Vietnam Journal of Mathematics
https://doi.org/10.1007/s10013-024-00697-9

ORIGINAL ARTICLE

Strong Quasi-nonexpansiveness of Solution Mappings of Equilibrium Problems

Pham Ngoc Anh[1]

Received: 11 September 2023 / Accepted: 6 April 2024

Abstract
In this work we introduce a new approach for solving equilibrium problems in a real Hilbert space. First, we propose a solution mapping and show its strong quasi-nonexpansiveness. Next, we apply the mapping to present an algorithm for solving equilibrium problems. Strong convergence of the algorithm is showed under quasimonotone and Lipschitz-type continuous assumptions of the cost bifunctions. Finally, we give some numerical results for the proposed algorithm and comparison with some other known methods using the solution mapping.

Keywords Equilibrium problems · Solution mapping · Quasi-nonexpansive · Subgradient method · Quasimonotone · Lipschitz-type condition

Mathematics Subject Classification (2010) 65K10 · 90C25 · 49J35 · 47J25 · 47J20 · 91B50

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its deduced norm $\|\cdot\|$. Denote $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}$ converges weakly to x, and $x^k \to x$ to indicate that the sequence $\{x^k\}$ converges strongly to x. Let C be a nonempty convex closed subset of $\mathcal{H}$ and a bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ such that the equilibrium condition $f(x, x) = 0$ holds for all $x \in \mathcal{H}$ and $f(x, \cdot)$ is lower semicontinuous, convex for each $x \in \mathcal{H}$. In this paper, we consider an equilibrium problem, shortly EPs(C, f), first suggested by Nikaido and Isoda in [29] for non-cooperative convex games and the term "equilibrium problem" first used by Blum and Oettli in [9] as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in C.$$

The solution set is denoted by Sol(C, f). In recently years, the problem EPs(C, f) is an attractive field that has been investigated in many research papers due to its applications in a large variety of fields arising in structural analysis, economics, optimization, opera-

Communicated by Christiane Tammer.

✉ Pham Ngoc Anh
anhpn@ptit.edu.vn

1 Department of Scientific Fundamentals, Posts and Telecommunications Institute of Technology, Hanoi, Vietnam

tions research and engineering sciences (see some interesting books [12, 13, 21] and many references therein).

It is worth to mention that during the recent years several solution algorithms came in the existence to solve the problem EPs(C, f) and its generalizations. The solution mapping $E : C \to C$ was first introduced by Moreau in [28] as the form:

$$Ex = \arg\min\left\{\lambda f(x, y) + \frac{1}{2}\|y - x\|^2 : y \in C\right\}, \quad \forall x \in C, \tag{1.1}$$

where $\lambda > 0$. Note that Ex is the unique solution of a strongly convex problem under proper lower semicontinuous and convex assumptions of f. It is originated by the fact that a point $x^* \in C$ is a solution of the problem EPs(C, f) if and only if it is a fixed point of the solution mapping E [25, Proposition 2.1]. The solution mapping has become a useful tool in solving the problem EPs(C, f). Under some conditions onto λ, the mapping E has some the following properties:

– If the cost bifunction f is strongly monotone and Lipschitz-type continuous, then E is contractive [14], [25, Proposition 2.1].

Then, by using Banach's contraction mapping principle, the unique solution of the problem EPs(C, f) is evaluated by the fixed point iteration sequence:

$$x^0 \in C, \quad x^{k+1} = Ex^k, \qquad \forall k \geq 0.$$

The authors showed that the sequence $\{x^k\}$ converges strongly to a unique solution of the problem EPs(C, f). To avoid strongly monotone assumption of f, as an extension of Kraikaew and Saejung in [22], Hieu used a solution mapping [15] as follows:

$$Fx = \arg\min\left\{\lambda f(Ex, y) + \frac{1}{2}\|y - x\|^2 : y \in C\right\}, \quad \forall x \in H_x,$$

where $H_x = \{w \in \mathcal{H} : \langle x - \lambda w_x - Ex, w - Ex\rangle \leq 0\}$, $w_x \in \partial f(x, \cdot)(x)$ and Ex is defined in (1.1). Then, for each $x^* \in \text{Sol}(C, f)$, we have

$$\|Fx - x^*\|^2 \leq \|x - x^*\|^2 - (1 - 2\lambda c_1)\|Ex - x\|^2 - (1 - 2\lambda c_2)\|Fx - Ex\|^2, \quad \forall x \in \mathcal{H},$$

where f is pseudomonotone and Lipschitz-type continuous. However, F can not be quasi-nonexpansive on $\mathcal{H}$ such as $\text{Sol}(C, f) \neq \text{Fix}(T)$.

