Non-Darcian flow to a well in a leaky aquifer using the Forchheimer equation

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Abstract Non-Darcian flow to a well in a leaky aquifer was investigated using a finite difference method. Flow in the leaky aquifer is assumed to be non-Darcian and horizontal, while flow in the aquitard is assumed to be Darcian and vertical. The Forchheimer equation was employed to describe the non-Darcian flow in the aquifer. The finite difference solution was compared with the solution of Birpinar and Sen (2004). The latter overestimates the drawdown at early times and underestimates the drawdown at late times; also, the impact of β_D on the drawdown depends on the value of B_D, where β_D is a dimensionless turbulent factor in the Forchheimer equation and B_D is the dimensionless leakage parameter. The impact of leakage on drawdown is similar to that of Darcian flow. A sensitivity analysis indicated that the drawdown is very sensitive to the change in the dimensionless well radius r_{cD} when B_D is relatively large, while it is sensitive to the change in B_D when B_D is relatively small. The numerical solution has been applied to analyze the pumping test data in Chaj-Doab area of Pakistan. Birpinar ME, Sen Z (2004) Forchheimer groundwater flow law type curves for leaky aquifers. J Hydrol Eng 9(1):51–59

Keywords Groundwater flow · Forchheimer equation · Finite difference method · Aquitard · Pakistan

Received: 12 July 2010 / Accepted: 30 January 2011
Published online: 19 February 2011

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Introduction

Darcy's law has been used for over one and a half centuries for solving groundwater problems (Wen et al. 2009). However, flow can be non-Darcian under certain conditions as long as the flow velocity is relatively high or low (Forchheimer 1901; Dudgeon 1966; Basak 1977; Sen 1990; Choi et al. 1997; Bordier and Zimmer 2000). The limitations of Darcy's law for solving flow problems have long been recognized and even Darcy himself realized that the linear relationship only worked for a certain range of grain size under a certain range of hydraulic gradient (Darcy 1856).

Many scientists have investigated the relationship between the hydraulic gradient and specific discharge for non-Darcian flow (e.g., Forchheimer 1901; Rose 1951; Polubarinova-Kochina 1962; Muskat 1937; Harr 1962; Izbash 1931; Escande 1953; Wilkinson 1956; Slepicka 1961), as summarized by Basak (1978). As can be seen from Table 1 in the paper of Basak (1978), all the relationships can be classified into two types: i.e., polynomial and power functions. Among these empirical (or theoretical) equations, two were commonly used. The first is the Forchheimer equation, which states that the hydraulic gradient is a second-order polynomial function of the specific discharge. The second is the Izbash equation which states that the hydraulic gradient is a power function of the specific discharge. Both equations have advantages and disadvantages. The Forchheimer equation has two terms, the first term represents the viscous term and the second term represents the inertial term. The physical meaning of the Forchheimer equation

 DOI 10.1007/s10040-011-0709-2

Table 1 Definition of the dimensionless variables

$r_D = \frac{r}{m}$	$r_{cD} = \frac{r_c}{m}$	$r_{wD} = \frac{r_w}{m}$	$q_D = -\frac{4\pi m^2}{Q}q$	$B_D = \frac{Bk}{m^2}$
$t_D = \frac{kt}{Sm}$	$s_D = \frac{4\pi km}{Q}s$	$s_{wD} = \frac{4\pi km}{Q}s_w$	$\beta_D = \frac{\beta Q}{4\pi m^2}$	

is evident and it has been validated theoretically by several scientists (Irmay 1958; Whitaker 1996; Giorgi 1997; Sorek et al. 2005). When the velocity is relatively low, the second term, as opposed to the first term, can be ignored. In this case, the Forchheimer equation becomes Darcy's law. The Izbash equation is a fully empirical equation based on numerous experimental data. However, because of the mathematical convenience, it also has been commonly used (e.g., Wen et al. 2006, 2008a, 2008b). In many cases, these two equations can describe non-Darcian flow equivalently well (Bordier and Zimmer 2000; Li et al. 2008).

Because of the high velocities, non-Darcian flow is likely to occur near pumping wells (Wu 2002a, b; Sen 1987, 1988, 1989, 1990; Wen et al. 2006, 2008a, b, c, 2009). Based on the Forchheimer or Izbash equations, many studies have been carried out to investigate the non-Darcian flow near the pumping well. A careful review of existing publications about the non-Darcian flow to a pumping well indicates that three methods have commonly been used to solve this type of non-linear problem: the Boltzmann transform, the linearization method and numerical modeling. Sen (1987, 1988, 1989, 1990) has done extensive studies on non-Darcian flow to a pumping well by using the Boltzmann transform method. Recently, Wen et al. (2008a, b) have proposed a linearization procedure for solving non-Darcian flow to a well based on the assumption that the flow can be described by the Izbash equation. Meanwhile, numerical methods have also been used to solve non-Darcian flow problems (e.g., Ewing et al. 1999; Ewing and Lin 2001; Wu 2002a, b; Mathias et al. 2008; Wen et al. 2009). As commented by many scientists (e.g., Camacho-V and Vasquez-C 1992; Mathias et al. 2008), the Boltzmann transform cannot be used to solve such non-Darcian flow problems in a rigorous mathematical sense. The linearization procedure also has some limitations such as the discrepancies associated with early-time solutions. From this viewpoint, it seems that the present analytical methods can not solve such non-Darcian problems very well and numerical modeling might be a good approach in this area.

Up to now, most models on non-Darcian flow to a well are for confined aquifers such as, e.g., Sen (1987, 1990) and Wen et al. (2006, 2008a). Research on non-Darcian flow in leaky aquifers is too limited. Sen (2000) used a volumetric approach to investigate non-Darcian flow in leaky aquifers with the Izbash equation, and that study has been extended by Birpinar and Sen (2004) to investigate a similar problem with the Forchheimer equation. These two studies were based on the solutions obtained by the Boltzmann transform which was found to be problematic in a rigorous mathematical sense (Camacho-V and Vasquez-C 1992; Mathias et al. 2008). Therefore, the solutions in these two studies might be questionable. Recently, Wen et al. (2008c) used a linearization procedure and finite difference method to solve non-Darcian flow in a leaky aquifer with the Izbash equation. As mentioned before, the Izbash equation has some disadvantages for describing radial non-Darcian flow. To the authors' knowledge, the investigations about non-Darcian flow in leaky aquifers with the Forchheimer equation are still very limited.

