}{u}\right),
\end{aligned} \tag{6.34}$$

and by (6.23) with $A_t = A_u$. Expressing the correction (6.33) relative to the Born matrix element (6.28), we obtain

$$\begin{aligned}
\delta^{\mathrm{SSC}}_{e^+_\kappa e^-_\kappa \to N^1_{\mathrm{T}} N^2_{\mathrm{T}}} &= \frac{\delta_{\kappa\mathrm{L}}}{s^2_{\mathrm{w}}} l(s) \sum_{k=1}^{2} \sum_{r=t,u} \frac{I^{N^k}_{W^-}}{I^{N^k}_{e^-_{\mathrm{L}}}} \\
&\quad \times \left(\frac{r'}{s} + \frac{r}{s}\frac{I^{N^{k'}}_{\nu_{\mathrm{L}}}}{I^{N^{k'}}_{e^-_{\mathrm{L}}}}\right) \log\frac{|r|}{s},
\end{aligned} \tag{6.35}$$

where $r' = (t,u)$ for $r = (u,t)$, and $k' = (1,2)$ for $k = (2,1)$.

Using (4.6) and (4.22) we obtain for the SL corrections relative to the Born matrix element

$$\delta^{\mathrm{C}}_{e^+_\kappa e^-_\kappa \to N^1_{\mathrm{T}} N^2_{\mathrm{T}}} = 3C^{\mathrm{ew}}_{e_\kappa} l_{\mathrm{C}} + 2l^{\mathrm{em}}(m_e^2) + \delta^{\mathrm{C}}_{N^1_{\mathrm{T}}} + \delta^{\mathrm{C}}_{N^2_{\mathrm{T}}}, \tag{6.36}$$

with

$$\begin{aligned}
\delta^{\mathrm{C}}_A &:= \delta^{\mathrm{C}}_{AA}(V_{\mathrm{T}}) = \frac{1}{2} b^{\mathrm{ew}}_{AA} l_{\mathrm{C}} - \delta Z^{\mathrm{em}}_e, \\
\delta^{\mathrm{C}}_Z &:= \delta^{\mathrm{C}}_{ZZ}(V_{\mathrm{T}}) + \delta^{\mathrm{C}}_{AZ}(V_{\mathrm{T}}) \frac{\mathcal{M}_0^{e^+_\kappa e^-_\kappa \to A_{\mathrm{T}} N_{\mathrm{T}}}}{\mathcal{M}_0^{e^+_\kappa e^-_\kappa \to Z_{\mathrm{T}} N_{\mathrm{T}}}} \\
&= \left[\frac{1}{2} b^{\mathrm{ew}}_{ZZ} + b^{\mathrm{ew}}_{AZ} \frac{I^A_{e^-_\kappa}}{I^Z_{e^-_\kappa}}\right] l_{\mathrm{C}}.
\end{aligned} \tag{6.37}$$

The PR logarithms result from the renormalization of (6.28). As shown in Appendix A, they are opposite to the collinear SL corrections up to pure electromagnetic logarithms. Relative to the Born matrix element they read

$$\delta^{\mathrm{PR}}_{e^+_\kappa e^-_\kappa \to N^1_{\mathrm{T}} N^2_{\mathrm{T}}} = \delta^{\mathrm{PR}}_{N^1_{\mathrm{T}}} + \delta^{\mathrm{PR}}_{N^2_{\mathrm{T}}}, \tag{6.38}$$

with

$$\delta_A^{\mathrm{PR}} := -\delta_A^{\mathrm{C}}, \quad \delta_Z^{\mathrm{PR}} := -\delta_Z^{\mathrm{C}} + \delta Z_e^{\mathrm{em}}. \tag{6.39}$$

For right-handed electrons, $\kappa = \mathrm{R}$, the various electroweak logarithmic contributions to the relative corrections $\delta^{\kappa\mathrm{T}}_{e^+e^-\to N^1N^2}$ give

$$\begin{aligned}
\delta^{\mathrm{RT,ew}}_{e^+e^-\to AA} &= -1.29L(s) + 0.15l_Z + 0.20l_{\mathrm{C}} + 3.67l_{\mathrm{PR}},\\
\delta^{\mathrm{RT,ew}}_{e^+e^-\to AZ} &= -1.29L(s) + 0.15l_Z - 11.3l_{\mathrm{C}} + 15.1l_{\mathrm{PR}},\\
\delta^{\mathrm{RT,ew}}_{e^+e^-\to ZZ} &= -1.29L(s) + 0.15l_Z - 22.8l_{\mathrm{C}} + 26.6l_{\mathrm{PR}}.
\end{aligned} \tag{6.40}$$

Note that there is no angular dependence. The PR contributions are numerically compensated by the SL and DL Sudakov contributions, and at $s^{1/2} = 1\,\mathrm{TeV}$ the electroweak logarithmic corrections are less than 1%. For left-handed electrons, we find

$$\begin{aligned}
\delta^{\mathrm{LT,ew}}_{e^+e^-\to AA} &= -8.15L(s) + 8.95F_1(t)l(s) + 0.22l_Z\\
&\quad + 7.36l_{\mathrm{C}} + 3.67l_{\mathrm{PR}}, \qquad (6.41)\\
\delta^{\mathrm{LT,ew}}_{e^+e^-\to AZ} &= -12.2L(s) + (17.0F_1(t) - 8.09F_2(t))l(s)\\
&\quad + 0.22l_Z + 28.1l_{\mathrm{C}} - 17.1l_{\mathrm{PR}},\\
\delta^{\mathrm{LT,ew}}_{e^+e^-\to ZZ} &= -16.2L(s) + (25.1F_1(t) - 45.4F_2(t))l(s)\\
&\quad + 0.22l_Z + 48.9l_{\mathrm{C}} - 37.9l_{\mathrm{PR}},
\end{aligned}$$

with the (t,u)-symmetric angular-dependent functions

$$\begin{aligned}
F_1(t) &:= \frac{u}{s}\log\frac{|t|}{s} + \frac{t}{s}\log\frac{|u|}{s},\\
F_2(t) &:= \frac{t}{s}\log\frac{|t|}{s} + \frac{u}{s}\log\frac{|u|}{s}.
\end{aligned} \tag{6.42}$$

