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On the validity of the Weibull failure model for brittle particles

Uwe Jansen, Dietrich Stoyan

Abstract The Weibull theory of material strength and fracture assumes that the Weibull modulus m is a material parameter, which does not depend on shape and size of the loaded object. Based on large data sets from single-particle fracture experiments with brittle materials (glass, clinker cement, limestone), the authors show that the Weibull modulus of nearly spherical particles seems to decrease with increasing particle diameter. A possible explanation is that the inner structure of the particles depends on their size so that small particles are much stronger than large ones.

Key words Weibull distribution, fracture probabilities, brittle particles, experimental data, Weibull modulus

1 Introduction

Usually, the Griffith [1] and Weibull [2] theories are used to describe fractures of particles; see, for example, [3–5]. They lead to the Weibull distribution function for particle strength, which has the form:

$$\begin{aligned} F(l) &= P(\text{fracture of object under load less than } l) \\ &= 1 - \exp(-\lambda l^z) \ . \end{aligned} \tag{1}$$

The term 'load' can be interpreted as external load onto the object investigated.

The two parameters λ and z describe in summarizing form all properties resulting from the material and the geometry of object and load in the experimental situation. The parameter z depends on the material; increasing z is related to increasing strength. It is closely related to the Weibull modulus m, see section 2. A fundamental assumption is that the Weibull modulus does not depend on the geometry (shape and size) of the object and also not on the load.

In the present paper the authors discuss the results of four series of single-grain fracture experiments with nearly spherical brittle particles. The data stem from earlier research of Schubert and his coworkers ([6–9]) in the 1970s and 1980s, who systematically investigated problems of particle breakage of brittle materials. At that time, these authors did not try to use the Weibull model for particle strength. Reconsidering their data, it turns out that the Weibull distribution is an excellent model for fixed particle size. However, and surprisingly, at first sight the assumption of a constant Weibull modulus does not seem to be true. For example, for (nearly) spherical clinker cement particles the Weibull distribution gives, for any fixed radius, an excellent fit of the experimental data, but the parameter z decreases linearly with increasing particle radius.

Nevertheless, the authors believe that this empirical result is not necessarily a contradiction to the classical assumption of a material-dependent and otherwise constant Weibull modulus. We simply recommend to consider 'particles of different size consisting of the same material', for example clinker cement, as objects of different materials because of different inner structures of small and large particles. This approach is in the spirit of Szabó [10], who emphasizes the idea of considering anisotropy and inhomogeneity of real grains.

2 The Weibull failure model

The Weibull failure model is based on the idea of the weakest link in a chain [3]. Therefore, the Weibull probability distribution belongs to the limit distributions in mathematical extreme value theory, see [11], which studies minima (or maxima) of a large number of random variables. This fact may ensure that the Weibull distribution is a good model even if the assumptions below appear to be not completely realistic.

In this paper we study statistically the dependence of the parameter z on particle size. The relationship of z to the parameter usually called Weibull modulus m is a non-trivial problem. Mathematically, it depends on mechanical assumptions, in particular on the assumption on the positions of (essential) flaws, in volume or surface, and on the relationship between external load l and internal stress

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U. Jansen(✉), D. Stoyan
Institute of Stochastics
Freiberg University of Mining and Technology
09596 Freiberg, Germany
e-mail: stoyan@orion.hrz.tu-freiberg.de

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σ, especially on traction and shearing components. In order to be concrete, we describe now two usual mechanical models.

In a nutshell, the Weibull failure model can be described as follows [3,12]:

In the grain under load, there is a Poisson point process of flaw locations $\vec{x}_1, \vec{x}_2, \ldots$. This process may be homogeneous or inhomogeneous, i.e. there may be a location-dependent density $\mu(\vec{x})$ of flaws in the grain. (The term 'flaw' stands here for all thinkable defects, in particular Griffith cracks.) The flaws have independent identically distributed sizes or strengths $A_1, A_2, \ldots$ that are independent of the locations. Their probability distribution function $H(a) = P(A_i \le a)$ reflects properties of the material and its processing. It is commonly assumed that H has approximately the form

$$H(a) = 1 - \left(\frac{a_0}{a}\right)^{\frac{m}{2}} \quad \text{for } a > a_0 \ , \tag{2}$$

where a_0 is some minimum strength and m is the Weibull modulus.