In this paper, a finite difference method will be used to investigate the non-Darcian flow to a pumping well in a leaky aquifer. The wellbore storage is also considered. The flow in the aquifer is assumed to be horizontal and non-Darcian, and the flow in the aquitard is assumed to be vertical and Darcian. The Forchheimer equation will be used to describe the non-Darcian flow in the aquifer. Sensitivity analysis for different parameters under non-Darcian flow conditions will be done in this study and the numerical solution will be applied to analyze the pumping test data in Chaj-Doab area of Pakistan. As Wen et al. (2008b) have investigated non-Darcian flow to a well in an aquifer-aquitard system with the Izbash equation, this study can be regarded as an extended work of Wen et al. (2008b).

Problem statement

Governing equations

The system discussed here is similar to that of Hantush and Jacob (1955), as shown in Fig. 1 (Wen et al. 2008b). The following assumptions have been used in this study to make the problem mathematically tractable. First, the aquifer and the bounded upper aquitard are assumed to be homogeneous, isotropic, and horizontally infinite. Second, the flow in the aquifer is assumed to be non-Darcian while the flow in the upper aquitard is assumed to be Darcian, and the flow direction in the aquifer is horizontal while the flow direction in the aquitard is vertical. Third, the aquitard storage is not considered and, fourth, the well fully penetrates the aquifer and the pumping rate is assumed to be constant. Under these assumptions, the mathematical model can be generated as follows:

$$\frac{1}{r}\frac{\partial[rq(r,t)]}{\partial r} - \frac{s(r,t)}{B} = \frac{S}{m}\frac{\partial s(r,t)}{\partial t}, \quad (1)$$

$$s(r,0) = 0, \quad (2)$$

$$s(\infty,t) = 0, \quad (3)$$

$$2\pi rmq(r,t)|_{r\to 0} = -Q, \quad (4)$$

 DOI 10.1007/s10040-011-0709-2

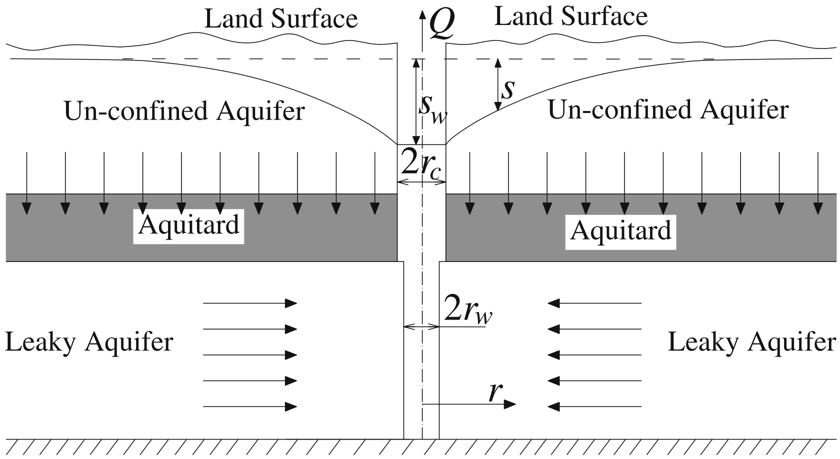

Fig. 1 A schematic diagram of the leaky confined aquifer system (Wen et al. 2008b, with permission from Elsevier)

in which r is radial distance from the center of the pumping well [L]; t is pumping time [T]; $q(r,t)$ is specific discharge [L/T]; $s(r,t)$ is drawdown [L]; S is storage coefficient of the aquifer; m is aquifer thickness [L]; Q is pumping rate, which is constant [L^3/T]; B is the leakage parameter defined as $m \times m_1/k_1$, [LT], where m_1 and k_1 are thickness and hydraulic conductivity of the aquitard, respectively. It is notable that a greater value of B, meaning that a larger aquitard thickness m_1 or a smaller aquitard hydraulic conductivity k_1, indicates smaller leakage effect. If the hydraulic conductivity of the aquitard k_1 goes to zero, the leakage parameter B goes to infinity, then the problem investigated here is similar to the Theis model for confined aquifers.

If the wellbore storage cannot be ignored, the boundary condition Eq. (4) should be replaced by

$$2\pi r_{\rm w} m q(r,t)|_{r=r_{\rm w}} - \pi r_{\rm c}^2 \frac{ds_{\rm w}(t)}{dt} = -Q, \tag{5}$$

where $r_{\rm w}$ is the radius of the well screen [L], $r_{\rm c}$ is the radius of the well casing [L]. In most cases, $r_{\rm c}$ is larger than, instead of equal to $r_{\rm w}$. $s_{\rm w}(t)$ is drawdown inside the well [L], which is dependent of the pumping time t.

The Forchheimer equation will be used to describe the non-Darcian flow in the aquifer. The Forchheimer equation can be expressed as:

$$q + \beta q|q| = k\frac{\partial s}{\partial r}, \tag{6}$$

in which β [T/L] is a non-Darcian factor representing the turbulence of the non-Darcian flow. If β is equal to zero, Eq. (6) becomes the well-known Darcy's law, k [L/T] can be regarded as the apparent hydraulic conductivity of the aquifer.