For left-handed electrons all contributions are larger than for right-handed electrons owing to the SU(2) interaction. The non-abelian effects are particularly strong for Z-boson pair production (see Figs. 7, 8), where the total corrections are almost -25% for $s^{1/2} = 1\,\mathrm{TeV}$ and $\theta = 90°$. The angular-dependent contribution is forward–backward symmetric, and for ZZ production it varies from $+15\%$ to -5% for scattering angles $30° < \theta < 90°$.

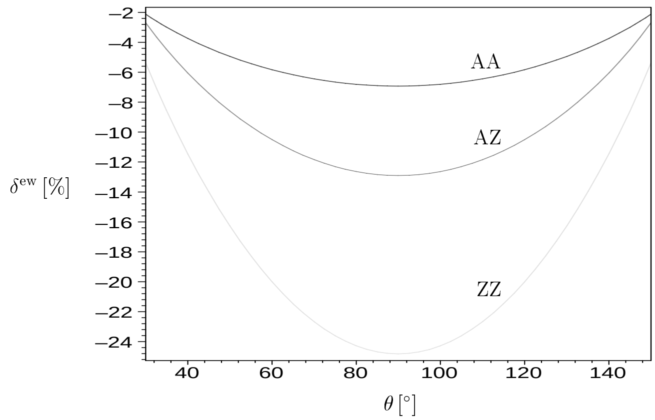


Fig. 7. Angular dependence of the electroweak corrections to $e^+_{\mathrm{L}}e^-_{\mathrm{L}} \to AA, AZ, ZZ$ at $s^{1/2} = 1\,\mathrm{TeV}$

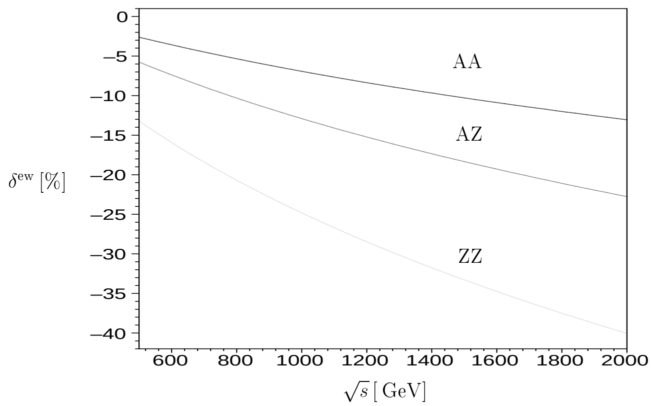


Fig. 8. Energy dependence of the electroweak corrections to $e^+_{\mathrm{L}}e^-_{\mathrm{L}} \to AA, AZ, ZZ$ at $\theta = 90°$

7 Conclusion

We have considered general electroweak processes at high energies. We have given recipes and explicit formulas for the extraction of the one-loop leading electroweak logarithms. Like the well-known soft–collinear double logarithms, also the collinear single logarithms can be expressed as simple correction factors that are associated with the external particles of the considered process. Up to electromagnetic terms, the collinear SL corrections for external longitudinal gauge bosons and for Higgs bosons are equal. The subleading single logarithms arising from the soft–collinear limit are angular-dependent and can be associated to pairs of external particles. Their evaluation requires in general all matrix elements that are linked to the lowest-order matrix element via global SU(2) rotations. Finally, the logarithms originating from coupling-constant renormalization are associated with the explicit dependence of the lowest-order matrix element on the coupling parameters.

Our results are applicable to general amplitudes that are not mass suppressed, as long as all invariants are large compared to the masses. As illustration, we have applied our general results to fermion–antifermion production and the pair production of charged and neutral gauge bosons. For processes involving resonances, like in $e^+e^- \to W^+W^- \to 4f$, the corresponding invariants are evidently not large and our results must be applied to the subprocesses $e^+e^- \to W^+W^-$ and $W^\pm \to 2f$.

Acknowledgements. We thank W. Beenakker, M. Melles, and A. Werthenbach for discussions.

Appendix A
Production of transverse gauge bosons in fermion–antifermion annihilation

For transverse W-pair production, we have observed that the symmetric-electroweak parts of the PR contributions and of the collinear SL corrections originating from the external gauge bosons cancel exactly. Here we illustrate how this cancellation takes place when arbitrarily many charged or neutral gauge bosons are produced in fermion–antifermion annihilation. To be specific, we consider electron–positron annihilation into n transverse charged gauge-boson pairs and m transverse neutral gauge bosons $N = A, Z$,

$$e^+_\kappa e^-_\kappa \to W^+_{1,\mathrm{T}} \ldots W^+_{n,\mathrm{T}} W^-_{1,\mathrm{T}} \ldots W^-_{n,\mathrm{T}} N_{1,\mathrm{T}} \ldots N_{m,\mathrm{T}}. \quad \text{(A.1)}$$