The load l on the grain causes a related stress distribution $\sigma(\vec{x}, S, T)$ within the grain or on the surface of the grain, respectively, where $\vec{x}$ is the location, S denotes the system of external loads, and T represents environmental variables such as, for example, temperature. It is assumed that

$$\sigma(\vec{x}, S, T) < \frac{K}{\sqrt{a_0}} = \sigma_0 \ ,$$

where K is some constant. The i-th flaw causes a fracture if

$$\sigma(\vec{x}_i, S, T) > \frac{K}{\sqrt{A_i}} \ ,$$

i.e. if the stress at $\vec{x}_i$ is too big. That is, the flaw which is weakest relative to its stress is responsible for failure. Consequently, the probability that the grain suffers fracture is

$$1 - \exp\left\{ - \int \left(\frac{\sigma(\vec{x}, S, T)}{\sigma_0} \right)^m \mu(\vec{x}) \mathrm{d}\vec{x} \right\} \ ,$$

where the integral extends over the whole grain or its surface [3]. The parameter m is called the Weibull modulus.

Several assumptions now lead to the Weibull distribution function

$$F(l) = 1 - \exp(-\lambda l^z) \quad \text{for } l > 0 \ . \tag{3}$$

Here, λ and z are model parameters and l characterizes the value of external load as in (1). Note that z is not necessarily equal to m. While m characterizes the reaction on local stress, z is related to external load.

In one approach [3] it is assumed that

$$\sigma(\vec{x}, S, T) = l \, g(\vec{x}, S, T) \ , \tag{4}$$

where $g(\vec{x}, S, T)$ is the stress distribution under unit load, i.e., the stresses are proportional the external load l. In this case it is $z = m$ and

$$\lambda = \int \left(\frac{g(\vec{x}, S, T)}{\sigma_0} \right)^m \mu(\vec{x}) \mathrm{d}\vec{x} \ .$$

For the calculation of fracture probabilities one has to determine $g(\vec{x}, S, T)$ and then to integrate. Papers in which such calculations are demonstrated by means of finite element methods are [13,14,17].

Another approach [5] is based on the Hertz theory of stress distribution at contact between a sphere and a flat surface and assumes that the flaws are located only in the particle surface. In this case it is

$$z = (2 + m)/5 \ . \tag{5}$$

For the present paper, which concentrates on the study of z, the parameter λ is uninteresting. If the grain is spherical, then according to both approaches the diameter has influence on λ, but not on z.

Tsoungui et al. [4,15] study the random variable l_{crit} (or F_{crit} in the notation of [4]), the fracture force or load required to break the grain. The distribution function of l_{crit} is just the F given by (3). These authors show that, in the case of a spherical grain of radius R, the distribution of l_{crit} includes a scaling factor c for the force values, which follows the power law

$$c = K_0 \, R^\alpha \ , \tag{6}$$

where α is a suitable exponent. It is given by

$$\alpha = \frac{2m - 3}{m} \tag{7}$$

and

$$\alpha = \frac{2m - 3}{m + 3} \tag{8}$$

in dependence of volume (7) or contact fracture (8). Again, m is here the Weibull modulus. A consequence of (6) is that also the mean and all quantiles of the distribution of l_{crit} satisfy a power law analogous to (6).

For clinker cement grains the value $\alpha = 1.5$ is given in [15]. It leads by (7) and (8) to the values of $m = 6$ and $m = 15$ respectively. Using $m = 15$ and (5) corresponding to the contact fracture approach the value $z = 3.2$ is obtained, which is close to our estimate $z = 3.1$ for the smallest grain size of 5 mm, see Table 1 below.

3 Estimation of Weibull parameters

A common method of construction of statistical parameter estimators is the maximum-likelihood method. In the case of the Weibull distribution it leads to two non-linear equations for the parameters λ and z:

$$\frac{1}{z} = \frac{\sum_{i=1}^n \ln(l_i) l_i^z}{\sum_{i=1}^n l_i^z} - \frac{1}{n} \sum_{i=1}^n \ln(l_i) \tag{9}$$

$$\frac{1}{\lambda} = \frac{1}{n} \sum_{i=1}^n l_i^z \ , \tag{10}$$

see, for example, [3]. Here the l_i are the observed loads or critical forces leading to fractures. The first non-linear equation for z is commonly solved numerically by a so-called zero-routine.

Unfortunately, the estimator of z is unbiased with the effect that the estimated values of z decrease with increasing number n of observations, since then there appear more frequently big l_i values.