For solving the monotone problem EPs(C, f), there are various instances of the computational algorithms with combining the solution mapping E with other iteration techniques. It is worth mentioning to very interesting results such as the extragradient algorithms proposed by Quoc et al. [31, 32], inexact proximal point methods of Iusem et al. [17, 18], extragradient-viscosity methods of Maingé and Moudafi [24], auxiliary principles of Mastroeni and Noor [25, 26, 30], extragradient methods of Anh et al. [2, 5, 6] and many other computational methods in [4, 7, 8, 10, 11, 16, 19, 20, 27, 33, 34] and the references cited therein.

Inspired and motivated by the ongoing research, we are aiming to suggest a new approach to the equilibrium problem EPs(C, f). *First*, we introduce a new solution mapping and prove its strongly quasi-nonexpansiveness. *Second*, we use the solution mapping for solving the problem EPs(C, f) via a Lipschitz continuous and strongly monotone mapping, and another Lipschitz continuous mapping. By the way, we can prove that the strong cluster point of the sequence constructed by our algorithm is the unique solution of a variational inequality problem where the constraint is the solution set of the problem EPs(C, f) under quasimonotone and Lipschitz continuous assumptions of the cost bifunction f. This constitutes a new approach which is called *solution mapping approach* and the fundamental difference of our algorithm with respect to current computational methods.

Our paper is organized as follows. In Section 2, we present some useful definitions, technique lemmas and a new solution mapping. A new algorithm and its convergent analysis for solving the problem $\text{EPs}(C, f)$ are presented in Section 3. In Section 4, several numerical experiments are provided to illustrate the efficiency and accuracy of our proposed algorithm.

2 Solution Mappings

For each $x \in \mathcal{H}$, the metric projection of x onto C is denoted by $\Pi_C(x)$ which is the unique solution to the strongly convex problem:

$$\min\{\|x - y\|^2 : y \in C\}.$$

Given a bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ and $\emptyset \neq K \subset \mathcal{H}$. The bifunction f is called to be:

- β-*strongly monotone* if

$$f(x, y) + f(y, x) \leq -\beta\|x - y\|^2, \quad \forall x, y \in \mathcal{H}.$$

- *monotone* if

$$f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in \mathcal{H}.$$

- *pseudomonotone* if

$$f(x, y) \geq 0 \quad \Rightarrow \quad f(y, x) \leq 0, \quad \forall x, y \in \mathcal{H}.$$

- η-strongly quasimonotone on K where $\eta > 0$ if

$$f(x, y) + f(y, x) \leq -\beta\|x - y\|^2, \quad \forall x \in K, y \in \mathcal{H}.$$

- quasimonotone on K if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \quad \forall x \in K, y \in \mathcal{H}.$$

- Lipschitz-type continuous on $\mathcal{H}$ with constants $c_1 > 0$ and $c_2 > 0$ if

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \quad \forall x, y, z \in \mathcal{H}.$$

Let a mapping $S : \mathcal{H} \to \mathcal{H}$ and the fixed point set of S be $\text{Fix}(S) := \{x \in \mathcal{H} : Sx = x\}$. The operator S is called to be:

- η-*strongly quasi-nonexpansive*, where $\eta > 0$, if

$$\|Sx - z\|^2 \leq \|x - z\|^2 - \eta\|Sx - x\|^2, \quad \forall x \in \mathcal{H}, \ z \in \text{Fix}(S).$$

- *quasi-nonexpansive* if

$$\|Sx - z\| \leq \|x - z\|, \quad \forall x \in \mathcal{H}, \ z \in \text{Fix}(S).$$

- *quasicontractive* with constant $\eta \in [0, 1)$ if

$$\|Sx - z\| \leq \eta\|x - z\|, \quad \forall x \in \mathcal{H}, \ z \in \text{Fix}(S).$$

When $\text{Fix}(S) = \mathcal{H}$, the mapping S is called *contractive* with constant η.

For each $x \in \mathcal{H}$ and $\xi > 0$, we consider the solution mapping $S : \mathcal{H} \to C$ of the problem EPs(C, f) defined in the form:

$$Sx = \arg\min \left\{ \xi f(x, y) + \frac{1}{2}\|x - y\|^2 : x \in C \right\}. \tag{2.1}$$