Collinear SL contributions give the correction factor

$$\delta^{\mathrm{C}} \mathcal{M}^{e^+e^- W_1^+ \ldots W_n^- N_1 \ldots N_m} = \left[2\delta^{\mathrm{C}}_{ee}(L_\kappa) + 2n\delta^{\mathrm{C}}_{WW}(V_{\mathrm{T}})\right] \mathcal{M}^{e^+e^- W_1^+ \ldots W_n^- N_1 \ldots N_m} + \sum_{k=1}^{m} \delta^{\mathrm{C}}_{N'_k N_k}(V_{\mathrm{T}}) \mathcal{M}_0^{e^+e^- W_1^+ \ldots W_n^- N_1 \ldots N'_k \ldots N_m}, \quad \text{(A.2)}$$

and owing to mixing we have a non-diagonal factor (4.22) in the neutral sector.

For the processes considered, it turns out that in the high-energy limit the contribution of coupling-constant renormalization can be written as a sum over the external gauge bosons. This can be easily shown, starting from the unbroken phase. If one considers the production of n W-boson pairs and m neutral gauge bosons $\tilde{N} = B, W^3$,

$$e^+_\kappa e^-_\kappa \to W^+_{1,\mathrm{T}} \ldots W^+_{n,\mathrm{T}} W^-_{1,\mathrm{T}} \ldots W^-_{n,\mathrm{T}} \tilde{N}_{1,\mathrm{T}} \ldots \tilde{N}_{m,\mathrm{T}}, \quad \text{(A.3)}$$

the Born matrix element receives a factor $g_W = g_2$ for each SU(2) gauge boson and a factor $g_B = g_1$ for each U(1) gauge boson, and neglecting masses in the propagators we arrive at

$$\tilde{\mathcal{M}}_0^{e^+e^- W_1^+ \ldots W_n^- \tilde{N}_1 \ldots \tilde{N}_m} = \left(g_2^{2n} \prod_{k=1}^{m} g_{\tilde{N}_k}\right) \tilde{A}_0^{e^+e^- W_1^+ \ldots W_n^- \tilde{N}_1 \ldots \tilde{N}_m}. \quad \text{(A.4)}$$

The coupling-constant renormalization gives

$$\delta^{\mathrm{PR}} = \left(2n\frac{\delta g_2}{g_2} + \sum_{k=1}^{m} \frac{\delta g_{\tilde{N}_k}}{g_{\tilde{N}_k}}\right)\Bigg|_{\mu^2=s}. \quad \text{(A.5)}$$

In the broken phase, the charged gauge bosons remain pure SU(2) eigenstates, and only the neutral gauge bosons mix. In the high-energy limit, if we neglect the gauge-boson masses in propagators, we can decompose the Born matrix elements into the symmetric amplitudes (A.4) using the Weinberg rotation. For the production of m neutral gauge bosons ($n = 0$), we find

$$\mathcal{M}_0^{e^+e^- N_1 \ldots N_m} = \tilde{\mathcal{M}}_0^{e^+e^- \tilde{N}_1 \ldots \tilde{N}_m} \left[\prod_{k=1}^{m} U^{-1}_{\tilde{N}_k N_k}(\theta_{\mathrm{w}})\right] = \tilde{A}_0^{e^+e^- \tilde{N}_1 \ldots \tilde{N}_m} \left[\prod_{k=1}^{m} U^{-1}_{\tilde{N}_k N_k}(\theta_{\mathrm{w}}) g_{\tilde{N}_k}\right], \quad \text{(A.6)}$$

where sums over $\tilde{N}_1 \ldots \tilde{N}_m$ are implicitly understood, and the renormalization of coupling constants and mixing angle gives

$$\delta^{\mathrm{PR}} \mathcal{M}^{e^+e^- N_1 \ldots N_m} = \tilde{A}_0^{e^+e^- \tilde{N}_1 \ldots \tilde{N}_m} \sum_{l=1}^{m} \left\{ \left(\delta U^{-1}_{\tilde{N}_l N_l}(\theta_{\mathrm{w}}) g_{\tilde{N}_l} + U^{-1}_{\tilde{N}_l N_l}(\theta_{\mathrm{w}}) \delta g_{\tilde{N}_l}\right) \times \left[\prod_{k=1, k\neq l}^{m} U^{-1}_{\tilde{N}_k N_k}(\theta_{\mathrm{w}}) g_{\tilde{N}_k}\right] \right\} = \sum_{l=1}^{m} \mathcal{M}_0^{e^+e^- N_1 \ldots N'_l \ldots N_m} \left(U_{N'_l \tilde{N}_l}(\theta_{\mathrm{w}}) \delta U^{-1}_{\tilde{N}_l N_l}(\theta_{\mathrm{w}}) + U_{N'_l \tilde{N}_l}(\theta_{\mathrm{w}}) \frac{\delta g_{\tilde{N}_l}}{g_{\tilde{N}_l}} U^{-1}_{\tilde{N}_l N_l}(\theta_{\mathrm{w}}) \right). \quad \text{(A.7)}$$

