Eur. Phys. J. C 17, 323–336 (2000)
Digital Object Identifier (DOI) 10.1007/s100520000459

THE EUROPEAN PHYSICAL JOURNAL C

The role of the input in Roy's equations for π–π scattering

G. Wanders

Institut de Physique Théorique, Université de Lausanne, 1015 Lausanne, Switzerland (e-mail: gerard.wanders@ipt.unil.ch)

Received: 9 May 2000 / Published online: 31 August 2000 – © Springer-Verlag 2000

Abstract. The Roy equations determine the S- and P-wave π–π phase shifts on a low energy interval. They allow the derivation of threshold parameters from experimental data. We examine the solutions of these equations that are in the neighborhood of a given solution by means of a linearization procedure. An updated survey of known results on the dimension of the manifold of solutions is presented. The solution is unique for a low energy interval with upper end at 800 MeV. We determine its response to small variations of the input: S-wave scattering lengths and absorptive parts above 800 MeV. We confirm the existence of a universal curve of solutions in the plane of the S-wave scattering lengths and provide a control of the decrease of the influence of the input absorptive parts with increasing energy. A general result on the suppression of unphysical singularities at the upper end of the low energy interval is established and its practical implications are discussed.

1 Introduction

Low-energy π–π scattering is a major testing ground of chiral perturbation theory [1]. Some of its coupling constants are directly related to the π–π threshold parameters. At present this relation is established at the two-loop level [1,2]. As it is impossible to measure π–π scattering at threshold, this relation cannot be exploited directly. A reliable extrapolation of the available experimental data down to threshold is required. Such an extrapolation is performed presently [3] with the aid of the Roy equations [4,5]. These equations are based on the analyticity, the crossing symmetry and the unitarity of the π–π partial wave amplitudes. The S- and P-wave Roy equations are solved in [3] by means of elaborate numerical methods. In this work we discuss aspects of the problem which allow an analytical approach and our effort is complementary to the work in [3].

The Roy equations contain as input the S-wave scattering lengths, the S- and P-wave absorptive parts above an energy E_0 (which will be called the "matching point") and driving terms coming from the higher partial waves. The Roy equations determine, at fixed elasticities, the S- and P-wave phase shifts below the matching point. This is a difficult, non-linear, problem that cannot be solved analytically. Here we restrict ourselves to questions that can be answered by linearization and which allow a partly analytic treatment. These concern the multiplicity of the solution and its sensitivity to small variations of the input.

Such questions have already been treated in [6,7] in conjunction with the early phenomenological applications of the Roy equations [8] and we can reduce the discussion of the multiplicity of the solution to the statement of our old results. The matching point used in [7] is at 1.13 GeV whereas the one used nowadays in [3] is at $E_0 = 800$ MeV. The answers to our questions depend on the choice of E_0. The solution is non-unique if $E_0 = 1.13$ GeV and becomes unique when $E_0 = 800$ MeV. The response to variations of the input also depends strongly on the position of the matching point and our analysis in [7] has to be updated.

The Roy equations with arbitrarily chosen input make up a well defined mathematical problem. A peculiar feature of this problem is that its solutions exhibit unphysical singularities at the matching point [we exclude throughout matching points coinciding with an inelastic threshold]. The physical input[1] is therefore a special one admitting at least one solution, the physical phase shifts, that is regular at E_0. Inputs with solutions regular at E_0 have been called "analytic inputs" in [9] in the context of simplified elastic one-channel Roy equations. The discussion of that class of inputs is extended here to the case of the complete coupled inelastic Roy equations. The main conclusion is that an analytic input admits a unique solution that is regular at the matching point. The non-uniqueness problem is thus circumvented.

Although non-uniqueness and singularities at E_0 are physically excluded, they show up in practical calculations because one is working with an approximate input which is not exactly an analytic one. An arbitrary variation of an analytic input produces a non-analytic one and induces singularities at E_0 even if the choice of E_0 guarantees uniqueness.

We find that one may stay close to an analytic input by correlating suitably the variations of two distinct

[1] By physical input we denote the input corresponding to scattering amplitudes which would be measured in the absence of isospin violation. Ideally, our physical amplitudes are those provided by QCD; in practice they are given by the available analyses of experimental data assuming isospin symmetry

pieces of the input. This comes mainly from the matching point at $E_0 = 800\,\mathrm{MeV}$ that is near the ρ-meson mass. For instance, singularities at E_0 are largely suppressed by correlating variations of the isospin 0 and 2 S-wave scattering lengths a_0^0 and a_0^2. This confirms the existence of a physically acceptable family of solutions along a "universal curve" in the (a_0^0, a_0^2)-plane [11]. Similar suppressions of singularities take place if a localized variation of an input absorptive part is combined with a variation of one of the scattering lengths, a_0^2 for instance. The response to such variations provides information on the sensitivity of the phase shifts to the uncertainties on the input absorptive parts. We find a very weak sensitivity to the uncertainties above 1 GeV. All our results are in qualitative and quantitative agreement with those obtained numerically in [3].

The coupling between the S- and P-wave channels built into the Roy equations is a manifestation of crossing symmetry. The practical implications of this symmetry are not well understood and the effects of variations of the input might be expected to provide some insight. We find that this is not really the case. In our framework the response to a change of the input absorptive part in one channel is largest in the same channel but the responses in the other channels are not much smaller. All we may say is that crossing symmetry produces a substantial coupling of the three S- and P-wave channels, but we do not recognize very striking features.

The paper is organized as follows. The linearization procedure developed in [6,7] is described, and the status of the uniqueness problem is outlined, in Sect. 2. Section 3 is devoted to the response to variations of the S-wave scattering lengths and the existence of a universal curve. The effects of correlated localized variations of input absorptive parts and variations of a scattering length are presented in Sect. 4. Variations of the driving terms are also briefly discussed in that section and our conclusions are displayed in Sect. 5. The fact that an analytic input admits only one solution that is regular at the matching point is a crucial result. We find it convenient to separate its proof from the presentation of phenomenological results and to explain it in Appendix A. Our approximation scheme for the determination of linear responses is described in Appendix B and Appendix C gives a list of the kernels entering the S- and P-wave Roy equations.

2 Solution manifold of the S- and P-wave Roy equations

To set the stage we recall the main features of the S- and P-wave Roy equations [4]. They relate the real and imaginary parts of the S- and P-wave π–π scattering amplitudes at low energies, below the matching point E_0:

$$\mathrm{Re} f_i(s) = (s-4)\sum_{j=0}^{2}\frac{1}{\pi}\int_4^{s_0}\mathrm{d}x\frac{1}{x-4}\left[\frac{\delta_{ij}}{x-s}+R_{ij}(s,x)\right] \times \mathrm{Im} f_j(x) + \phi_i(s), \tag{2.1}$$

$i = 0,1,2$. To lighten the writing, our notation differs from the standard one: f_0 and f_2 are the isospin $I = 0$ and $I = 2$ S-wave amplitudes and f_1 is the isospin $I = 1$ P-wave. We return to the conventional notation f_l^I in the presentation of final results. The variables s and x are squared total CM energies in units of M_π^2 (M_π = pion mass, $s_0 = (E_0/M_\pi)^2$). Equations (2.1) contain singular diagonal Cauchy kernels and regular kernels R_{ij} which are displayed in Appendix C.

The ϕ_i are input functions

$$\phi_i(s) = a_i + (s-4)\Big\{c_i(2a_0 - 5a_2) + \sum_{j=0}^{2}\frac{1}{\pi}\int_{s_0}^{\infty}\mathrm{d}x\frac{1}{x-4}\left[\frac{\delta_{ij}}{x-s}+R_{ij}(s,x)\right]A_j(x) + \psi_i(s)\Big\}. \tag{2.2}$$

In this equation a_0 and a_2 are the isospin 0 and 2 S-wave scattering lengths, $a_1 = 0$ here and

$$c_0 = \frac{1}{12}, \quad c_1 = \frac{1}{72}, \ c_2 = -\frac{1}{24}; \tag{2.3}$$

the A_i are the absorptive parts above the matching point:

$$A_i(s) = \mathrm{Im} f_i(s), \quad s \geq s_0, \tag{2.4}$$

and the ψ_i are so-called driving terms describing the contributions of the higher partial waves ($l \geq 2$). They have partial wave expansions converging in $[4, s_0]$ as long as $s_0 < 125.31$ [5]. The (2.1) constrain the S- and P-waves on $[4, s_0]$ at given input (a_i, A_i, ψ_i). Unitarity implies

$$f_i(s) = \frac{1}{2i\sigma(s)}\left(\eta_i(s)\mathrm{e}^{2\mathrm{i}\delta_i(s)} - 1\right), \quad \sigma(s) = \sqrt{1-\frac{4}{s}}, \tag{2.5}$$

where δ_i is the channel i phase shift and η_i is the elasticity parameter ($0 \leq \eta_i \leq 1$) which we incorporate into the input.

At given input (2.1) are coupled non-linear integral equations for the phase shifts δ_i on the interval $[4, s_0]$. To be acceptable, a solution of these equations has to provide absorptive parts below s_0 that join continuously the inputs A_i at that point:

$$\lim_{s \nearrow s_0}\frac{1}{2\sigma(s)}\left(1 - \eta_i(s)\cos(2\delta_i(s))\right) = A_i(s_0). \tag{2.6}$$

This boundary condition has to be added to (2.1).

The Roy equations being singular, the uniqueness of their solution is by no means guaranteed. We sum up the discussion of that point using the technique developed in [6,7]. This technique will be our main tool throughout this article.

We assume we have a set of phase shifts δ_i satisfying the (2.1) and (2.6), the amplitudes f_i being given by (2.5). We ask if these equations have other solutions δ_i' with the same input. If the δ_i' are infinitesimally close to the δ_i the differences $(\delta_i' - \delta_i)$ obey the linearized coupled equations

$$\cos(2\delta_i(s))h_i(s) = \sum_j \frac{1}{\pi}\int_4^{s_0}\mathrm{d}x\frac{1}{x-4}\left[\frac{\delta_{ij}}{x-s}+R_{ij}(s,x)\right] \times \sin(2\delta_j(s))h_j(s), \tag{2.7}$$

where

$$h_i(s) = \frac{1}{\sigma(s)}\eta_i(s)(\delta_i'(s) - \delta_i(s)). \tag{2.8}$$

The boundary conditions (2.6) imply

$$h_i(s_0) = 0, \tag{2.9}$$

i.e. $\delta_i'(s_0) = \delta_i(s_0)$. The homogeneous equations (2.7) with boundary conditions (2.9) may have non-trivial solutions because of the presence of Cauchy kernels. The uniqueness or non-uniqueness of the δ_i depends on the existence of such solutions.

If the regular kernels R_{ij} are omitted, (2.7) decouple and one recovers the one-channel problem discussed in [9]. The existence of non-trivial solutions of this problem depends on the value of the phase shift δ_i at the matching point s_0. We assume that $\delta_i(s_0) > -\pi/2$. There is no solution if $-\pi/2 < \delta_i(s_0) < \pi/2$. If $\delta_i(s_0) > \pi/2$, the general solution is

$$h_i(s) = (s-4)G_i(s)P_i(s), \tag{2.10}$$

where

$$G_i(s) = \left(\frac{s_0}{s_0 - s}\right)^{m_i} \exp\left[\frac{2}{\pi}\int_4^{s_0} \mathrm{d}x \frac{\delta_i(x)}{x-s}\right] \tag{2.11}$$

with

$$m_i = \left[\frac{2}{\pi}\delta_i(s_0)\right]. \tag{2.12}$$

$[x]$ is the greatest integer smaller than x (as in [9], s_0 is chosen in such a way that $\delta_i(s_0)$ is not an integral multiple of $\pi/2$). The last factor P_i in the r.h.s. of (2.10) is an arbitrary polynomial of degree $m_i - 1$.