Dimensionless transform

Similar to the analysis of the Theis type curves for Darcian flow, one can define the following dimensionless variables, as listed in Table 1. Notice that a minus sign was included in the definition of the dimensionless specific discharge $q_{\rm D}$. It is necessary to emphasize two important dimensionless variables, i.e., $B_{\rm D}$ and $\beta_{\rm D}$. $B_{\rm D}$ is a dimensionless variable representing the leakage and a larger $B_{\rm D}$ means smaller leakage. $\beta_{\rm D}$ is a dimensionless parameter representing the non-Darcian effect and a larger $\beta_{\rm D}$ indicates greater turbulence. With these definitions, the problem proposed here can be expressed in dimensionless form as:

$$-\frac{\partial[r_{\rm D}q_{\rm D}]}{r_{\rm D}\partial r_{\rm D}} - \frac{s_{\rm D}}{B_{\rm D}} = \frac{\partial s_{\rm D}}{\partial t_{\rm D}}, \tag{7}$$

$$s_{\rm D}(r_{\rm D}, 0) = 0, \tag{8}$$

$$s_{\rm D}(\infty, t_{\rm D}) = 0, \tag{9}$$

$$\frac{1}{2}(r_{\rm D}q_{\rm D})|_{r_{\rm D}\to 0} = 1. \tag{10}$$

The boundary condition considering wellbore storage can also be expressed in dimensionless format as:

$$\frac{1}{2}(r_{\rm D}q_{\rm D})|_{r_{\rm D}=r_{\rm wD}} + \frac{r_{cD}^2}{4S}\frac{ds_{\rm wD}(t_{\rm D})}{dt_{\rm D}} = 1. \tag{11}$$

The Forchheimer equation will be changed to:

$$q_{\rm D} + \beta_{\rm D} q_D^2 = -\frac{\partial s_{\rm D}}{\partial r_{\rm D}}. \tag{12}$$

Solutions for the problem

Numerical solution

The finite difference method was used to simulate the problem investigated here. It should be pointed out that

DOI 10.1007/s10040-011-0709-2

only the case with wellbore storage was considered when doing the simulation. The case without wellbore storage can be easily solved by setting very small values (close to zero) to the dimensionless radii r_w and r_c. First, one needs to discretize the dimensionless spatial domain $[r_{wD}, r_{eD}]$, where r_{eD} is a very large value which is used to approximate the outer boundary condition at which the drawdown is equal to zero. The value of r_D at the ith node can be referred as r_i, where the subscript represents the node of interest. For any node of r_i, $r_{wD}<r_i<r_{eD}$, i=1, 2,..., N, and $r_0=r_{wD}$ and $r_{n+1} = r_{eD}$. As the drawdown changes rapidly near the pumping well, the dimensionless domain was discretized logarithmically (Wu 2002a; Mathias et al. 2008):

$$r_i = \left(r_{i-1/2} + r_{i+1/2}\right)/2, \quad i = 1, 2, \ldots, N, \tag{13}$$

with respect to

$$\log_{10}\left(r_{i+1/2}\right) = \log_{10}(r_{wD}) + i\left[\frac{\log_{10}(r_{eD}) - \log_{10}(r_{wD})}{N}\right], \quad i = 0,\ 1, \ldots, N. \tag{14}$$

The governing equation, Eq. (7), can also be reduced to the following differential equation with respect to the dimensionless time t_D:

$$\frac{ds_i}{dt_D} \approx \frac{r_{i-1/2}q_{i-1/2} - r_{i+1/2}q_{i+1/2}}{r_i\left(r_{i+1/2} - r_{i-1/2}\right)} - \frac{s_i}{B_D}, \quad i = 1,\ 2, \ldots, N, \tag{15}$$

in which s_i and q_i are the dimensionless drawdown s_D and dimensionless specific discharge q_D at node i, respectively. From the Forchheimer equation, one has:

$$q_D = \frac{1}{2\beta_D}\left\{-1 + \left[1 - 4\beta_D \frac{\partial s_D}{\partial r_D}\right]^{1/2}\right\}. \tag{16}$$

Therefore, the dimensionless specific discharge that goes into and out of each cell can be expressed as:

$$q_{i-1/2} \approx \frac{1}{2\beta_D}\left\{-1 + \left[1 + 4\beta_D\left(\frac{s_{i-1} - s_i}{r_i - r_{i-1}}\right)\right]^{1/2}\right\}, \ i = 2,\ 3, \ldots, N, \tag{17}$$

$$q_{i+1/2} \approx \frac{1}{2\beta_D}\left\{-1 + \left[1 + 4\beta_D\left(\frac{s_i - s_{i+1}}{r_{i+1} - r_i}\right)\right]^{1/2}\right\}, \ i = 1, 2, \ldots, N-1, \tag{18}$$

. At the inner and outer boundaries, one has:

$$q_{1-1/2} \approx \frac{1}{2\beta_D}\left\{-1 + \left[1 + 4\beta_D\left(\frac{s_{wD} - s_1}{r_1 - r_{wD}}\right)\right]^{1/2}\right\}, \tag{19}$$

$$q_{N+1/2} \approx \frac{1}{2\beta_D}\left\{-1 + \left[1 + 4\beta_D\left(\frac{s_N - 0}{r_{eD} - r_N}\right)\right]^{1/2}\right\}, \tag{20}$$

where s_{wD} is the dimensionless drawdown inside the well, which can be approximated as follows by considering Eq. (11):

$$\frac{ds_{wD}}{dt_D} \approx \frac{4S}{r_{cD}^2}\left(1 - \frac{1}{2} r_{wD} q_{1-1/2}\right). \tag{21}$$

After these preparations, the problem can be solved with the MATLAB software by using the stiff integrator ODE15s (Mathias et al. 2008). A MATLAB program named LeakyForch was developed to do this calculation. For all the simulations, N was chosen to be 5000 and r_{eD} was chosen to be 10^8. These values are found to be sufficiently large for this study.

