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ORIGINAL ARTICLE

New Existence Results for Evolutionary Quasi-Hemi-Variational Inequalities with Semi-monotone Maps and Applications to Optimal Control Problems

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Abstract
The paper presents new results for the existence of solutions for an evolutionary quasi-hemi-variational inequality, which involves a semi-monotone and a pseudo-monotone map. A crucial aspect of this research lies in the utilization of variational selection to disentangle the monotonic and pseudo-monotonic components. By employing this technique, we can bypass the commonly made assumption that the sum of the two maps must be monotone. Consequently, we establish various new existence results for evolutionary quasi-hemi-variational inequalities. Furthermore, we demonstrate applications of these results in solving optimal control problems.

Keywords Evolutionary quasi-variational-hemivariational inequality · Variational selection · Generalized solutions · Optimal control

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1 Introduction

Let X and Y be reflexive Banach spaces with X^* and Y^* as their dual spaces, respectively. The duality pairing of a Banach space Z and its dual space Z^* will be denoted by $\langle \cdot, \cdot \rangle_Z$, and by $\| \cdot \|_Z$ we denote the norm on Z. Given a Banach space Z, the domain and the graph of a multi-valued map $F : Z \rightrightarrows Z^*$ are denoted by $D(F) := \{u \in Z \mid F(u) \neq \emptyset\}$ and

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$G(F) := \{(u, w) \mid u \in D(F), w \in F(u)\}$, respectively. Let $\gamma : X \to Y$ be a compact, linear map, and let $L : D(L) \subseteq X \to X^*$ be a linear, maximal monotone map (so, in particular, densely defined and closed). Let $A : X \rightrightarrows X^*$ and $B : Y \rightrightarrows Y^*$ be multi-valued maps, let K be a closed and convex subset of X, let $C : K \rightrightarrows K$ be a multi-valued map such that for any $z \in K$, $C(z)$ is a nonempty, closed, and convex subset of K, and let $f \in X^*$. Let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper functional (i.e., $\not\equiv +\infty$), with its (effective) domain defined by $D(\Phi) := \{u \in X \mid \Phi(u) < +\infty\}$.

In this work, our primary focus is on the following evolutionary quasi-hemi-variational inequality of finding $u \in C(u) \cap D(L) \cap D(\Phi)$ such that for some $u^* \in A(u)$ and $v^* \in B(\gamma u)$, we have

$$\langle L(u) + u^* - f, z - u\rangle_X + \langle v^*, \gamma z - \gamma u\rangle_Y \geq \Phi(u) - \Phi(z) \quad \text{for all } z \in C(u) \cap D(L). \quad (1)$$

Due to the presence of the unbounded operator L, which is the prototype of the time derivative $L(u) = u'$ on a Banach space $X = L^p(0, \tau, V)$, with $1 < p < +\infty$, $\tau > 0$, and a reflexive Banach space V, the quasi-hemi-variational inequality (1) is of evolutionary nature. Before proceeding further, we mention a few relevant special cases of (1). If $B = 0$ in (1), we recover the evolutionary quasi-variational inequality of finding $u \in C(u) \cap D(L) \cap D(\Phi)$ such that for some $u^* \in A(u)$, we have

$$\langle L(u) + u^* - f, z - u\rangle_X \geq \Phi(u) - \Phi(z) \quad \text{for all } z \in C(u) \cap D(L). \quad (2)$$

Furthermore, if additionally the map A is single-valued with $D(A) = X$, $\Phi = 0$, and $L = 0$, then (2) becomes the quasi-variational inequality studied by Bensoussan and Lions [9]:

$$\langle A(u) - f, z - u\rangle_X \geq 0 \quad \text{for every } z \in C(u). \quad (3)$$

If $C(u) = K$, for every $u \in K$, then (3) recovers the celebrated variational inequality of finding $u \in K$ such that

$$\langle A(u) - f, z - u\rangle_X \geq 0 \quad \text{for every } z \in K.$$

Over the past sixty years, variational inequalities have been extensively researched due to their wide-ranging applications across various fields and deep mathematical relevance. This exploration of diverse applied models within a variational framework has paved the way for the emergence of several generalizations of variational inequalities. Notably, two variations that have garnered substantial attention are quasi-variational inequalities and hemi-variational inequalities. However, unlike variational inequalities, quasi-variational inequalities possess a distinct characteristic wherein the underlying constraint set varies and is dependent on the unknown solution. To be specific, notice the dependence of the constraint set $C(u)$ in (1) on the solution u, which presents a significant challenge when dealing with quasi-variational inequalities. As a result, only a fraction of the theoretical and computational tools available for variational inequalities have been adapted to address the realm of quasi-variational inequalities. Interestingly, many practical models involve dynamic or moving constraints similar to those found in (1). To highlight the wide range of applied models that can be effectively represented as quasi-variational inequalities, we mention economic growth models [31], elastohydrodynamics [27], energy production management [11], equilibrium problems [5], frictional elastostatic contact [16, 46], frictionless quasistatic contact with history-dependent stiffness [51], temperature dependent velocity constraint [19], image processing [39], Nash game equilibrium [25, 52] and multiobjective elliptic control [10, 17, 26], reaction diffusion [47], sandpiles formation [8], semiconductor and transistor design [38, 50], shape optimization [1], superconductivity models [7, 49].

We also observe that the set-valued map B in (1) aims to include a generalized derivative of a nonconvex functional. This enables (1) to encompass a wider range of problems, including hemi-variational inequalities as a special case. At this juncture, we also note that the notion of hemivariational inequalities was initially introduced by P.G. Panagiotopoulos in the early 1980s to address contact mechanics problems involving non-monotone and multivalued friction laws, see, e.g., [48]. Unlike variational inequalities, which primarily deal with problems featuring convex potentials, hemi-variational inequalities emerge from models with nonconvex and nonsmooth superpotentials associated with locally Lipschitz functions. In recent years, hemi-variational inequalities have gained significant attention due to their broad applications in various fields, such as engineering, transportation, and economics, as well as mathematical novelty. As a result, they have been extensively expanded and explored in numerous directions, see e.g., [6, 13, 18, 20, 24, 40, 43].

Since the introduction of quasi-variational inequalities, the predominant approach for their solvability has involved defining a set-valued map known as the variational selection. This method entails temporarily fixing the varying constraints as an arbitrary parameter, solving the resultant parametric variational inequality to obtain the image of the variational selection, and subsequently seeking a fixed point of the variational selection.

To elucidate, we fix an element $w \in K$ arbitrarily, and consider the parametric evolutionary hemi-variational inequality of finding $u \in C(w) \cap D(L) \cap D(\Phi)$ such that for some $u^* \in A(u)$ and $v^* \in B(\gamma u)$, we have

$$\langle L(u) + u^* - f, z - u\rangle_X + \langle v^*, \gamma z - \gamma u\rangle_Y \geq \Phi(u) - \Phi(z) \quad \text{for all } z \in C(w) \cap D(L). \tag{4}$$

Thus, we can define a set-valued map, the so-called variational selection, $\widehat{\mathbb{S}} : K \rightrightarrows K$ such that for any $v \in K$, the set $\widehat{\mathbb{S}}(v)$ is the set of all solutions of (4) for $w = v$. It is evident that if u is a fixed point of the map $\widehat{\mathbb{S}}$, then u is a solution of (1).

However, a major limitation of the aforementioned approach is the imposition of strict conditions on the associated variational selection $\widehat{\mathbb{S}}$. The most demanding requirement is that the solution set $\widehat{\mathbb{S}}(w)$ must be convex for all $w \in K$. In situations where a monotone map defines the parametric variational inequalities, the convexity of the solution set can be established through the application of the linearization technique, commonly known as the Minty formulation. This, in the context of (2), means that the map A is monotone. However, when we deviate from the monotone framework and consider a more general map defining the parametric variational inequalities, such as (1), the convexity of the solution set cannot be established, in general. This challenge becomes especially prominent when tackling quasi-hemi-variational inequalities. To overcome this technical constraint, numerous researchers studying quasi-hemi-variational inequalities, which involve the summation of two operators - often one monotone and the other pseudo-monotone - adopt a broad assumption that the overall sum is monotone. For example, in the study of (1), it has been a common assumption that $(A + B)$ is monotone. In this context, it is worth mentioning the insightful work of Kano, Kenmochi, and Murase [32], who have successfully extended the application of variational selection to quasi-variational inequalities involving more general semi-monotone maps. Semimontone maps, defined on a product space and denoted as $A(u, v)$, incorporate certain monotonicity properties in the second argument. The key concept employed in [32] is to freeze the nonmonotonicity in the first argument employing the variational selection and consider a parametric variational inequality with a monotone map. Recently, Chadli, Li, and Mohapatra [15] independently proposed a similar approach for quasi-hemi-variational inequalities, removing the requirement for the sum of the two operators involved to be monotone.