The general solution of the complete set of coupled equations (2.7) has a form similar to (2.10):

$$h_i(s) = (s-4)G_i(s)\left[P_i(s) + H_i(s)\right] \tag{2.13}$$

with corrections H_i [7].

The P_i are again arbitrary polynomials of degree $m_i - 1$: m_i is given by (2.12) if $\delta_i(s_0) > \pi/2$; $m_i = 0$ and $P_i = 0$ if $|\delta_i(s_0)| < \pi/2$. The functions H_i are regular on $[4, s_0]$ and are solutions of a set of coupled non-singular integral equations:

$$\begin{aligned}\delta_{m_i,0}H_i(s) &- \frac{1}{\pi}\int_4^{s_0} \mathrm{d}x G_i(x) \\ &\times \sin(2\delta_i(x))\frac{H_i(x) - H_i(s)}{x-s} \\ &= \sum_j \frac{1}{\pi}\int_4^{s_0} \mathrm{d}x R_{ij}(s,x)G_j(x) \\ &\times \sin(2\delta_j(x))[P_j(x) + H_j(x)].\end{aligned} \tag{2.14}$$

According to definition (2.11) we have

$$G_i(s) \sim (s_0 - s)^{\gamma_i} \tag{2.15}$$

for $s \sim s_0$ with $\gamma_i = (2/\pi)\delta_i(s_0) - m_i$. This shows that G_i vanishes at s_0 if $\delta_i(s_0) > 0$ and diverges at that point if $\delta_i(s_0) < 0$. Due to the regularity of H_i at s_0 the boundary condition (2.9) is automatically fulfilled if $\delta_i(s_0) > 0$. If $-\pi/2 < \delta_i(s_0) < 0$, H_i has to vanish at s_0 (remember that $P_i = 0$ in this case).

We now apply these results to the uniqueness problem of the physical π–π S- and P-waves as solutions of the Roy equations (2.1). The input $(a_i, A_i, \psi_i, \eta_i)$ is identified with the physical one and we take the physical phase shifts as our master solution δ_i of the Roy equations. The physical isospin 0 S-wave and isospin 1 P-wave phase shifts being positive [12], we have $m_0 \geq 0$, $m_1 \geq 0$ and $\gamma_0 > 0$, $\gamma_1 > 0$. On the other hand, the isospin 2 S-wave phase shift δ_2 is negative ($-\pi/2 < \delta_2 \leq 0$), $m_2 = 0$, $P_2 = 0$ and $\gamma_2 < 0$. Consequently, the three boundary conditions (2.9) are satisfied if

$$H_2(s_0) = 0. \tag{2.16}$$

As solutions of (2.14), the H_i are linear functionals of the polynomials P_0 and P_1. Condition (2.16) gives a homogeneous linear equation relating the coefficients of these polynomials and reduces by one the number of free parameters. If $m_0 + m_1 > 1$ we are left with $m_0 + m_1 - 1$ free parameters. There is no non-trivial solution if $m_0 + m_1 \leq 1$. If $m_0 + m_1 > 1$, the physical phase shifts are embedded in a d-dimensional manifold of solutions of the Roy equations with $d = m_0 + m_1 - 1$. If $m_0 + m_1 \leq 1$, they form an isolated solution of these equations.

The actual values of m_0 and m_1 depend on the choice of the matching point s_0. Taking into account the known behavior of the physical phase shifts [12], one finds four different situations when $E_0 = s_0^{1/2}M_\pi$ is lowered from 1.15 GeV to threshold.

(1) *1 GeV*$< E_0 <$ *1.15 GeV.* In that interval, $\pi < \delta_0(s_0) < 3\pi/2$, $\pi/2 < \delta_1(s_0) < \pi$. This gives $m_0 = 2$, $m_1 = 1$ and $d = 2$. The physical S- and P-waves are members of a two-parameter family of solutions of the Roy equations at fixed physical input and fixed phase shifts at s_0. The physical solution can be selected by imposing the physical values of the position and width of the ρ-meson.

(2) *860 MeV*$< E_0 <$ *1 GeV.* We now have $\pi/2 < \delta_i(s_0) < \pi$, $i = 0, 1$ and $m_0 = m_1 = 1$, $d = 1$. The polynomials P_0 and P_1 reduce to constants related by (2.16). The physical amplitudes belong to a one-parameter family of solutions. The position of the ρ-meson can be used as a parameter.

(3) *780 MeV*$< E_0 <$ *860 MeV.* In this interval $m_0 = 0$, $m_1 = 1$ and $d = 0$ because $0 < \delta_0(s_0) < \pi/2$, $\pi/2 < \delta_1(s_0) < \pi$. The polynomial P_0 vanishes and P_1 is a constant which is set equal to zero by condition (2.16). The physical amplitudes form an isolated solution of the Roy equations. Position and shape of the ρ-resonance are determined by the input.

(4) *280 MeV*$< E_0 <$ *780 MeV.* Here $0 < \delta_i(s_0) < \pi/2$, $i = 0, 1$, $m_0 = m_1 = 0$ and both P_0 and P_1 vanish. The physical amplitudes again define an isolated solution.

The above results concern the mathematical problem defined by (2.1), (2.5) and (2.6). Due to the behavior (2.15) of the G_i at s_0, the representation (2.13) implies that if there are solutions δ_i' in the neighborhood of δ_i

they are singular at s_0 and exhibit cusps at that point. These singularities are unphysical because the choice of s_0 is arbitrary. In fact, all solutions of the Roy equations with arbitrary input are singular at s_0. The physical amplitudes being regular at s_0, the physical input has to be such that the corresponding Roy equations have at least one solution which is non-singular at s_0 and coincides with the physical amplitudes. Among all possible inputs the physical input is a very special one: it is an analytic input in the sense of Ref. [9]. It has been shown there that in simplified one-channel Roy equations with analytic input there is only one solution which is regular at s_0. This crucial result is extended to the present realistic case in Appendix A.

We see that there is no non-uniqueness problem when working with the exact physical input. For instance in case 1 above, one could vary the position and width of the ρ-resonance in the two-parameter family of solutions. The singularities at s_0 would disappear at the physical values of these parameters. In practice, however, the physical input is only known approximately and one is not really working with an analytic input. Therefore singularities are present at s_0 and non-uniqueness cannot be avoided if $E_0 > 860\,\mathrm{MeV}$. We have to put up with these unpleasant features which are merely consequences of a deficient knowledge of the physical input.

From now on we choose the matching point used in the low energy extrapolation based on the Roy equations performed in [3]: $E_0 = 800\,\mathrm{MeV}$ ($s_0 = 33$). Non-uniqueness is avoided but there are unwanted cusps at the matching point. It turns out that some of these cusps are in fact a helpful tool. Their suppression provides insights into the correlations constraining the scattering lengths of an analytic input. This will be illustrated repeatedly in this paper.

3 Varying the S-wave scattering lengths: universal curve

We come now to our main topic, the linear response to small variations of the input. We proceed along the same lines as in the previous section. Starting from the solution δ_i with input $(a_i, A_i, \psi_i, \eta_i)$, we determine the solution δ_i' produced by a slightly modified input in linear approximation. To obtain quantitative results, we need a model for the δ_i which provides an acceptable representation of the physical phase shifts. We use the Schenk parametrization [13]:

$$\delta_i(s) = \tan^{-1}\left\{\sigma(s)\frac{4-z_i}{s-z_i}\left[a_i + b_i q^2 + c_i q^4\right] f_i(s)\right\}, \tag{3.1}$$

where

$$q^2 = \frac{s}{4} - 1, \quad f_i = \begin{cases} 1 & \text{for} i = 0, 2, \\ q^2 & \text{for} i = 1. \end{cases} \tag{3.2}$$

The values of the parameters are given in Table 1 and the phase shifts are shown in Fig. 1.

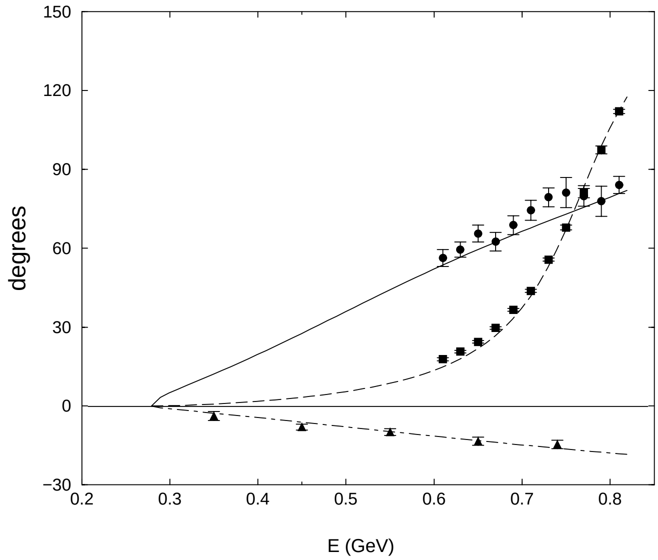


Fig. 1. The π–π S- and P-wave phase shifts according to the Ansatz (3.1) and data points obtained from analyses of experiments: δ_0^0 and δ_1^1 from [14] and δ_0^2 from [15] —— δ_0^0, ----- δ_1^1, -·-·- δ_0^2

Table 1. Values of the coefficients in the parametrization (3.1) of the physical S- and P-wave phase shifts

i	a_i	b_i	c_i	z_i
0	0.200	0.245	-0.0177	39.3
1	0.035	$2.76{\cdot}10^{-4}$	$-6.9 \cdot 10^{-5}$	31.1
2	-0.041	-0.0730	$-3.2 \cdot 10^{-4}$	-37.3

The elasticities η_i are very close to 1 below our matching point and will be set equal to 1 in all our numerical results.

In the present section, we vary only the S-wave scattering lengths:

$$a_i \to a_i' = a_i + \delta a_i, \quad i = 0, 2. \tag{3.3}$$

One of the main goals of the low energy extrapolation of the experimental data lies in the determination of these scattering lengths. This cannot be achieved directly by solving the Roy equations because the scattering lengths enter into the input of these equations. However, they can be predicted in an indirect way because the physical input is an analytic one. Consequently, the scattering lengths are not independent of the other pieces of the input. If we know the physical A_i, ψ_i and η_i we may solve the Roy equations for arbitrary scattering lengths a_i. According to Proposition 1 in Appendix A, their physical values are obtained by varying these a_i until one arrives at a solution which is regular at s_0. In practice, when working with an approximation of the physical A_i, ψ_i and η_i, the scattering lengths have to be varied until the corresponding solution of the Roy equations can be declared a good approximation of the solution of the problem with exact input. This is precisely the procedure used in [3] and it is instructive to have an explicit control of the response to the variations (3.3).