The solution of Birpinar and Sen (2004)

Birpinar and Sen (2004) proposed an analytical solution for the same problem by using the Forchheimer equation. They obtained the analytical solution for the type curves in leaky aquifers by using the volumetric approach proposed by Sen (1985), and on the basis of their previous study (Sen 1989). It is notable that the solution of Sen (1989) for the type curves in confined aquifers was obtained by the Boltzmann transform. Here the point is that the Boltzmann transform cannot be used to derive the type curves for non-Darcian flow problems, as pointed out by several scientists (e.g., Camacho-V and Vasquez-C 1992; Mathias et al. 2008; Wen et al. 2008a). In this study, the numerical solution will be used to verify the analytical solution obtained by Birpinar and Sen (2004) for the same problem. As stated by Birpinar and Sen (2004), the analytical solution (see Eq. (11) in their study) obtained by the volumetric approach can be changed to the following equation by using the dimensionless variables defined in this study:

$$s_D = \int_{\frac{r_D^2}{4t_D}}^{\infty} \frac{\exp\left(-x - \frac{r_D^2}{4B_D}\frac{1}{x}\right)}{\left(1 + \frac{2\sqrt{\pi}\beta_D\sqrt{x}}{r_D}\right)x} dx + \frac{2\beta_D}{r_D}\int_{\frac{r_D^2}{4t_D}}^{\infty} \frac{\exp\left(-2x - \frac{r_D^2}{4B_D}\frac{1}{x}\right)}{\left(1 + \frac{2\sqrt{\pi}\beta_D\sqrt{x}}{r_D}\right)^2 x} dx, \tag{22}$$

in which x is a dummy variable. The aforementioned equation can be integrated easily by using a MATLAB program.

 DOI 10.1007/s10040-011-0709-2

Results and discussion

In order to check the numerical solution proposed in this study, first the numerical solution was compared with the analytical solution of Hantush and Jacob (1955) for the Darcian flow case (β_D=0). Three typical dimensionless distances were chosen as r_D=0.5, 1, and 5 and the dimensionless leakage parameter was given as 10. As shown in Fig. 2, the numerical solution agrees perfectly with the analytical solution during the entire pumping period. This indicates that the numerical error caused by the finite difference method probably can be ignored and the numerical solution in this study is sufficiently accurate.

Dimensionless drawdown without wellbore storage

Figure 3 shows the comparison of the numerical solution used in this study and the solution of Birpinar and Sen (2004). The analytical solution of Birpinar and Sen (2004) was calculated from Eq. (22). The parameters used in Fig. 3 were given as β_D=10, B_D=10, r_D=0.5 and 5. As can be seen from Fig. 3, it is evident that the solution obtained by Birpinar and Sen (2004) is not reliable. The method of the volumetric approach associated with the Boltzmann transform overestimates the dimensionless drawdown at early times and underestimates the dimensionless drawdown at late times. It can also be found that when the dimensionless distance is larger, the discrepancy between the analytical solution and the numerical solution is smaller, as expected.

The sensitivity analysis of the dimensionless turbulent factor β_D on the dimensionless drawdown is shown in Fig. 4. Two typical dimensionless leakage parameters B_D=100 and 1 were used in this figure, as shown in Fig. 4a and b. The other parameters were given as: r_D= 0.5, β_D=1, 10 and 50. The solution of Hantush and Jacob (1955) has also been depicted as a reference. Interestingly enough, the features of these two figures are quite

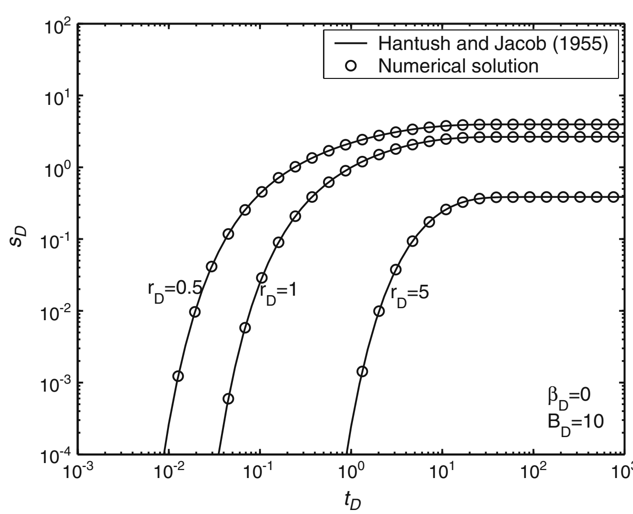

Fig. 2 Comparison of the numerical solution and the solution of Hantush and Jacob (1955) for the Darcian flow case

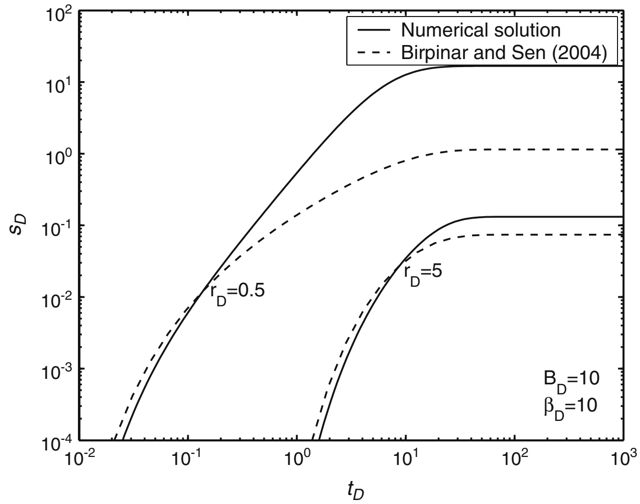

Fig. 3 Comparison of the numerical solution and the solution of Birpinar and Sen (2004) with B_D=10, β_D=10, r_D=0.5 and 5