Therefore, the complete PR correction can be written as

$$\delta^{\mathrm{PR}} \mathcal{M}^{e^+e^- W_1^+ \ldots W_n^- N_1 \ldots N_m} = 2n\delta^{\mathrm{PR}}_{WW} \mathcal{M}^{e^+e^- W_1^+ \ldots W_n^- N_1 \ldots N_m} + \sum_{k=1}^{m} \delta^{\mathrm{PR}}_{N'_k N_k} \mathcal{M}_0^{e^+e^- W_1^+ \ldots W_n^- N_1 \ldots N'_k \ldots N_m}, \quad \text{(A.8)}$$

with the correction factors

$$\delta^{\mathrm{PR}}_{WW} = \frac{\delta g_2}{g_2}\bigg|_{\mu^2=s} = -\frac{1}{2} b^{\mathrm{ew}}_{W} l(s) + \delta Z^{\mathrm{em}}_e,$$
$$\delta^{\mathrm{PR}}_{N'N} = \left\{ \left[U(\theta_{\mathrm{w}}) \delta U^{-1}(\theta_{\mathrm{w}})\right]_{N'N} + \left(\frac{\delta g}{g}\right)_{N'N} \right\}\bigg|_{\mu^2=s} = -\frac{1}{2}\left[E_{N'N} b^{\mathrm{ew}}_{AZ} + b^{\mathrm{ew}}_{N'N}\right] l(s) + \delta Z^{\mathrm{em}}_e \delta_{N'N}, \quad \text{(A.9)}$$

where we have used (4.16), $U\delta U^{-1} = -\delta U U^{-1}$, and the coupling-renormalization matrix

$$\left(\frac{\delta g}{g}\right)_{N'N} := U_{N'\tilde{N}}(\theta_{\mathrm{w}}) \frac{\delta g_{\tilde{N}}}{g_{\tilde{N}}} U^{-1}_{\tilde{N}N}(\theta_{\mathrm{w}}) = -\frac{1}{2} b^{\mathrm{ew}}_{N'N} l(\mu^2) + \delta Z^{\mathrm{em}}_e \delta_{N'N}, \quad \text{(A.10)}$$

generated by Weinberg rotation of the two gauge-coupling counterterms (5.11).

Comparing the PR and collinear SL contributions for transverse gauge bosons, (A.9), (4.10) and (4.22), we find that all symmetric-electroweak logarithms are related to the β-function and cancel in the sum so that only large logarithms of pure electromagnetic origin contribute to the complete SL corrections,

$$\begin{aligned}\delta^{\mathrm{C}}_{WW}(V_{\mathrm{T}})+\delta^{\mathrm{PR}}_{WW}&=\delta Z_e^{\mathrm{em}}+Q_W^2 l^{\mathrm{em}}(M_W^2),\\ \delta^{\mathrm{C}}_{N'N}(V_{\mathrm{T}})+\delta^{\mathrm{PR}}_{N'N}&=\delta Z_e^{\mathrm{em}}\delta_{N'Z}\delta_{NZ}.\end{aligned}\tag{A.11}$$

Note that this cancellation between PR and collinear logarithms occurs already in the symmetric basis and is a consequence of Ward identities, like the identity between the electric charge and the photonic FRC in QED. In the physical basis, both coupling and field-renormalization constants receive additional terms owing to mixing of the neutral gauge bosons, but also these terms cancel. The same relation holds for an arbitrary fermion–antifermion pair in the initial state.

Appendix B
Representation of $\mathrm{SU}(2)\times\mathrm{U}(1)$ operators

Generators of the gauge group and various group-theoretical matrices used in the article are presented in detail. Our notation for the components of such matrices is

$$M_{\varphi_i\varphi_{i'}}(\varphi),\tag{B.1}$$

where the argument φ represents a multiplet and fixes the representation for the matrix M, whereas φ_i are the components of the multiplet. In this appendix we give explicit representations for left- and right-handed fermions ($\varphi=f^{\mathrm{L}},f^{\mathrm{R}},\bar{f}^{\mathrm{L}},\bar{f}^{\mathrm{R}}$), for gauge bosons ($\varphi=V$) and for the scalar doublet ($\varphi=\Phi$). Where the representation is implicit, the argument φ is omitted. For the eigenvalues of diagonal matrices we write

$$M_{\varphi_i\varphi_{i'}}=\delta_{\varphi_i\varphi_{i'}}M_{\varphi_i}.\tag{B.2}$$

Symmetric and physical gauge fields and gauge couplings

For gauge bosons we take special care of the effect of Weinberg rotation (mixing). The symmetric basis $\tilde{V}_a=B,W^1,W^2,W^3$, is formed by the U(1) and SU(2) gauge bosons, which transform as a singlet and a triplet, respectively, and quantities in this basis are denoted by a tilde. The physical basis is given by the charge and mass eigenstates $V_a=A,Z,W^+,W^-$. The physical charged gauge bosons,

$$W^{\pm}=\frac{W^1\mp\mathrm{i}W^2}{\sqrt{2}},\tag{B.3}$$

are pure SU(2) states, whereas in the neutral sector the SU(2) and U(1) components mix, and the physical fields $N=A,Z$ are related to the symmetric fields $\tilde{N}=B,W^3$ by the Weinberg rotation,

$$N=U_{N\tilde{N}}(\theta_{\mathrm{w}})\tilde{N},\quad U(\theta_{\mathrm{w}})=\begin{pmatrix}c_{\mathrm{w}} & -s_{\mathrm{w}}\\ s_{\mathrm{w}} & c_{\mathrm{w}}\end{pmatrix},\tag{B.4}$$

with $c_{\mathrm{w}}=\cos\theta_{\mathrm{w}}$ and $s_{\mathrm{w}}=\sin\theta_{\mathrm{w}}$. In the on-shell renormalization scheme the Weinberg angle is fixed by

$$c_{\mathrm{w}}=\frac{M_W}{M_Z}.\tag{B.5}$$

The gauge couplings are given by the generators of global gauge transformations (2.4). In the symmetric basis, they read

