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THEORETICAL ADVANCES

Entropy and similarity measure of Atanassov's intuitionistic fuzzy sets and their application to pattern recognition based on fuzzy measures

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Abstract In this study, we first examine entropy and similarity measure of Atanassov's intuitionistic fuzzy sets, and define a new entropy. Meanwhile, a construction approach to get the similarity measure of Atanassov's intuitionistic fuzzy sets is introduced, which is based on entropy. Since the independence of elements in a set is usually violated, it is not suitable to aggregate the values for patterns by additive measures. Based on the given entropy and similarity measure, we study their application to Atanassov's intuitionistic fuzzy pattern recognition problems under fuzzy measures, where the interactions between features are considered. To overall reflect the interactive characteristics between them, we define three Shapley-weighted similarity measures. Furthermore, if the information about the weights of features is incompletely known, models for the optimal fuzzy measure on feature set are established. Moreover, an approach to pattern recognition under Atanassov's intuitionistic fuzzy environment is developed.

Keywords Pattern recognition · Atanassov's intuitionistic fuzzy set · Entropy · Similarity measure · Fuzzy measure

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1 Introduction

Since the theory of fuzzy sets (FSs) is introduced by Zadeh [1], it has been successfully used in various fields. Later, several extension forms are proposed such as interval-valued fuzzy sets [2], Atanassov's intuitionistic fuzzy sets (IFSs) [3] and vague sets [4]. In 1996, Bustince and Burillo [5] showed the notion of vague sets coincides with that of IFSs. As an extension of FSs, IFSs are characterized by a membership degree, a non-membership degree and a hesitancy degree. So vagueness in real applications [6–13].

As two important information measures in the theory of fuzzy sets, entropy and similarity measure of IFSs have been widely investigated by many researchers from different point of views. Burillo and Bustince [14] introduced the notion of entropy of IFSs to measure the degree of intuitionism of an IFS. Bustince et al. [15] presented fuzzy entropy of IFSs by using the fuzzy implication operators. Szmidt and Kacprzyk [16] extended the axioms given by De Luca and Termini [17] and proposed another axiomatic definition for the entropy of IFSs. Szmidt and Kacprzyk [16], Wang and Lei [18] and Huang and Liu [19], respectively, gave an entropy of IFSs from different point of views, which are shown equivalently by Wei et al. [20]. On the other hand, the similarity measures of IFSs are also studied by many researchers [10, 21–25], whilst the application of similarity measures in digital image processing is considered in the literature [26, 27].

However, all the above researches are based on the assumption that the elements in a set are independent, and each intuitionistic fuzzy value (IFV) has the same importance. However, in many practical situations, the elements in a set are usually correlative [28, 29]. The fuzzy measure, introduced by Sugeno [30], has been shown a very effective tool for modeling the correlative characteristics among elements [31–35], and has been successfully used to deal with decision problems [36–41]. As far as we know, however, there is less investigation on entropy and similarity measure of Atanassov's intuitionistic fuzzy sets by using fuzzy measures. In the study we first introduce an entropy of IFSs, and give a construction method to obtain the similarity measure of IFSs. To overall reflect the interactive characteristics between features, we further define three Shapley-weighted similarity measures. If the information about the weights of features is partly known, models for the optimal fuzzy measure on feature set are constructed. Since the fuzzy measure is defined on the power set, it makes the problem exponentially complex. Thus, it is not easy to obtain the fuzzy measure of each combination in a set when it is large. The λ-fuzzy measure proposed by Sugeno [30] seems to well deal with this issue, which only needs n variables to determine a λ-fuzzy measure on a set with n elements. For this reason, we further research Atanassov's intuitionistic fuzzy pattern recognition problems under λ-fuzzy measures.

This paper is organized as follows: in Sect. 2, we review some basic concepts about IFSs, entropy and similarity measure. In Sect. 3, we propose a new entropy of IFSs, and present a construction method to obtain the similarity measure of IFSs by using entropy. Based on the Shapley function with respect to fuzzy measures, we propose three Shapley-weighted similarity measures, which reflect the interactive between elements. In Sect. 4, models for the optimal fuzzy measure on feature set are established, and an approach to pattern recognition under Atanassov's intuitionistic fuzzy environment is developed. To simplify the complexity of solving a fuzzy measure, we further study Atanassov's intuitionistic fuzzy pattern recognition under λ-fuzzy measures. Moreover, the corresponding examples are given to illustrate the developed procedure. In Sect. 5, the conclusions are made.

2 Preliminaries

2.1 Some basic concepts

By extending Zadeh's fuzzy sets, Atanassov [3] introduced the concept of Atanassov's intuitionistic fuzzy sets (IFSs) as follows:

Definition 1 [3] Let X be a no empty finite set. An IFS A in X is expressed as

$A = \{\langle x, u_A(x), v_A(x)\rangle | x \in X\}$,

where $u_A(x) \in [0,1]$ and $v_A(x) \in [0,1]$, respectively, denote the degrees of membership and non-membership of element $x \in X$ with the condition $u_A(x) + v_A(x) \leq 1$. The hesitancy degree is denoted by $\pi_A(x) = 1 - u_A(x) - v_A(x)$.

When $u_A(x) = 1 - v_A(x)$ for each $x \in X$, we get a fuzzy set, expressed by $A = \{\langle x, [u_A(x), 1 - u_A(x),]\rangle | x \in X\}$. The set of all IFSs in X is denoted by IFS(X).

Definition 2 Let $A = \{\langle x, u_A(x), v_A(x)\rangle | x \in X\}$ and $B = \{\langle x, u_B(x), v_B(x)\rangle | x \in X\}$ be two IFSs in X, then

(1) $A \subseteq B$ if and only if $u_A(x) \leq u_B(x), v_A(x) \geq v_B(x)$,
(2) $A = B$ if and only if $A \subseteq B$ and $A \supseteq B$,
(3) $A^C = \{\langle x, v_A(x), u_A(x)\rangle | x \in X\}$.

Definition 3 [16] A real-valued function E: IFS(X) $\rightarrow$ [0,1] is called an entropy measure of IFSs, if it satisfies the following axiomatic requirements:

(E1) $E(A) = 0$ if and only if A is a crisp set;
(E2) $E(A) = 1$ if and only if $u_A(x) = v_A(x)$ for each $x \in X$;
(E3) $E(A) = E(A^C)$;
(E4) $E(A) \leq E(B)$ if $A \subseteq B$ with $u_B(x) \leq v_B(x)$ for each $x \in X$, or $A \supseteq B$ with $u_B(x) \geq v_B(x)$ for each $x \in X$.