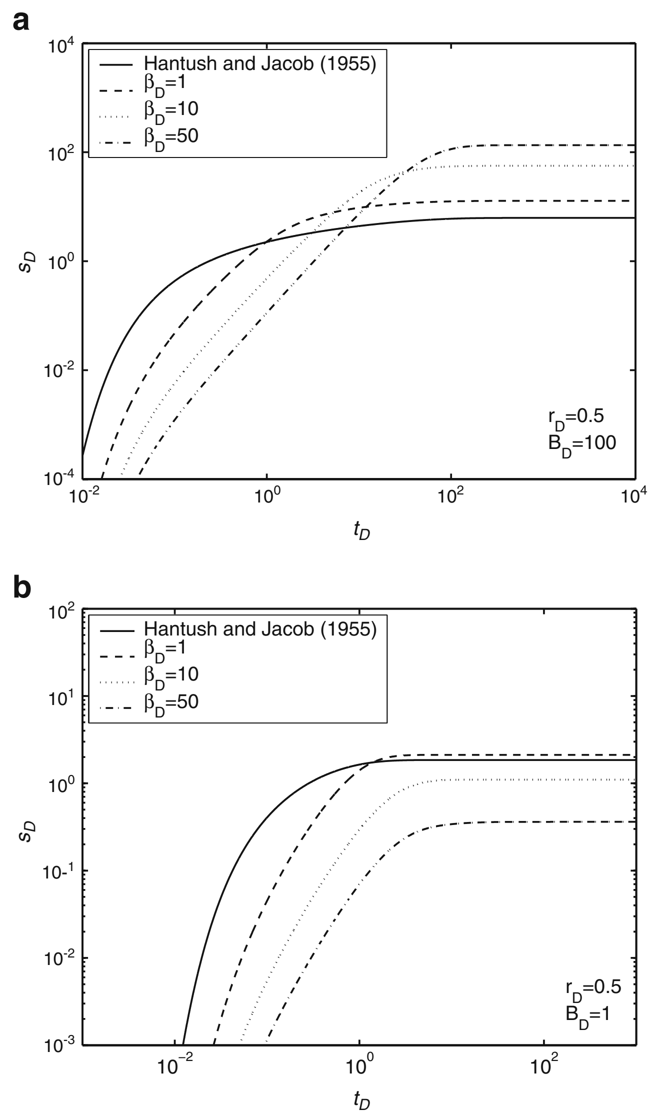

Fig. 4 Drawdown-time behavior for different values of the turbulent factor β_D with r_D=0.5, β_D = 1, 10 and 50. **a** B_D=100; **b** B_D=1

DOI 10.1007/s10040-011-0709-2

different. For the case B_D=100, a relatively large value for the dimensionless leaky parameter which indicates a relatively small leakage effect, a larger β_D results in a smaller dimensionless drawdown at early times and leads to a larger dimensionless drawdown at late times. This phenomenon can be explained as follows. The Forchheimer equation can be rewritten as: $q = K_\beta(\partial s/\partial x)$, in which $K_\beta = k/(1+\beta|q|)$, and K_β can be regarded as apparent "Darcian" hydraulic conductivity. If β_D is larger, which also means a larger β, then K_β is smaller. At early times, the flow has not approached the steady state and the specific discharge q increases from zero to the stationary q_0 at steady state. A smaller K_β indicates that it will take a longer time for the flow to approach the steady state. Thus, at the same time, a smaller K_β has a smaller specific discharge. Consequently, a smaller drawdown can be found from the equation $q = K_\beta(\partial s/\partial x)$. At late times, the flow approaches the steady state and the specific discharge does not change any more, which is only proportional to the pumping rate Q. From the equation $q = K_\beta(\partial s/\partial x)$, one can see a smaller K_β will result in a larger drawdown. As reflected in Fig. 4a, a larger K_β results in a larger dimensionless drawdown at late times. For the case B_D=1 shown in Fig. 4b, a relatively small value for the dimensionless leaky parameter which indicates a relatively great leakage effect, a larger β_D results in a smaller dimensionless drawdown during the entire pumping period. This might be because a large leakage from the aquitard will result in the system approaching the quasi steady state earlier. Therefore, the "early-time" behavior of the type curve is very short and not long enough to be shown in the figure. Figure 4b might only reflect the "large-time" behavior of the type curve; therefore, it is no wonder that a larger β_D results in a smaller drawdown during the entire pumping period, as shown in Fig. 4b.

It is also worthwhile to analyze the effect of the leakage on the drawdown. Figure 5 is about the dimensionless drawdown versus dimensionless time for different dimensionless leakage parameters. The parameters were given as: r_D=0.5, β_D=10, B_D=1, 10 and 100. The case without the leakage (Mathias et al. 2008) has also been depicted in this figure as a reference. As shown in Fig. 5, all the curves for different B_D values approach the same asymptotic value at early times, as well as the case of Mathias et al. (2008). This is because water from the leakage has not arrived at the confined aquifer at early times; therefore, the leakage has little impact on the drawdown at early times. As the pumping time increases, great differences have been found for different B_D. When B_D is larger, the dimensionless drawdown is larger. This is because a larger B_D means a smaller leakage effect, thus a larger dimensionless drawdown at late times. It can also been found that when B_D is larger, the curve is closer to that obtained by Mathias et al. (2008), as expected.

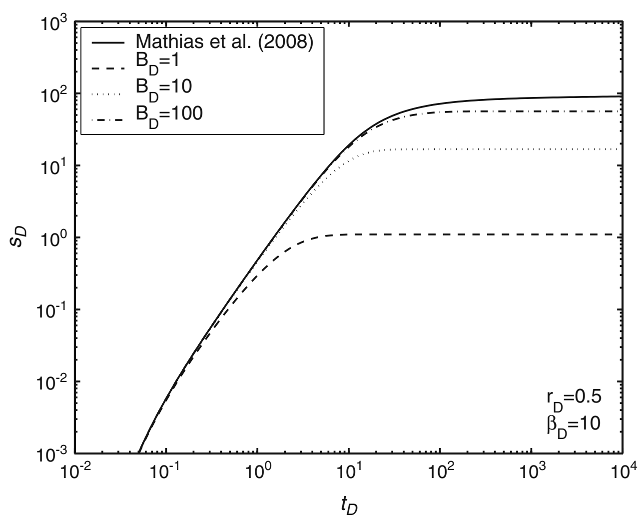

Fig. 5 Drawdown-time behavior for different values of the leakage factor B_D with r_D=0.5, β_D = 10, B_D=1, 10 and 100