Our task is to determine functions h_i defined as in (2.8). In the linearized scheme the representation (2.13) is replaced by [7]

$$h_i(s) = G_i(s)\left\{\frac{\delta a_i}{G_i(4)} + (s-4)\right. \times \left.\left[(-s)^{m_i} c_i(2\delta a_0 - 5\delta a_2) + p\delta_{i,1} + H_i(s)\right]\right\}. \tag{3.4}$$

We recall that $m_0 = m_2 = 0$, $m_1 = 1$, $P_0 = P_2 = 0$ and P_1 is a constant which we call p. The functions H_i are the solutions of inhomogeneous extensions of (2.14). They have the form of (B.1) with

$$Z_i = \sum_{j=0}^{2} Z_{ij},$$

$$Z_{ij}(s) = \frac{1}{\pi}\int_4^{s_0} \mathrm{d}x R_{ij}(s,x) G_j(x)\sin(2\delta_j(x)) \times \left\{\frac{1}{(x-4)}\frac{\delta a_j}{G_j(4)} + [(-x)^{m_j} c_i(2\delta a_0 - 5\delta a_2) + p\delta_{j,1}]\right\}. \tag{3.5}$$

The solutions H_i depend linearly on p and this constant is fixed in such a way that $h_2(s_0) = 0$, i.e.

$$\frac{\delta a_2}{G_2(4)} + (s_0 - 4)\left[c_2(2\delta a_0 - 5\delta a_2) + H_2(s_0)\right] = 0. \tag{3.6}$$

We refer the reader to [7] for a derivation of (3.5) and (B.1).

To obtain the h_i we have to evaluate the modulating functions G_i on the right-hand side of (3.5) and find the solution H_i of (B.1) and (3.5). The functions G_i defined in (2.11) and obtained from the model (3.1) are shown in Fig. 2. The exponents γ_i appearing in (2.15) are

$$\gamma_0 = 0.89, \quad \gamma_1 = 0.19, \quad \gamma_2 = -0.20. \tag{3.7}$$

The small value of γ_1 comes from the fact that s_0 is close to z_1, the position of the ρ-resonance. This leads to the spectacular cusp of G_1 at s_0, seen in Fig. 2. As h_2 behaves as $(s_0 - s)G_2$ at s_0, it is this product which is relevant and shown in Fig. 2. The exponents γ_0 and $\gamma_2 + 1$ are close to 1 and the cusps of G_0 and $(s_0 - s)G_2$ are not visible in the figure.

The H_i are slowly varying and (3.5) tells us that h_1 has a sharp cusp at s_0. The δ_i defined in (3.1) are models of the physical phase shifts: they are regular at s_0 and meant to be produced by an analytic input $(a_i, A_i, \psi_i, \eta_i)$. We see that δ_1' obtained from (2.8) has a sharp cusp at s_0. This singular behavior is a visible signal that the modified input $(a_i + \delta a_i, A_i, \psi_i, \eta_i)$ is no longer an analytic one. The differences $(\delta_i' - \delta_i)$ are linear in δa_0 and δa_2:

$$\delta_i'(s) - \delta_i(s) = G_i(s)\left[f_{i0}(s)\delta a_0 + f_{i2}(s)\delta a_2\right], \tag{3.8}$$

where f_{i0} and f_{i2} are regular at s_0. We see that the cusp of δ_1' is suppressed if

$$f_{10}(s_0)\delta a_0 + f_{12}(s_0)\delta a_2 = 0. \tag{3.9}$$

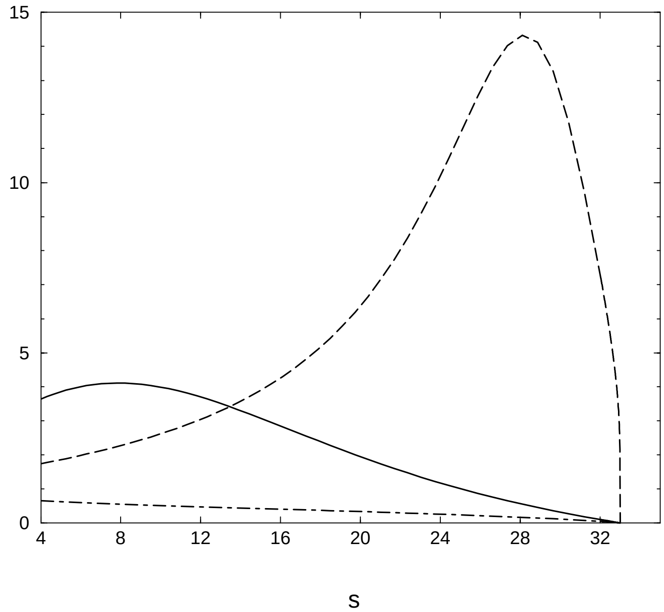


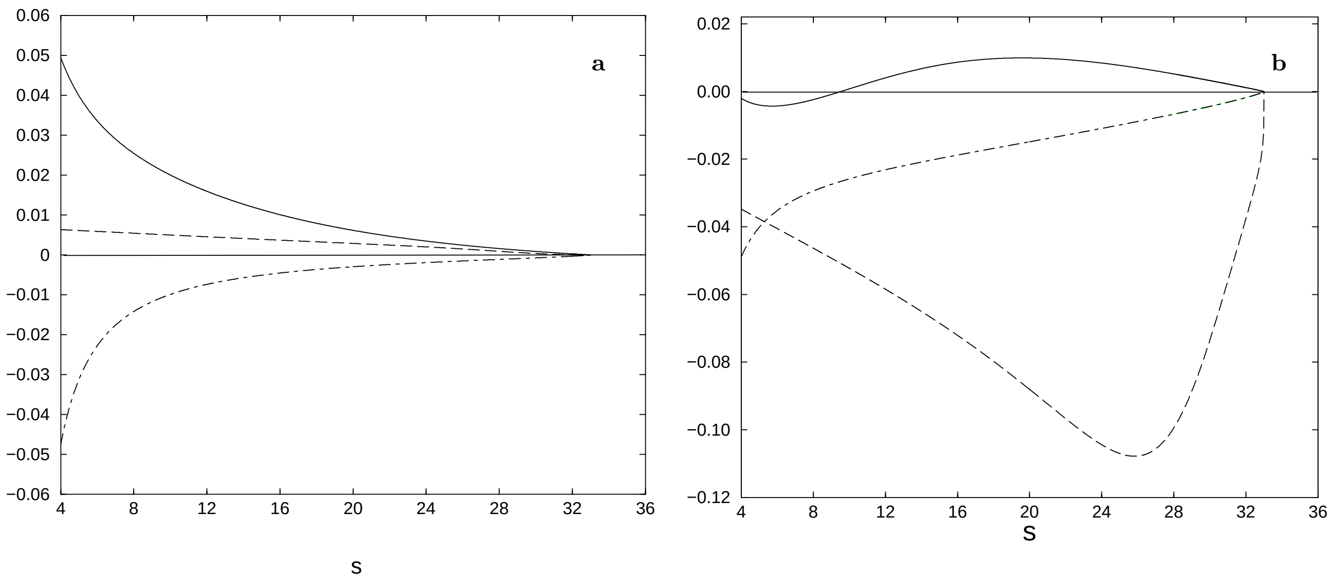
Fig. 2. The functions G_i defined in (2.11) appearing as factors in the responses to variations of the input —— G_0, ----- G_1, -·-·- $\frac{s_0 - s}{s_0} G_2$

There is a direction in the (a_0, a_2)-plane along which δ_1' has no cusp. There is still a singularity at s_0 but no infinite slope. This indicates that the input $(a_i + \delta a_i, A_i, \psi_i, \eta_i)$ is close to an analytic input if condition (3.9) is satisfied. An analytic input is not an isolated one: it is transformed into new analytic inputs by suitably correlated variations of its ingredients. Our finding and the fact that δ_0' and δ_2' have no visible cusps show that the physical input is transformed into nearly analytic inputs by variations of the scattering lengths obeying (3.9). One can show that variations of the scattering lengths alone cannot transform an analytic input exactly into an analytic one. A movement along a direction in the (a_0, a_2)-plane has to be accompanied by modifications of the remaining pieces of the input if one wants to keep it exactly analytic. Our results show that these modifications are small and we confirm at the local level the existence of a one-parameter family of nearly analytic inputs along a universal curve, $a_2 = a_2(a_0)$ in the (a_0, a_2)-plane [11].

To go beyond qualitative results, (B.1) have to be solved and we apply the approximation scheme described in Appendix B. In this scheme, the H_i have the form

$$H_i(s) \simeq \left(\frac{s}{s_0}\right)^{m_i}\left(\hat{H}_{i,0}(s)\delta a_0 + \hat{H}_{i,2}(s)\delta a_2 + \hat{H}_{i,3}(s)p\right), \tag{3.10}$$

where the $\hat{H}_{i,k}$ are second-degree polynomials. Once we have determined $\hat{H}_{i,k}$ and when p has been fixed by (3.6), we find that the ratio $\delta a_2/\delta a_0$ defined in (3.9), for which the cusp in δ_1' is suppressed, is equal to 0.197. In fact the ratio $f_{1,0}(s)/f_{1,2}(s)$ is nearly constant and equal to its value at s_0 on the whole interval $[4, s_0]$. We identify the ratio 0.197 with the slope of the universal curve at



—— $(\delta_0^{0'} - \delta_0^0)_\parallel/\delta_0^0$,
----- $(\delta_1^{1'} - \delta_1^1)_\parallel/\delta_1^1$,
-·-·-· $(\delta_0^{2'} - \delta_0^2)_\parallel/\delta_0^2$.

—— $(\delta_0^{0'} - \delta_0^0)_\perp/\delta_0^0$,
----- $(\delta_1^{1'} - \delta_1^1)_\perp/\delta_1^1$,
-·-·-· $(\delta_0^{2'} - \delta_0^2)_\perp/\delta_0^2$.

Fig. 3a,b. Relative responses $(\delta_l^{I'} - \delta_l^I)/\delta_l^I$ to variations of the S-wave scattering lengths. **a** Displacement (3.11) along the universal curve, $\delta a_\parallel = 0.05 a_0^2$. **b** Displacement orthogonal to the universal curve, $\delta a_\perp = 0.05|a_0^2|$

Table 2. Accuracy of the approximate values of $(\delta_i' - \delta_i)_\parallel$ and $(\delta_i' - \delta_i)_\perp$. The mean relative quadratic discrepancies χ_i are defined in (B.10)

i	0	1	2
$\chi_i^\parallel$	$3.7 \cdot 10^{-4}$	$1.9 \cdot 10^{-3}$	$2.9 \cdot 10^{-4}$
$\chi_i^\perp$	$9.5 \cdot 10^{-3}$	$3.1 \cdot 10^{-2}$	$1.8 \cdot 10^{-2}$

(a_0, a_2): it coincides with the slope found in [3]. There is a strong compensation of the two terms in the right-hand side of (3.8) when $i = 1$ if one moves along the universal curve. This compensation is maximally removed in the orthogonal direction where $\delta a_2/\delta a_0 = -5.08$. This is illustrated in Fig. 3 which displays the relative phase-shift differences $(\delta_i' - \delta_i)_\parallel/\delta_i$ and $(\delta_i' - \delta_i)_\perp/\delta_i$. The differences $(\delta_i' - \delta_i)_\parallel$ are obtained when the point (a_0, a_2) moves along the universal curve

$$\delta a_0 = \delta a_\parallel \cos\theta_\parallel, \quad \delta a_2 = \delta a_\parallel \sin\theta_\parallel, \tag{3.11}$$

with $\theta_\parallel = \tan^{-1} 0.197 = 11°$. The differences $(\delta_i' - \delta_i)_\perp$ are obtained in response to a displacement $(\delta a_0, \delta a_2)$ normal to the universal curve, δa_0 and δa_2 being given by (3.11) with $\delta a_\parallel$ replaced by $\delta a_\perp$ and $\theta_\parallel$ replaced by $\theta_\perp = 101°$.