An alternative method is based on the empirical distribution function $\hat{F}$ of critical load. $\hat{F}(l)$ is simply the fraction of grains in the sample which have failed under load l. Obviously, the Weibull distribution function can be transformed into a linear function:

$$\ln\left(\ln\left(\frac{1}{1-F(l)}\right)\right) = z \ln l + \ln \lambda \ , \tag{11}$$

i.e., the Weibull distribution function becomes a straight line in a coordinate system with correspondingly scaled axes. A straight line fitted to the transformed empirical distribution function leads to estimates of the parameters. The slope estimates z, while the parameter λ can be found by means of

$$\lambda = \frac{1}{l_{cut}^z} \ , \tag{12}$$

where l_{cut}^z is given by $\hat{F}(l_{cut}) = 0.632$, which sets the left side of (11) to zero.

The fit of a straight line to $\hat{F}$ is carried out numerically by means of linear regression, preferably by a weighted-least-square method, where quantiles near to the median get a higher weight than distant ones. This results from the observation that in ordinary least squares the slope, and consequently z, is essentially determined by values of the empirical distribution function corresponding to high and low probabilities, but just they can be estimated only with low accuracy. Furthermore, in the sum of squares the squared *relative* residuals with respect to the given loads or forces were taken instead of the absolute residuals.

4
Experimental conditions

The fracture experiments of Schubert and his coworkers are described in detail in [8]. They used a material testing machine of typ ZD 10/90 of the former East German 'Wergzeugmaschinenkombinat Fritz Heckert' with a maximum force of 100 kN. It was used for single grains in a pressure pot, which is an experiment mechanically analogous to diametral compression between two plates. The pot was used for collecting all fracture parts of the grain. The plates consisted of special alloys of hard metals called HG10 for glass and HG80 for the other materials. The critical forces were taken from force-way-diagrams automatically registered during the experiments.

The velocity of the plates was constant, nearly 0.13 mm/s for the materials clinker cement and limestone. For the material glass the velocity differed with size, the speeds were 0.096 mm/s for spheres of diameter 3 mm and 5 mm and 0.016 mm/s for the other diameters between 8 mm and 16 mm. As May [8] showed the results of the experiments depend on the velocity of the plates. He carried out fracture experiments with different speeds and found that the empirical distribution functions for different speeds show for fixed size only a parallel shift of the curves, which means that the Weibull modulus z is not altered. Since the velocities used are small, the authors believe that correct results for z were obtained.

Only for the material glass true spheres were used as specimens. In the other cases 'nearly spherical grains' were selected for the experiments, the corresponding grain sizes are approximating diameters.

Furthermore, the specimens of a fixed size and material are selected with respect to their masses. Thus, the form of the grains does not influence the stability of the experimental conditions in a critical manner. The masses in a fixed class differ in a range from 5% to 20% in dependence of material and size.

For each material and each grain size at least 1000 specimens were used. Only for the largest grains of cement clinker of size 20 mm and limestone of size 15 mm, only 300 and 500 specimens could be used, respectively. The large samples guarantee a correct representation of the probability laws by empirical distribution functions.

In the experiments the term 'fracture' means an essential breakage of the grain. Thus, breaking-off of corners and other little pieces on the surface of the grains was not considered as a fracture. The force-way-diagrams showed in these cases only little decreases of force.

In order to check the quality of the data of Schubert and his coworkers, the authors looked for hints that the empirical distribution functions behave like mixtures or superpositions of Weibull distributions. If there were mixtures, then this would point to changing experimental conditions, while superpositions would point to different flaw types, especially volume or surface flaws.

In the latter case, the distribution function of grain strength would take a form such as

$$F(l) = 1 - \exp\left(-\sum_i \lambda_i l^{z_i}\right) \ , \tag{13}$$

where the indices i correspond to different flaw types [3]. If the parameters z_i would show big differences, then the curves of the distribution functions would show characteristic salient points, see the examples in [3, p. 153].

Between a superposition and a mixture of several Weibull distributions there is an essential difference. The distribution function of a mixture is

$$F(l) = 1 - \sum_i w_i \exp\bigl(-\lambda_i l^{z_i}\bigr) \tag{14}$$

with

$$\sum_i w_i = 1 \ , \tag{15}$$

where the weights w_i belong to parts of the experiment under fixed conditions.

The authors did not find any evidence of superpositions and only weak ones of mixtures. Thus they fitted usual Weibull distribution functions to the data.