Type curves

It is useful to generate a series of type curves for the hydrology community to conduct well testing analysis. When conducting a pumping test, the position of the observation well r and the aquifer thickness m are usually known. This means one can obtain r_D before analyzing the pumping test data. For a specific value of r_D, one can obtain a series of type curves for different dimensionless parameters. Taking r_D=0.5 as an example, the corresponding type curves for different values of the dimensionless parameters are shown in Fig. 6. For a specific pumping test, if r_D happens to equal 0.5, one can use these type curves to estimate the aquifer parameters (an example will be given in the following section). The type curves for different r_D values can also be obtained by the MATLAB program, but it is impossible to report all the type curves in this paper. For the hydrology community use, one can get the MATLAB program upon request.

Dimensionless drawdown with wellbore storage

When the wellbore storage is considered, the dimensionless drawdowns inside the well for different turbulent factor β_D and different leakage parameter B_D were analyzed. As shown in Figs. 7 and 8, all the curves for different turbulent factor β_D and different leakage parameter B_D approach the same asymptotic value at early times while significant differences have been found at late times. Similar to the features of the drawdowns in the aquifer without the wellbore storage, a larger β_D (or B_D) results in a larger drawdown inside the well. The drawdowns in the aquifer when considering the wellbore storage have also been analyzed. As the features are similar to those without the wellbore storage, it is not repeated here.

Sensitivity analysis

In order to assess the influence of different parameters on the results, the sensitivity analysis was done on different parameters, i.e., B_D, β_D, r_{wD}, and r_{cD}. This analysis is

 DOI 10.1007/s10040-011-0709-2

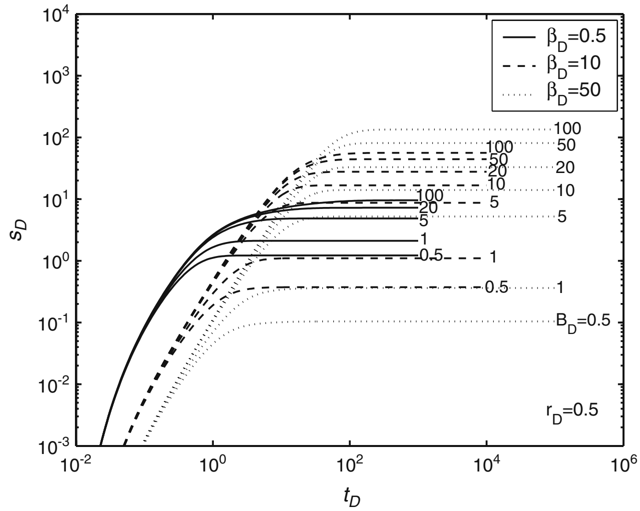

Fig. 6 Type curves for r_D=0.5, B_D=0.5, β_D = 0.5, 10 and 50

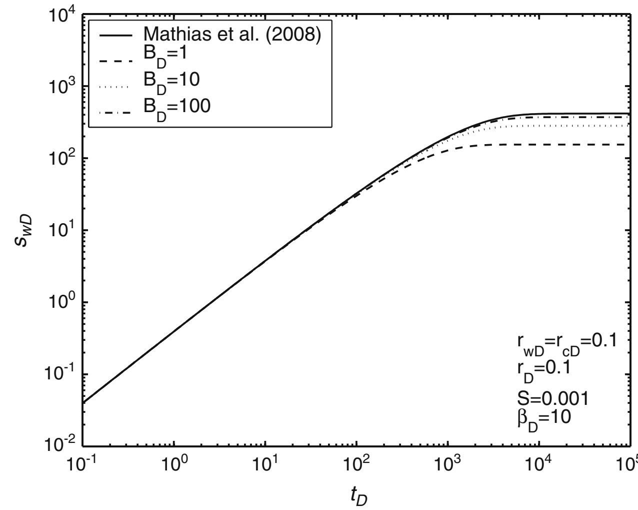

Fig. 8 Drawdown-time behavior inside the well when considering the wellbore storage for different values of the leakage factor B_D with r_D=0.1, r_{wD}=r_{cD}=0.1, S=0.001, β_D = 10, B_D = 1, 10 and 100

useful in assessing how a conceptual model responds to the change in certain variables. As proposed by Liou and Yeh (1997) and Huang and Yeh (2007), the sensitivity of a dependent variable can be defined as:

$$X_{\mathrm{i,j}} = \frac{\partial R_{\mathrm{i}}}{\partial P_{\mathrm{j}}}, \tag{23}$$

in which $X_{\mathrm{i,j}}$ is the sensitivity coefficient of the jth parameter P_{j} at the ith time. R_{i} is the dependent variable at the ith time, which is the dimensionless drawdown in this study. Huang and Yeh (2007) have proposed a normalized sensitivity method to assess the effect of different variables on the dependent variable, which is defined as:

$$X'_{\mathrm{i,j}} = P_{\mathrm{j}} \frac{\partial R_{\mathrm{i}}}{\partial P_{\mathrm{j}}}, \tag{24}$$

