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Sensitivity to initial conditions in the Bak-Sneppen model of biological evolution

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Abstract. We consider biological evolution as described within the Bak and Sneppen 1993 model. We exhibit, at the self-organized critical state, a power-law sensitivity to the initial conditions, calculate the associated exponent, and relate it to the recently introduced nonextensive thermostatistics. The scenario which here emerges *without tuning* strongly reminds of that of the *tuned* onset of chaos in say logistic-like one-dimensional maps. We also calculate the dynamical exponent z.

PACS. 05.20.-y Statistical mechanics – 05.45.+b Theory and models of chaotic systems – 05.70.Ln Nonequilibrium thermodynamics, irreversible processes

There is nowadays a massive evidence of fractals and scale invariant phenomena in nature. They appear in an impressive variety of inanimate systems such as the geological (*e.g.*, earthquakes) or climatic (*e.g.*, atmospheric turbulence) ones, as well as of biological or living systems (*e.g.*, biological evolution, cell growth, economic phenomena, among others). In most of the naturally occurring cases, no particular tuning is perceived. Per Bak and collaborators have advanced [1,2] the hypothesis that, for many if not all the cases, this is so because the microscopic dynamics of the system makes it to spontaneously evolve towards a critical, scale-invariant, state. This is largely known today as *self-organized criticality* (SOC). Models have been formulated and experiments have been performed which profusely exhibit this interesting type of behaviour in sandpiles, ricepiles, earthquakes and others (see for instance [3–5] and references therein). One such model is that introduced in 1993 by Bak and Sneppen (BS) [6] to paradigmatically describe biological species evolution. This is the model that we focus on here. The system is self–critical in the sense that, after a transient, it attains a stationary state in which there are avalanches of activity of all temporal sizes. Even more, both temporal and spatial correlations functions decay as power laws with non–trivial exponents.

In what concerns to evolutionary theory, this was the first statistical model of biological evolution which displayed punctuated equilibrium. This concept, introduced in 1977 by Gould [7] refers to the fact that evolution seems to take place not through gradual and continuous process, but rather by means of burst of strong activity, separating long periods of quasi stability of the species during which almost nothing changes.

Although the BS model has been vastly studied during the last years, there is one important point that has never been addressed as much as we know. Moreover, it has received little attention in the general context of SOC systems. This is the *sensitivity to the initial conditions*, which is known to be most relevant in nonlinear dynamical systems (quantities intensively studied such as Lyapunov exponents, spread of damage, are in fact nothing but specific expressions of this concept). The study of this important property is the basic aim of the present work. It is worth to stress here that one should not confuse criticality with chaos, as Bak and Sneppen properly pointed out in their original paper [6]. There are examples of *critical* systems which are not *chaotic* [8] showing that the last concept is not necessarily related to the former.

Before describing our particular approach of BS model, let us introduce some preliminary notions by using, as a simple illustration, the following one-dimensional logistic-like map (see [9] and references therein)

$$x_{t+1} = 1 - a\,x_t^2, \;\; (t = 0, 1, 2, \ldots). \tag{1}$$

There is a critical value $a_c = 1.4011\ldots$ such that, for $a < a_c$, we observe a regular evolution (finite-cycle attractors), whereas, for $a > a_c$, chaos becomes possible. Approaching a_c from below we can see the celebrated doubling-period road to chaos, with its successive bifurcations. If we consider, at $t = 0$, two values of x_0 which

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slightly differ by $\Delta x(0)$ and follow their time evolution, we typically observe the following exponential behaviour for $\Delta x(t)$

$$\lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} = \exp[\lambda t] \tag{2}$$

where λ is known as the Lyapunov exponent. If $\lambda < 0$ (which is in fact the case for most values of *a below* a_c) we shall say that the system is *strongly insensitive* to the initial conditions. If $\lambda > 0$ (which is in fact the case for most values of *a above* a_c) we shall say that the system is *strongly sensitive* to the initial conditions. Finally, if λ vanishes we shall speak of a *marginal* case. This is what happens, in particular, at the *onset of chaos*. For this value of a, the sensitivity is not characterized by an exponential-law, but rather by a power-law [10,11]

$$\lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} \sim C t^{\delta} \qquad (t \gg 1). \tag{3}$$

The $\delta > 0$ and the $\delta < 0$ cases respectively correspond to *weakly sensitive* and *weakly insensitive* to the initial conditions [10].