To assess the quality of the results displayed in Fig. 3, Table 2 gives the values of the χ_i defined in (B.10) in the parallel and orthogonal directions. The values of the $\chi_i^\perp$ are acceptable whereas those for $\chi_i^\parallel$ are surprisingly small.

The relative variations of the phase shifts in the direction of the universal curve are decreasing functions of s. The S-waves have peaks at threshold, whose sizes are dictated by the values of δa_0 andδa_2. The pattern in the orthogonal direction is different and more complicated. The effects of the variation of the scattering length spread over the whole interval $[4, s_0]$ and $(\delta_1' - \delta_1)_\perp/\delta_1$ has a cusp which cannot be overlooked.

The overall size of the variations $(\delta_i' - \delta_i)_\perp$ is significantly larger than that of the corresponding $(\delta_i' - \delta_i)_\parallel$. To characterize this fact quantitatively we evaluate the mean values of the absolute ratios over the interval $[4, s_0 - 2]$ ($s_0 - 2$ instead of s_0 as upper limit to avoid effects of the cusp in $(\delta_1' - \delta_1)_\perp$)

$$\rho_i = \left\langle \left| \frac{(\delta_i' - \delta_i)_\perp}{(\delta_i' - \delta_i)_\parallel} \right| \right\rangle . \tag{3.12}$$

We take $\delta a_\perp = \delta a_\parallel$, i.e. we assume that the same distance is covered along and perpendicularly to the universal curve, and find

$$\rho_0 = 15.7, \quad \rho_1 = 217, \quad \rho_2 = 18.6. \tag{3.13}$$

These large values reflect the sharp definition of the universal curve obtained in [3].

4 Combined variations of input absorptive parts, driving terms and scattering length

This section is mainly devoted to variations δA_i of the absorptive parts A_i above the matching point. As shown in [7] the response is obtained from the following Ansatz for the functions h_i in (2.8):

$$h_i(s) = G_i(s)(s - 4)[p\delta_{i,1} + F_i(s) + H_i(s)], \tag{4.1}$$

where

$$F_i(s) = \frac{s^{m_i}}{\pi} \int_{s_0}^{\infty} \mathrm{d}x \frac{1}{x^{m_i}} \frac{1}{x-4} \frac{1}{x-s} \frac{\delta A_i(x)}{G_i(x)}. \tag{4.2}$$

The functions H_i on the right-hand side of (4.1) are solutions of (B.1) with

$$Z_i(s) = \sum_{j=0}^{2} Z_{ij}(s), \tag{4.3}$$

where

$$Z_{ij}(s) = Y_{ij}(s) + \frac{1}{\pi} \int_4^{s_0} \mathrm{d}x R_{ij}(s,x) \sin(2\delta_j(x)) \times G_j(x) \left[p\delta_{j,1} + F_j(x)\right] \tag{4.4}$$

and

$$Y_{ij}(s) = \frac{1}{\pi} \int_{s_0}^{\infty} \mathrm{d}x \frac{1}{x-4} R_{ij}(s,x) \delta A_j(x). \tag{4.5}$$

Equations (4.2)–(4.5) have been derived in [7]. For simplicity we assume that the δA_i vanish at s_0 and the boundary condition (2.9) remains unchanged: $h_i(s_0) = 0$.

Using the analyticity properties of the kernels R_{ij} referred to in Appendix B, the integral in the right-hand side of (4.4) can be transformed to give

$$Z_{ij}(s) = pQ_i(s)\delta_{j,1} + \int_{s_0}^{\infty} \mathrm{d}u M_{ij}(s,u) \frac{1}{u-4} \frac{\delta A_j(u)}{G_j(u)}, \tag{4.6}$$

where Q_i is a known polynomial and

$$M_{ij}(s,u) = -\frac{1}{2\mathrm{i}\pi} \oint_{\Gamma'} \mathrm{d}x R_{ij}(s,x) \bar{G}_j(x) \left(\frac{x}{u}\right)^{m_i} \frac{1}{x-u}. \tag{4.7}$$

The contour Γ' encircles the segment $[-(s-4), 0]$. Formula (4.7) affords an explicit evaluation of M_{ij} once the $\bar{G}_j$ have been approximated by polynomials, as in Appendix B. The functions H_i are determined by applying the method of that appendix. The condition $h_2(s_0) = 0$ fixes p as a linear functional of the δA_i and the differences $\delta_i' - \delta_i$ resulting from (2.8) and (4.1) can be written as

$$\delta_i'(s) - \delta_i(s) = \sum_{j=0}^{2} \int_{s_0}^{\infty} \mathrm{d}u K_{ij}(s,u) \delta A_j(u). \tag{4.8}$$

The kernel $K_{ij}(u)$ gives the effect on the channel i phase shift of a variation of the channel j absorptive part at point u ($u > s_0$). It follows from (4.1) that $K_{ij}(s,u)$ is proportional to $G_i(s)$ and $K_{1,j}$ exhibits, as a function of s, a sharp cusp at $s = s_0$. An arbitrary variation of the input absorptive parts transforms an analytic input into a non-analytic one. We correct this partly and stay in the vicinity of an analytic input by modifying simultaneously the scattering length a_2 and choosing δa_2 in such a way that the cusp in $K_{1,j}$ is suppressed. This variation δa_2 is a linear functional of the δA_i:

$$\delta a_2 = \sum_{i=0}^{2} \int_{s_0}^{\infty} \mathrm{d}u \kappa_i(u) \delta A_i(u) \tag{4.9}$$

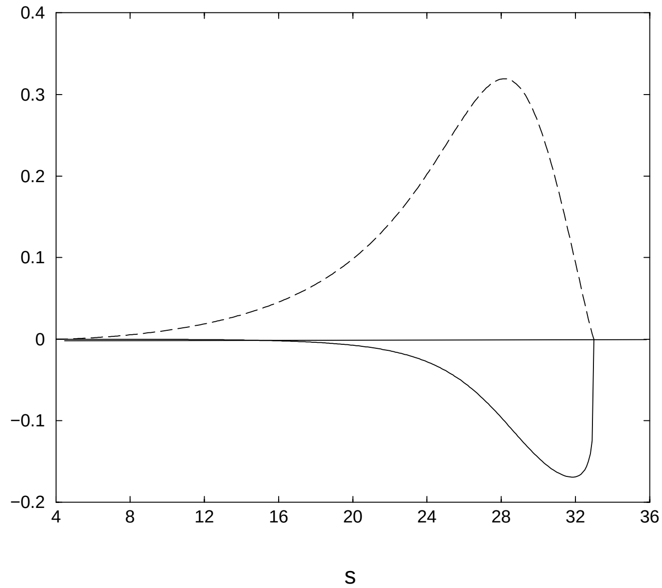


Fig. 4. The kernels K_{11} and $\hat{K}_{11}$ at $u = 35$ as functions of s: $\hat{K}_{11}$ includes the effect of a variation of a_0^2 suppressing the cusp in K_{11} —— K_{11}, ----- $\hat{K}_{11}$

and Sect. 3 tells us that equations of the form (4.8) are still valid if the K_{ij} are replaced by new kernels $\hat{K}_{ij}$.

The kernels K_{ij} and $\hat{K}_{ij}$ have been evaluated as functions of s for 3 values of u: $u_1 = 35$ ($E_1 = 828\,\mathrm{MeV}$) close to the matching point, $u_2 = 51$ ($E_2 = 1\,\mathrm{GeV}$) and $u_3 = 100$ ($E_3 = 1.4\,\mathrm{GeV}$).

The passage from K_{11} to $\hat{K}_{11}$ at $u = u_1 = 35$, slightly above the matching point, is illustrated in Fig. 4. As must be the case, the large cusp in K_{11} has disappeared in $\hat{K}_{11}$. The effect of the induced variation of a_2 ($\kappa_1(u_1) = -0.0085$) dominates the response to the variation of A_1 outside the neighborhood of s_0. The fact that $\hat{K}_{11}$ is larger than K_{11} is peculiar: $\hat{K}_{10}$ and $\hat{K}_{12}$ are much smaller than K_{10} and K_{12}.

The values of the kernels $\hat{K}_{ij}$ at $u = u_1$ determine the responses to small variations δA_j concentrated around that point. If δA_j is sufficiently small and narrow (4.8) and (4.9) give

$$\delta_i'(s) - \delta_i(s) \simeq \sum_j \hat{K}_{ij}(s,u_1) \Delta A_j, \quad \delta a_2 \simeq \sum_i \kappa_i(u_1) \Delta A_i, \tag{4.10}$$

with

$$\Delta A_i = \int \mathrm{d}u \delta A_i(u). \tag{4.11}$$

The relative phase-shift differences produced by such variations of the input absorptive parts with corresponding variations of the scattering length a_2 are displayed in Figs. 5, 6 and 7. The ΔA_i have been chosen in such a way that the responses are of the order of a few percent. Our linearization should be reliable under these circumstances. To describe the situation in physical terms we can imagine that the ΔA_j are produced by the insertion of fictitious narrow elastic resonances of width Γ_j at u_1.