Motivated by the recent advancements, notably the work of [15], along with the seminal ideas presented in [32], we embark on an extensive investigation of quasi-variational inequalities, with a specific emphasis on the following innovative facets:

1. We focus on parabolic quasi-hemi-variational inequalities with semi-monotone maps. This extends the results in [15] from the elliptic case to the parabolic case and for a more general map even for the elliptic case.
2. As a particular case, we derive new results for parabolic quasi-variational inequalities, extending the results in [32] from the elliptic case to the parabolic framework. The study of the generalized solutions in our paper is also new for parabolic problems.
3. Although the key idea of defining the variational selection is the same as in [15], our proofs are quite different and much more concise. To be specific, we bypass the use of elliptic regularization.
4. We also consider a novel application to optimal control problem governed by evolutionary quasi-hemi-variational inequalities.

We organize the contents of this paper into six sections. Section 2 collects the necessary background material. In Section 3, we give the main existence results for the considered evolutionary hemi-quasi variational inequality. Section 4 is devoted to the study of generalized solutions, whereas Section 5 gives an application to optimal control problems governed by an evolutionary hemi-quasi variational inequality. The paper concludes with some remarks and open questions in Section 6.

2 Preliminaries

In the following, we collect some necessary background material. For details, see [53].

Definition 2.1 Let X be a real Banach space with X^* as its dual and let $F : X \rightrightarrows X^*$ be a set-valued map.

(a) The map F is called monotone, if $\langle u - v, x - y\rangle_X \geq 0$ for every $(x, u), (y, v) \in G(F)$.
(b) The map F is called strictly monotone, if $\langle u - v, x - y\rangle_X > 0$ for every $(x, u), (y, v) \in G(F), u \neq v$.
(c) The map F is called maximal monotone, if F is monotone and $\langle u - v, x - y\rangle_X \geq 0$ for every $(y, v) \in G(F)$ implies $(x, u) \in G(F)$.
(d) The map F is called generalized pseudo-monotone if for any sequence $\{(x_n, w_n)\} \subset G(F)$ with $x_n \rightharpoonup x$ and $w_n \rightharpoonup w$ such that $\limsup_{n\to\infty}\langle w_n, x_n - x\rangle_X \leq 0$, we have $w \in F(x)$ and $\langle w_n, x_n\rangle_X \to \langle w, x\rangle_X$.

Definition 2.2 Let X be a real Banach space with X^* as its dual and let $F : X \rightrightarrows X^*$ be a set-valued map. The set-valued map $F : X \rightrightarrows X^*$ on a Banach space X is called pseudo-monotone if it satisfies the conditions:

(PM1) For each $x \in X$, the set $F(x)$ is nonempty, bounded, closed, and convex in X^*.
(PM2) For any sequence $\{(x_n, w_n)\} \subset G(F)$ such that $x_n \rightharpoonup x$ and $\limsup_{n\to\infty}\langle w_n, x_n - x\rangle_X \leq 0$, then for each $y \in X$ there exists $w(y) \in F(x)$ satisfying $\liminf_{n\to\infty}\langle w_n, x_n - y\rangle_X \geq \langle w(y), x - y\rangle_X$.
(PM3) F is upper semicontinuous from each finite-dimensional subspace of X to the weak-topology of X^*.

Remark 2.3 A map F satisfying the condition (PM3) is often called finitely continuous. Kenmochi [33, Lemma 1.2] noticed that a set-valued map $F : X \rightrightarrows X^*$ with $D(F) = X$ on a reflexive Banach space X that satisfies conditions (PM1) and (PM2) of Definition 2.2 is pseudo-monotone, provided that it satisfies the following condition:

(PM4) For each $x \in X$ and for each bounded subset B of X, there exists a constant $c(B, x)$ such that for every $(z, u) \in G(F)$ with $z \in B$, it holds $\langle u, z - x\rangle_X \geq c(B, x)$.

The above condition is satisfied by any monotone, single-valued map F with $D(F) = X$ and by bounded maps.

Remark 2.4 A pseudo-monotone map $F : X \rightrightarrows X^*$ on a reflexive Banach space X is generalized pseudo-monotone. Conversely, a generalized pseudo-monotone and bounded map $F : X \rightrightarrows X^*$ on a reflexive Banach space X satisfying (PM1) is pseudo-monotone (see [13, Propositions 2.122, 2.123]).

Definition 2.5 A map $\tilde{A} : X \times X \rightrightarrows X^*$ is called semi-monotone, if $D(\tilde{A}) = X \times X$ and the following conditions hold:

(SM1) For any fixed $v \in X$, the map $u \to \tilde{A}(v, u)$ is maximal monotone.
(SM2) Let u be an element of X and let $\{v_n\} \subset X$ be a sequence with $v_n \rightharpoonup v$ in X. Then, for every $u^* \in \tilde{A}(v, u)$, there exists a sequence $\{u_n^*\}$ such that $u_n^* \in \tilde{A}(v_n, u)$ and $u_n^* \to u^*$.

Given a semi-monotone map $\tilde{A} : X \times X \rightrightarrows X^*$, the map $A : X \rightrightarrows X^*$ defined by $A(u) = \tilde{A}(u, u)$, for all $u \in X$ is called the map generated by $\tilde{A}$.

The utility of the above concept is reflected in the following result [32]:

Theorem 2.6 *Let $\tilde{A} : X \times X \rightrightarrows X^*$ be a semi-monotone map, and let $A : X \rightrightarrows X^*$ be the map generated by $\tilde{A}$. Then the following properties are satisfied:*

(a) *For any $v, u \in X$, $\tilde{A}(v, u)$ is a nonempty, closed, bounded, and convex subset of X^*.*
(b) *Let $\{u_n\}$ and $\{v_n\}$ be sequences in X such that $u_n \rightharpoonup u$ in X and $v_n \rightharpoonup v$ in X. If $u_n^* \in \tilde{A}(v_n, u_n)$, and $u_n^* \rightharpoonup u^*$ in X^*, and $\limsup_{n\to\infty}\langle u_n^*, u_n\rangle \leq \langle u^*, u\rangle$, then $u^* \in \tilde{A}(v, u)$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.*

We recall the following result from Kenmochi [33, Proposition 4.1].

Theorem 2.7 *Let $F : X \rightrightarrows X^*$ be a set-valued map on a reflexive Banach space X satisfying (PM1), (PM2), and (PM4), let C be a nonempty, closed, convex, and bounded subset of X, let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous, convex, with $C \cap D(\Phi) \neq \emptyset$, and let $f \in X^*$. Then there exists $x \in C \cap D(\Phi)$ such that for some $w \in F(x)$ we have*

$$\langle w - f, z - x\rangle_X \geq \Phi(x) - \Phi(z) \quad \textit{for every } z \in C.$$

We also recall the set convergence introduced by Mosco [44, 45].

Definition 2.8 Let X be a reflexive Banach space, let K be a nonempty, closed, and convex subset of X, and let $C : K \rightrightarrows K$ be a set-valued map. Then map C is termed as M-continuous if the following conditions hold:

(M1) For any sequence $\{x_n\} \subset K$ with $x_n \rightharpoonup x$, and for each $y \in C(x)$, there exists a sequence $\{y_n\}$ such that $y_n \in C(x_n)$ and $y_n \to y$.

(M2) For $y_n \in C(x_n)$ with $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, we have $y \in C(x)$.

Remark 2.9 We will make use of an M-continuity property of a map $C : K \rightrightarrows K$ relative to a function $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ which is $\not\equiv +\infty$, see condition $[\mathbb{H}_M]$.