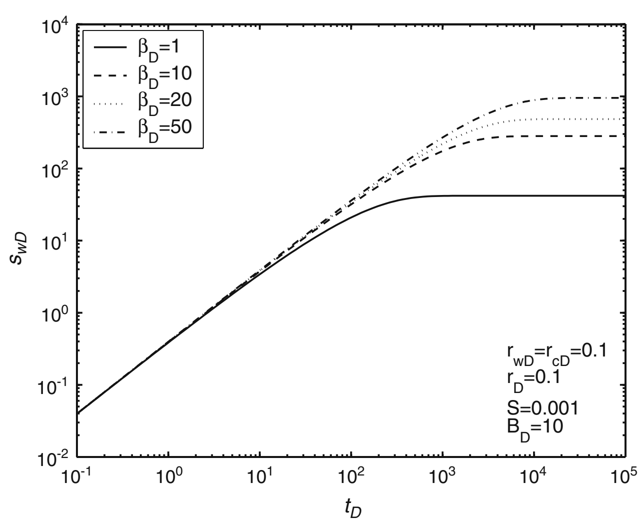

Fig. 7 Drawdown-time behavior inside the well when considering the wellbore storage for different values of the turbulent factor β_D with r_D=0.1, r_{wD}=r_{cD}=0.1, S=0.001, B_D=10, β_D = 1, 10, 20 and 50

where $X'_{i,j}$ is the normalized sensitivity of the jth parameter P_{j} at the ith time. Notice that there is a partial derivative on the right hand of the Eq. (24), which is always difficult to obtain for a specific case. A finite difference formula will be used to approximate this differentiation (Yeh 1987), that is

$$\frac{\partial R_{\mathrm{i}}}{\partial P_{\mathrm{j}}} = \frac{R_{\mathrm{i}}\left(P_{\mathrm{j}} + \Delta P_{\mathrm{j}}\right) - R_{\mathrm{i}}\left(P_{\mathrm{j}}\right)}{\Delta P_{\mathrm{j}}}, \tag{25}$$

in which ΔP_{j} is a small increment, which will be chosen as $10^{-2} \times P_{\mathrm{j}}$ (Yang and Yeh 2009). The two most important dimensionless parameters used in this study are β_{D} and B_{D}. When the wellbore storage is considered, r_{cD} is also an important parameter which reflects the water stored in the wellbore. Thus, normalized sensitivities of these parameters will be analyzed in the following. The hypothetical data used were β_D = 10, r_{wD}=r_{cD} = 0.1, r_{D}=1, S= 0.001, and B_{D}=100 (or 1). The reason for not analyzing the sensitivities of r_{wD} and S are as follows: (1) r_{wD} is the dimensionless radius of the well screen which is assumed to be equal to r_{cD} in this study; (2) the storage coefficient S has been used in the definition of the dimensionless variable t_{D}.

Figure 9a and b plot the dimensionless time-drawdown curve and the normalized sensitivities of the parameters β_{D}, B_{D} and r_{cD} in semi-log scales. Figure 9a is for a case with a relatively small leakage B_{D}=100, while Fig. 9b is for a case with a relatively large leakage B_{D}=1. It can be seen that the normalized sensitivity of the drawdown with respect to B_{D} has a positive effect from both parts a and b of Fig. 9. This feature is quite distinct in Fig. 9b and this is because when B_{D} is larger, the leaky rate is smaller, resulting in a larger drawdown. For B_{D}=1, the leakage is very large, then this positive phenomenon is evident. A relative change in β_{D} has a positive effect for B_{D}=100 in Fig. 9a, while it has a negative effect for B_{D}=1 in Fig. 9b. This is consistent with the features of Fig. 4a and b. The

DOI 10.1007/s10040-011-0709-2

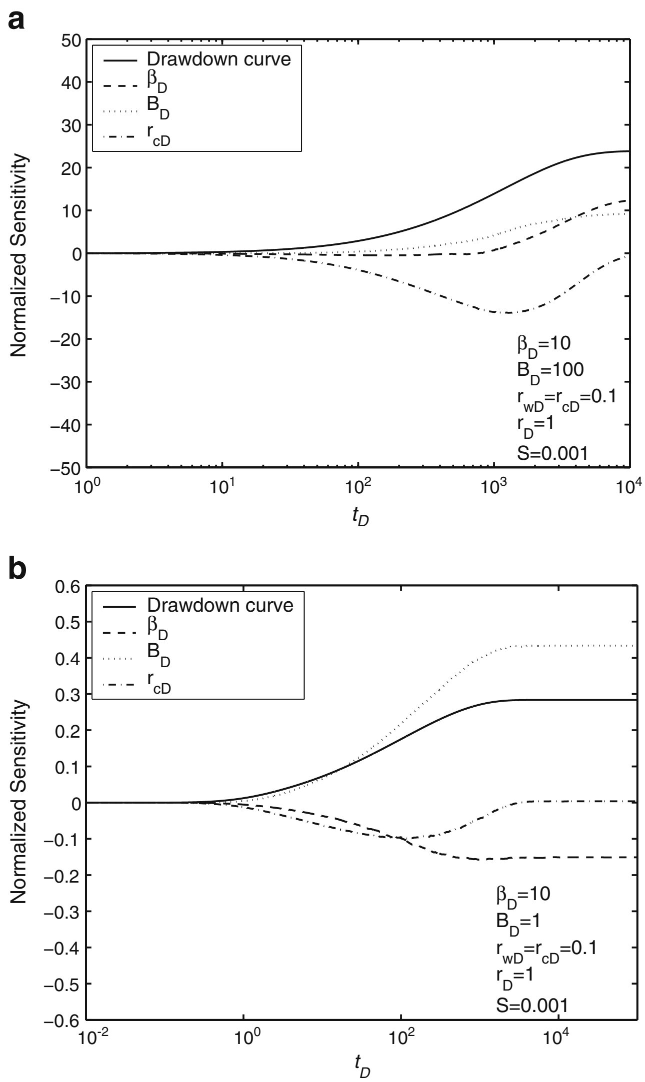

Fig. 9 Drawdown-time curve and the normalized sensitivities of the parameters β_D, B_D and r_{cD}. **a** B_D=100; **b** B_D=1

normalized sensitivity with respect to r_{cD} produces a negative effect at early times and stabilizes at zero after a relatively large dimensionless time t_{D} (say, 10^4). This can be easily understood by considering the physics of the wellbore storage. Comparing parts a and b of Fig. 9, one can see that r_{cD} produces the largest normalized sensitivity in the magnitude when B_{D} is relatively large (B_{D}=100), while B_{D} produces the largest normalized sensitivity in the magnitude when B_{D} is relatively small (B_{D}=1). Those results indicate that the drawdown is very sensitive to the change in r_{cD} when B_{D} is large, while it is sensitive to the change in B_{D} when B_{D} is relatively small.