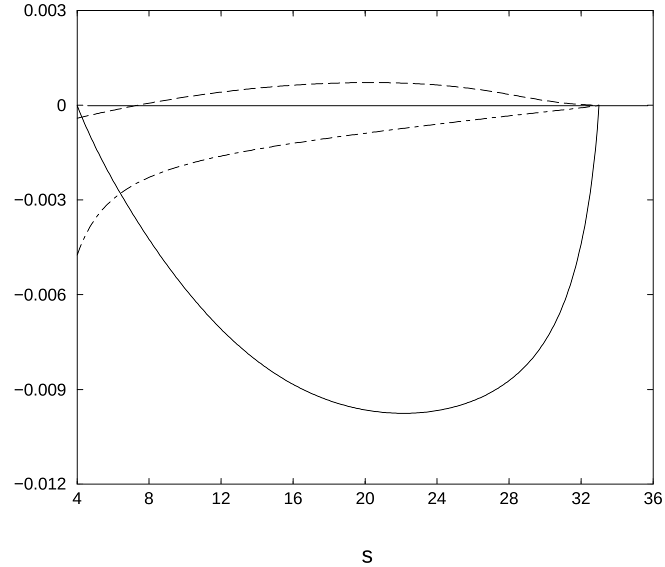


Fig. 5. Relative responses $(\delta_l^{I'} - \delta_l^I)/\delta_l^I$ to a variation of A_0^0 concentrated on $u = 35$, $\Delta A_0^0 = -0.1$ —— $(\delta_0^{0'} - \delta_0^0)/\delta_0^0$, ----- $(\delta_1^{1'} - \delta_1^1)/\delta_1^1$, -·-·- $(\delta_0^{2'} - \delta_0^2)/\delta_0^2$

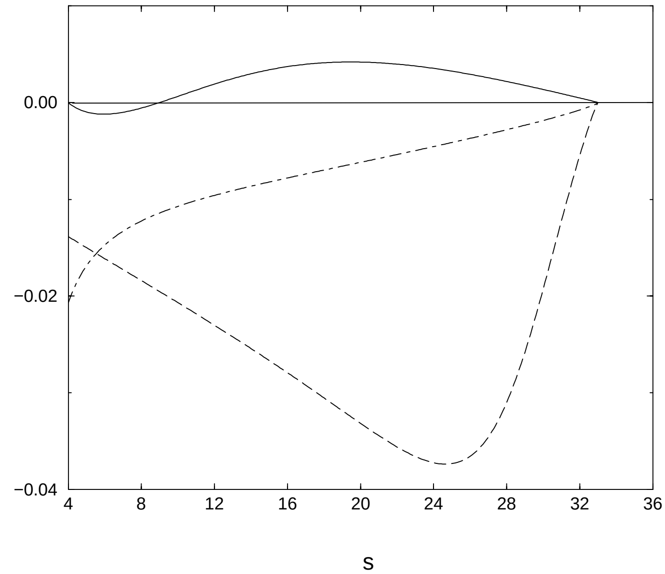


Fig. 6. Relative responses $(\delta_l^{I'} - \delta_l^I)/\delta_l^I$ to a variation of A_1^1 concentrated on $u = 35$, $\Delta A_1^1 = -0.1$ —— $(\delta_0^{0'} - \delta_0^0)/\delta_0^0$, ----- $(\delta_1^{1'} - \delta_1^1)/\delta_1^1$, -·-·- $(\delta_0^{2'} - \delta_0^2)/\delta_0^2$

The values of the ΔA_j used in Figs. 5, 6 and 7 correspond to $\Gamma_0 = 0.76\,\mathrm{MeV}$, $\Gamma_1 = 1.56\,\mathrm{MeV}$, $\Gamma_2 = 0.92\,\mathrm{MeV}$. The effects of these resonances sitting just above the matching point spread over the whole interval $[4, s_0]$. The induced variation of a_2 produces a modest peaking of $(\delta_2' - \delta_2)/\delta_2$ at threshold. The responses to a variation ΔA_0 in the isospin 0 S-wave are globally smaller than the effects of variations ΔA_1 and ΔA_2 of the same size in the other channels. A variation ΔA_i in channel i produces a response in the same channel that is enhanced near s_0 and dominates the responses in the other channels. This dominance is significant but not very strong in the case of ΔA_2. Apart from these observations we do not discover any striking feature characterizing qualitatively the coupling of the S- and P-waves.

The $\hat{K}_{ij}$ are decreasing functions of u without significant change in their shape as functions of s. The decrease is rapid just above the matching point. For instance, the $\hat{K}_{i0}$ are scaled down at $u = 36$ to 70% of their values at $u = 35$.

To characterize the decrease of the responses when variations ΔA_j are shifted to higher energies, we compute averages ρ_{ij} of the absolute values of the relative phase-shift differences at $u_1 = 35$, $u_2 = 51$ and $u_3 = 100$. According to (4.10) these are given by

$$\rho_{ij}(u_k) = \frac{1}{s_0 - 4} \int_4^{s_0} \mathrm{d}s \left| \frac{\hat{K}_{ij}(s, u_k)}{\delta_i(s)} \Delta A_j \right|. \qquad (4.12)$$

Approximate values of the $\rho_{ij}(u_1)$ are given in Table 3. The ratios $\rho_{ij}(u_2)/\rho_{ij}(u_1)$ and $\rho_{ij}(u_3)/\rho_{ij}(u_1)$ show the decrease of the responses at higher energies. None of the mean responses to variations located at u_2 exceed 11% of the corresponding responses at u_1. This percentage is

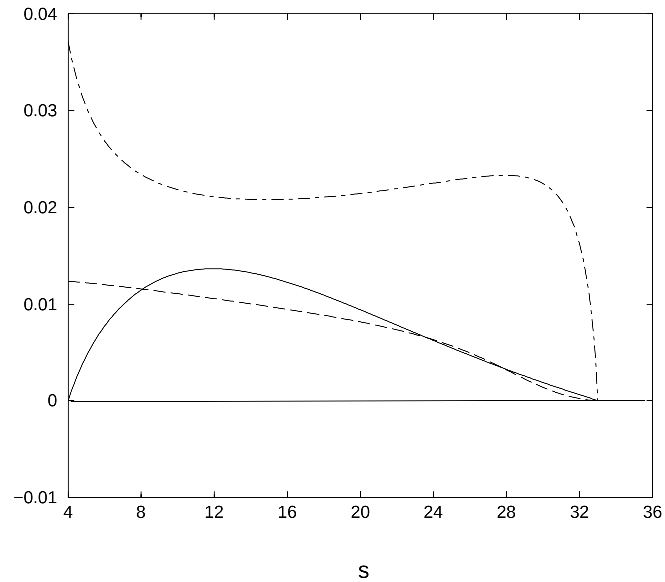


Fig. 7. Relative responses $(\delta_l^{I'} - \delta_l^I)/\delta_l^I$ to a variation of A_0^2 concentrated on $u = 35$, $\Delta A_0^2 = 0.1$ —— $(\delta_0^{0'} - \delta_0^0)/\delta_0^0$, ----- $(\delta_1^{1'} - \delta_1^1)/\delta_1^1$, -·-·- $(\delta_0^{2'} - \delta_0^2)/\delta_0^2$

reduced to 1.2% when u_2 is replaced by u_3. Table 4 gives the values of the variations δa_2 coming from (4.10).

We conclude that the solution of the Roy equations is quite insensitive to the errors on the input absorptive parts above $E_3 = u_3^{1/2} M_\pi = 1.4\,\mathrm{GeV}$. The solution of the Roy equations is most sensitive to the input absorptive parts close to the matching point. According to Table 4, the uncertainty in a_2, associated in our scheme with an error on the absorptive parts at u_3, is less than 1% of the uncertainty due to the same error at u_1.

Table 3. Mean relative responses $\rho_{ij}(u_1)$ defined in (4.12) for $|\Delta A_j| = 0.1$ and ratios of mean relative responses at u_2 and u_3 versus responses at u_1, $u_1^{1/2}M_\pi = 828\,\text{MeV}$, $u_2^{1/2}M_\pi = 1\,\text{GeV}$, $u_3^{1/2}M_\pi = 1.4\,\text{GeV}$

(i,j)	$\rho_{ij}(u_1)$	$\frac{\rho_{ij}(u_2)}{\rho_{ij}(u_1)}$	$\frac{\rho_{ij}(u_3)}{\rho_{ij}(u_1)}$
$(0,0)$	$7.5 \cdot 10^{-3}$	0.061	0.0049
$(1,0)$	$4.2 \cdot 10^{-4}$	0.10	0.010
$(2,0)$	$1.1 \cdot 10^{-3}$	0.10	0.010
$(0,1)$	$2.4 \cdot 10^{-3}$	0.10	0.012
$(1,1)$	$2.6 \cdot 10^{-2}$	0.073	0.0069
$(2,1)$	$6.8 \cdot 10^{-3}$	0.11	0.012
$(0,2)$	$8.3 \cdot 10^{-3}$	0.080	0.0069
$(1,2)$	$7.5 \cdot 10^{-3}$	0.079	0.0069
$(2,2)$	$2.2 \cdot 10^{-2}$	0.052	0.0035

Table 4. Relative variations of the scattering length a_2 induced according to (4.10) by variations of the input absorptive parts A_j at u_1, $|\Delta A_j| = 0.1$ and ratios of variations at u_2 and u_3 versus variations at u_1, $u_1^{1/2}M_\pi = 828\,\text{MeV}$, $u_2^{1/2}M_\pi = 1\,\text{GeV}$, $u_3^{1/2}M_\pi = 1.4\,\text{GeV}$

j	$\frac{\delta a_2(u_1)}{a_2}$	$\frac{\delta a_2(u_2)}{\delta a_2(u_1)}$	$\frac{\delta a_2(u_3)}{\delta a_2(u_1)}$
0	$-4.8 \cdot 10^{-3}$	0.09	0.009
1	$-2.1 \cdot 10^{-2}$	0.10	0.011
2	$3.6 \cdot 10^{-2}$	0.07	0.006

Table 5. Total discrepancies χ_j defined in (B.11) to variations of the input absorptive part A_j at u_k and the correlated variation of the scattering length a_2, $u_1^{1/2}M_\pi = 828\,\text{MeV}$, $u_2^{1/2}M_\pi = 1\,\text{GeV}$, $u_3^{1/2}M_\pi = 1.4\,\text{GeV}$

	u_1	u_2	u_3
χ_0	0.012	0.012	0.005
χ_1	0.024	0.057	0.016
χ_2	0.017	0.019	0.014

We close the discussion of variations of the input absorptive parts with an assessment of the accuracy of our results. The errors come from our functions H_i. These form an approximate solution of (B.1) with inhomogeneous terms Z_i containing a component (4.3) coming from variations of the A_j at u_k and a component (3.5) due to the corresponding variation of a_2. Let $\chi_j(u_k)$ be the total discrepancy between left- and right-hand sides of (B.1) defined in (B.11). These quantities are listed in Table 5. All equations (B.1) are verified at least at the percent level, which is sufficient for our purpose.

We close this section with a survey of the response to variations of the driving terms ψ_i in (2.2). The ψ_i are small and approximated by polynomials on $[4, s_0]$ in [3]. We consider variations of these polynomials. The Ansatz for the functions h_i defined in (2.8) becomes

$$h_i(s) = (s-4)G_i(s)\,(p\delta_{i,1} + H_i(s))\,, \tag{4.13}$$

where p is a constant and the H_i form a solution of the (B.1) with

$$Z_i(s) = \delta\psi_i(s) + \delta_{i,1}p\int_4^{s_0} \mathrm{d}x R_{i1}(s,x)\sin(2\delta_1(x))G_1(x). \tag{4.14}$$

As before, the variations of the driving terms are combined with variations of a_2 such that h_1 has no cusp at s_0. The result shows that large relative variations of the driving terms affect only weakly the phase shifts below s_0. For instance, a reduction of the size of ψ_0 or ψ_2 by 50% changes the δ_i by less than 5%. In the case of a 50% reduction of ψ_1 the response is smaller than 0.5%.

5 Summary and conclusions

We have developed an approximation scheme to determine the linear response of the solution of the S- and P-wave Roy equations with matching point $s_0 = 33$ to small variations of their input (S-wave scattering lengths, S- and P-wave absorptive parts above s_0, and driving terms). Our results are precise at the percent level, which is sufficient for a qualitative insight. Our problem has been solved long ago, in a different way, for a higher matching point $s_0 = 70$ in [7]. At $s_0 = 33$ the solution of the Roy equations is unique, entirely determined by their input.

An arbitrary input leads to a solution that is singular at s_0. As the physical amplitudes are regular at s_0, the physical input belongs to the restricted class of our analytic inputs producing a solution that is non-singular at s_0. We prove that under legitimate assumptions an analytic input has in fact only one solution regular at s_0 (Appendix A).