We will also use the following result by Alber and Notik [2] concerning monotone maps.

Lemma 2.10 *Let Z be a reflexive Banach space with Z^* as its dual. Let $A : Z \rightrightarrows Z^*$ be a monotone map with $\bar{x} \in \operatorname{int}(D(A))$. Then there exists a constant $r = r(\bar{x}) > 0$ such that for every $(x, w) \in G(A)$ we have*

$$\langle w, x - \bar{x} \rangle \geq r\|w\| - (\|x - \bar{x}\| + r)c,$$

with $c := \sup\{\|w'\| \mid \|x' - \bar{x}\| \leq r$ and $w' \in A(x')\} < \infty$.

The following fixed point theorem by Kluge [37] will play an important role.

Theorem 2.11 *Let Z be a reflexive Banach space and let $D \subset Z$ be nonempty, convex, bounded, and closed. Assume that $P : D \rightrightarrows D$ is a set-valued map such that for every $u \in D$, the set $P(u)$ is nonempty, closed, and convex, and its graph $G(P)$ is sequentially weakly closed. Then P has a fixed point.*

Remark 2.12 In Theorem 2.11, the hypothesis on the set D to be bounded in Z can be replaced by requiring that the image $P(D)$ be bounded. For this it is sufficient to apply Theorem 2.11 to the closed convex hull $\overline{\text{co}}(P(D))$ of $P(D)$ in place of D.

Finally, we recall the following existence result from Asfaw and Kartsatos [3, Corollary].

Lemma 2.13 *Let X be a reflexive Banach space with X^* as its dual. Let $T : D(T) \subseteq X \rightrightarrows X^*$ be a maximal monotone map with $0_X \in D(T)$ and let $S : X \rightrightarrows X^*$ be finitely continuous, generalized pseudomonotone, and satisfy condition (PM4). Assume that the following coercivity condition holds:*

$$\inf_{w^* \in Su, u^* \in Tu} \frac{\langle u^* + w^*, u \rangle}{\|u\|} \to \infty \quad \text{as } \|u\| \to \infty. \tag{5}$$

Then $R(T + S) = X^*$.

3 Existence Results for Evolutionary Quasi-Hemi-Variational Inequalities

We begin with the following existence result for evolutionary quasi-hemi-variational inequalities.

Theorem 3.1 *Let X and Y be reflexive Banach spaces with X^* and Y^* as their dual spaces, let $\gamma : X \to Y$ be a compact linear map, and let $\gamma^* : Y^* \to X^*$ be the adjoint of γ. Let $K \subset X$ be a closed and convex set. We formulate the following hypotheses:*

$[\mathbb{H}_L]$ *$L : D(L) \subset X \to X^*$ is a linear, maximal monotone map.*

$[\mathbb{H}_A]$ *$\tilde{A} : X \times X \rightrightarrows X^*$ is a bounded semi-monotone map with $0_X \in \tilde{A}(\cdot, 0_X)$ that generates the map $A : X \rightrightarrows X^*$. There are constants $m > 0$ and $p > 1$ such that for each $(w, v) \in X \times X$, we have*

$$\inf_{v^* \in \tilde{A}(w,v)} \langle v^*, v \rangle \geq m\|v\|_X^p. \tag{6}$$

[$\mathbb{H}_B$] *$B : Y \rightrightarrows Y^*$ is a bounded map with sequentially strongly-weakly closed graph and nonempty, closed, and convex values. Moreover, there are constants $\tau > 0$ and $\mu > 0$ such that the following growth condition holds:*

$$\|v^*\|_{Y^*} \le \tau \|\gamma v\|_Y^{p-1} + \mu \quad \text{for every } v^* \in B(\gamma v),\ v \in X. \tag{7}$$

[$\mathbb{H}_S$] *The following compatibility condition relates the maps A and B:*

$$m > \tau \|\gamma\|^p. \tag{8}$$

[$\mathbb{H}_\Phi$] *$\Phi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, lower semicontinuous function with $K \subset \mathrm{int}(D(\Phi))$, $\Phi \ge 0$, and $\Phi(0_X) = 0$.*

[$\mathbb{H}_C$] *$C : K \rightrightarrows K$ is such that for any $w \in K$, $C(w)$ is closed, and convex with $0_X \in \mathrm{int}(\cap_{w \in K} C(w))$.*

[$\mathbb{H}_M$] *The following Mosco-type continuity properties hold:*

(a) *If $\{w_n\} \subset K$ and $u_n \in C(w_n) \cap D(L) \cap D(\Phi)$ satisfy $w_n \rightharpoonup w$ in X and $u_n \rightharpoonup u$ in X, with $u \in D(L)$ and $L(u_n) \rightharpoonup L(u)$ in X^*, then $u \in C(w) \cap D(\Phi)$.*

(b) *For every sequence $\{w_n\} \subset K \cap D(L)$ with $w_n \rightharpoonup w$ in X and for every $v \in C(w) \cap D(L)$, there exist a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and a sequence $v_k \in C(w_{n_k}) \cap D(L)$ with $v_k \to v$ in X and $\Phi(v_k) \to \Phi(v)$.*

Then the evolutionary quasi-hemi-variational inequality (1) is solvable, that is, there is at least one $u \in C(u) \cap D(L) \cap D(\Phi)$ such that for some $u^* \in A(u)$ and for some $v^* \in B(\gamma u)$, we have

$$\langle L(u) + u^* - f, z - u\rangle_X + \langle v^*, \gamma z - \gamma u\rangle_Y \ge \Phi(u) - \Phi(z) \quad \text{for all } z \in C(u) \cap D(L).$$

Proof We will divide the proof into five parts.

Step 1. We fix an element $w \in K$ arbitrarily and consider the parametric evolutionary variational inequality of finding $u \in C(w) \cap D(L) \cap D(\Phi)$ such that for some $u^* \in \tilde{A}(w, u)$ and $v^* \in B(\gamma u)$, we have

$$\langle L(u)+u^*-f, z-u\rangle_X+\langle v^*, \gamma z-\gamma u\rangle_Y \ge \Phi(u)-\Phi(z) \quad \text{for every } z \in C(w)\cap D(L). \tag{9}$$

For the solvability of (9), with the fixed $w \in K$, we define a functional $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ by

$$\Psi(u) = \begin{cases} \Phi(u) & \text{if } u \in C(w), \\ +\infty & \text{otherwise.} \end{cases}$$

Then (9) seeks $u \in C(w) \cap D(L) \cap D(\Psi)$ such that for some $u^* \in \tilde{A}(w, u)$ and $v^* \in B(\gamma u)$, we have

$$\langle L(u) + u^* - f, z - u\rangle_X + \langle v^*, \gamma z - \gamma u\rangle_Y \ge \Psi(u) - \Psi(z) \quad \text{for every } z \in X. \tag{10}$$

It follows that (10) is equivalent to finding $u \in D(L) \cap D(\partial\Psi)$ such that

$$f \in L(u) + \tilde{A}(w, u) + \gamma^* B(\gamma u) + \partial\Psi(u), \tag{11}$$

where $\partial\Psi$ stands for the convex subdifferential of Ψ.

We define a set-valued map $T : X \rightrightarrows X^*$ by $T(u) := L(u) + \tilde{A}(w, u) + \partial\Psi(u)$, which is a maximal monotone map because $0_X \in D(L) \cap \mathrm{int}(D(\partial\Phi))$ that follows from [$\mathbb{H}_C$] and [$\mathbb{H}_\Phi$]. In particular, it holds $0_X \in D(T)$. Moreover, define a set-valued map $S : X \rightrightarrows X^*$ by $S(u) = \gamma^* B(\gamma u)$. Due to (6), (7), (8), and [$\mathbb{H}_\Phi$], the operator $T + S$ satisfies the coercivity condition (5). The solvability of (11) follows from Lemma 2.13, which confirms the solvability of (9). Thus, we can define a set-valued map $\mathbb{P} : K \rightrightarrows K$ such that for each $w \in K$, $\mathbb{P}(w)$ is the solution set of (9), which we have proven to be nonempty.