5
Results of estimation

The statistical analysis started with empirical distribution functions such as those shown in Fig. 1, which is a copy of [8].

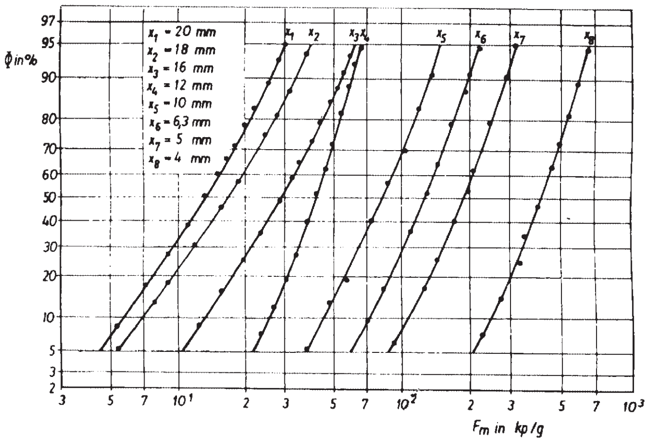

Fig. 1. Empirical distribution functions of critical forces F_m for clinker cement (working speed $v \approx 0.13$ mm/s), see [8]

The curves analyzed correspond to the three materials clinker cement, limestone and glass and are taken from [6,8]. The differences between the results for the material clinker cement are negligible for the same sizes with some irregularity for the size 12 mm, which we cannot explain. Similarly as in [15], it is possible to assume that the 12 mm particles come from another charge of cement.

We take the values l_j as quantiles corresponding to the probabilities p_j of the critical forces F_m for the corresponding grain size. The values of the forces are always normalized by the masses of the grains, i.e., the values of l_j (and of the residuals δ_j explained below) were originally measured in the today obsolete unit [kp/g], which corresponds to [9.81 N/g]. Clearly, this form of scaling does not have any influence on the estimation of the parameter z.

For each empirical distribution function $\hat{F}$ seven quantiles were estimated for a probability range between 0.05 to 0.95. Then the parameter estimation procedure described in Section 3 was carried out. For example, Table 1 shows the quantiles and estimation results for clinker cement. It gives first the empirical data, namely the empirical quantiles l_j corresponding to the probabilities p_j given by $\hat{F}(l_j) = p_j$. Second, it presents the estimated parameters and residuals δ_j characterizing the quality of the fit by the Weibull distribution. The δ_j are defined as

$$F_{\hat{z}}(l_j + \delta_j) = p_j \ , \tag{16}$$

where $F_{\hat{z}}$ is the fitted Weibull distribution function with parameter $\hat{z}$. Finally, it gives the corresponding values for a Weibull distribution with a global z close to the average of the various z-values.

The results for the other materials are similar and can be obtained from the authors upon request. The statistical analysis has yielded two clear results:

(1) For each material and grain size, the Weibull distribution fits the data very well.
(2) The parameter depends on the grain size as shown in Fig. 2.

A clear size dependence is obtained for the materials limestone and clinker cement from the samples in [6]. Here the relationship is nearly linear, z decreases with increasing grain diameter. The differences between the z-values for small and large sizes are so big that an average z would lead to essential errors in the calculation of fracture probabilities, see the residuals in the last column of Table 1.

Table 1. Statistical data for clinker cement (using Schubert [6])

Size	j	p_j	l_j	$\hat{z}$	δ_j (for $\hat{z}$)	z	δ_j (for z)
20	1	0.05	4.3	1.972	−0.549	2.5	0.841
	2	0.10	5.4		0.003		1.456
	3	0.20	7.8		0.105		1.456
	4	0.50	13.0		1.044		1.565
	5	0.70	18.0		0.582		0.165
	6	0.90	26.0		−0.184		−2.456
	7	0.95	30.0		−0.499		−3.843
16	1	0.05	10.0	2.197	−0.630	2.5	1.098
	2	0.10	13.0		0.001		1.801
	3	0.20	18.0		0.294		1.983
	4	0.50	29.5		1.142		1.946
	5	0.70	39.0		0.395		0.217
	6	0.90	54.0		−1.082		−3.170
	7	0.95	60.0		−0.350		−3.528
10	1	0.05	37.0	2.748	−3.128	2.5	−6.951
	2	0.10	43.0		1.015		−2.924
	3	0.20	57.0		0.836		−2.894
	4	0.50	85.0		2.362		0.141
	5	0.70	105.0		1.803		1.183
	6	0.90	135.0		0.225		2.625
	7	0.95	150.0		−1.185		2.901
5	1	0.05	85.0	3.123	−0.753	2.5	−20.686
	2	0.10	105.0		1.088		−19.227
	3	0.20	135.0		−1.093		−19.197
	4	0.50	195.0		−1.060		−12.774
	5	0.70	230.0		1.450		−2.738
	6	0.90	285.0		−0.138		9.555
	7	0.95	310.0		−0.091		17.251