Definition 4 [23] A real-valued function S: IFS(X) $\times$ IFS(X) $\rightarrow$ [0,1] is called a similarity measure of IFS(X), if it satisfies the following conditions:

(S1) $0 \leq S(A,B) \leq 1$;
(S2) $S(A,B) = 1$ if and only if A = B;
(S3) $S(A,B) = S(B,A)$;
(S4) If $A \subseteq B \subseteq C$, then $S(A,C) \leq S(A,B)$ and $S(A,C) \leq S(B,C)$.

In the rest parts, without special explanation, we always assume that the universe X is a finite set, denoted by $\{x_1, x_2, \ldots, x_n\}$.

2.2 Several entropy of IFSs

Entropy, as an information measure, plays an important role in uncertain theory. Burillo and Bustince [5] defined the distance measure between Atanassov's intuitionistic fuzzy sets and gave an axiomatic definition of Atanassov's intuitionistic fuzzy entropy and a theorem which characterizes it. Furthermore, Burillo and Bustince [5] proposed the entropy that measures the distance from the considered set to IFSs rather than crisp sets, denoted by.

$$E_{BB}(A) = \sum_{i=1}^{n} (1 - \Phi(u_A(x), v_A(x))), \qquad (1)$$

where the function Φ:[0,1] $\times$ [0,1] $\rightarrow$ [0,1] satisfies the properties (1) $\Phi(x,y) = 1$ iff $x + y = 1$, and $\Phi(x,y) = 0$ iff $x = y = 0$; (2) $\Phi(x,y) = \Phi(y,x)$; (3) $\Phi(x,y) \leq \Phi(x',y')$ for $x \leq x', y \leq y'$.

Burillo and Bustince [5] gave us a good example of how to define an entropy measure from a theoretical point of view. The special cases given by Burillo and Bustince [5] did not consider hesitancy information.

Later, Szmidt and Kacprzyk [16] introduced an entropy measure of IFSs, which is based on the biggest cardinality (max-sigma-count) of IFSs, denoted by.

$$E_{SK}(A)=\frac{1}{n}\sum_{i=1}^{n}\frac{\max\mathrm{Count}(\mathrm{A_i}\cap \mathrm{A_i^C})}{\max\mathrm{Count}(A_i\cup A_i^C)},\tag{2}$$

where for each i, A_i denotes the single-element IFS corresponding to the element $x_i\in X$, described as $A_i=\langle x_i,u_A(x_i),v_A(x_i)\rangle$, and $\max\mathrm{Count}(A_i\cap A_i^C)=\langle x_i,\min\{u_A(x_i),v_A(x_i)\},\max\{u_A(x_i),v_A(x_i)\}\rangle$, $\max\mathrm{Count}(A_i\cup A_i^C)=\langle x_i,\max\{u_A(x_i),v_A(x_i)\},\min\{u_A(x_i),v_A(x_i)\}\rangle$.

Later, Huang and Liu [19] proposed an entropy measure of vague sets. Since the vague sets are IFSs [5]. When we apply it in the setting of IFSs, for any $A\in \mathrm{IFS}(X)$, it can be transformed as follows [20]:

$$E_{HL}(A)=\frac{1}{n}\sum_{i=1}^{n}\frac{1-|u_A(x_i)-v_A(x_i)|+\pi_A(x_i)}{1+|u_A(x_i)-v_A(x_i)|+\pi_A(x_i)}.\tag{3}$$

Recently, Wang and Lei [18] gave another entropy measure of IFSs, for any $A\in \mathrm{IFS}(X)$, defined by.

$$E_{WL}(A)=\frac{1}{n}\sum_{i=1}^{n}\frac{\min\{\mathrm{u_A(x_i)},\mathrm{v_A(x_i)}\}+\pi_{\mathrm{A}}(\mathrm{x_i})}{\max\{u_A(x_i),v_A(x_i)\}+\pi_A(x_i)}.\tag{4}$$

Wei et al. [20] pointed out that the entropy measures defined by Szmidt and Kacprzyk [16], Wang and Lei [18] and Huang and Liu [19] are equivalent, which can be seen as the arithmetic mean of the ratio of each corresponding item.

Different from the above-mentioned entropy, Li et al. [42] gave another entropy measure of IFSs, expressed by:

$$E_L(A)=\frac{\sum_{i=1}^{n}(u_A(x_i)\wedge v_A(x_i))}{\sum_{i=1}^{n}(u_A(x_i)\vee v_A(x_i))}.\tag{5}$$

E_L is the ratio of the sum of different corresponding items, but it does not consider the hesitancy information.

2.3 Similarity measures of IFSs

Similarity measure, as another important information measure, is applied to denote the similarity degree of fuzzy sets and has received considerable attention. Li and Cheng [23] introduced a concept of similarity measure of IFSs and applied it to pattern recognition. Later, Liang and Shi [24] pointed out that Li and Cheng's method is not always reasonable in some examples, and gave an improved entropy of IFSs, defined by:

$$S_{LS}(A,B)=1-\sqrt[p]{\frac{\sum_{i=1}^{n}(f_{u_{AB}}(i)+f_{v_{AB}}(i))^p}{n}},\tag{6}$$

where $f_{v_{AB}}(i)=|(1-v_A(x_i))/2-(1-v_B(x_i))/2|$, $f_{u_{AB}}(i)=|u_A(x_i)-u_B(x_i)|/2$, and $A,B\in \mathrm{IFS}(X)$.

Mitchell [10] adopted a statistical approach and interpreted IFSs as ensembles of ordered fuzzy sets to modify Li and Cheng's similarity measure. Let $S_u(A,B)$ and $S_v(A,B)$ denote the similarity measures between the "low" membership functions u_A and u_B and between the "high" membership functions $1-v_A$ and $1-v_B$, respectively, as follows:

$$S_u(A,B)=1-\sqrt[p]{\frac{\sum_{i=1}^{n}|u_A(x_i)-u_B(x_i)|^p}{n}},$$

$$S_v(A,B)=1-\sqrt[p]{\frac{\sum_{i=1}^{n}|v_A(x_i)-v_B(x_i)|^p}{n}}.$$

They then defined the following modified similarity measure of IFSs.

$$S_M(A,B)=\frac{1}{2}(S_u(A,B)+S_v(A,B))\tag{7}$$

for IFSs $A,B\in \mathrm{IFS}(X)$.