Let us now return to the BS evolution model. The model consists in a N-site ring (linear chain with periodic boundary conditions); on each site ($j = 1, 2, ..., N$) we locate a real variable B_j ($0 \le B_j \le 1, \forall j$) which corresponds to a "fitness barrier" separating two connected (first-neighboring) "species of living organisms". We start by randomly and independently attributing the set of values $\{B_j\}$. At each successive *elementary* time step we identify the *smallest* B_j, and randomly change ("mutate") it as well as its two nearest neighbors. After some transient (that depends on the size of the system), a peculiar self-organized state emerges [6], rich in avalanches of all sizes and other scale invariant properties which makes the system to exhibit a variety of power laws.

In the present work we focused the sensitivity to the initial conditions of the Bak and Sneppen model. We shall exhibit that, at its self-organized critical state, weak sensitivity to the initial conditions (*i.e.*, a power-law of the type indicated in Eq. (3)) occurs very similarly to the one just described at the onset of chaos of the map (1). To do that we apply the spreading of damage technique, which is especially useful for systems with a large number of degrees of freedom like the present model. It basically consists in studying the time evolution (under the same realization of the noise) of the Hamming distance between two replicas of the system with slightly different initial conditions. If the Hamming distance or *damage* spreads, we say that the system presents *sensitivity to initial conditions*. As in the case of the logistic map (1), we can classify the sensitivity as weak or strong depending on whether the damage spreads as a power or exponential law.

The procedure is the following: once SOC has been achieved, we consider that system as replica 1 ($\{B_j^{(1)}\}$) and create a replica 2 ($\{B_j^{(2)}\}$) by randomly choosing one of its N sites, and exchanging the value associated with

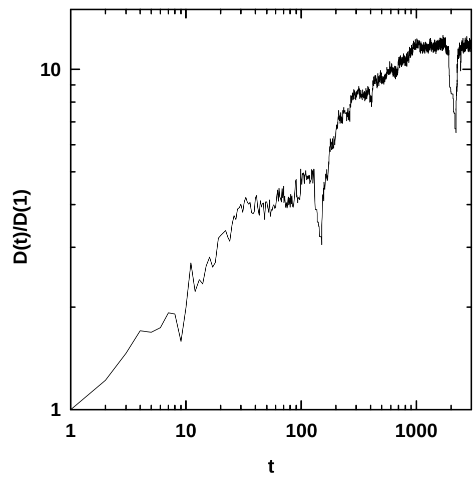

Fig. 1. Time evolution of the damage associated with one realization of a typical system with $N = 1000$.

this site and that with the smallest barrier; hence the initial damage is of order $1/N$ so, for $N \to \infty$ we satisfy the restriction that the initial perturbation be vanishingly small (see Eq.(2)). We consider this moment as the *collective* time step $t = 1$ (we define a collective time step as N times the elementary time step, *i.e.*, each site is going to be updated only once in average during a unit collective time step). From now on, we apply, for both replicas, the rules of identifying the smallest barrier and updating that particular one and its two first-neighbors. We use the *same random numbers for both replicas* (hence, three different and independent random numbers are involved in the operation).