It seems that a relative change of these three parameters has no effect on the drawdown at early times (say, $t_{\mathrm{D}}<10^0$), as shown in Fig. 9a. The truth is that the value of the dimensionless drawdown at early times is very small (see Fig. 4), consequently, the corresponding normalized sensitivity is a very small value around zero.

Application

The numerical solution developed in this study can be used to determine the aquifer parameters associated with the pumping test data. In order to show the applications of the solutions, the data from Ahmad (1998) was used, who did a pumping test in the Chaj-Doab area near Gujrat water distributory in Pakistan. The pumping rate is fixed at Q=3.77m^3/min, the thickness of the aquifer is measured as m=76 m, and the observation well is about r=122 m away from the pumping well. Therefore, the corresponding dimensionless distance r_{D} is equal to 1.61. As stated by Ahmad (1998), the aquifer in that area can be classified as a mixture of leaky and unconfined aquifers (Birpinar and Sen 2004). Therefore, it is possible to use the type curves developed in this study to estimate the aquifer parameters. The observation data of time-drawdown was plotted in log-log scale as shown in Fig. 10. After a trial and error process, the curve with r_{D}=1.6053, B_{D}=25 and β_{D} = 0.4 matched the observation data perfectly, as shown in Fig. 10. A matching point was chosen in the common area so that one could obtain the corresponding coordinates: s=0.1 m, t=10 min, s_{D}=0.36 and t_{D}=5.68. When the matching procedure is done, one can do the following calculations.

1. From the definitions of the dimensionless drawdown $s_D = \frac{4\pi km}{Q}s$, one can obtain the apparent hydraulic conductivity $k = \frac{s_D Q}{4\pi ms} = \frac{0.36\times3.77}{4\times3.14\times76\times0.1} = 0.014[m/min]$.
2. When k is known, the storage coefficient S can be calculated as: $S = \frac{kt}{mt_D} = \frac{0.014\times10}{5.68\times76} = 3.24\times10^{-4}$.
3. With the dimensionless definition $\beta_D = \frac{\beta Q}{4\pi m^2}$, one has: $\beta = \frac{4\pi m^2\beta_D}{Q} = \frac{4\times3.14\times76^2\times0.4}{3.77} = 7697[min/m]$.
4. Finally, the leakage parameter B can be calculated as: $B = \frac{B_D m^2}{k} = \frac{25\times76^2}{0.014} = 1.031\times10^7$. As B is defined as $m\times m_1/k_1$, if the thickness of the aquitard is known, the hydraulic conductivity of the aquitard can be obtained subsequently.

As discussed in section Type curves, for a specific given distance r_{D}, one can generate a series of type curves. When

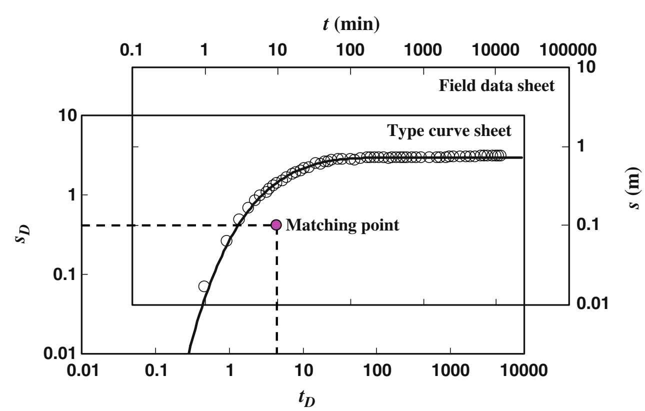

Fig. 10 Type curve and field data matching sheets. The field data were obtained from Birpinar and Sen (2004)

 DOI 10.1007/s10040-011-0709-2

the pumping data are available, one can estimate the aquifer parameters through the matching processes described in the previous with the type curves and the observed data.

Summary and conclusions

A numerical solution for non-Darcian flow to a well in leaky aquifers has been obtained by using the finite difference method in this study. The Forchheimer equation has been used to describe the flow in the aquifer. The impacts of the turbulent factor and the leakage parameter on the drawdowns have been analyzed. The results were also compared with the solution of Birpinar and Sen (2004) who investigated a similar problem with a volumetric approach. The solutions obtained in this study can also be used to estimate the aquifer parameters when the groundwater flow is non-Darcian. Several findings can be presented from this study:

1. The method of the volumetric approach associated with the Boltzmann transform (Birpinar and Sen 2004) overestimates the dimensionless drawdown at early times and underestimates the dimensionless drawdown at late times. When the dimensionless distance is larger, the discrepancy between the analytical solution and the numerical solution becomes smaller.
2. For a relatively larger dimensionless leakage parameter, e.g., B_D=100, a larger turbulent factor β_D results in a smaller drawdown at early times and a larger drawdown at late times. While for a relatively smaller dimensionless leakage parameter, e.g., B_D=1, a larger β_D results in a smaller drawdown during the entire pumping period.
3. The impact of the leakage on the drawdown is similar to that of Darcian flow; the leakage has little impact on the drawdown at early times, while a greater leakage parameter B_D leads to a larger drawdown at late times.
4. The drawdown is very sensitive to the change in r_{cD} reflecting the importance of wellbore storage when B_D is large, while it is sensitive to the change in B_D when B_D is relatively small.

Acknowledgements This research was partially supported by the National Natural Science Foundation of China (Grant Nos. 41002082, 50779067), the National Basic Research Program of China (Grant No. 2010CB428802), and the Special Fund for Basic Scientific Research of Central Colleges, China University of Geosciences (Wuhan; Grant No. CUGL090301). The constructive comments of two anonymous reviewers and the Editor are also gratefully acknowledged. The authors also sincerely thank the Technical Editorial Advisor, Sue Duncan, for carefully checking the manuscript.

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