p_j – probability
l_j – estimated quantile (cp. F_m in Fig. 1) [kp/g]
δ_j – residual (see text for explanation) [kp/g]
$\hat{z}$ – estimated parameter
z – average parameter

For the material glass the relationship is ambiguous. One interpretation of Fig. 2 is that also the material glass shows a tendency of decreasing z in the whole range of diameters, which is, however, not a linear relationship. An alternative interpretation is that z is constant with the exception of very small diameters.

It is interesting to compare the results in Fig. 2 with [5,16]. According to [5], the parameter z and thus also the Weibull modulus is nearly independent of sphere diameter. But, while Weichert [5] says that the Weibull modulus is constant for glass spheres. His Fig. 2 shows that just the smallest glass spheres (of diameter 3 mm, as in our Fig. 2) have a Weibull modulus greater than the other spheres. While he explains this observation by altered basic material properties, the authors believe that also here a size effect may play a role. Also Fig. 4 in [5], which corresponds to particles of quartz, can be seen as a hint that very small particles like those of diameter 1 mm have a greater z. Furthermore, Weichert [5] remarks that the properties of

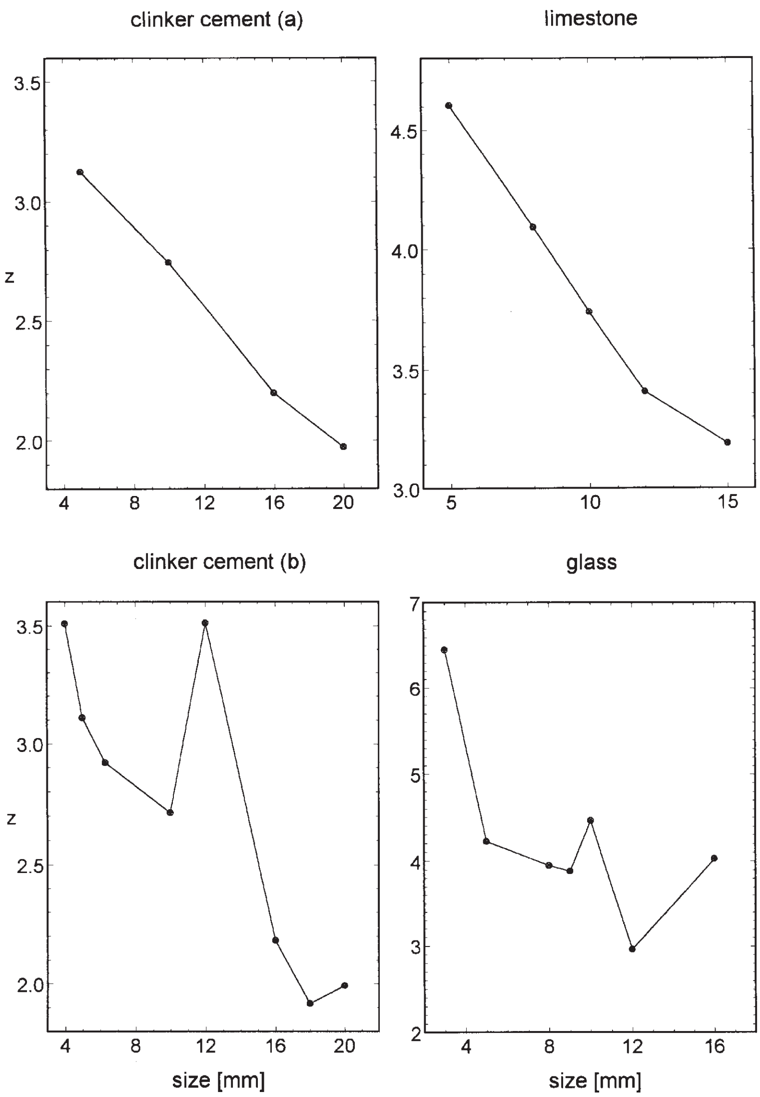

Fig. 2. Estimated Weibull parameters z in dependence on size and material

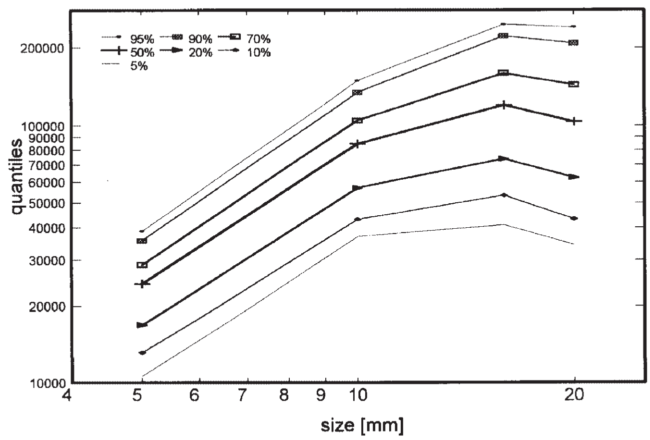

Fig. 3. Dependence between quantiles of forces (arranged from top (95%) to bottom (5%)) and sizes (forces not normalized by masses)

the surface, e.g. rough or smooth, influence the results essentially. In close relation to this, we observe clear linear relationships Fig. 2 just for the cases of clinker cement and limestone, where rough surfaces are typical and non-spherical grains were used in the fracture experiments.

Following [4], the authors also studied the relationship between critical force and grain size. While the scaling relation (6) leads to a linear relationship for the logarithm of the quantiles l_j corresponding to fixed probability p_j and the logarithm of the sizes, in the case of a size-dependent z a non-linear relationship has to be expected. Figure 3 shows the corresponding relations for all seven probabilities p_j used in Table 1. It is remarkable that in fine agreement with the theory in [4] the curves are nearly parallel. However, the relation obtained by using in (7) a linear relationship between the Weibull modulus and size does not follow these curves exactly. Obviously, the inhomogeneity of the material influences both parameters m (or z) and λ of the Weibull distribution in a more complex and dependent way as in the Griffith theory.

6 Conclusions