Step 2. We will show that for each $w \in K$, the set $\mathbb{P}(w)$ of all solutions of (9) is bounded. Moreover, the following containment holds:

$$\cup_{w\in K}\, \mathbb{P}(w) \subset B_X(0_X, c_0), \tag{12}$$

where $B_X(0_X, c_0)$ is the closed ball in X, which is centered at the origin 0_X and has radius c_0.

To prove (12), we set $z = 0_X$ in (9), and rearrange the resulting inequality to obtain

$$\langle f, u\rangle_X - \langle v^*, \gamma u\rangle_Y \geq \langle L(u) + u^*, u\rangle_X + \Phi(u) - \Phi(0_X) \geq m\|u\|_X^p, \tag{13}$$

where we used the monotonicity of L, the coercivity condition (6), and assumption $[\mathbb{H}_\Phi]$. Using the growth condition (7) in (13), we obtain

$$m\|u\|_X^p \leq \|f\|_{X^*}\|u\|_X + \tau\|\gamma\|^p\|u\|_X^p + \mu\|\gamma\|\|u\|_X.$$

Taking into account assumption $[\mathbb{H}_S]$ and that $p > 1$, the above estimate leads to the bound:

$$\|u\|_X \leq (\|f\|_{X^*} + \mu\|\gamma\|)^{\frac{1}{p-1}}(m - \tau\|\gamma\|^p)^{-\frac{1}{p-1}} =: c_0. \tag{14}$$

Since $w \in K$ was chosen arbitrarily, the bound (14), which is independent of w, confirms that (12) holds.

Step 3. Let a closed ball $B_{Y^*}(0_{Y^*}, c)$ in Y^*, centered at the origin 0_{Y^*} and of radius $c > 0$ that will be fixed later. For fixed $w \in K$ and $\chi^* \in B_{Y^*}(0_{Y^*}, c)$, we consider the two-parameter evolutionary variational inequality of finding $u \in C(w) \cap D(L) \cap D(\Phi)$ such that for some $u^* \in \tilde{A}(w, u)$, we have

$$\langle L(u)+u^*-f, z-u\rangle_X+\langle \chi^*, \gamma z-\gamma u\rangle_Y \geq \Phi(u)-\Phi(z) \quad \text{for every } z \in C(w)\cap D(L). \tag{15}$$

Note that (15) can be written in the form

$$\langle L(u) + u^* - f + \gamma^*\chi^*, z - u\rangle_X \geq \Phi(u) - \Phi(z) \quad \text{for every } z \in C(w) \cap D(L), \tag{16}$$

ensuring that (15) is an evolutionary monotone variational inequality that can be solved by Step 1 arguments.

Step 4. We define another variational selection $\mathbb{S} : K \times B_{Y^*}(0_{Y^*}, c) \to K$ that associates to each $(w, \chi^*) \in K \times B_{Y^*}(0_{Y^*}, c)$, the set $\mathbb{S}(w, \chi^*)$ of all solutions of (16). For each $(w, \chi^*) \in K \times B_{Y^*}(0_{Y^*}, c)$, the solution set $\mathbb{S}(w, \chi^*)$ of (16) is closed and convex, as can be seen from [34].

Step 5. We begin with computing an upper bound on the solution set of (15). Setting $z = 0_X$ in (15) and using assumption $[\mathbb{H}_\Phi]$ result in the inequality

$$\langle L(u) + u^* - f, -u\rangle_X + \langle \chi^*, -\gamma u\rangle_Y \geq 0.$$

Proceeding as in Step 2, we infer that

$$\|u\|_X \leq m^{-\frac{1}{p-1}}(\|f\|_{X^*} + c\|\gamma\|)^{\frac{1}{p-1}}. \tag{17}$$

Next, from $[\mathbb{H}_B]$ and (17), we have for any $v^* \in B(\gamma u)$ that

$$\|v^*\|_{Y^*} \leq \tau\|\gamma u\|_Y^{p-1} + \mu \leq \tau\|\gamma\|^{p-1}m^{-1}(\|f\|_{X^*} + c\|\gamma\|) + \mu.$$

At this point, we determine c from the equation

$$c = \tau\|\gamma\|^{p-1}m^{-1}(\|f\|_{X^*} + c\|\gamma\|) + \mu,$$

which gives

$$c = (m - \tau\|\gamma\|^p)^{-1}(\tau\|\gamma\|^{p-1}\|f\|_{X^*} + m\mu).$$

For $\mathbb{D} := K_{c_0} \times B_{Y^*}(0_{Y^*}, c)$, where $K_{c_0} := K \cap B_X(0_X, c_0)$ for c and c_0 as given above, we define the set-valued map $\mathbb{Q} : \mathbb{D} \rightrightarrows \mathbb{D}$ by

$$\mathbb{Q}(w, \chi^*) = (\mathbb{S}(w, \chi^*), B(\gamma w)).$$

The preceding reasoning ensures that the range of $\mathbb{Q}$ is in $\mathbb{D}$.

By the aid of Theorem 2.11, we will prove that the map $\mathbb{Q}$ has a fixed point. We already know that $\mathbb{Q}(\mathbb{D})$ is bounded.

Since the map $\mathbb{S}(w, \chi^*)$ is closed and convex valued by Step 4, and since the map B is closed and convex valued by the assumption, the map $\mathbb{Q}$ is closed and convex valued. Therefore, to ensure that $\mathbb{Q}$ has a fixed point by Theorem 2.11, it suffices to show that the graph of $\mathbb{Q}$ is sequentially weakly closed.

Let $\{(h_n, g_n)\} \subset G(\mathbb{Q})$ be such that $h_n \rightharpoonup h$ and $g_n \rightharpoonup g$, where $h_n := (w_n, \chi_n)$, $g_n := (u_n, \xi_n)$, $h = (w, \chi)$, and $g = (u, \xi)$. The containment $(u_n, \xi_n) \in \mathbb{Q}(w_n, \chi_n)$ implies two things. Firstly, $\chi_n \in B(\gamma u_n)$. Secondly, $w_n \in K$, $u_n \in C(w_n) \cap D(L) \cap D(\Phi)$, and for some $u_n^* \in \tilde{A}(w_n, u_n)$, we have

$$\langle L(u_n) + u_n^* - f, z - u_n\rangle_X + \langle \chi_n, \gamma z - \gamma u_n\rangle_Y \geq \Phi(u_n) - \Phi(z) \quad \text{for all } z \in C(w_n) \cap D(L). \tag{18}$$

Leveraging the boundedness of the map $\tilde{A}$, up to a subsequence, we have $u_n^* \rightharpoonup u^*$ in X^*. To prove that $\{L(u_n)\}$ is bounded, we note that due to the assumption $[\mathbb{H}_C]$, we can choose a ball $\mathcal{B}$ in X such that $\mathcal{B} \subset C(w_n)$, for all n. Then by using (18) and the density of $D(L)$ in X, we deduce that the sequence $\{L(u_n)\}$ is bounded from below on the ball $\mathcal{B}$, which implies that $\{L(u_n)\}$ is bounded in X^*. Since L is linear, maximal monotone, it has a weakly closed graph, and hence we have $u \in D(L)$ and $L(u_n) \rightharpoonup L(u)$ in X^*. Moreover, due to assumption $[\mathbb{H}_M]$, we have that $u \in C(w) \cap D(\Phi)$.