Based on the Hausdorff distance, Huang and Yang [43] defined three similarity measures of IFSs, denoted by

$$S_{HY}^1(A,B)=1-d_H(A,B),\tag{8}$$

$$S_{HY}^2(A,B)=\frac{e^{-d_H(A,B)}-e^{-1}}{1-e^{-1}},\tag{9}$$

$$S_{HY}^3(A,B)=\frac{1-d_H(A,B)}{1+d_H(A,B)},\tag{10}$$

where $d_H(A,B)=\frac{1}{n}\sum_{i=1}^{n}\max\{|u_A(x_i)-u_B(x_i)|,|v_A(x_i)-v_B(x_i)|\}$ and $A,B\in \mathrm{IFS}(X)$.

3 New entropy and similarity measure of IFSs

Based on analysis above, this section gives a new entropy and similarity measure of IFSs.

3.1 A new entropy of IFSs

Based on the definitions of IFSs and entropy, we introduce the following entropy of IFSs:

$$GE(A)=\frac{\sum_{i=1}^{n}f_1(u_A(x_i),v_A(x_i))}{\sum_{i=\cdot}^{n}f_2(u_A(x_i),v_A(x_i))},$$

where $A\in \mathrm{IFS}(X)$, the functions $f_1:[0,1]\times[0,1]\to[0,1]$ and $f_2:[0,1]\times[0,1]\to[0.5,1]$ satisfy the properties (i) $f_1(x,y)=0$ and $f_2(x,y)=1$ iff $x=0$, $y=1$, or $x=1$, $y=0$; (ii) $f_1(x,y)=f_2(x,y)$ iff $x=y$, otherwise, $f_1(x,y)<f_2(x,y)$; (iii) $f_k(x,y)=f_k(y,x)$, $k=1,\ 2$; (iv) $\frac{f_1(x,y)}{f_2(x,y)}\le\frac{f_1(x',y')}{f_2(x',y')}$ for $x\le x'\le y'\le y$, or $y\le y'\le x'\le x$.

Theorem 1 *The mapping E: IFS(X) →[0,1], defined by GE, is an entropy of IFSs.*

Proof To prove GE is an entropy of IFSs, it only needs to show that GE satisfies (E1)–(E4) given in Definition 3.

(E1): If A is a crisp set, then we have $u_A(x_i)=0$, $v_A(x_i)=1$ or $u_A(x_i)=1$, $v_A(x_i)=0$. It gets $f_1(u_A(x_i),v_A(x_i))=0$ and $f_2(u_A(x_i),v_A(x_i))=1$ for each $x_i\in X$. Thus, $GE(A)=0$. On the other hand, if $GE(A)=0$, by f_1:[0,1] × [0,1] → [0,1] and f_2:[0,1] × [0,1] → [0.5,1], it has $f_1(u_A(x_i),v_A(x_i))=0$ for each $x_i\in X$. Namely, $u_A(x_i)=0$, $v_A(x_i)=1$ or $u_A(x_i)=1$, $v_A(x_i)=0$. Thus, A is a crisp set.

(E2): When $u_A(x_i)=v_A(x_i)$ for each $x_i\in X$, we have $f_1(u_A(x_i),v_A(x_i))=f_2(u_A(x_i),v_A(x_i))$. Then, $GE(A)=1$. On the other hand, suppose that $GE(A)=1$. Since $f_1(u_A(x_i),v_A(x_i))\le f_2(u_A(x_i),v_A(x_i))$ for each $x_i\in X$, we get $f_1(u_A(x_i),v_A(x_i))=f_2(u_A(x_i),v_A(x_i))$. Namely, $u_A(x_i)=v_A(x_i)$, $x_i\in X$.

(E3): From $A^C=\{\langle x_i,v_A(x_i),u_A(x_i)\rangle | x_i\in X\}$ and $f_k(x,y)=f_k(y,x)$, $k=1,2$, one easily gets $GE(A)=GE(A^C)$.

(E4): When $A\subseteq B$ and $u_B(x_i)\le v_B(x_i)$ for each $x_i\in X$, we have

$$u_A(x_i)\le u_B(x_i)\le v_B(x_i)\le v_A(x_i)$$

for each $x_i\in X$.

Thus,

$$\frac{f_1(u_A(x_i),v_A(x_i))}{f_2(u_A(x_i),v_A(x_i))}\le\frac{f_1(u_B(x_i),v_B(x_i))}{f_2(u_B(x_i),v_B(x_i))}\ \forall x_i\in X.$$

By GE, it gets $GE(A)\le GE(B)$.

Similarly, when $A\supseteq B$ with $u_B(x_i)\ge v_B(x_i)$ for each $x_i\in X$, one can also prove $GE(A)\le GE(B)$.

Next, let us pay more attention to a special case. To get more information on IFSs, combining the entropy given by Szmidt and Kacprzyk [16], Wang and Lei [18], Huang and Liu [19] and Li et al. [42], we define the following entropy of IFSs:

$$E_N(A)=\frac{\sum_{i=1}^{n}(u_A(x_i)\wedge v_A(x_i)+\pi_A(x_i))}{\sum_{i=1}^{n}(u_A(x_i)\vee v_A(x_i)+\pi_A(x_i))},\tag{11}$$

where $A\in$ IFS(X).

In E_N, when we delete the hesitancy information of each element, it reduces to be the entropy given by Li et al. [42].

Corollary 1 The mapping E: IFS(X)→[0,1], defined by E_N, is an entropy of IFSs.