$$\tilde{I}^B=-\frac{1}{c_{\mathrm{w}}}\frac{Y}{2},\quad \tilde{I}^{W^a}=\frac{1}{s_{\mathrm{w}}}T^a,\quad a=1,2,3,\tag{B.6}$$

where Y is the weak hypercharge and T^a are the components of the weak isospin. In the physical basis we have

$$\begin{aligned}I^A&=-Q,\quad I^Z=\frac{T^3-s_{\mathrm{w}}^2Q}{s_{\mathrm{w}}c_{\mathrm{w}}},\\ I^{\pm}&=\frac{1}{s_{\mathrm{w}}}T^{\pm}=\frac{1}{s_{\mathrm{w}}}\frac{T^1\pm\mathrm{i}T^2}{\sqrt{2}},\end{aligned}\tag{B.7}$$

with $Q=T^3+Y/2$.

Casimir operators

The SU(2) Casimir operator is defined by

$$C=\sum_{a=1}^{3}(T^a)^2.\tag{B.8}$$

Loops involving charged gauge bosons are often associated with the product of the non-abelian charges

$$(I^W)^2:=\sum_{\sigma=\pm}\left[I^\sigma I^{-\sigma}\right]=\left[\frac{C-(T^3)^2}{s_{\mathrm{w}}^2}\right],\tag{B.9}$$

and if one includes the contributions of neutral gauge bosons, one obtains the effective electroweak Casimir operator

$$C^{\mathrm{ew}}:=\sum_{V_a=A,Z,W^{\pm}}I^{V_a}I^{\bar{V}_a}=\frac{1}{c_{\mathrm{w}}^2}\left(\frac{Y}{2}\right)^2+\frac{1}{s_{\mathrm{w}}^2}C.\tag{B.10}$$

For irreducible representations (fermions and scalars) with isospin T_φ, the SU(2) Casimir operator is proportional to the identity and reads

$$C_{\varphi_i\varphi_{i'}}(\varphi)=\delta_{\varphi_i\varphi_{i'}}C_\varphi,\quad C_\varphi=T_\varphi[T_\varphi+1].\tag{B.11}$$

For gauge bosons we have a reducible representation. In the symmetric basis $\tilde{C}(V)$ is a diagonal 4×4 matrix

$$\tilde{C}_{\tilde{V}_a\tilde{V}_b}=\delta_{ab}\tilde{C}_{\tilde{V}_a},\tag{B.12}$$

with U(1) and SU(2) eigenvalues

$$\tilde{C}_B=0,\quad \tilde{C}_{W^a}=2.\tag{B.13}$$

The transformation of a matrix like (B.12) to the physical basis, yields a 4×4 matrix with diagonal 2×2 block structure, i.e. without mixing between the charged sector ($W^\pm$) and the neutral sector ($N = A, Z$). In the neutral sector $C(V)$ becomes non-diagonal owing to mixing of the U(1) and SU(2) eigenvalues,

$$C_{NN'} = \left[U(\theta_\mathrm{w})\tilde{C}U^{-1}(\theta_\mathrm{w})\right]_{NN'} = 2\begin{pmatrix} s_\mathrm{w}^2 & -s_\mathrm{w}c_\mathrm{w} \\ -s_\mathrm{w}c_\mathrm{w} & c_\mathrm{w}^2 \end{pmatrix}, \tag{B.14}$$

whereas in the charged sector it remains diagonal,

$$C_{W^\sigma W^{\sigma'}} = 2\delta_{\sigma\sigma'}. \tag{B.15}$$

Explicit values for Y, Q, T^3, C, $(I^A)^2$, $(I^Z)^2$, $(I^W)^2$, C^ew, and $I^\pm$

Here we list the eigenvalues (or components) of the operators Y, Q, T^3, C, $(I^A)^2$, $(I^Z)^2$, $(I^W)^2$, C^ew, and $I^\pm$, that have to be inserted in our general results. For incoming particles or outgoing antiparticles the values for the particles have to be used, for incoming antiparticles or outgoing particles the values of the antiparticles.

Fermions

The fermionic doublets $f^\kappa = (f^\kappa_+, f^\kappa_-)^\mathrm{T}$ transform according to the fundamental or trivial representations, depending on the chirality $\kappa = \mathrm{L}, \mathrm{R}$. Except for $I^\pm$, the above operators are diagonal. For lepton and quark doublets, $L^\kappa = (\nu^\kappa, l^\kappa)^\mathrm{T}$ and $Q^\kappa = (u^\kappa, d^\kappa)^\mathrm{T}$, their eigenvalues are (see (B.16) on top of the next page). For left-handed fermions, $I^\pm(f^\mathrm{L})$ have the non-vanishing components

$$I^\sigma_{f_{\sigma'}f_{-\sigma'}}(f^\mathrm{L}) = -I^\sigma_{\bar{f}_{-\sigma'}\bar{f}_{\sigma'}}(\bar{f}^\mathrm{L}) = \frac{\delta_{\sigma\sigma'}}{\sqrt{2}s_\mathrm{w}}, \tag{B.17}$$

whereas for right-handed fermions $I^\pm(f^\mathrm{R}) = 0$.