It is well-known that $x \in C$ is a solution of the problem EPs(C, f) if and only if it is a fixed point of the mapping S [25, Proposition 2.1]. Let $\gamma > 0$. We introduce a new half space as follows:

$$H_x = \left\{ w \in \mathcal{H} : \langle x - \xi w_x - Sx, w - Sx\rangle \leq \gamma \|x - Sx\|^2 \right\}, \tag{2.2}$$

where $w_x \in \partial f(x, \cdot)(Sx)$. Using the well-known necessary and sufficient condition for optimality of the convex programing (2.1), we see that Sx solves the strongly convex program

$$\min \left\{ \xi f(x, y) + \frac{1}{2}\|x - y\|^2 : x \in C \right\}$$

if and only if

$$0 \in \xi \partial f(x, \cdot)(Sx) + Sx - x + N_C(Sx),$$

where $N_C(Sx)$ is the (outward) normal cone of C at $Sx \in C$. Since $f(x, \cdot)$ is subdifferentiable for each $x \in \mathcal{H}$, so there exists $w_x \in \partial f(x, \cdot)(Sx)$ such that

$$x - Sx - \xi w_x \in N_C(Sx), \quad \forall x \in \mathcal{H}.$$

Consequently

$$\langle x - Sx - \xi w_x, y - Sx\rangle \leq 0, \quad \forall y \in C.$$

From $\gamma > 0$, it follows that

$$\langle x - \xi w_x - Sx, y - Sx\rangle \leq \gamma \|x - Sx\|^2, \quad \forall y \in C.$$

By (2.2), it yields $C \subset H_x$ for all $x \in \mathcal{H}$.

Now we propose a *new solution mapping* $T : \mathcal{H} \to \mathcal{H}$ for the problem EPs(C, f) as follows:

$$Tx = \arg\min \left\{ \nu\xi f(Sx, y) + \frac{1}{2}\|y - x\|^2 : y \in H_x \right\}, \tag{2.3}$$

where regular parameter $\nu > 0$ is very important for strongly quasi-nonexpansiveness of T. In the case $f(x, y) = \langle F(x), y - x\rangle$, set $z = x - \nu\xi F(x)$. It is easy to evaluate that Tx is the projection of z onto H_x and presented in an explicit formula:

$$Tx = \Pi_{H_x}(z) = \begin{cases} z - \frac{\langle d_x, z - Sx\rangle - \gamma\|x - Sx\|^2}{\|d_x\|^2} d_x & \text{if } z \notin H_x, \\ z & \text{otherwise}, \end{cases}$$

where $d_x = x - \xi w_x - Sx$. Note that, if $d_x = 0$ then obviously $z \in H_x$ and $Tx = x - \nu\xi F(x)$.

The following result will present some important properties of the operators T and S that will be needed in the sequel.

Lemma 2.1 *Suppose that f is Lipschitz-type continuous with constants $c_1 > 0$ and $c_2 > 0$. Under conditions $\xi > 0$, $\nu > 0$ and $\gamma > 0$, the following inequality holds*

$$\begin{aligned} \|Tx - t\|^2 \leq \|t - x\|^2 - (1 - \nu)\|x - Tx\|^2 - \nu(1 - 2\gamma - 2\xi c_1)\|x - Sx\|^2 \\ -\nu(1 - 2\xi c_2)\|Tx - Sx\|^2 + 2\nu\xi f(Sx, t), \quad \forall t \in C, x \in \mathcal{H}. \end{aligned}$$

Proof By using the definition of $w_x \in \partial f(x, \cdot)(Sx)$, we get

$$f(x, z) - f(x, Sx) \geq \langle w_x, z - Sx \rangle, \quad \forall z \in \mathcal{H}. \tag{2.4}$$

Combining (2.4) and $Tx \in H_x$, it implies

$$\begin{aligned}\langle x - Sx, Tx - Sx \rangle - \gamma \|x - Sx\|^2 &\leq \xi \langle w_x, Tx - Sx \rangle \\ &\leq \xi [f(x, Tx) - f(x, Sx)].\end{aligned}$$

From the necessary and sufficient condition for the strongly convex problem (2.3), there exists $v_x \in \partial f(Sx, \cdot)(Tx)$ such that

$$0 \in \nu \xi v_x + Tx - x + N_{H_x}(Tx).$$

Thus,

$$\langle \nu \xi v_x + Tx - x, t - Tx \rangle \geq 0, \quad \forall t \in H_x,$$

and hence

$$\nu \xi \langle v_x, t - Tx \rangle \geq \langle x - Tx, t - Tx \rangle, \quad \forall t \in H_x. \tag{2.5}$$

By the definition of $v_x \in \partial f(Sx, \cdot)(Tx)$ and $\nu \xi > 0$, it implies

$$f(Sx, y) - f(Sx, Tx) \geq \langle v_x, y - Tx \rangle, \quad \forall y \in \mathcal{H}.$$

This together with (2.5) implies

$$\begin{aligned}\langle x - Tx, t - Tx \rangle &\leq \nu \xi \langle v_x, t - Tx \rangle \\ &\leq \nu \xi [f(Sx, t) - f(Sx, Tx)], \quad \forall t \in H_x.\end{aligned} \tag{2.6}$$