Theorem 2 *Let E_N: IFS(X) →[0,1] be an entropy measure given as (11), then E_N can be equivalently expressed by.*

$$E'(A)=\frac{\sum_{i=1}^{n}\max \mathrm{Count}(A_i\cap A_i^C)}{\sum_{i=1}^{n}\max \mathrm{Count}(A_i\cup A_i^C)}\tag{12}$$

and

$$E''(A)=\frac{\sum_{i=1}^{n}(1-|u_A(x_i)-v_A(x_i)|+\pi_A(x_i))}{\sum_{i=1}^{n}(1+|u_A(x_i)-v_A(x_i)|+\pi_A(x_i))}\tag{13}$$

for any $A\in$ *IFS(X).*

Proof We first show (11) can be equivalently expressed by (12). For (12): By.

$$\max Count(A_i)=u_A(x_i)+\pi_A(x_i)$$

for any $i=1,2,\ldots,n$, it has.

$$E_N(A)=\frac{\sum_{i=1}^{n}(u_A(x_i)\wedge v_A(x_i)+\pi_A(x_i))}{\sum_{i=1}^{n}(u_A(x_i)\vee v_A(x_i)+\pi_A(x_i))}$$
$$=\frac{\sum_{i=1}^{n}(\min\{u_A(x_i),v_A(x_i)\}+\pi_A(x_i))}{\sum_{i=1}^{n}(\max\{u_A(x_i),v_A(x_i)\}+\pi_A(x_i))}$$
$$=\frac{\sum_{i=1}^{n}\max\mathrm{Count}(A_i\cap A_i^C)}{\sum_{i=1}^{n1}\max\mathrm{Count}(A_i\cup A_i^C)}$$

For (13): suppose that $u_A(x_i)\ge v_A(x_i)$ for some x_i, then.

$$1-|u_A(x_i)-v_A(x_i)|+\pi_A(x_i)=2(v_A(x_i)+\pi_A(x_i))$$
$$=2(u_A(x_i)\wedge v_A(x_i)+\pi_A(x_i)),$$
$$1+|u_A(x_i)-v_A(x_i)|+\pi_A(x_i)=2(u_A(x_i)+\pi_A(x_i))$$
$$=2(u_A(x_i)\vee v_A(x_i)+\pi_A(x_i)).$$

Similarly, when $u_A(x_i)\le v_A(x_i)$ for some x_i, then.

$$1-|u_A(x_i)-v_A(x_i)|+\pi_A(x_i)=2(u_A(x_i)+\pi_A(x_i))$$
$$=2(u_A(x_i)\wedge v_A(x_i)+\pi_A(x_i)),$$
$$1+|u_A(x_i)-v_A(x_i)|+\pi_A(x_i)=2(v_A(x_i)+\pi_A(x_i))$$
$$=2(u_A(x_i)\vee v_A(x_i)+\pi_A(x_i)).$$

From (13), it gets.

$$E''(A)=\frac{\sum_{i=1}^{n}(1-|u_A(x_i)-v_A(x_i)|+\pi_A(x_i))}{\sum_{i=1}^{n}(1+|u_A(x_i)-v_A(x_i)|+\pi_A(x_i))}$$
$$=\frac{\sum_{i=1}^{n}2(u_A(x_i)\wedge v_A(x_i)+\pi_A(x_i))}{\sum_{i=1}^{n}2(u_A(x_i)\vee v_A(x_i)+\pi_A(x_i))}$$
$$=\frac{\sum_{i=1}^{n}(u_A(x_i)\wedge v_A(x_i)+\pi_A(x_i))}{\sum_{i=1}^{n}(u_A(x_i)\vee v_A(x_i)+\pi_A(x_i))}.$$

That's $E_N=E''$.

3.2 A new similarity measure of IFSs

Different from the similarity measures of IFSs introduced in Sect. 2.3, we here give a construction approach to get the similarity measure of IFSs by using entropy.

Let A, $B\in$ IFS(X). For each $x_i\in X$, define

$$u_{AB}(x_i) = \frac{1 + \min\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}}{2}, \tag{14}$$

$$v_{AB}(x_i) = \frac{1 - \max\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}}{2}. \tag{15}$$

Let

$G(A, B) = \{\langle x_i, (u_{AB}(x_i), v_{AB}(x_i))\rangle | x_i \in X\}$.

It is not difficult to get $G(A,B) \in$ IFS(X).

Theorem 3 *Let E be an entropy of IFS(X), then the mapping S: IFS(X) × IFS(X) →[0,1], defined by S(A,B) = E(G(A,B)) for each pair of IFSs A and B, is a similarity measure of IFSs.*

Proof **(S1)**: Since $E(A) \in [0, 1]$ for any $A \in$ IFS(X), and $G(A,B)$ is an IFS in X, we have $E(G(A, B)) \in [0, 1]$.

(S2): From the definition of the entropy measure of IFSs, we have $E(G(A, B)) = 1$ if and only if $u_{AB}(x_i) = v_{AB}(x_i)$ for each $x_i \in X$. From (14) and (15), we know $u_A(x_i) = u_B(x_i) = v_A(x_i) = v_B(x_i) = 0$ for each $x_i \in X$. Thus, $A = B$.

(S3): from the construction of $G(A,B)$, it is obvious that $G(A,B) = G(B,A)$. Thus, $E(G(A, B)) = E(G(B, A))$.

(S4): when $A \subseteq B \subseteq C$, it has $u_A(x_i) \le u_B(x_i) \le u_C(x_i)$ and $v_A(x_i) \ge v_B(x_i) \ge v_C(x_i)$ for each $x_i \in X$. Namely, $|u_A(x_i) - u_C(x_i)| \ge |u_A(x_i) - u_B(x_i)|$ and $|v_A(x_i) - v_C(x_i)| \ge |v_A(x_i) - v_B(x_i)|$ for each $x_i \in X$. Thus, $G(A, B) \subseteq G(A, C)$.

Since $u_{AB}(x_i) \ge \frac{1}{2} \ge v_{AB}(x_i)$ for each $x_i \in X$, it follows from Definition 3 that $E(G(A, B)) \ge E(G(A, C))$.

Similarly, one can also prove $E(G(B, C)) \ge E(G(A, C))$.

Corollary 2 Let E_N be an entropy measure of IFSs defined by (11), then the mapping S_N, given in Theorem 3, i.e., $S_N(A, B) = E_N(G(A, B))$ for each pair of IFSs A and B, is a similarity measure and can be denoted by

$$S_N(A, B) = \frac{\sum_{i=1}^{n} (1 - \min\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\})}{\sum_{i=1}^{n} (1 + \max\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\})}.$$

Corollary 3 Let E_L be an entropy measure of IFSs defined by (5), then the mapping S_L, given in Theorem 3, i.e., $S_L(A, B) = E_L(G(A, B))$ for each pair of IFSs A and B, is a similarity measure and can be denoted by

$$S_L(A, B) = \frac{\sum_{i=1}^{n} (1 - \max\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\})}{\sum_{i=1}^{n} (1 + \min\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\})}.$$

Corollary 4 Let E_{SK} be an entropy measure of IFSs defined by (2), then the mapping S_{SK}, given in Theorem 3, i.e., $S_{SK}(A, B) = E_{SK}(G(A, B))$ for each pair of IFSs A and B, is a similarity measure and can be denoted by

$$S_{SK}(A, B) = \frac{1}{n}\sum_{i=1}^{n} \frac{1 - \min\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}}{1 + \max\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}}.$$

From Corollaries 2–4, we know they are more easily calculated than some existing similarity measures [10, 24], as well as consider more information than some existing similarity measures [43].