Scalar fields

The symmetric scalar doublet, $\Phi = (\phi^+, \phi_0)^\mathrm{T}$, $\Phi^* = (\phi^-, \phi_0^*)^\mathrm{T}$, transforms according to the fundamental representation, and its quantum numbers correspond to those of left-handed leptons (B.16) with

$$\begin{aligned} &\phi^+ \leftrightarrow \bar{l}^\mathrm{L}, \\ &\phi_0 \leftrightarrow \bar{\nu}^\mathrm{L}, \quad \phi^- \leftrightarrow l^\mathrm{L}, \quad \phi_0^* \leftrightarrow \nu^\mathrm{L}. \end{aligned} \tag{B.18}$$

After symmetry breaking the neutral scalar fields are parameterized by the mass eigenstates

$$\phi_0 = \frac{1}{\sqrt{2}}(v + H + \mathrm{i}\chi). \tag{B.19}$$

With respect to this basis $S = (H, \chi)$ the operators Q, C, $(I^N)^2$, and C^ew remain unchanged, while T^3 and Y become non-diagonal in the neutral components

$$T^3_{SS'} = -\left(\frac{Y}{2}\right)_{SS'} = -\frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{B.20}$$

and

$$I^Z_{H\chi} = -I^Z_{\chi H} = \frac{-\mathrm{i}}{2s_\mathrm{w}c_\mathrm{w}}. \tag{B.21}$$

The $W^\pm$ couplings read

$$I^\sigma_{S\phi^{-\sigma'}} = -I^\sigma_{\phi^{\sigma'}S} = \delta_{\sigma\sigma'}I^\sigma_S, \tag{B.22}$$

with

$$I^\sigma_H := -\frac{\sigma}{2s_\mathrm{w}}, \quad I^\sigma_\chi := -\frac{\mathrm{i}}{2s_\mathrm{w}}. \tag{B.23}$$

Gauge fields

For transversely polarized external gauge bosons we have to use the adjoint representation. In the symmetric basis the diagonal operators have eigenvalues

	$Y/2$	Q	T^3	C	$(I^A)^2$	$(I^Z)^2$	$(I^W)^2$	C^ew
$W^\pm$	0	±1	±1	2	1	$\frac{c_\mathrm{w}^2}{s_\mathrm{w}^2}$	$\frac{1}{s_\mathrm{w}^2}$	$\frac{2}{s_\mathrm{w}^2}$
W^3	0	0	0	2	0	0	$\frac{2}{s_\mathrm{w}^2}$	$\frac{2}{s_\mathrm{w}^2}$
B	0	0	0	0	0	0	0	0

(B.24)

In the neutral sector, owing to the Weinberg rotation, the non-trivial operators C^ew, C and $(I^W)^2$ become non-diagonal in the physical basis $N = A, Z$, with components

$$C^\mathrm{ew}_{NN'} = \frac{1}{s_\mathrm{w}^2}C_{NN'} = (I^W)^2_{NN'} = \frac{2}{s_\mathrm{w}^2}\begin{pmatrix} s_\mathrm{w}^2 & -s_\mathrm{w}c_\mathrm{w} \\ -s_\mathrm{w}c_\mathrm{w} & c_\mathrm{w}^2 \end{pmatrix}, \tag{B.25}$$

whereas the trivial operators $Y/2 = Q = T^3 = (I^A)^2 = (I^Z)^2 = 0$ remain unchanged. In the physical basis the non-vanishing components of the $I^\pm$ couplings are

$$I^\sigma_{NW^{-\sigma'}} = -I^\sigma_{W^{\sigma'}N} = \delta_{\sigma\sigma'}I^\sigma_N, \tag{B.26}$$

with

$$I^\sigma_A = -\sigma, \quad I^\sigma_Z = \sigma\frac{c_\mathrm{w}}{s_\mathrm{w}}. \tag{B.27}$$

Dynkin operator

The group-theoretical object appearing in gauge-boson self-energies is the Dynkin operator

$$D^\mathrm{ew}_{ab}(\varphi) := \mathrm{Tr}\left\{I^a(\varphi)I^b(\varphi)\right\}. \tag{B.28}$$

The indices a, b are those of the gauge group, and the trace is over the isospin doublet for $\varphi = \Phi, f^\mathrm{L}, f^\mathrm{R}$ and over the gauge group for $\varphi = V$. In the latter case the Dynkin operator corresponds to the electroweak Casimir operator,

$$D^\mathrm{ew}_{ab}(V) = C^\mathrm{ew}_{ab}(V). \tag{B.29}$$

In the symmetric basis $\tilde{D}^\mathrm{ew}$ is diagonal,

$$\tilde{D}^\mathrm{ew}_{ab}(\varphi) = \delta_{ab}\tilde{D}^\mathrm{ew}_a(\varphi). \tag{B.30}$$