Since $u \in C(w) \cap D(L) \cap D(\Phi)$, by assumption $[\mathbb{H}_M]$(b), we obtain a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and a sequence $\{z_k\}$ with $z_k \in C(w_{n_k}) \cap D(L)$ such that $z_k \to u$ in X and $\Phi(z_k) \to \Phi(u)$. We return to (18) with $n = n_k$ and $z = z_k$ to get

$$\langle L(u_{n_k}) + u_{n_k}^* - f, z_k - u_{n_k}\rangle_X + \langle \chi_{n_k}, \gamma z_k - \gamma u_{n_k}\rangle_Y \geq \Phi(u_{n_k}) - \Phi(z_k),$$

which can be written as

$$\begin{aligned}\langle u_{n_k}^*, u_{n_k} - u\rangle_X \leq{} & \langle u_{n_k}^*, z_k - u\rangle_X + \langle L(u_{n_k}) - f, z_k - u_{n_k}\rangle_X \\ & + \langle \chi_{n_k}, \gamma z_k - \gamma u_{n_k}\rangle_Y + \Phi(z_k) - \Phi(u_{n_k}).\end{aligned} \tag{19}$$

The monotonicity of L, combined with the convergence $u_{n_k} \rightharpoonup u$ in X and $L(u_{n_k}) \rightharpoonup L(u)$ in X^*, yield

$$\liminf_{k\to\infty} \langle L(u_{n_k}), u_{n_k}\rangle_X \geq \langle L(u), u\rangle_X. \tag{20}$$

Equipped with (20), we return to (19), and use $[\mathbb{H}_\Phi]$ and the compactness of the map γ, to obtain

$$\limsup_{k\to\infty} \langle u_{n_k}^*, u_{n_k} - u\rangle_X \leq 0,$$

which further leads to

$$\limsup_{k\to\infty}\langle u_{n_k}^*, u_{n_k}\rangle = \limsup_{k\to\infty}\left[\langle u_{n_k}^*, u_{n_k}-u\rangle + \langle u_{n_k}^*, u\rangle\right] \le \langle u^*, u\rangle.$$

Therefore, we can invoke Theorem 2.6, which ensures that

$$u^* \in \tilde{A}(w,u) \quad \text{and} \quad \langle u_{n_k}^*, u_{n_k}\rangle_X \to \langle u^*, u\rangle_X. \tag{21}$$

Let $v \in C(w)\cap D(L)$ be arbitrary. By assumption $[\mathbb{H}_M]$(b), there exists $v_k \in C(w_{n_k})\cap D(L)$ (up to a subsequence of $\{w_{n_k}\}$) such that $v_k \to v$ in X and $\Phi(v_k)\to\Phi(v)$. We return to use (18) with $z = v_k$ and $n = n_k$ and obtain

$$\langle L(u_{n_k}) + u_{n_k}^* - f, v_k - u_{n_k}\rangle_X + \langle \chi_{n_k}, \gamma v_k - \gamma u_{n_k}\rangle_Y \ge \Phi(u_{n_k}) - \Phi(v_k). \tag{22}$$

Letting $k\to\infty$ in (22) and using (20) and (21) lead to

$$\langle L(u) + u^* - f, v-u\rangle_X + \langle \chi, \gamma v - \gamma u\rangle_Y \ge \Phi(u) - \Phi(v). \tag{23}$$

We have shown that for $u \in C(w)\cap D(L)\cap D(\Phi)$, there exists $u^* \in \tilde{A}(w,u)$ such that for every $v\in C(w)\cap D(L)$, (23) holds.

On the other hand, since $\xi_n \in B(\gamma u_n)$, $\xi_n \rightharpoonup \xi$ and $u_n \rightharpoonup u$, we have $\xi \in B(\gamma u)$, thanks to hypothesis $[\mathbb{H}_B]$ and to the fact that γ is compact. Consequently, we have shown that the graph of $\mathbb{Q}$ is sequentially weakly closed. Therefore, we have verified all the conditions of Theorem 2.11, and as a result, the map $\mathbb{Q}$ admits a fixed point. In other words, there exists an element $(u,\xi)\in\mathbb{D}$ such that $(u,\xi)\in\mathbb{Q}(u,\xi)$. This confirms that $u\in C(u)\cap D(L)\cap D(\Phi)$ and for some $u^*\in A(u) = \tilde{A}(u,u)$ and some $\xi\in B(\gamma u)$, we have

$$\langle L(u) + u^* - f, v-u\rangle_X + \langle \xi, \gamma v - \gamma u\rangle_Y \ge \Phi(u) - \Phi(v) \quad \text{for every } v\in C(u)\cap D(L).$$

The proof is thus complete. □

Remark 3.2 When $B = 0$, we can deduce the solvability of evolutionary quasi-variational inequalities extending results in [32] from elliptic to parabolic case. Notably, in this scenario, we can also replace the coercivity condition with a much weaker recessivity condition introduced by Khan and Motreanu [34]. We note that when A is monotone, then the boundedness restriction can be dropped by the aid of Lemma 2.10 in Theorem 4.3 in the context of the generalized solutions. See also Khan and Motreanu [34]. This would extend the results in [15] from elliptic to parabolic and improve the results, even for the elliptic case.

4 Generalized Solution of Evolutionary Quasi-Hemi-Variational Inequalities

This section investigates the practicality of generalized solutions in the context of evolutionary quasi-hemi-variational inequalities. In the preceding section, we demonstrated the existence of solutions by seeking fixed points of the associated variational selection, leading to the imposition of stringent data assumptions. In contrast, we propose a different approach introducing generalized solutions through a well-formulated minimization problem, effectively addressed using a variant of the classical Weierstrass theorem. Remarkably, this alternative method requires fewer restrictive assumptions on the data. Consequently, we present a new existence theorem for evolutionary quasi-hemi-variational inequalities to emphasize the significant disparity between classical and generalized solutions.

We continue studying the evolutionary quasi-hemi-variational inequality that seeks $u \in C(u) \cap D(L) \cap D(\Phi)$ such that for some $u^* \in A(u)$ and $v^* \in B(\gamma u)$, we have (1). We also recall the notion of variational selection. For a fixed element $w \in K$ arbitrarily, we consider the parametric evolutionary variational inequality of finding $u \in C(w) \cap D(L) \cap D(\Phi)$ such that for some $u^* \in A(u)$ and $v^* \in B(\gamma u)$, we have (4). The variational selection $\widehat{\mathbb{S}} : K \rightrightarrows K$ associates to each $v \in K$, the set $\widehat{\mathbb{S}}(v)$ of all solutions of (4).

We now introduce the following optimization problem: find $(u, w) \in G(\widehat{\mathbb{S}})$ such that

$$\|u - w\|_X^2 \leq \|v - s\|_X^2 \quad \text{for every } (v, s) \in G(\widehat{\mathbb{S}}). \tag{24}$$

If $(u, w) \in G(\widehat{\mathbb{S}})$ is a minimizer of (24), then the element $u \in K$ is called a generalized solution of the evolutionary quasi-hemi-variational inequality (1).

The following connections between a generalized solution and a classical solution are readily apparent:

- If (24) has a solution (u, w) such that $\|u - w\|_X = 0$, then (1) has a solution.
- If (1) has a solution, then (24) also has a solution, and the solution sets of the two coincide.

Remark 4.1 Considering the preceding observation and adopting established terminology (see [23, Chapter 5]), the function employed in (24) may aptly be referred to as a "gap function". Nevertheless, it is important to note a departure from the conventional usage of gap functions. In our current context, our focus diverges from pinpointing the optimal value of this function to attain a solution for the quasi-variational inequalities. Instead, we aim to find a generalized solution that is typically different from the actual solution.

Many researchers have directed their focus towards (24) instead of (1) to harness the benefits of a minimization formulation. While the roots of this technique can be traced back to Mosco's original work [45], it is important to recognize the valuable contributions made by Bruckner [12], who investigated the relationship between simpler quasi-variational inequalities and the associated least-squares optimization problem, see [4, 28]. It should be noted that another advantage of (24) is its compatibility with numerous numerical procedures designed for seeking solutions to quasi-variational inequalities by minimizing the gap $\|u - w\|_X$ for $(u, w) \in G(\widehat{\mathbb{S}})$.

Based on the classical Weierstrass theorem, we can establish a general existence result for (24).

Lemma 4.2 [12] *Assume that there exists $(\tilde{u}, \tilde{w}) \in G(\widehat{\mathbb{S}})$ such that the set*

$$\mathbb{M} = \{(u, w) \in G(\widehat{\mathbb{S}}) : \ \|u - w\|_X \leq \|\tilde{u} - \tilde{w}\|_X\} \tag{25}$$

is sequentially weakly closed and bounded. Then (24) *has a nonempty solution set.*

A sufficient condition for the set $\mathbb{M} \subset X \times X$ to be bounded is that the following set is bounded in X for any constant c,

$$\widehat{\mathbb{M}} = \{u \in X : \text{there exists } w \in K \text{ such that } (u, w) \in G(\widehat{\mathbb{S}}) \text{ and } \|u - w\|_X \leq c\}. \tag{26}$$

Indeed setting $c = \|\tilde{u} - \tilde{w}\|_X$, we notice that if $\widehat{\mathbb{M}}$ in (26) is bounded, then $\mathbb{M}$ introduced in (25) is bounded.