We define now the *Hamming distance* between the two replicas as

$$D(t) \equiv \frac{1}{N} \sum_{j=1}^{N} \left| B_j^{(1)}(t) - B_j^{(2)}(t) \right|. \tag{4}$$

One such realization is shown in Figure 1 for $N = 1000$. We then average N_r realizations (we have used typically $N_r N = 10^5$) and obtain $\langle D \rangle(t)$, which is presented in Figure 2 in a double logarithmic plot for different sizes. It is then clear that a power law spread of the damage emerges (and therefore $\lambda = 0$) indicating a weak sensitivity to initial conditions. The observed saturation of the curves for very long times is a consequence of the finite size of the systems. These results enable the determination of the slope $\delta = 0.32\ldots$ to be compared, for instance, with the logistic map value $1.31\ldots$ [10].

For fixed N, we denote by τ the value of t at which the increasing regime *crosses over* onto the saturation regime (intersection, in Fig. 2, of two straight lines, namely those defined by the linearly increasing branch of the curve and the horizontal branch). The proportionality $\tau(N) \propto N^z$ defines the dynamical exponent z ([12] and references therein). We obtained (see Fig. 3) $z = 1.6 \pm 0.1$, to be compared, for instance, with 2.16 obtained [12] for the square-lattice Ising ferromagnet.

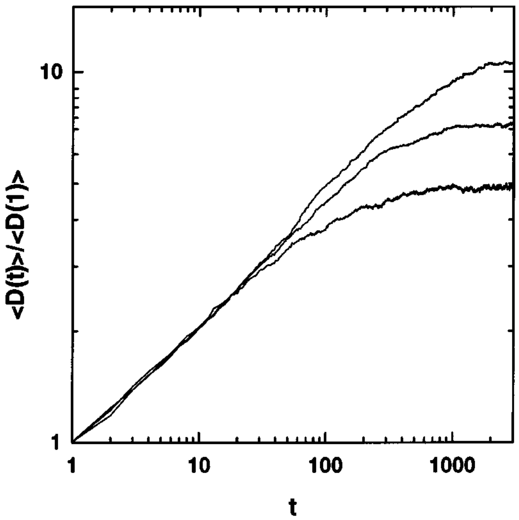

Fig. 2. Average of N_r realizations such as those of Figure 1 for three different sizes: $N = 1000$ (top), $N = 500$ (middle) and $N = 250$ (bottom). The slope δ equals 0.32 ± 0.01.

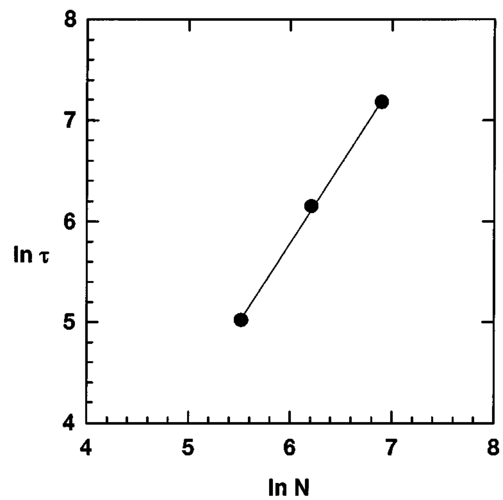

Fig. 3. Log-log plot of τ *versus* N for the three curves of Figure 2.

Finally, we see from the data collapse of Figure 4 that the normalized Hamming distance $D(N,t) \equiv \langle D(t)\rangle / \langle D(1)\rangle$ presents the following finite size scaling behavior:

$$D(N,t) \sim N^{\beta}\, F\left(\frac{t}{N^{\gamma}}\right), \tag{5}$$

with $\beta = 0.54$ and $\gamma = 1.7$, which agree, within the error bars, with the previous scalings by setting $\beta = \gamma\delta$ and $\gamma = z$.