An arbitrary variation of the input transforms an analytic input into a non-analytic one and induces responses that are singular at s_0. Due to the fact that our s_0 is close to M_ρ^2 (M_ρ = ρ-meson mass), the sharpest singularities show up as cusps in the isospin 1 P-wave responses. These cusps are suppressed by correlating suitably the variations of two pieces of the input. We choose to associate in this way variations of the isospin 2 S-wave scattering length a_0^2 to arbitrary variations of other components of the input.

It is instructive to compare our strategy with the procedure used in [3] when solving the Roy equations themselves. The solution is parametrized in [3] by an Ansatz that is regular at s_0. As one is working with a non-analytic approximation of the physical input, the solution is singular at s_0 and cannot be fitted exactly by the Ansatz. An approximate solution is constructed by a least square procedure tuning simultaneously the parameters in the Ansatz and the scattering length a_0^2 in the input. In this way the input is brought close to an analytic one and the Ansatz gives a model of the corresponding solution. In some of its features this machinery resembles our simple strategy. In fact, their equivalence for the computation of responses to small variations of the input has been checked in the case of the variation of the isospin 0 S-wave absorptive parts displayed in Fig. 5. The response obtained by solving the full Roy equations coincides with our result within a few percents. This confirms that the main factor tuning a_0^2 in [3] is the avoidance of a cusp in the isospin 1 P-wave phase shift.

Our technique shows that one stays in the vicinity of an analytic input when moving infinitesimally along a given direction of the (a_0^0, a_0^2) plane without changing the other pieces of the input. This confirms the existence of a so-called universal curve at the linear response level.

We have determined the response to localized variations of the input absorptive parts above the matching point. It spreads over the whole interval $[4, s_0]$ and illustrates the intricate coupling of the S- and P-waves produced by crossing symmetry. It shows that the sensitivity to the errors in the input absorptive parts decreases rapidly with increasing energy.

Acknowledgements. I thank G. Colangelo, J. Gasser and H. Leutwyler for their continual interest in the present work. I am especially grateful to J. Gasser for his help in the preparation of the figures.

Appendix A. Analytic input and uniqueness

An analytic input $(a_i, A_i, \psi_i, \eta_i)$ is defined as an input admitting at least one solution of the Roy equations which is regular at the matching point. A precise definition is given below. In any case it is an indirect definition: as shown at the end of this appendix, we know how to construct analytic inputs but we are unable to identify an analytic input by direct inspection. Its components are correlated: in particular, the scattering lengths depend on the A_i, ψ_i and η_i. Analytic inputs are relevant objects because the physical input belongs to that class. The aim of this appendix is to prove that an analytic input has only one solution regular at s_0. The requirement of regularity at s_0 eliminates in principle the uniqueness problem. This result has already been established in [9] for simplified one-channel Roy equations.

To establish our result we need general analyticity properties of the partial wave amplitudes f_i. Let f be one of them. It is known to be the boundary value of an analytic function F on the interval $[4, 125.31]$ [10]. This function is holomorphic in a complex domain Δ extending on the real axis from $s_\mathrm{L} = -28$ to $s_\mathrm{R} = 125.31$ and provided with a left-hand cut $[s_\mathrm{L}, 0]$ and a right-hand cut $[4, s_\mathrm{R}]$. We have

$$f(s) = \lim_{\epsilon\searrow 0} F(s + \mathrm{i}\epsilon), \quad s \in [4, s_\mathrm{R}]. \tag{A.1}$$

Our matching point s_0 being above the first inelastic threshold $i_1 = 16$, we need properties characterizing the elasticity parameters η which enter into an analytic input. According to (2.5), η is equal to the modulus of the S-matrix element $(1 + 2\mathrm{i}\sigma f)$ which is the boundary value of

$$S(z) = 1 - 2\sqrt{\frac{4 - z}{z}}F(z). \tag{A.2}$$

This function is regular in Δ. Using the relation $\bar{S}(z) = S(\bar{z})$ we write

$$\begin{aligned}\eta^2(s) &= \lim_{\epsilon\searrow 0} S(s + \mathrm{i}\epsilon)\bar{S}(s + \mathrm{i}\epsilon)\\ &= \lim_{\epsilon\searrow 0} S(s + \mathrm{i}\epsilon)S(s - \mathrm{i}\epsilon), \quad s \in [i_1, s_\mathrm{R}].\end{aligned} \tag{A.3}$$

Although it cannot be derived from first principles [16], it is legitimate to assume that the inelastic thresholds i_k $(k = 1, 2, \ldots)$ are the only singularities of f on $[4, s_\mathrm{R}]$ and that S has an analytic continuation S_{II} into the sheet reached by crossing the cut $[4, s_\mathrm{R}]$ from below between two successive inelastic thresholds i_k and i_{k+1} [S_{II} depends on the pair (i_k, i_{k+1})]. Equation (A.3) gives

$$\eta^2(s) = \lim_{\epsilon\searrow 0} S(s + \mathrm{i}\epsilon)S_{II}(s + \mathrm{i}\epsilon), \quad s \in (i_k, i_{k+1}), \tag{A.4}$$

as long as $i_{k+1} < s_\mathrm{R}$.

We assume that S_{II} is regular in the upper half-plane, in a neighborhood D of the segment (i_k, i_{k+1}). Equation (A.4) tells us that the real-valued η^2 is the boundary value on (i_k, i_{k+1}) of a function holomorphic in D and we apply the following general result.

Lemma 1 *Let w be a real-valued function defined on the interval (i_k, i_{k+1}). If w is the boundary value of an analytic function W holomorphic in D, it is the restriction to (i_k, i_{k+1}) of a function regular in the domain $D \cup \bar{D}$ where $\bar{D}$ is the mirror domain of D: $\bar{D} = \{z | \bar{z} \in D\}$.*

A proof of this Lemma is given at the end of this appendix. It implies that η^2 has an analytic continuation regular in a complex neighborhood of (i_k, i_{k+1}). We assume that the possible complex zeros of η^2 are at a finite distance from (i_k, i_{k+1}). We choose D sufficiently narrow so that η^2 is non-vanishing on D and we have

Lemma 2 *If the above conditions are fulfilled η has a holomorphic continuation from each interval (i_k, i_{k+1}) with $i_{k+1} < s_\mathrm{R}$ into a complex neighborhood of that interval with i_k and i_{k+1} on its boundary.*

We turn now to properties of the full amplitude f and establish

Lemma 3 *The real and imaginary parts of f are separately holomorphic in a complex neighborhood of each interval (i_k, i_{k+1}) $(i_{k+1} < s_\mathrm{R})$ with i_k and i_{k+1} on its boundary. Here $k = 0, 1, 2, \ldots$ with $i_0 = 4$.*

This is a well known result in the case of the interval $[4, i_1]$ [17]. For any interval we define the function

$$V = \frac{1}{\mathrm{i}\sigma}\frac{1 - \eta + 2\mathrm{i}\sigma f}{1 + \eta + 2\mathrm{i}\sigma f} \tag{A.5}$$

on (i_k, i_{k+1}) [$\eta = 1$ on $(4, i_1)$]. According to Lemma 2, V has a regular analytic continuation into a domain N in the upper half-plane – $[i_k, i_{k+1}]$ belongs to the boundary of N – except for poles at the possible zeros of the denominator. Using unitarity,

$$\mathrm{Im}f = \sigma|f|^2 + \frac{1}{4\sigma}(1-\eta^2), \tag{A.6}$$

we find that $\mathrm{Im}V = 0$ on (i_k, i_{k+1}). Lemma 1 is easily extended to the case of meromorphic functions and one concludes that V has a meromorphic continuation into $N \cup \bar{N}$. The definition (A.5) gives

$$\mathrm{Re}f = \frac{\eta V}{1+\sigma^2 V^2}, \quad \mathrm{Im}f = \frac{\sigma\eta V^2}{1+\sigma^2 V^2} + \frac{1}{2\sigma}(1-\eta). \tag{A.7}$$

We assume again that the zeros of the denominators are at a finite distance from the real axis and discover that $\mathrm{Re}f$ and $\mathrm{Im}f$ are indeed separately holomorphic in a neighborhood of (i_k, i_{k+1}) contained in $N \cup \bar{N}$. The phase shift δ is also regular in a neighborhood of each (i_k, i_{k+1}).

We close our preliminaries with the structure of f at an inelastic threshold i_k with square root singularity. There are four functions a, b, c and d that are regular in a circle C_k with center i_k and radius ρ such that

$$\begin{aligned} \eta_>(s) &= \exp\left[-2\left(a(s) + \sqrt{s-i_k}\,b(s)\right)\right], \\ \delta_>(s) &= c(s) + \sqrt{s-i_k}\,d(s) \end{aligned} \tag{A.8}$$

for $s \in (i_k, i_{k+1})$: $\eta_>$ and $\delta_>$ designate respectively the elasticity parameter and the phase shift above i_k. The amplitude f can be written on (i_k, i_{k+1}) in terms of a complex phase shift $\tilde{\delta}_>$ as

$$\begin{aligned} f &= \tfrac{1}{2\mathrm{i}\sigma}\left(\mathrm{e}^{2\mathrm{i}\tilde{\delta}_>} - 1\right), \\ \tilde{\delta}_>(s) &= c(s) + \mathrm{i}a(s) + \sqrt{s-i_k}\,(d(s) + \mathrm{i}b(s)). \end{aligned} \tag{A.9}$$

The value of f below i_k, on $(i_k - \rho, i_k)$, is obtained through analytic continuation of the expression (A.9) in the upper half-plane along curves contained in C_k. The outcome is determined by a complex phase shift

$$\tilde{\delta}_<(s) = c(s) + \mathrm{i}a(s) - \sqrt{i_k - s}\,(b(s) - \mathrm{i}d(s)). \tag{A.10}$$

The elasticity parameter $\eta_<$ and the phase shift $\delta_<$ below i_k are given by

$$\begin{aligned} \eta_<(s) &= \exp\left[-2\left(a(s) + \sqrt{i_k - s}\,d(s)\right)\right], \\ \delta_<(s) &= c(s) - \sqrt{i_k - s}\,b(s), \end{aligned} \tag{A.11}$$

on $(i_k - \rho, i_k)$. The functions b and d interchange their roles when we cross i_k. Below i_1, $\eta_<$ is equal to 1. This implies

$$a = d = 0 \tag{A.12}$$

in the case $k = 1$.

We summarize our findings in

Lemma 4 *The structure of f at a square root inelastic threshold i_k is described by formulas (A.8) and (A.11). Equation (A.12) holds at i_1.*

After this lengthy preparation we are ready for a complete definition of an analytic input.

Definition 1. An analytic input $(a_i, A_i, \psi_i, \eta_i)$ contains elasticities fulfilling Lemma 2 on $(4, s_0)$. It admits at least one solution f_i, $i = 0, 1, 2$, of the S- and P-wave Roy equations with $\mathrm{Re}f_i$ regular at s_0 in the sense that they are holomorphic in a circle $C_{s_0} : |s - s_0| < \epsilon$. These f_i satisfy Lemmas 3 and 4.

We establish the following

Proposition 1 *Let f_i, $i = 0, 1, 2$, form a solution of the S- and P-wave Roy equations with analytic input (a_i, A_i, ψ_i, η_i) that is regular at s_0 and verifies Lemmas 3 and 4. A second solution f'_i of these equations, $f'_i \neq f_i$, is singular at s_0.*

This proposition is an extension of Proposition 4 in [9] to the realistic situation. In the following proof we assume $i_1 < s_0 < i_2$, which is true for our $s_0 = 33$.