We begin with the following existence result for evolutionary quasi-hemi-variational inequalities.

Theorem 4.3 *Let X and Y be reflexive Banach spaces with X^* and Y^* as their dual spaces, let $\gamma : X \to Y$ be a compact linear map, and let $\gamma^* : Y^* \to X^*$ be the adjoint of γ. Let $K \subset X$ be a closed and convex set. We assume the conditions:* $[\mathbb{H}_L]$, $[\mathbb{H}_\Phi]$, $[\mathbb{H}_C]$, $[\mathbb{H}_M]$, *and*

$[\mathbb{H}_A]'$ *$A : X \rightrightarrows X^*$ is a maximal monotone map with $0_X \in A(0_X)$ and $K \subset \operatorname{int}(D(A))$.*
$[\mathbb{H}_B]'$ *$B : Y \rightrightarrows Y^*$ is a bounded map with sequentially strongly-weakly closed graph and nonempty, closed, and convex values.*
$[\mathbb{H}_S]'$ *The maps A and $S := \gamma^* B \gamma$ fulfill*

$$\inf_{w^* \in Au, u^* \in Su} \frac{\langle u^* + w^*, u\rangle_X}{\|u\|_X} \to \infty \quad \text{as } \|u\|_X \to \infty,\ u \in K. \tag{27}$$

Then the evolutionary quasi-hemi-variational inequality (1) has a generalized solution, that is, the minimization problem (24) defined through the variational solution associated to (1) has a solution.

Proof We will break down the proof into three parts, demonstrating that the set $\mathbb{M}$ defined in (25) is nonempty, sequentially weakly closedness, and bounded, which will prove the existence of a minimizer of (24), leveraging the application of Lemma 4.2. This proves the existence of a generalized solution of (1).

Step 1. As done in the first step of Theorem 3.1, with a fixed $w \in K$, we define a functional $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ by

$$\Psi(u) = \begin{cases} \Phi(u) & \text{if } u \in C(w), \\ +\infty & \text{otherwise,} \end{cases}$$

and pose, for a fixed parameter w, the inclusion of finding $u \in D(L) \cap D(\partial\Psi)$ such that

$$f \in L(u) + A(u) + \gamma^* B(\gamma u) + \partial\Psi(u).$$

We also define the set-valued map $T : X \rightrightarrows X^*$ by $T(u) := L(u) + A(u) + \partial\Psi(u)$, which is a maximal monotone map with $0 \in D(T)$. Due to the coercivity condition (27), the solvability of (1) follows from Lemma 2.13. Thus, for any $w \in K$, the solution set $\widehat{\mathbb{S}}(w)$ of (1) is nonempty.

Step 2. We claim that the graph $G(\widehat{\mathbb{S}})$ of $\widehat{\mathbb{S}}$ is sequentially weakly closed in $X \times X$. To verify the claim, assume that $\{(u_n, w_n)\} \subset G(\widehat{\mathbb{S}})$ is a sequence converging weakly to (w, u) in $X \times X$. Then, $w_n \in K$, $u_n \in C(w_n) \cap D(L) \cap D(\Phi)$ and for some $u_n^* \in A(u_n)$ and for some $s_n^* \in B(\gamma u_n)$, we have

$$\langle L(u_n) + u_n^* - f, z - u_n\rangle_X + \langle s_n^*, \gamma z - \gamma u_n\rangle_Y \ge \Phi(u_n) - \Phi(z) \quad \text{for all } z \in C(w_n) \cap D(L). \tag{28}$$

Assumption $[\mathbb{H}_C]$ allows us to choose a ball $\mathcal{B}$ in X such that $\mathbb{B} \subset C(w_n)$ for all n. Utilizing (28), and proceeding as before in the proof of Theorem 3.1, we infer the boundedness of the sequence $\{L(u_n)\}$ in X^*, that $u \in D(L)$, and $L(u_n) \rightharpoonup L(u)$ in X^*. Moreover, by assumption $[\mathbb{H}_M]$ (b), we have $u \in C(w)$.

We will now prove that the sequence $\{u_n^*\}$ is bounded in X^*. Because $u \in C(w) \cap D(L)$, we invoke assumption $[\mathbb{H}_M]$ (b) and obtain a relabeled subsequence of $\{w_n\}$ and a sequence $z_n \in C(w_n) \cap D(L)$ with $z_n \to u$ in X and $\Phi(z_n) \to \Phi(u)$. We insert $z = z_n$ in (28) to get

$$\langle L(u_n) + u_n^* - f, z_n - u_n\rangle_X + \langle s_n^*, \gamma z_n - \gamma u_n\rangle_Y + \Phi(z_n) - \Phi(u_n) \ge 0.$$

We employ Lemma 2.10 for the operator $A - f$, and for certain constants $c > 0$ and $r > 0$ we obtain

$$\begin{aligned} r\|u_n^* - f\|_X &\leq \langle u_n^* - f, u_n - u\rangle_X + c(r + \|u_n - u\|_X) \\ &= \langle u_n^* - f, u_n - z_n\rangle_X + \langle u_n^* - f, z_n - u\rangle_X + c(r + \|u_n - u\|_X) \\ &\leq \|u_n^* - f\|_{X^*}\|z_n - u\|_X + c(r + \|u_n - u\|_X) \\ &\quad + \langle L(u_n), z_n - u_n\rangle_X + \Phi(z_n) - \Phi(u_n) + \langle s_n^*, \gamma z_n - \gamma u_n\rangle_Y, \end{aligned}$$

which can be written as follows:

$$\begin{aligned} [r - \|z_n - u\|_X]\|u_n^* - f\|_{X^*} &\leq c(r + \|u_n - u\|_X) + \langle L(u_n), z_n - u_n\rangle_X \\ &\quad + \Phi(z_n) - \Phi(u_n) + \langle s_n^*, \gamma z_n - \gamma u_n\rangle_Y. \end{aligned} \tag{29}$$

Recalling that $z_n \to u$ in X and noting that the right-hand side in (29) is bounded, we obtain that $\{u_n^*\}$ is bounded.

Moreover, since the map $\gamma : X \to Y$ is compact, we have that $\gamma u_n \to \gamma u$ in Y. Also, leveraging $[\mathbb{H}_B]'$, along a relabeled sequence, we have $s_n^* \rightharpoonup s^*$ in Y^* with $s^* \in B(\gamma u)$.

Let $v \in C(w) \cap D(L)$. Invoking assumption $[\mathbb{H}_M]$ (b), there exists $v_k \in C(w_{n_k}) \cap D(L)$ such that $v_k \to v$ in X and $\Phi(v_k) \to \Phi(v)$. From (28), with $z = v_k$, $u_{n_k}^* \in A(u_{n_k})$ and $s_{n_k}^* \in B(\gamma u_{n_k})$, we obtain

$$\langle L(u_{n_k}) + u_{n_k}^* - f, v_k - u_{n_k}\rangle_X + \langle s_{n_k}^*, \gamma v_k - \gamma u_{n_k}\rangle_Y + \Phi(v_k) - \Phi(u_{n_k}) \geq 0.$$

Then, for any $v^* \in A(v)$, the above inequality can be written as follows:

$$\begin{aligned} \langle L(v) + v^*, u_{n_k} - v\rangle_X &\leq \langle L(v) + v^*, u_{n_k} - v\rangle_X + \langle L(u_{n_k}) + u_{n_k}^* - f, v_k - u_{n_k}\rangle_X \\ &\quad + \Phi(v_k) - \Phi(u_{n_k}) + \langle s_{n_k}^*, \gamma v_k - \gamma u_{n_k}\rangle_Y \\ &= \langle L(u_{n_k}) + u_{n_k}^*, v_k - v\rangle_X + \langle L(u_{n_k}) + u_{n_k}^* - L(v) - v^*, v - u_{n_k}\rangle_X \\ &\quad + \langle s_{n_k}^*, \gamma v_k - \gamma u_{n_k}\rangle_Y + \langle f, u_{n_k} - v_k\rangle_X + \Phi(v_k) - \Phi(u_{n_k}) \\ &\leq \langle L(u_{n_k}) + u_{n_k}^*, v_k - v\rangle_X + \langle f, u_{n_k} - v_k\rangle_X \\ &\quad + \Phi(v_k) - \Phi(u_{n_k}) + \langle s_{n_k}^*, \gamma v_k - \gamma u_{n_k}\rangle_Y, \end{aligned}$$

where we used the monotonicity of A and L. In the limit, the above inequality reads as follows:

$$\langle L(v) + v^*, u - v\rangle_X \leq \Phi(v) - \Phi(u) + \langle f, u - v\rangle_X + \langle s^*, \gamma v - \gamma u\rangle_Y.$$