Example 1 Assume that there are four kinds of minerals $A = \{A_1, A_2, A_3, A_4\}$, and a recognized sample ε, which are represented by IFSs in the feature space $C = \{c_1, c_2, c_3\}$. Suppose we have the following data.

$A_1 = \{\langle c_1, 0.4, 0.4\rangle, \langle c_2, 0.3, 0.5\rangle, \langle c_3, 0.6, 0.3\rangle\}$,

$A_2 = \{\langle c_1, 0.2, 0.6\rangle, \langle c_2, 0.3, 0.5\rangle, \langle c_3, 0.4, 0.6\rangle\}$,

$A_3 = \{\langle c_1, 0.2, 0.4,\rangle, \langle c_2, 0.4, 0.5\rangle, \langle c_3, 0.3, 0.5\rangle\}$,

$A_4 = \{\langle c_1, 0.2, 0.5\rangle, \langle c_2, 0.4, 0.4,\rangle, \langle c_3, 0.4, 0.3\rangle\}$,

$\varepsilon = \{\langle c_1, 0.2, 0.6\rangle, \langle c_2, 0.2, 0.5\rangle, \langle c_3, 0.4, 0.3\rangle\}$.

Calculate the similarity measure between A_i (i = 1,2,3,4) and ε, the results are presented in Table 1.

From the boldface letters in Table 1, it shows that the sample ε belongs to the pattern A_4 according to the similarity measures (8)–(10) and S_L. Furthermore, the sample ε belongs to the pattern A_2 according to the similarity measures S_N and S_{SK}. However, the similarity measures S_{LS} and S_M cannot classify this sample.

3.3 The Shapley-weighted similarity measures of IFSs

Although there are many similarity measures of IFSs, they are all based on the assumptions that the elements in a set are independent, and each feature has the same importance. In most situations, these assumptions do not hold, for example, we give the following classical example: "we are to evaluate a set of different brands of cars in relation to

Table 1 The results with respect to the different similarity measures (p = 1)

	S_N	S_L	S_{SK}	S_{LS}	S_M	S_{HY}^1	S_{HY}^2	S_{HY}^3
A_1	0.80	0.78	0.80	0.77	0.88	0.83	0.76	0.71
A_2	**0.88**	0.86	**0.89**	**0.87**	**0.93**	0.87	0.80	0.76
A_3	0.81	0.77	0.81	0.77	0.88	0.80	0.71	0.67
A_4	0.87	**0.87**	0.88	**0.87**	**0.93**	**0.90**	**0.85**	**0.82**

three subjects: {security, service, price}, we want to give more importance to security than to service or price, but on the other hand we want to give some advantage to cars that are good in security and in any of service and price". In this situation, it is not suitable to endow their weights by using additive measures. Fuzzy measures [30] as a powerful tool for modeling the interaction among elements can well deal with this issue.

Definition 5 [30] A fuzzy measure on a finite set $N = \{1,2,\ldots,n\}$ is a set function μ: $P(N) \to [0, 1]$ satisfying $\mu(\phi) = 0$, $\mu(N) = 1$, If $A, B \in P(N)$ and $A \subseteq B$ then $\mu(A) \le \mu(B)$,

where $P(N)$ denotes the power set of N.

In the pattern recognition, $\mu(A)$ can be viewed as the importance degree of feature set A. Especially, if $A = \{i\}$, then $\mu(i)$ is the importance degree of the feature i. When we have $\mu(A) = \sum_{i\in A} \mu(i)$ for any $A \in P(N)$, the fuzzy measure μ degenerates to be an additive measure.

When there are inter-dependent or interactive phenomena among features, the importance of each feature is not only determined by itself, but also receives the influence from other features. In order to overall reflect the interaction between features, we shall use their Shapley values as their weights. The Shapley function [44] as one of the most important payoff indices has been deeply researched in game theory, which satisfies several reasonable axioms, denoted by

$$\varphi_i(\mu,N) = \sum_{S\subseteq N\backslash i} \frac{(n-s-1)!s!}{n!}(\mu(S\cup i) - \mu(S)),\ \forall i \in N \tag{16}$$

where μ is a fuzzy measure as given in Definition 5, s and n denote the cardinalities of S and N, respectively.

From Definition 5 and the Shapley function, it is not difficult to get $\varphi_i(\mu,N) \ge 0$ for each $i \in N$ and $\sum_{i=1}^{n} \varphi_i(\mu,N) = \mu(N)$. Thus, $\{\varphi_i(\mu,N)\}_{i\in N}$ is a weight vector. Further, if the fuzzy measure μ is an additive measure, namely, there is no interaction between features; their Shapley values are equal to the importance of themselves. That's $\varphi_i(\mu,N) = \mu(i)$ for any $i \in N$. Based on the introduced similarity measures in Sect. 3.2, we give the following Shapley-weighted similarity measures of IFSs.

$$S_{SN}(A,B) = \frac{\sum_{i=1}^{n} \varphi_i(\mu,N)(1 - \min\{|\mathrm{u_A(x_i) - u_B(x_i)}|, |\mathrm{v_A(x_i) - v_B(x_i)}|\})}{\sum_{i=1}^{n} \varphi_i(\mu,N)(1 + \max\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\})} \tag{17}$$

$$S_{SL}(A,B) = \frac{\sum_{i=1}^{n} \varphi_i(\mu,N)(1 - \max\{|\mathrm{u_A(x_i) - u_B(x_i)}|, |\mathrm{v_A(x_i) - v_B(x_i)}|\})}{\sum_{i=1}^{n} \varphi_i(\mu,N)(1 + \min\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\})} \tag{18}$$

$$S_{SSK}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \varphi_i(\mu,N)\frac{1 - \min\{|\mathrm{u_A(x_i) - u_B(x_i)}|, |\mathrm{v_A(x_i) - v_B(x_i)}|\}}{1 + \max\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}} \tag{19}$$

where $\varphi_i(\mu,N)$ as given (16).

Similarly, we have the following Shapley-weighted similarity measures of IFSs.

(1) The Shapley-weighted similarity measure [24]

$$S_{SLS}(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n} (\varphi_i(\mu,N)(f_{u_{AB}}(i) + f_{v_{AB}}(i)))^p}{n}}, \tag{20}$$

where $f_{u_{AB}}(i)$ and $f_{v_{AB}}(i)$ as given in (6).