The proof of Proposition 1 is based on Lemmas 2, 3 and 4. To make sure that the Roy equations (2.1) guarantee the required analyticity of the f_i we rewrite these equations as follows:

$$\mathrm{Re}\Phi_i(s) = (s-4) \times \frac{1}{\pi}\,\mathrm{P}\!\!\int_4^\infty \mathrm{d}x \frac{1}{(x-4)(x-s)} \mathrm{Im}\Phi_i(x), \tag{A.13}$$

where

$$\Phi_i(s) = f_i(s) - a_i - (s-4)\left\{c_i(2a_0 - 5a_2) + \sum_{j=0}^{2} \frac{1}{\pi}\int_4^\infty \mathrm{d}x R_{ij}(s,x)\mathrm{Im}f_j(x) + \psi_i(s)\right\}. \tag{A.14}$$

In these equations $\mathrm{Im}f_i(s) = A_i(s)$ for $s \geq s_0$, and the ψ_i are the driving terms appearing in (2.2). The fact that $\mathrm{Im}\Phi_i = \mathrm{Im}f_i$ on $[4, \infty)$ has been used. Equations (A.13) ensure that the Φ_i are boundary values of analytic functions holomorphic in $\mathbb{C}\backslash[4, \infty)$. For $x \in [4, \infty)$ the kernels $R_{ij}(s, x)$ are holomorphic functions of s in $\mathbb{C}\backslash(-\infty, 0]$ and the driving terms are regular in the domain Δ without right-hand cut [10]. Taking all this into account, (A.14) provides a representation of the f_i ensuring that they are indeed boundary values of functions, holomorphic in the domain Δ, with right- and left-hand cuts. The same conclusion holds for the f'_i.

To establish Proposition 1, we show that the f'_i have to coincide with the f_i if the $\mathrm{Re}f'_i$ are regular at s_0. Inversion of the dispersion relations (A.13) gives

$$\begin{aligned} \mathrm{Im}\Phi_i(s) &= \mathrm{Im}f_i(s) \\ &= -(s-4)\frac{1}{\pi}\,\mathrm{P}\!\!\int_{\mathbb{R}} \frac{\mathrm{d}x}{x-4}\frac{\mathrm{Re}\Phi_i(x)}{x-s}, \quad s \in [4, \infty). \end{aligned} \tag{A.15}$$

The regularity of $\mathrm{Re}f_i$ at s_0 implies that $\mathrm{Re}\Phi_i$ is regular at that point and it follows from (A.15) that $\mathrm{Im}f_i$ is holomorphic in C_{s_0}. The relation (A.15) holds true for Φ'_i defined by f'_i and the assumed regularity of $\mathrm{Re}f'_i$ at s_0 implies the analyticity of $\mathrm{Im}f'_i$ in C_{s_0}. According to Definition 1,

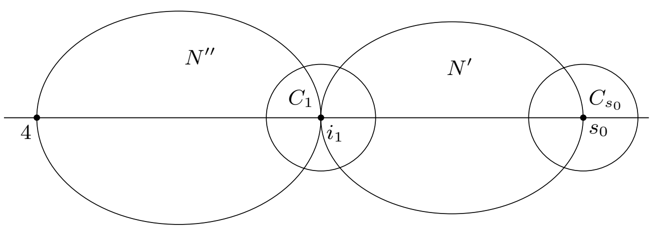


Fig. 8. Domains of the complex s-plane used in the proof of Proposition 1

$\mathrm{Im} f_i$ and $\mathrm{Im} f_i'$ have analytic continuations holomorphic in a complex neighborhood N' of the interval $[s, s_0]$ shown in Fig. 8. We conclude that $\mathrm{Im} f_i$ and $\mathrm{Im} f_i'$ are holomorphic in $N' \cup C_{s_0}$. As

$$\mathrm{Im} f_i'(s) = \mathrm{Im} f_i(s) = A_i(s) \mathrm{for} s \in [s_0, s_0 + \epsilon), \tag{A.16}$$

$\mathrm{Im} f_i'$ and $\mathrm{Im} f$ coincide in $N' \cup C_{s_0}$ and

$$\mathrm{Im} f_i(s) = \mathrm{Im} f_i'(s) \mathrm{for} s \in (i_1, s_0]. \tag{A.17}$$

To complete our proof, we have to extend the equality (A.16) below the inelastic threshold i_1. The discussion of Lemma 4 shows that phase shifts are required to go through i_1. The equality of imaginary parts above i_1 at fixed η_i implies

$$\delta_i'(s) = \pm \delta_i(s) \mod \pi, \quad s \in (i_1, s_0]. \tag{A.18}$$

We show that the minus sign must be rejected. If, for a given i, $\delta_i' = -\delta_i \mod \pi$, (A.8) gives

$$\delta_{i>}'(s) = c'(s) + \sqrt{s - i_1} d'(s), \tag{A.19}$$

with $c'(s) = -c(s) \mod \pi$, $d'(s) = -d(s)$. According to (A.11) this would produce an elasticity $\eta_{i<}'$ below i_1 that would differ from the input elasticity $\eta_{i<}$.

In terms of the functions b and c appearing at i_1, we now have

$$c_i' = c_i \mod \pi \tag{A.20}$$

whereas $b_i' = b_i$ is given, η_i being a member of the input. Equations (A.10) and (A.12) give

$$\delta_{i<}'(s) = \left(c_i(s) - \sqrt{i_k - s} b_i(s) \right) \mod \pi, \quad s \in (i_1 - \rho, i_1). \tag{A.21}$$

Therefore we have

$$\mathrm{Im} f_i'(s) = \mathrm{Im} f_i(s), \quad s \in (i_1 - \rho, i_1). \tag{A.22}$$

As both sides of this equations have analytic extensions regular in a neighborhood N'' of $(4, i_1)$ the equality (A.22) extends to $(4, i_1)$. Thus we have established that f_i' and f_i have the same imaginary parts on $[4, s_0]$ if $i_1 < s_0 < i_2$ and the Roy equations imply the full equality of these two amplitudes. This result extends to arbitrary choices of the matching point. The proof becomes easier if $s_0 < i_1$; it requires more steps if $s_0 > i_2$.

For completeness we prove Lemma 1. We define a function $\hat{W}$ by $\hat{W}(z) = \bar{W}(\bar{z})$. It is holomorphic in the mirror domain $\bar{D}$ and, w being real, we have $w(s) = \lim_{\epsilon \searrow 0} \hat{W}(s - \mathrm{i}\epsilon)$, $s \in [i_k, i_{k+1}]$. We write for $z \in D$

$$W(z) = \frac{1}{2\mathrm{i}\pi} \oint_{\partial D} \mathrm{d}x \frac{W(x)}{x - z} + \frac{1}{2\mathrm{i}\pi} \oint_{\partial \bar{D}} \mathrm{d}x \frac{\hat{W}(x)}{x - z}. \tag{A.23}$$

The first term is the Cauchy representation of W and the second integral vanishes because $z \notin \bar{D}$. The contributions of the segment $[i_k, i_{k+1}]$ to both integrals cancel and one is left with the Cauchy representation of a function holomorphic in $D \cup \bar{D}$.

Three remarks close this appendix.

(1) If $s_0^{1/2} M_\pi = 800\,\mathrm{MeV}$, we know, according to Sect. 2, that the physical solution of the Roy equations is an isolated one. The relevance of Proposition 1 comes from the possible existence of other solutions with $\delta_i'(s_0) = \delta_i(s_0) + n_i \pi$ resulting from CDD-pole ambiguities [19]. These solutions are singular at s_0.
(2) The proof of Proposition 1 tells us that the absorptive parts A_i of an analytic input are regular on some interval $[s_0, s_0')$ above the matching point ($s_0' \geq s_0 + \epsilon$) and are the analytic continuation of $\mathrm{Im} f_i$ below s_0 on that interval. The Roy equations (2.1) define real parts $\mathrm{Re} f_i$ above s_0. On $[s_0, s_0')$ they are the analytic continuations of the $\mathrm{Re} f_i$ below s_0. As the interval $[s_0, s_0')$ cannot contain an inelastic threshold, all the ingredients of the unitarity condition (A.6) have analytic continuations from below s_0 onto $[s_0, s_0')$. This implies that (A.6) holds on $[s_0, s_0')$: $\mathrm{Re} f_i$ and A_i are the real and imaginary parts of amplitudes verifying unitarity on that interval. This means that they fulfill, at least on $[s_0, s_0')$, a consistency condition discussed in [3].
(3) Although we have no direct way of checking whether a given input is an analytic one, we have a recipe for the construction of such inputs. Take a matching point s_0' above s_0 ($s_0' < 125.31$) and choose arbitrarily an input $(a_i', A_i', \psi_i', \eta_i')$. Let f_i' be a solution of the Roy equations with that input, verifying Lemmas 2, 3 and 4. These f_i' are expected to be singular at s_0' but they are regular at s_0. Define a new Ansatz $(a_i, A_i, \psi_i, \eta_i)$ with matching point s_0:

$$\begin{aligned} a_i &= a_i', \\ \psi_i &= \psi_i', \quad \eta_i = \eta_i' \quad \mathrm{on} [4, s_0], \\ A_i(s) &= \begin{cases} \mathrm{Im} f_i'(s) \ \mathrm{for} s_0 \leq s \leq s_0', \\ A_i'(s) \quad \mathrm{for} s > s_0'. \end{cases} \end{aligned} \tag{A.24}$$

The f_i' define a solution f_i of this new problem,

$$f_i(s) = f_i'(s) \mathrm{for} 4 \leq s \leq s_0. \tag{A.25}$$

This solution is regular at s_0 and the Ansatz $(a_i, A_i, \psi_i, \eta_i)$ is an analytic one.

Our recipe is of no practical use because it requires the explicit resolution of the Roy equations with matching point s_0'. The important point is that we recognize that an analytic input with matching point s_0 is unconstrained above some s_0', $s_0' > s_0$. It is the behavior of the A_i on $[s_0, s_0']$ which is constrained and s_0' can be close to s_0.

In our definition an input is analytic with respect to its matching point s_0. The physical input is special because it generates inputs with matching points $s_0' > s_0$ ($s_0' < 125.31$) that are analytic with respect to s_0'.

Appendix B. Approximation scheme

We write the equations we have to solve in Sects. 3 and 4 in the following way:

$$\sum_{j=0}^{2} X_{ij}[H_j](s) = Z_i(s). \tag{B.1}$$

Each X_{ij} is a linear and homogeneous functional of the unknown H_j,

$$\begin{aligned} X_{ij}[H_j](s) = \delta_{i,j} \Bigg\{ &\delta_{m_j,0} H_j(s) - \frac{1}{\pi}\int_4^{s_0} \mathrm{d}x G_j(x) \\ &\times \sin(2\delta_j(x)) \frac{H_j(x) - H_j(s)}{x - s} \Bigg\} \\ &- \frac{1}{\pi}\int_4^{s_0} \mathrm{d}x R_{ij}(s,x) G_j(x) \sin(2\delta_j(x)) H_j(x). \end{aligned} \tag{B.2}$$

The Z_i are known functions determined by the variation of the input under consideration. The unknown H_i are regular and slowly varying on $[4, s_0]$ and we approximate them by polynomials

$$H_i(s) = s^{m_i} \sum_{n=0}^{N} c_{i,n} s^n. \tag{B.3}$$

We have to determine the coefficients $c_{i,n}$. The X_{ij} become

$$X_{ij}[H_j](s) = \sum_{n=0}^{N} X_{ij}^{(n)}(s) c_{j,n} \tag{B.4}$$

where the $X_{ij}^{(n)}$ are known functions obtained by replacing $H_j(x)$ by $x^{(m_j+n)}$ in the right-hand side of (B.2).