Therefore, we have shown that $u \in C(w) \cap D(L) \cap D(\Phi)$ and for every $v^* \in A(v)$ and for some $s^* \in B(\gamma u)$, we have

$$\langle L(v) + v^* - f, v - u\rangle_X + \langle s^*, \gamma v - \gamma u\rangle_Y \geq \Phi(u) - \Phi(v) \quad \text{for every } v \in C(w) \cap D(L). \tag{30}$$

We claim that (30) implies that for $u \in C(u) \cap D(L) \cap D(\Phi)$ there exists $u^* \in A(u)$ such that

$$\langle L(u) + u^* - f, v - u\rangle_X + \langle s^*, \gamma v - \gamma u\rangle_Y \geq \Phi(u) - \Phi(v) \quad \text{for all } v \in C(w) \cap D(L). \tag{31}$$

Indeed, by the definition of the convex subdifferential, for all $v \in C(u) \subset D(\partial\Phi)$, $w^* \in \partial\Phi(v)$, and $z \in X$, we have

$$\Phi(z) - \Phi(v) \geq \langle w^*, z - v\rangle_X.$$

We set $z = u$ in the above inequality and combine it with (30) to deduce that for any $v \in C(w) \cap D(L)$, for any $w^* \in \partial \Phi(v)$, and for any $v^* \in A(v)$, we have

$$\langle L(v) + v^* + w^* + \gamma^* s^* - f, v - u \rangle_X \geq 0. \tag{32}$$

We define a set-valued map $T : X \rightrightarrows X^*$ by $T := L + A + N_{C(w)} + \partial \Phi$, where $N_{C(w)}$ is the normal cone to $C(w)$ (i.e., the subdifferential of the indicator function of $C(w)$). Then T is a maximal monotone map with $D(T) = C(w) \cap D(L)$ (see, e.g., [53, Theorem 32.1]). Thus, for any $v \in C(w) \cap D(L)$ and for any $t^* \in N_{C(w)}(v)$ by (32), we have

$$\begin{aligned} &\langle L(v) + v^* + w^* + t^* + \gamma^* s^* - f, v - u \rangle_X \\ &= \langle L(v) + v^* + w^* + \gamma^* s^* - f, v - u \rangle_X + \langle t^*, v - u \rangle_X \geq 0, \end{aligned}$$

which, due to the fact that T is maximal monotone, yields $f - \gamma^* s^* \in (L + A + \partial \Phi + N_{C(w)})(u)$.

As a consequence, there are $u^* \in A(u)$, $\bar{t}^* \in N_{C(w)}(u)$, and $\bar{s}^* \in \partial \Phi(u)$ such that

$$L(u) + u^* + \bar{t}^* + \bar{s}^* + \gamma^* s^* - f = 0.$$

Then the inequalities $\langle \bar{t}^*, v - u \rangle_X \leq 0$ and $\Phi(v) - \Phi(u) \geq \langle \bar{s}^*, v - u \rangle_X$ for all $v \in C(w) \cap D(L)$ imply

$$\langle L(u) + u^* - f, v - u \rangle_X + \langle s^*, \gamma v - \gamma u \rangle_Y = \langle \bar{t}^*, u - v \rangle_X + \langle \bar{s}^*, u - v \rangle_X \geq \Phi(u) - \Phi(v),$$

and hence (31) holds. This proves that $(u, w) \in G(\widehat{\mathbb{S}})$, so the graph $G(\widehat{\mathbb{S}})$ is sequentially weakly closed, thereby the set $\mathbb{M}$ in (25) is sequentially weakly closed.

Step 3. We claim that the set $\mathbb{M}$ in (25) is bounded. For this, by Lemma 4.2 it suffices to show that the set $\widehat{\mathbb{M}}$, defined in (26), is bounded. Suppose by contradiction that the set $\widehat{\mathbb{M}}$ is unbounded. Hence, there exists a sequence $\{u_n\} \subset \widehat{\mathbb{M}}$ such that $\|u_n\|_X \to \infty$ as $n \to \infty$.

Let $w_n \in K$ be such that $u_n \in \widehat{\mathbb{S}}(w_n)$ and $\|u_n - w_n\|_X \leq c$ for all n, with a constant $c > 0$. Thus, $u_n \in C(w_n) \cap D(L) \cap D(\Phi)$ and there are $u_n^* \in A(u_n)$ and $s_n^* \in B(\gamma u_n)$ such that (28) holds. We set $z = 0_X$ and arrive at

$$\langle u_n^*, u_n \rangle_X + \langle s_n^*, \gamma u_n \rangle_Y \leq C \|u_n\|_X,$$

where $C > 0$ is a constant, contradicting the coercivity condition (27). Therefore, the set $\widehat{\mathbb{M}}$ ought to be bounded.

Given the above three steps, all the conditions of Lemma 4.2 are met, ensuring the existence of a generalized solution for (1). The proof is thus complete. □

Remark 4.4 We note that the above existence result for generalized solutions does not require the compatibility assumption that was used to prove the existence of solutions.

5 An Application to a Nonlinear Optimal Control Problem

We now focus on the optimal control of evolutionary quasi-hemi variational inequality (1). For this, let the control space V be a reflexive Banach space, and let $U \subset V$ be a nonempty, closed, and convex set of admissible controls. Given a control $w \in U$ and a compact map $G : V \to X^*$, the associated state $u(w) = u$ is a solution of the following evolutionary quasi-hemi-variational inequality of finding $u \in C(u) \cap D(L) \cap D(\Phi)$ such that for some

$u^* \in A(u)$ and $v^* \in B(\gamma u)$, we have

$$\begin{aligned}&\langle L(u)+u^*-f, z-u\rangle_X+\langle v^*, \gamma z-\gamma u\rangle_Y\\&\geq\langle G(w), z-u\rangle_X+\Phi(u)-\Phi(z) \quad \text{for all } z\in C(u)\cap D(L).\end{aligned} \tag{33}$$

Given another Banach space, the so-called observation space, W, a compact map $\Gamma : V \to W$, and a target $\Upsilon \in W$, we introduce the following cost function:

$$J(w) := \|\Gamma(u(w)) - \Upsilon\|_W^2 + \varepsilon\|w\|_V^2.$$

Here $\|\cdot\|_W$ is the norm of the observation space W, $\|\cdot\|_V$ is the norm of the control space V, $\varepsilon > 0$ is the regularization parameter, and $u(w)$ is a solution of the quasi variational inequality (33) for control w.

We formulate the optimal control problem that seeks $w \in U$ by solving the minimization problem:

$$\min_{w\in U} J(w).$$

The above optimal control problem encompasses several noteworthy optimal control problems as specific instances. Indeed, one can derive optimal control problems for single-valued quasi-variational inequalities, set-valued and single-valued variational inequalities, and complementarity problems by appropriately adjusting the data.