	$Y/2$	Q	T^3	C	$(I^A)^2$	$(I^Z)^2$	$(I^W)^2$	C^{ew}
$\nu^{\mathrm{L}}, \bar{\nu}^{\mathrm{L}}$	$\mp\frac{1}{2}$	0	$\pm\frac{1}{2}$	$\frac{3}{4}$	0	$\frac{1}{4s_{\mathrm{w}}^2 c_{\mathrm{w}}^2}$	$\frac{1}{2s_{\mathrm{w}}^2}$	$\frac{1+2c_{\mathrm{w}}^2}{4s_{\mathrm{w}}^2 c_{\mathrm{w}}^2}$
$l^{\mathrm{L}}, \bar{l}^{\mathrm{L}}$	$\mp\frac{1}{2}$	∓ 1	$\mp\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{(c_{\mathrm{w}}^2-s_{\mathrm{w}}^2)^2}{4s_{\mathrm{w}}^2 c_{\mathrm{w}}^2}$	$\frac{1}{2s_{\mathrm{w}}^2}$	$\frac{1+2c_{\mathrm{w}}^2}{4s_{\mathrm{w}}^2 c_{\mathrm{w}}^2}$
$l^{\mathrm{R}}, \bar{l}^{\mathrm{R}}$	∓ 1	∓ 1	0	0	1	$\frac{s_{\mathrm{w}}^2}{c_{\mathrm{w}}^2}$	0	$\frac{1}{c_{\mathrm{w}}^2}$
$u^{\mathrm{L}}, \bar{u}^{\mathrm{L}}$	$\pm\frac{1}{6}$	$\pm\frac{2}{3}$	$\pm\frac{1}{2}$	$\frac{3}{4}$	$\frac{4}{9}$	$\frac{(3c_{\mathrm{w}}^2-s_{\mathrm{w}}^2)^2}{36s_{\mathrm{w}}^2 c_{\mathrm{w}}^2}$	$\frac{1}{2s_{\mathrm{w}}^2}$	$\frac{s_{\mathrm{w}}^2+27c_{\mathrm{w}}^2}{36c_{\mathrm{w}}^2 s_{\mathrm{w}}^2}$
$d^{\mathrm{L}}, \bar{d}^{\mathrm{L}}$	$\pm\frac{1}{6}$	$\mp\frac{1}{3}$	$\mp\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{9}$	$\frac{(3c_{\mathrm{w}}^2+s_{\mathrm{w}}^2)^2}{36s_{\mathrm{w}}^2 c_{\mathrm{w}}^2}$	$\frac{1}{2s_{\mathrm{w}}^2}$	$\frac{s_{\mathrm{w}}^2+27c_{\mathrm{w}}^2}{36c_{\mathrm{w}}^2 s_{\mathrm{w}}^2}$
$u^{\mathrm{R}}, \bar{u}^{\mathrm{R}}$	$\pm\frac{2}{3}$	$\pm\frac{2}{3}$	0	0	$\frac{4}{9}$	$\frac{4}{9}\frac{s_{\mathrm{w}}^2}{c_{\mathrm{w}}^2}$	0	$\frac{4}{9c_{\mathrm{w}}^2}$
$d^{\mathrm{R}}, \bar{d}^{\mathrm{R}}$	$\mp\frac{1}{3}$	$\mp\frac{1}{3}$	0	0	$\frac{1}{9}$	$\frac{1}{9}\frac{s_{\mathrm{w}}^2}{c_{\mathrm{w}}^2}$	0	$\frac{1}{9c_{\mathrm{w}}^2}$

(B.16)

The SU(2) and U(1) eigenvalues of the fundamental representation, $\varphi = \Phi, f^{\mathrm{L}}$, read

$$\tilde{D}^{\mathrm{ew}}_B(\varphi) = \frac{Y_\varphi^2}{2c_{\mathrm{w}}^2}, \quad \tilde{D}^{\mathrm{ew}}_W(\varphi) = \frac{1}{2s_{\mathrm{w}}^2}, \tag{B.31}$$

while for right-handed fermions

$$\tilde{D}^{\mathrm{ew}}_B(f^{\mathrm{R}}) = \frac{Y_{f_+^{\mathrm{R}}}^2 + Y_{f_-^{\mathrm{R}}}^2}{4c_{\mathrm{w}}^2}, \quad \tilde{D}^{\mathrm{ew}}_W(f^{\mathrm{R}}) = 0. \tag{B.32}$$

In the physical basis we have

$$D^{\mathrm{ew}}_{W^\sigma W^{\sigma'}}(\varphi) = \delta_{\sigma\sigma'} \tilde{D}^{\mathrm{ew}}_W(\varphi) \tag{B.33}$$

for the charged components, whereas in the neutral sector the U(1) and SU(2) eigenvalues mix, resulting in

$$D^{\mathrm{ew}}_{NN'}(\varphi) = \left[U(\theta_{\mathrm{w}})\tilde{D}^{\mathrm{ew}}(\varphi)U^{-1}(\theta_{\mathrm{w}})\right]_{NN'} = \frac{1}{2}\begin{pmatrix} 1+Y_\varphi^2 & \frac{Y_\varphi^2 s_{\mathrm{w}}^2 - c_{\mathrm{w}}^2}{s_{\mathrm{w}}c_{\mathrm{w}}} \\ \frac{Y_\varphi^2 s_{\mathrm{w}}^2 - c_{\mathrm{w}}^2}{s_{\mathrm{w}}c_{\mathrm{w}}} & \frac{Y_\varphi^2 s_{\mathrm{w}}^4 + c_{\mathrm{w}}^4}{s_{\mathrm{w}}^2 c_{\mathrm{w}}^2} \end{pmatrix} \tag{B.34}$$

for $\varphi = \Phi, f^{\mathrm{L}}$, and

$$D^{\mathrm{ew}}_{NN'}(f^{\mathrm{R}}) = \frac{Y_{f_+^{\mathrm{R}}}^2 + Y_{f_-^{\mathrm{R}}}^2}{4c_{\mathrm{w}}^2}\begin{pmatrix} c_{\mathrm{w}}^2 & c_{\mathrm{w}}s_{\mathrm{w}} \\ c_{\mathrm{w}}s_{\mathrm{w}} & s_{\mathrm{w}}^2 \end{pmatrix}. \tag{B.35}$$

The explicit values of the components of the Dynkin operator for the leptonic doublets (and for the scalar doublet) are