To evaluate these functions we define auxiliary analytic functions $\bar{G}_i$, holomorphic in $\mathbb{C}\backslash[4, s_0]$

$$\bar{G}_i(z) = \left(\frac{s_0}{s_0 - z}\right)^{m_i} \exp\left[\frac{2}{\pi}\int_4^{s_0} \mathrm{d}x \frac{\delta_i(x)}{x - z}\right]. \tag{B.5}$$

They are related to the G_i defined in (2.11) by their discontinuity $\mathrm{Disc}\bar{G}_i$ across the cut $[4, s_0]$,

$$\frac{1}{2\mathrm{i}} \mathrm{Disc}\bar{G}_i(s) = G_i(s) \sin(2\delta_i(s)), \quad 4 \leq s \leq s_0. \tag{B.6}$$

The contribution to $X_{ij}^{(n)}$ coming from the first integral in the right-hand side of (B.2) is transformed into a sum of integrals along a closed contour Γ surrounding the segment $[4, s_0]$:

$$-\frac{1}{2\mathrm{i}\pi} \sum_{m=0}^{n-1} \oint_\Gamma \mathrm{d}z \bar{G}_i(z) z^m s^{n-m-1} = -\sum_{m=0}^{n-1} g_{i,m+1} s^{n-m-1}, \tag{B.7}$$

where the $g_{i,p}$ are the coefficients of the Laurent series of $\bar{G}_i$,

$$\bar{G}_i(z) = \sum_{p=0}^{\infty} g_{i,p} \frac{1}{z^p}. \tag{B.8}$$

The second integral in the right-hand side of (B.2) is evaluated in a similar way by exploiting the analyticity properties of the kernels R_{ij}. At fixed real s, $s \geq 4$, these are analytic functions of x, holomorphic in $\mathbb{C}\backslash[-(s-4), 0]$. Deforming the contour Γ, we get

$$\begin{aligned} &\int_4^{s_0} \mathrm{d}x R_{ij}(s,x) \bar{G}_j(x) \sin(2\delta_j(x)) x^n \\ &= -\frac{1}{2\mathrm{i}\pi}\int_{-(s-4)}^{0} \mathrm{d}x \mathrm{Disc} R_{ij}(s,x) \bar{G}_j(x) x^n + \text{polynomial}. \end{aligned} \tag{B.9}$$

The polynomial is determined by the asymptotic behavior in z of the product $R_{ij}(s,z)G_j(z)z^n$. The discontinuity $\mathrm{Disc}R_{ij}$ of R_{ij} across $[-(s-4), 0]$ being known, we need the $\bar{G}_j$ on that interval. These smooth functions are approximated by third-degree polynomials at a level smaller than 1%. This allows the explicit evaluation of the integral in the right-hand side of (B.9) and the result is a polynomial in s. The $X_{ij}^{(n)}$ are thus approximated by polynomial $\tilde{X}_{ij}^{(n)}$ of degree ≤ 6.

Evaluated along the same lines, the inhomogeneous terms Z_i in (B.1) become known functions $\tilde{Z}_i$. According to (B.1), $\tilde{X}_i$ and $\tilde{Z}_i$ have to be made approximately equal on $[4, s_0]$ by adjusting the $3(N+1)$ coefficients $c_{n,i}$ in (B.4) [$\tilde{X}_i$ is obtained by substituting $\tilde{X}_{ij}^{(n)}$ for $X_{ij}^{(n)}$ in (B.4) and inserting the result into (B.1)]. We keep our calculations simple by using polynomials of low degree for the H_i in (B.3) and choose $N = 2$. To determine the nine coefficients $c_{i,n}$, the $\tilde{X}_i$ and $\tilde{Z}_i$ are approximated on $[4, s_0]$ by second-degree polynomials using a χ^2 technique, and these polynomials are set equal. This gives nine equations for the nine unknowns (in $\tilde{X}_i$, each $\tilde{X}_{ij}^{(n)}$ is replaced by a polynomial of degree 2). The whole procedure is legitimate because the $\tilde{X}_i$ and $\tilde{Z}_i$ are slowly varying.

The $\tilde{X}_{ij}^{(n)}$ and $\tilde{Z}_i$ are close to the $X_{ij}^{(n)}$ and Z_i, the differences coming only from the replacement of the $\bar{G}_i$ by third-degree polynomials on $[-(s-4), 0]$. Thus, in view of (B.1), the $c_{i,n}$ we obtain must be such that $\tilde{X}_i$ and $\tilde{Z}_i$ are close to each other on $[4, s_0]$. This can be checked by evaluating the mean relative quadratic discrepancies of $\tilde{X}_i$ and $\tilde{Z}_i$

$$\chi_i = \left[\frac{1}{(s_0 - 4)}\int_4^{s_0} \mathrm{d}s \frac{\left(\tilde{X}_i(s) - \tilde{Z}_i(s)\right)^2}{\left(\tilde{Z}_i(s)\right)^2}\right]^{1/2}. \tag{B.10}$$

We can also define a total discrepancy

$$\chi = \left[\frac{1}{3}\sum_{i=0}^{2} \chi_i^2\right]^{1/2}. \tag{B.11}$$

The various values we obtain for these quantities are quoted in Sects. 3 and 4.

Appendix C. The kernels R_{ij}

Our technique makes extensive use of the analyticity properties of the regular kernels R_{ij} in (2.1) and (2.2). It is therefore convenient to display them explicitly. They are obtained from 4 functions L_k, $k = 1, \ldots, 4$:

$$L_1(s,x) = \frac{1}{x(s-4)}\left[\frac{1}{2}s - x + 2 + (x-4)\frac{x}{s-4}\ln\left(1+\frac{s-4}{x}\right)\right], \tag{C.1}$$

$$L_2(s,x) = \frac{1}{x(s-4)}\left[-\frac{3}{2}s - x + 2 + (2s+x-4)\frac{x}{s-4}\ln\left(1+\frac{s-4}{x}\right)\right],$$

$$L_3(s,x) = \frac{1}{x(s-4)^2}\left\{-\frac{1}{6}\left[s^2 - 8s + 4(3x^2-12x+4)\right] + (2s+x-4)(x-4)\frac{x}{s-4}\ln\left(1+\frac{s-4}{x}\right)\right\},$$

$$L_4(s,x) = \frac{1}{x(s-4)^2} \times \left\{-\frac{1}{6}\left[s^2 + 8s(3x-1) + 4(3x^2-12x+4)\right] + (2x+s-4)(2s+x-4)(x-4) \times \frac{x}{s-4}\ln\left(1+\frac{s-4}{x}\right)\right\}.$$

The R_{ij} are given by

$$\begin{aligned}
R_{00}(s,x) &= \frac{2}{3}L_1(s,x) - \frac{1}{x}, & R_{02}(s,x) &= \frac{10}{3}L_1(s,x),\\
R_{20}(s,x) &= \frac{2}{3}L_1(s,x), & R_{22}(s,x) &= \frac{1}{3}L_1(s,x) - \frac{1}{x},\\
R_{01}(s,x) &= 6L_2(s,x), & R_{21}(s,x) &= -3L_2(s,x),\\
R_{10}(s,x) &= \frac{2}{3}L_3(s,x), & R_{12}(s,x) &= -\frac{5}{3}L_3(s,x),\\
R_{11}(s,x) &= 3L_4(s,x) - \frac{1}{x}.
\end{aligned} \tag{C.2}$$

References

1. S. Weinberg, Physica A **96**, 327 (1979); J. Gasser, H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984); Nucl. Phys. B **250**, 465 (1985)
2. B. Knecht, B. Moussalam, J. Stern, N.H. Fuchs, Nucl. Phys. B **457**, 513 (1995); J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, M.E. Sainio, Phys. Lett. B **374**, 210 (1996); Nucl. Phys. B **508**, 263 (1997)
3. B. Ananthanarayan, G. Colangelo, J. Gasser, H. Leutwyler, work in progress
4. S.M. Roy, Phys. Lett. B **36**, 353 (1971)
5. G. Mahoux, S.M. Roy, G. Wanders, Nucl. Phys. B **70**, 297 (1974)
6. C. Pomponiu, G. Wanders, Nucl. Phys. B **103**, 172 (1976)
7. L. Epele, G. Wanders, Phys. Lett. B **72**, 390 (1978); Nucl. Phys. B **137**, 521 (1978)
8. J.L. Basdevant, J.C. Le Guillou, H. Navelet, Nuovo Cimento A **7**, 363 (1972); M.R. Pennington, S.D. Protopopescu, Phys. Rev. D **7**, 1429 (1973); ibid. D **7**, 2591 (1973); J.L. Basdevant, C.D. Froggatt, J.L. Petersen, Phys. Lett. B **41** 173 (1972); ibid. B **41**, 178 (1972); Nucl. Phys. B **72**, 413 (1974); J.L. Petersen, Acta Phys. Austriaca Suppl. **13**, 291 (1974); C.D. Froggatt, J.L. Petersen, Nucl. Phys. B **91**, 454 (1975); ibid. B **104**, E186 (1976); J.L. Petersen, The π–π interaction, Lectures given in the Academic Training Program of CERN, 1975–1976, Yellow CERN report CERN 77-04; C.D. Froggatt, J.L. Petersen, Nucl. Phys. B **129**, 89 (1977)
9. J. Gasser, G. Wanders, Eur. Phys. J. C **10**, 159 (1999)
10. S.M. Roy, G. Wanders, Phys. Lett. B **74**, 347 (1978); Nucl. Phys. B **141**, 220 (1978)
11. D. Morgan, G. Shaw, Nucl. Phys. B **10**, 261 (1968); Phys. Rev. D **2**, 520 (1970)
12. D. Morgan, M.R. Pennington, in The second DAPHNE physics handbook, edited by L. Maiani, C. Pancheri, N. Paver (INFN-LNF-Divisione Ricerca, SIS-Ufficio Pubblicazioni, Frascati 1995, ISBN 88-86409-02-8)
13. A. Schenk, Nucl. Phys. B **363**, 97 (1991)
14. W. Ochs, Ph.D. thesis, Ludwig-Maximilian-Universität, München, 1973; B. Hyams et al., Nucl. Phys. B **64**, 134 (1973)
15. W. Hoogland et al., Nucl. Phys. B **126**, 109 (1977), solution A
16. A. Martin, in Problems in Theoretical Physics, Essays dedicated to N.N. Bogoliubov on the occasion of his 60th birthday, edited by A.I. Blokintseff et al. (Nauka, Moscow, 1969)
17. W. Zimmermann, Nuovo Cimento **51**, 249 (1961) and references cited therein.
18. J. Gasser, private communication.
19. D. Atkinson, R.L. Warnock, Phys. Rev. D **16**, 1948 (1977)