(2) The Shapley-weighted similarity measure [10]

$$S_{SM}(A,B) = \frac{1}{2}(S_{Su}(A,B) + S_{Sv}(A,B)), \tag{21}$$

where

$$S_u(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n} (\varphi_i(\mu,N)|u_A(x_i) - u_B(x_i)|)^p}{n}},$$

$$S_v(A,B) = 1 - \sqrt[p]{\frac{\sum_{i=1}^{n} (\varphi_i(\mu,N)|v_A(x_i) - v_B(x_i)|)^p}{n}}.$$

(3) The Shapley-weighted similarity measures [43]

$$S_{SHY1}(A,B) = 1 - d_{SH}(A,B), \tag{22}$$

$$S_{SHY2}(A,B) = \frac{e^{-d_{SH}(A,B)} - e^{-1}}{1 - e^{-1}}, \tag{23}$$

$$S_{SHY3}(A,B) = \frac{1 - d_{SH}(A,B)}{1 + d_{SH}(A,B)}, \tag{24}$$

where $d_{SH}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \varphi_i(\mu,N)\max\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}$ and $A, B \in$ IFS(X).

If there is no interaction among elements in N, then we get their corresponding weighted similarity measures.

4 Approaches to pattern recognition based on fuzzy measures

4.1 A general case

If the fuzzy measure of each combination in feature set is given, then we give the following decision procedure to pattern recognition under Atanassov's intuitionistic fuzzy environment.

Step 1: Suppose that there exist m patterns $A = \{A_1, A_2,\ldots,A_m\}$ and n features $C = \{c_1,c_2,\ldots,c_n\}$. The evaluation of each pattern A_i w.r.t. each feature c_j is an Atanassov's IFV

$$A_i = \{\langle c_j, a_{ij}, b_{ij}\rangle | j = 1,2,\ldots,n\} \quad i = 1,2,\ldots,m.$$

Furthermore, assume that there is a sample ε to be recognized, which is represented by an IFS $\varepsilon = \{\langle c_j, e_j, f_j \rangle | j = 1, 2, \ldots, n\}$.

Step 2: Calculate the Shapley value of each feature using (16).

Step 3: Calculate the Shapley-weighted similarity measure between A_i $(i = 1,2,\ldots,m)$ and ε using (17) or (18, 19), and then select the best one.

Step 4: End.

According to the entropy theory, if the entropy value for a feature is small across patterns, it can provide decision-makers with useful information. Therefore, the feature should be assigned a bigger weight; otherwise, such a feature will be judged unimportant by most decision-makers. In other words, such a feature should be evaluated as a very small weight. If the information about the weights of features is incompletely known, the following linear programming model for the optimal λ-fuzzy measure on feature set C is built.

$$\min \sum_{j=1}^{n} \sum_{i=1}^{m} E(A_{ij}) \varphi_j(\mu, C)$$

$$\text{s.t.} \begin{cases} \mu(\emptyset) = 0, \quad \mu(C) = 1, \\ \mu(c_j) \in H_j, \quad j = 1, 2, \ldots, n, \\ \mu(S) \le \mu(T)\ \forall S, \quad T \subseteq C,\ S \subseteq T, \end{cases} \tag{25}$$

where $\varphi_j(\mu, C)$ is the Shapley value of the feature c_j, and $H_j = [h_j^-, h_j^+]$ is its range.

Since (28) is a linear programming, we can easily get its solution by using Simplex method. If there are no interactive characteristics among elements in a set, then we get the corresponding model for the optimal weight vector.

Example 2 In Example 1, if the importance of features is different, which is, respectively, given by [0.4,0.6], [0.3,0.5] and [0.6,0.8]. Then the main steps are given as follows:

Step 1: Calculate the fuzzy measures of all combinations in feature set C. Let $E = E_{BB}$ and $\Phi(u_A(x), v_A(x)) = u_A(x) + v_A(x)$. From the model (25), the following linear programming model is established.

$$\min 0.12(\mu^{BB}(c_1) - \mu^{BB}(c_2, c_3)) - 0.03(\mu^{BB}(c_2) - \mu^{BB}(c_1, c_3)) \ - 0.08(\mu^{BB}(c_3) - \mu^{BB}(c_1, c_2)) + 0.73$$

$$\text{s.t.} \begin{cases} \mu^{BB}(S) \le \mu^{BB}(T)\ S, T \subseteq \{c_1, c_2, c_3\},\ S \subseteq T \\ \mu^{BB}(c_1) \in [0.4, 0.6], \mu^{BB}(c_2) \in [0.3, 0.5], \\ \mu^{BB}(c_3) \in [0.6, 0.8] \end{cases} .$$

Solve above linear programming, it gets

$\mu^{BB}(c_1) = \mu^{BB}(c_2) = \mu^{BB}(c_1, c_2) = 0.4, \mu^{BB}(c_3) = \mu^{BB}(c_1, c_3) = 0.8, \mu^{BB}(c_2, c_3) = \mu^{BB}(c_1, c_2, c_3) = 1;$

Step 2: By (16), calculate the feature Shapley values w.r.t. μ^{BB}, it has

$$\varphi_1(\mu^{BB}, C) = 0.13, \varphi_2(\mu^{BB}, C) = 0.23, \varphi_3(\mu^{BB}, C) = 0.63;$$

Step 3: Calculate the Shapley-weighted similarity measure between A_i $(i = 1,2,3,4)$ and ε, the results are presented in Table 2.

Similarly, when $E = E_N$ or $E = E_L$, the results are presented in Table 3.

Furthermore, when $E = E_{SK}$, the results are presented in Table 4.

According to Tables 2, 3 and 4, the different ranking results are obtained by using the Shapley-weighted similarity measures (17)–(24). But all ranking results show that the recognized sample ε belongs to the fourth kind of minerals (A_4), see the boldface letters in Tables 2, 3 and 4.