The statistical analysis seems to show that for the particles considered the parameter z and, if one of the models of Section 2 is accepted, the Weibull modulus m depend on particle size. Nevertheless, the authors believe that this result is not necessarily a contradiction to the traditional assumption of a material-dependent constant Weibull modulus. In order to retain this assumption, they recommend to consider 'particles of different size of the same material' as being of 'different materials' because of the different 'inner architecture' of small and large particles.

For explaining the observed dependence of the Weibull modulus on the particle size, the authors offer the following possibilities. First, the inner architecture of small and large grains may be different. Grains like cement clinker are formed in a growth process and have a hard mantle and a softer interior; for smaller grains, the relative fraction of the hard mantle may be bigger. In the case of clinker cement, for grains of a size considered here, it is known that the mantle is pressure-strengthened while the interior is traction-strengthened. This results from the technology of producing the grains, which includes an abrupt cooling process. Second, in some cases the grains consist of subgrains, the number of which increases with grain size. It may be easier to divide a grain into subgrains than to destroy a subgrain. This may explain the size-dependence of the Weibull modulus for cement clinker and limestone and the size-independence in the case of glass. These first two arguments are in the spirit of Szabó [10] who emphasizes the idea to consider anisotropy and inhomogeneity of real grains and shows their influence on strength. One of the anonymous referees wrote that the parameters 'particle size' or 'diameter' is a summary parameter which describes in some form structural properties of the particles as well as load conditions.

Third, for larger particles the assumption of independent micro-cracks causing fracture is perhaps incorrect. From acoustic records accompanying fracture experiments it is known that the breakage is a time-dependent process consisting of many little steps preparing the main fracture, see also the statistical model in [18]. In large particles, crack propagation may be a collective phenomenon of micro-cracks accelerating the fracture process. Also the so-called R-curve effect ([3], chapter 5) in crack propagation may be a hint that the assumption of a size independent defect behaviour is too simple.

While it is a very satisfactory observation that the Weibull distribution turns out to be an excellent approximation for the grains studied in this paper, the question is still open whether there is a physical and mathematical model that explains the size dependence of the Weibull modulus. Probably, the simple Griffith-Weibull theory has to be refined.

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