Note that, as proposed in [10,11], the concept of Lyapunov exponent can be generalized by the following expression:

$$\lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} \sim [1 + (1-q)\lambda_q t]^{\frac{1}{1-q}} \qquad (q \in \mathcal{R}) \tag{6}$$

in such a way that for $q = 1$ we recover the usual definition and for $q \neq 1$ and $t \to \infty$ we take into account the marginal cases ($\lambda_1 = 0$) described by the following power law:

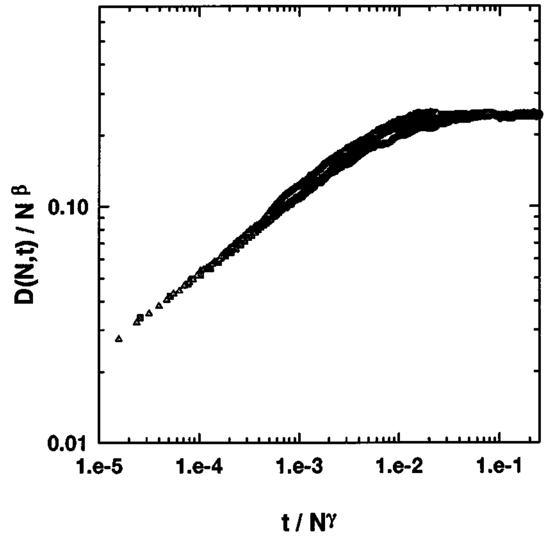

Fig. 4. Data collapse of $D(N,t)/N^{\beta}$ *versus* N^{γ} for the three curves of Figure 2.

$$\lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} \sim [(1-q)\lambda_q]^{\frac{1}{1-q}} t^{\frac{1}{1-q}} \quad (t \to \infty). \tag{7}$$

Therefore we can relate q to the δ exponent as

$$\delta = \frac{1}{1-q}. \tag{8}$$

In the present case this relation implies $q = -2.1$. The coefficient λ_q is the generalized Lyapunov exponent, and appears to satisfy [10,11] $K_q = \lambda_q$ if $\lambda_q \geq 0$ and $K_q = 0$ if $\lambda_q < 0$ (generalization, for arbitrary q, of the well known Pesin equality), where the generalized Kolmogorov-Sinai entropy K_q is defined [10,11] analogously to the usual Kolmogorov-Sinai entropy (K_1 herein). More precisely, in the same way K_1 essentially is the increase per unit time of the Boltzmann-Gibbs-Shannon entropy $S_1 \equiv -\sum_i p_i \ln p_i$, K_q essentially is the increase per unit time of the generalized, nonextensive, entropic form [13]

$$S_q \equiv \frac{1 - \sum_i p_i^q}{q-1} \quad (q \in \mathcal{R}). \tag{9}$$

This entropy (5) has led to a generalized thermostatistics [13,14] (which recovers the usual, extensive, Boltzmann-Gibbs statistics as the $q = 1$ particular case), and has received applications in a variety of situations such as self-gravitating systems [15], two-dimensional-like turbulence in pure-electron plasma [16,17], Lévy-like [18], correlated-like [19] anomalous diffusions, among others [20]. Recently it has been shown [11] that the index q can be related to the fractal dimension of the attractor at the onset of chaos in one dimensional maps. In particular, the Boltzmann Gibbs limit $q = 1$ corresponds to the case when the attractor has an Euclidean (nonfractal) dimension $d_{\rm f}$, while $q \neq 1$ reveals the existence of a fractal attractor. In this sense, it would be interesting to analyze in more detail the possible connections between sensibility to initial conditions in self critical systems and the fractal dimension of the SOC attractors. Since $(q-1)$ measures the degree of entropy nonextensivity ($S_q(\mathrm{A}+\mathrm{B}) = S_q(\mathrm{A}) + S_q(\mathrm{B}) + (1-q)S_q(\mathrm{A})S_q(\mathrm{B})$ for two independent systems A and B), it is an important index to be analyzed whenever discussing universality classes. Consistently, the determination of q for other SOC models would be very welcome. Indeed, it will provide an insight on the fractal nature of the attractor towards which the system is spontaneously driven.

Concluding, we have shown in this work that the SOC state of the BS model displays, like *the onset of chaos* in one dimensional maps, weak sensitivity to initial conditions, a property which is believed to be crucial for the evolutionary process [21].

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