Example 3 [45] Let us consider a set of diagnoses $Q = \{Q_1(\text{Viral fever}), Q_2(\text{Malaria}), Q_3(\text{Typhoid})\}$, and a set of symptoms $S = \{s_1(\text{Temperature}), s_2(\text{Headache}), s_3(\text{Cough})\}$. Suppose a patient, with respect to all the symptoms, can be represented by the following IFS:

Table 2 The results for the Shapley-weighted similarity measures ($p = 1$)

	$S_{SN}^{\mu^{BB}}$	$S_{SL}^{\mu^{BB}}$	$S_{SSK}^{\mu^{BB}}$	$S_{SLS}^{\mu^{BB}}$	$S_{SM}^{\mu^{BB}}$	$S_{SHY1}^{\mu^{BB}}$	$S_{SHY2}^{\mu^{BB}}$	$S_{SHY3}^{\mu^{BB}}$
A_1	0.83	0.80	0.27	0.97	0.97	0.94	0.91	0.89
A_2	0.82	0.79	0.27	0.96	0.96	0.93	0.89	0.87
A_3	0.78	0.85	0.29	0.96	0.96	0.93	0.90	0.88
A_4	**0.92**	**0.92**	**0.32**	**0.99**	**0.99**	**0.98**	**0.97**	**0.96**

Table 3 The results for the Shapley-weighted similarity measures ($p = 1$)

	S_{SN}^{μ}	S_{SL}^{μ}	S_{SSK}^{μ}	S_{SLS}^{μ}	S_{SM}^{μ}	S_{SHY1}^{μ}	S_{SHY2}^{μ}	S_{SHY3}^{μ}
A_1	0.76	0.74	0.26	0.95	0.95	0.94	0.90	0.88
A_2	0.87	0.86	0.30	0.98	0.98	0.95	0.93	0.91
A_3	0.80	0.77	0.29	0.96	0.96	0.93	0.90	0.87
A_4	**0.93**	**0.93**	**0.32**	**0.99**	**0.99**	**0.98**	**0.97**	**0.96**

Table 4 The results for the Shapley-weighted similarity measures ($p = 1$)

	$S_{SN}^{\mu^{SK}}$	$S_{SL}^{\mu^{SK}}$	$S_{SSK}^{\mu^{SK}}$	$S_{SLS}^{\mu^{SK}}$	$S_{SM}^{\mu^{SK}}$	$S_{SHY1}^{\mu^{SK}}$	$S_{SHY2}^{\mu^{SK}}$	$S_{SHY3}^{\mu^{SK}}$
A_1	0.81	0.77	0.27	0.96	0.96	0.94	0.90	0.88
A_2	0.83	0.8	0.28	0.97	0.97	0.93	0.90	0.87
A_3	0.78	0.86	0.26	0.96	0.96	0.93	0. 90	0.87
A_4	**0.95**	**0.95**	**0.32**	**0.99**	**0.99**	**0.99**	**0.98**	**0.97**

P (patient) = {<s_1, 0.6, 0.1>, <s_2, 0.5, 0.4>, <s_3, 0.7, 0.2>};

And assume each diagnosis Q_i (i = 1,2,3) can also be viewed as an IFS with respect to all the symptoms as follows:

Q_1 (Viral fever) = {<s_1, 0.4, 0>, <s_2, 0.3, 0.5>, <s_3, 0.4, 0.3>};

Q_2 (Malaria) = {<s_1, 0.7, 0>, <s_2, 0.2, 0.6>, <s_3, 0.7, 0>};

Q_3 (Typhoid) = {<s_1, 0.3, 0.3>, <s_2, 0.6, 0.1> , <s_3, 0.2, 0.6>}.

The importance of symptoms is different, which is, respectively, given by [0.4,0.5], [0.3,0.4] and [0.2,0.3]. Our aim is to classify the patient P to one of the diagnoses Q_1, Q_2 and Q_3. Similar to Example 2, if $E = E_{BB}$, $E = E_N$ or $E = E_{SK}$ the results are presented in Table 5.

When $E = E_L$, the results are presented in Table 6.

According to Table 5, the same ranking results are obtained, and from Table 6, we get the different ranking results. However, all ranking results show that the patient belongs to the diagnosis Q_2 (Malaria), see the boldface letters in Tables 5 and 6.

4.2 A special case

As we know, the fuzzy measure is defined on the power set, which makes the problem exponentially complex. Thus, it is not easy to get the fuzzy measure of each combination in a set when it is large. For this reason, we further research pattern recognition under λ-fuzzy measures, which will largely simplify the complexity of solving a fuzzy measure.

Definition 6 [30] Let g_λ: $P(N) \to [0, 1]$ be a fuzzy measure.g_λ is called a λ-fuzzy measure if

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B)$$

for any $A, B \subseteq N$ with $A \cap B = \emptyset$, where $\lambda > -1$.

Table 5 The results for the Shapley-weighted similarity measures (p = 1)

	S^{μ}_{SN}	S^{μ}_{SL}	S^{μ}_{SSK}	S^{μ}_{SLS}	S^{μ}_{SM}	S^{μ}_{SHY1}	S^{μ}_{SHY2}	S^{μ}_{SHY3}
Q_1	0.73	0.69	0.24	0.94	0.94	0.92	0.88	0.85
Q_2	**0.73**	**0.70**	**0.24**	**0.94**	**0.94**	**0.92**	**0.88**	**0.86**
Q_3	0.55	0.50	0.19	0.90	0.90	0.87	0.81	0.77

Table 6 The results for the Shapley-weighted similarity measures (p = 1)

	$S^{\mu^L}_{SN}$	$S^{\mu^L}_{SL}$	$S^{\mu^L}_{SSK}$	$S^{\mu^L}_{SLS}$	$S^{\mu^L}_{SM}$	$S^{\mu^L}_{SHY1}$	$S^{\mu^L}_{SHY2}$	$S^{\mu^L}_{SHY3}$
Q_1	0.74	0.71	0.25	0.84	0.84	0.93	0.89	0.87
Q_2	**0.77**	**0.76**	**0.26**	**0.96**	**0.96**	**0.95**	**0.92**	**0.90**
Q_3	0.59	0.55	0.20	0.91	0.91	0.89	0.84	0.80

Table 7 The results for the Shapley-weighted similarity measures (p = 1)

	$S^{g_\lambda}_{SN}$	$S^{g_\lambda}_{SL}$	$S^{g_\lambda}_{SSK}$	$S^{g_\lambda}_{SLS}$	$S^{g_\lambda}_{SM}$	$S^{g_\lambda}_{SHY1}$	$S^{g_\lambda}_{SHY2}$	$S^{g_\lambda}_{SHY3}$
A_1	0.80	0.78	0.27	0.96	0.96	0.94	0.91	0.89
A_2	0.86	0.84	0.29	0.97	0.97	0.95	0.92	0.90
A_3	0.80	0.77	0.27	0.96	0.96	0.93	0.90	0.87
A_4	**0.90**	**0.90**	**0.30**	**0.98**	**0.98**	**0.97**	**0.96**	**0.95**