	D^{ew}_{AA}	D^{ew}_{AZ}	D^{ew}_{ZZ}	D^{ew}_{W}
L^{L}, Φ	1	$\frac{s_{\mathrm{w}}^2-c_{\mathrm{w}}^2}{2s_{\mathrm{w}}c_{\mathrm{w}}}$	$\frac{s_{\mathrm{w}}^4+c_{\mathrm{w}}^4}{2s_{\mathrm{w}}^2c_{\mathrm{w}}^2}$	$\frac{1}{2s_{\mathrm{w}}^2}$
L^{R}	1	$\frac{s_{\mathrm{w}}}{c_{\mathrm{w}}}$	$\frac{s_{\mathrm{w}}^2}{c_{\mathrm{w}}^2}$	0
$L^{\mathrm{L}}+L^{\mathrm{R}}$	2	$\frac{3s_{\mathrm{w}}^2-c_{\mathrm{w}}^2}{2s_{\mathrm{w}}c_{\mathrm{w}}}$	$\frac{3s_{\mathrm{w}}^4+c_{\mathrm{w}}^4}{2s_{\mathrm{w}}^2c_{\mathrm{w}}^2}$	$\frac{1}{2s_{\mathrm{w}}^2}$

(B.36)

and for the quark doublets

	D^{ew}_{AA}	D^{ew}_{AZ}	D^{ew}_{ZZ}	D^{ew}_{W}
Q^{L}	$\frac{5}{9}$	$\frac{s_{\mathrm{w}}^2-9c_{\mathrm{w}}^2}{18s_{\mathrm{w}}c_{\mathrm{w}}}$	$\frac{s_{\mathrm{w}}^4+9c_{\mathrm{w}}^4}{18s_{\mathrm{w}}^2c_{\mathrm{w}}^2}$	$\frac{1}{2s_{\mathrm{w}}^2}$
Q^{R}	$\frac{5}{9}$	$\frac{5}{9}\frac{s_{\mathrm{w}}}{c_{\mathrm{w}}}$	$\frac{5}{9}\frac{s_{\mathrm{w}}^2}{c_{\mathrm{w}}^2}$	0
$Q^{\mathrm{L}}+Q^{\mathrm{R}}$	$\frac{10}{9}$	$\frac{11s_{\mathrm{w}}^2-9c_{\mathrm{w}}^2}{18s_{\mathrm{w}}c_{\mathrm{w}}}$	$\frac{11s_{\mathrm{w}}^4+9c_{\mathrm{w}}^4}{18s_{\mathrm{w}}^2c_{\mathrm{w}}^2}$	$\frac{1}{2s_{\mathrm{w}}^2}$

(B.37)

β-function coefficients

In gauge-boson self-energies and mixing energies, the sums of gauge-boson, scalar, and fermionic loops give the following combination of Dynkin operators

$$b^{\mathrm{ew}}_{ab} := \frac{11}{3}C^{\mathrm{ew}}_{ab}(V) - \frac{1}{3}D^{\mathrm{ew}}_{ab}(\phi) - \frac{2}{3}\sum_{f,i} N_{\mathrm{C}}^f \sum_\lambda D^{\mathrm{ew}}_{ab}(f^\lambda), \tag{B.38}$$

which is proportional to the one-loop coefficients of the β-function. The fermionic sum runs over the generations $i = 1,2,3$ for leptons and quarks $f = l, q$. In the symmetric basis $\tilde{b}^{\mathrm{ew}}_{ab}$ is diagonal, and its eigenvalues

$$\tilde{b}^{\mathrm{ew}}_B = -\frac{41}{6c_{\mathrm{w}}^2}, \quad \tilde{b}^{\mathrm{ew}}_W = \frac{19}{6s_{\mathrm{w}}^2}, \tag{B.39}$$

describe the running of the U(1) and SU(2) coupling constants. In the physical basis b^{ew}_{ab} remains diagonal in the charged sector

$$b^{\mathrm{ew}}_{W^\sigma W^{\sigma'}} = \delta_{\sigma\sigma'} b^{\mathrm{ew}}_W, \tag{B.40}$$

with $b^{\mathrm{ew}}_W = \tilde{b}^{\mathrm{ew}}_W$, whereas the neutral components

$$b^{\mathrm{ew}}_{NN'} = \left[U(\theta_{\mathrm{w}})\tilde{b}^{\mathrm{ew}}U^{-1}(\theta_{\mathrm{w}})\right]_{NN'} \tag{B.41}$$

are

$$\begin{aligned} b^{\mathrm{ew}}_{AA} &= c_{\mathrm{w}}^2\tilde{b}^{\mathrm{ew}}_B + s_{\mathrm{w}}^2\tilde{b}^{\mathrm{ew}}_W = -\frac{11}{3}, \\ b^{\mathrm{ew}}_{AZ} &= c_{\mathrm{w}}s_{\mathrm{w}}(\tilde{b}^{\mathrm{ew}}_B - \tilde{b}^{\mathrm{ew}}_W) = -\frac{19+22s_{\mathrm{w}}^2}{6s_{\mathrm{w}}c_{\mathrm{w}}}, \\ b^{\mathrm{ew}}_{ZZ} &= s_{\mathrm{w}}^2\tilde{b}^{\mathrm{ew}}_B + c_{\mathrm{w}}^2\tilde{b}^{\mathrm{ew}}_W = \frac{19-38s_{\mathrm{w}}^2-22s_{\mathrm{w}}^4}{6s_{\mathrm{w}}^2c_{\mathrm{w}}^2}. \end{aligned} \tag{B.42}$$

The AA component determines the running of the electric charge, and the AZ component is associated with the running of the weak-mixing angle [cf. (5.7) and (5.6)].

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