We have the following existence results for the optimal control problem:

Theorem 5.1 *Assume that the hypotheses of Theorem 3.1 hold. Then the optimal control problem (33) has at least one solution.*

Proof Since the functional J is positive and coercive, there exists a minimizing sequence $\{w_n\} \subset U$, such that

$$\lim_{n\to\infty} J(w_n) = \lim_{n\to\infty} \left\{\|\Gamma(u_n) - \Upsilon\|_W^2 + \varepsilon\|w_n\|_V^2\right\} = \inf\{J(w) : w \in U\},$$

where u_n is chosen from the set of all solutions of (33) that correspond to the control w_n, that is, $u_n = u(w_n)$. Consequently, $u_n \in C(u_n) \cap D(L) \cap D(\Phi)$ such that for some $u_n^* \in A(u_n)$ and $v_n^* \in B(\gamma u_n)$, we have

$$\begin{aligned}&\langle L(u_n)+u_n^*-f, z-u_n\rangle_X+\langle v_n^*, \gamma z-\gamma u_n\rangle_Y\\&\geq\langle G(w_n), z-u_n\rangle_X+\Phi(u_n)-\Phi(z) \quad \text{for all } z\in C(u_n)\cap D(L).\end{aligned} \tag{34}$$

Since $\varepsilon\|w_n\|^2 \leq J(w_n)$, the sequence $\{w_n\}$ is bounded in the reflexive Banach space V, and hence we can extract a weakly convergent subsequence from $\{w_n\}$. Using the same notations for all the subsequences as well, let $\{w_n\}$ be a subsequence that converges weakly to some $\bar{w} \in V$. Moreover, the set U, being closed and convex is weakly closed, and hence, we have $\bar{w} \in U$.

We next claim that the sequence $\{u_n\}$ is bounded. For this, we insert $v = 0_X$ into (34), and by performing a simple computation, obtain

$$\begin{aligned}m\|u_n\|_X^p &\leq \|f\|_{X^*}\|u_n\|_X + \|G(w_n)\|_{X^*}\|u_n\|_X + (\tau\|\gamma u_n\|_X^{p-1} + \mu)\|\gamma\|\|u_n\|_X\\&\leq [\|f\|_{X^*} + \|G(w_n)\|_{X^*}]\,\|u_n\|_X + \tau\|\gamma\|^p\|u_n\|_X^p + \mu\|\gamma\|\|u_n\|_X,\end{aligned}$$

and consequently

$$\|u_n\|_X \leq \left[\frac{\|f\|_{X^*} + \|G(w_n)\|_{X^*} + \mu\|\gamma\|}{m - \tau\|\gamma\|^p}\right]^{\frac{1}{p-1}},$$

which proves that $\{u_n\}$ is bounded. Since X is a reflexive Banach space, we can extract a weakly convergent subsequence from $\{u_n\}$. Keeping the same notations for subsequences, let $\{u_n\}$ be such a subsequence that converges weakly to some $\bar{u}$. We will show that $\bar{u} = u(\bar{w})$.

We first note that due to $[\mathbb{H}_M]$(a), we have $\bar{u} \in C(\bar{u}) \cap D(L) \cap D(\Phi)$. Furthermore, as in the proof of Theorem 3.1, we choose a ball $\mathcal{B}$ in X such that $\mathcal{B} \subset C(u_n)$ for all n. Then (34) and the density of $D(L)$ in X ensure that the sequence $\{L(u_n)\}$ is uniformly bounded from below on the ball $\mathcal{B}$, which guarantees that $\{L(u_n)\}$ is bounded in X^*. Since L is linear and maximal monotone, it has a weakly closed graph, which implies that $\bar{u} \in D(L)$ and $L(u_n) \rightharpoonup L(\bar{u})$ in X^*. Then according to assumption $[\mathbb{H}_M]$(a) we have $\bar{u} \in V(w)$. Note that $\bar{u} \in D(\Phi)$.

Since $\bar{u} \in C(\bar{u}) \cap D(L) \cap D(\Phi)$, we employ assumption corresponding to which there exists a sequence $\{z_n\}$ with $z_n \in C(u_n) \cap D(L)$, $z_n \to \bar{u}$ in X and $\Phi(z_n) \to \Phi(\bar{u})$. We substitute $v = z_n$ in (34) to get

$$\langle L(u_n) + u_n^* - f, z_n - u_n\rangle_X + \langle v_n^*, \gamma z_n - \gamma u_n\rangle_Y \geq \langle G(w_n), z_n - u_n\rangle_X + \Phi(u_n) - \Phi(z_n),$$

which can be rearranged as

$$\begin{aligned}\langle u_n^*, u_n - \bar{u}\rangle_X \leq \langle u_n^*, z_n \bar{u}\rangle_X &+ \langle L(u_n) - f - G(w_n), z_n - u_n\rangle_X \\ &+ \langle v_n^*, \gamma z_n - \gamma u_n\rangle_Y + \Phi(z_n) - \Phi(u_n).\end{aligned} \tag{35}$$

Leveraging the monotonicity of L, coupled with $u_n \rightharpoonup \bar{u}$ in X and $L(u_n) \rightharpoonup L(\bar{u})$ in X^*, we obtain

$$\liminf_{n\to\infty} \langle L(u_n), u_n\rangle_X \geq \langle L(\bar{u}), \bar{u}\rangle_X.$$

Since the map $\gamma : X \to Y$ is compact, we have that $\gamma u_n \to \gamma u$ in Y. Moreover, due to $[\mathbb{H}_B]$, along a relabeled sequence, we have $v_n^* \rightharpoonup v^*$ in Y^* with $v^* \in B(\gamma\bar{u})$. Furthermore, since the map A is bounded by assumption, up to a subsequence, we have $u_n^* \rightharpoonup u^*$ in X^*. Consequently, it follows from (35) that

$$\limsup_{n\to\infty} \langle u_n^*, u_n - \bar{u}\rangle_X \leq 0.$$

We can now invoke the key characterization of the semi-monotone maps given in Theorem 2.6 to ensure that

$$\bar{u} \in A(\bar{u}) \quad \text{and} \quad \langle u_n^*, u_n\rangle_X \to \langle u^*, \bar{u}\rangle_X.$$

For any $v \in C(u) \cap D(L)$, assumption $[\mathbb{H}_M]$(b) permits us to find a sequence $\{v_n\}$ with $v_n \in C(u_n) \cap D(L)$ satisfying $v_n \to v$ in X and $\Phi(v_n) \to \Phi(v)$. Inserting $z = v_n$ into (35), we have

$$\langle L(u_n) + u_n^* - f, v_n - u_n\rangle_X + \langle v_n^*, \gamma v_n - \gamma u_n\rangle_Y \geq \langle G(w_n), v_n - u_n\rangle_X + \Phi(u_n) - \Phi(v_n),$$

which when passed to the limit $n \to \infty$ implies that

$$\begin{aligned}&\langle L(\bar{u}) + u^*, v - \bar{u}\rangle_X + \langle f, v - \bar{u}\rangle_X + \langle v^*, \gamma v - \gamma\bar{u}\rangle_Y \\ &\geq \langle G(\bar{w}), v - \bar{u}\rangle_X + \Phi(\bar{u}) - \Phi(\bar{v}) \quad \text{for all } v \in C(\bar{u}) \cap D(L),\end{aligned}$$

proving that $\bar{u} = u(\bar{w})$.

Finally, we have

$$\begin{aligned} J(\bar{w}) &= \|\varGamma(u(\bar{w})) - \varUpsilon\|^2 + \epsilon\|\bar{w}\|^2 \\ &\le \liminf_{n\to\infty} \|\varGamma(u_n) - \varUpsilon\|^2 + \liminf_{n\to\infty} \epsilon\|w_n\|^2 \\ &\le \liminf_{n\to\infty} J(w_n) \\ &= \lim_{n\to\infty} J(w_n) \\ &= \inf\{J(w) :\ w \in U\}, \end{aligned}$$

ensuring that $\bar{w}$ is a solution. This completes the proof. □

6 Concluding Remarks

We gave new existence results for solutions and generalized solutions for evolutionary hemi-quasi variational inequalities involving the sum of a semi-monotone and a pseudo-monotone map. The main contribution is to define the variational selection to disentangle the monotonic and pseudo-monotonic components, which allowed us to circumvent the frequently assumed requirement that the sum of the two maps must be monotone. Additionally, we also gave a new application to optimal control problems governed by evolutionary hemi-quasi variational inequalities. It is of great interest to extend the developed theory to the inverse problem of parameter identification in quasi-hemi variational inequalities. Regarding the recent advancements in inverse problems research, see [14, 21, 22, 29, 30, 35, 36, 41, 42, 54–56], as well as the relevant references cited within them.

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