Table 8 The results for the Shapley-weighted similarity measures (p = 1)

	S^{μ}_{SN}	S^{μ}_{SL}	S^{μ}_{SSK}	S^{μ}_{SLS}	S^{μ}_{SM}	S^{μ}_{SHY1}	S^{μ}_{SHY2}	S^{μ}_{SHY3}
Q_1	0.73	0.70	0.24	0.95	0.94	0.92	0.88	0.86
Q_2	**0.74**	**0.72**	**0.25**	**0.95**	**0.95**	**0.93**	**0.90**	**0.87**
Q_3	0.58	0.53	0.20	0.91	0.91	0.88	0.82	0.79

It is evident if $\lambda = 0$, then g_λ is an additive measure, which means there is no interaction between subsets A and B. If $\lambda > 0$, then g_λ is a superadditive measure, which indicates there exists complementary interaction between subsets A and B. If $-1 < \lambda < 0$, then $g_\lambda$is a subadditive measure, which shows there exists redundancy interaction between subsets A and B.

For a finite set N, the λ-fuzzy measure g_λ can be equivalently expressed by

$$g_\lambda(A) = \begin{cases} \dfrac{1}{\lambda}\left(\mathop{\Pi}\limits_{i \in A}[1 + \lambda g_\lambda(i)] - 1\right) & \text{if } \lambda \neq 0 \\ \sum\limits_{i \in A} g_\lambda(i) & \text{if } \lambda = 0 \end{cases}. \tag{26}$$

Since $\mu(N) = 1$, we know λ is determined by $\mathop{\Pi}\limits_{i \in N}[1 + \lambda g_\lambda(i)] = 1 + \lambda$. So when each $g_\lambda(i)$ is given, one can get the value of λ. From (26), for a set with n elements it only needs n values to get a $\lambda-$ fuzzy measure.

When the information about the weights of features is partly known, the following linear programming model for the optimal $\lambda-$ fuzzy measure on feature set C is built.

$$\min \sum_{j=1}^{n} \sum_{i=1}^{m} E(A_{ij}) \varphi_j(g_\lambda, C)$$

$$\text{s.t.} \begin{cases} g_y(\emptyset) = 0, g_y(C) = 1, \\ g_y(c_j) \in H_j (j = 1, 2, \ldots, n) \\ \lambda > -1 \end{cases} \tag{27}$$

where $\varphi_j(\mu, N)$ is the Shapley value of the feature c_j (j = 1, 2,…,n), and $H_j = [h_j^-, h_j^+]$ is its range.

Similar to the decision procedure given in Sect. 4.1, the main steps to Atanassov's intuitionistic fuzzy pattern recognition under λ-fuzzy measures are given as follows:

Step 1′: See Step 1.

Step 2′: When the fuzzy measure of each combination in feature set is given, calculate their Shapley values by (16);

Table 9 The results for the Shapley-weighted similarity measures ($p = 1$)

	$S_{SN}^{\mu^L}$	$S_{SL}^{\mu^L}$	$S_{SSK}^{\mu^L}$	$S_{SLS}^{\mu^L}$	$S_{SM}^{\mu^L}$	$S_{SHY1}^{\mu^L}$	$S_{SHY2}^{\mu^L}$	$S_{SHY3}^{\mu^L}$
Q_1	0.74	0.71	0.24	0.95	0.95	0.93	0.88	0.86
Q_2	**0.75**	**0.74**	**0.25**	**0.95**	**0.95**	**0.94**	**0.91**	**0.89**
Q_3	0.59	0.55	0.19	0.91	0.91	0.89	0.83	0.80

otherwise, use the model (27) to get the optimal λ-fuzzy measure on feature set C. Then, calculate their Shapley values using (16).

Step 3′: Calculate the Shapley-weighted similarity measures between A_i ($i = 1,2,\ldots,m$) and ε using (17) [or (18), (19)], and then select the best one.

Step 4′: End.

Example 4 In Example 2, if we make a decision by using the λ-fuzzy measures, then the main steps are given as follows:

Step I-1′: Calculate the optimal λ-fuzzy measure on feature set C, it gets the optimal λ-fuzzy measure on feature set C is $(g_\lambda(c_j))_{j\in\{1,2,3\}} = (0.4, 0.3, 0.6)$, where $\lambda = -0.9439$.

Step II-2′: By (16), it gets the feature Shapley values

$$\varphi_1(g_\lambda, C) = 0.3032, \varphi_2(g_\lambda, C) = 0.2409, \varphi_3(g_\lambda, C) = 0.456.$$

Step II-3′: Calculate the Shapley-weighted similarity measure between A_i ($i = 1,2,3,4$) and ε, the results are presented in Table 7.

In Example 4, the same ranking results are obtaining by using the Shapley-weighted similarity measures (17)–(24), and all ranking results show that the recognized sample ε belongs to the fourth kind of minerals (A_4), see the boldface letters in Table 7.

Example 5 In Example 3, if we make a decision by using λ-fuzzy measures, Similar to Example 4, if $E = E_{BB}$, $E = E_N$ or $E = E_{SK}$; the results are presented in Table 8.

Similarly, if $E = E_L$; the results are presented in Table 9.

According to Tables 8 and 9, the same ranking results are obtained, and the ranking results all show that the patient to the diagnosis Q_2 (malaria), see the boldface letters in Tables 8 and 9.

5 Conclusions

We have researched entropy and similarity measures of IFSs and proposed a new entropy of IFSs. Furthermore, we give a construction method to get the similarity measure of IFSs by using entropy. To deal with the interactive characteristics among features, we further introduce three Shapley-weighted similarity measures of IFSs, which can be seen an extension of some weighted similarity measures. When the information about the weights of features is incompletely known, the model for the optimal fuzzy measure on feature set is built. As we know, the fuzzy measure is defined on the power set, which makes the problem exponentially complex. In order to simplify the complexity of solving a fuzzy measure and reflect the interaction among features, we further research the pattern recognition problems under λ-fuzzy measures. Based on the introduced entropy, similarity measure and the model for the optimal fuzzy measure on feature set, we develop an approach to pattern recognition problems under Atanassov's intuitionistic fuzzy environment.

Similar to the application of the Shapley-weighted similarity measures in pattern recognition, we can also apply them in some other fields, such as digital image processing, clustering analysis and decision-making.

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