Since f is Lipschitz-type continuous with c_1 and c_2, we deduce

$$f(x, Sx) + f(Sx, Tx) \geq f(x, Tx) - c_1 \|x - Sx\|^2 - c_2 \|Tx - Sx\|^2.$$

Combining this and (2.6), we get that, for each $t \in \mathcal{H}_x$,

$$\begin{aligned}&\langle x - Tx, t - Tx \rangle - \nu \xi f(Sx, t) \\ &\leq -\nu \xi f(Sx, Tx) \\ &\leq \nu \xi \left[f(x, Sx) - f(x, Tx) + c_1 \|x - Sx\|^2 + c_2 \|Tx - Sx\|^2\right], \quad \forall t \in \mathcal{H}_x \\ &\leq \nu \langle Sx - x, Tx - Sx \rangle + \nu \gamma \|x - Sx\|^2 + \nu \xi c_1 \|x - Sx\|^2 + \nu \xi c_2 \|Tx - Sx\|^2. \end{aligned} \tag{2.7}$$

By using the relation

$$2\langle a, b \rangle = \|a\|^2 + \|b\|^2 - \|a - b\|^2, \quad \forall a, b \in \mathcal{H},$$

we obtain

$$\begin{aligned}2\langle x - Tx, t - Tx \rangle &= \|x - Tx\|^2 + \|t - Tx\|^2 - \|t - x\|^2, \\ 2\langle Sx - x, Tx - Sx \rangle &= \|x - Tx\|^2 - \|Sx - x\|^2 - \|Tx - Sx\|^2.\end{aligned}$$

This together with (2.7) implies that

$$\begin{aligned}\|Tx - t\|^2 \leq \|t - x\|^2 - (1 - \nu)\|x - Tx\|^2 - \nu(1 - 2\gamma - 2\xi c_1)\|x - Sx\|^2 \\ -\nu(1 - 2\xi c_2)\|Tx - Sx\|^2 + 2\nu \xi f(Sx, t), \quad \forall t \in C.\end{aligned}$$

The proof is complete. □

Lemma 2.2 *Assume that f is Lipschitz-type continuous with constants $c_1 > 0$ and $c_2 > 0$. Let parameters ν, ξ and γ satisfy the following conditions:*

$$\xi \in \left(0, \frac{1}{c_1 + c_2}\right), \quad \gamma \in (0, 1 - \xi(c_1 + c_2)), \quad \nu \in \left(0, \frac{1}{\xi c_2}\right). \tag{2.8}$$

Then, $x^* \in C$ is a solution of the problem EPs(C, f) *if and only if it is a fixed point of the solution mapping T.*

Proof Assume that $\bar{x} \in C$ is a fixed point of T, i.e., $T\bar{x} = \bar{x}$. Substituting $x = \bar{x}$ into Lemma 2.1, we obtain

$$\begin{aligned}\|\bar{x} - t\|^2 \le \|t - \bar{x}\|^2 - (1-\nu)\|\bar{x} - T\bar{x}\|^2 - \nu(1 - 2\gamma - 2\xi c_1)\|\bar{x} - S\bar{x}\|^2 \\ -\nu(1 - 2\xi c_2)\|T\bar{x} - S\bar{x}\|^2 + 2\nu\xi f(S\bar{x}, t), \quad \forall t \in C.\end{aligned}$$

Consequently

$$\xi f(S\bar{x}, t) \ge (1 - \gamma - \xi c_1 - \xi c_2)\|\bar{x} - S\bar{x}\|^2, \quad \forall t \in C.$$

From (2.8), it follows

$$f(S\bar{x}, t) \ge 0, \quad \forall t \in C.$$

Thus, $S\bar{x} \in C$ is a solution of the problem EPs(C, f). Since $x \in C$ is a solution of the problem EPs(C, f) if and only it is a fixed point of S, so $\bar{x} = S\bar{x} \in \text{Sol}(C, f)$.

Now we assume $\hat{x} \in \text{Sol}(C, f)$. Then, $S\hat{x} = \hat{x}$. Substituting $t = \hat{x}$ and $x = \hat{x}$ into Lemma 2.1 and using $f(x, x) = 0$ for all $x \in C$, we get

$$(1 - \nu\xi c_2)\|T\hat{x} - \hat{x}\|^2 \le 0.$$

By (2.8), it yields $T\hat{x} = \hat{x}$. Thus, the solution $\hat{x} \in \text{Sol}(C, f)$ is a fixed point of T. This implies the proof. □

Lemma 2.3 *Suppose that f is quasimonotone on* Sol(C, f) *and Lipschitz-type continuous with constants $c_1 > 0$, $c_2 > 0$. The parameters satisfy the following restrictions:*

$$\begin{cases} \xi \in \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right), \\ \gamma \in (0, 1 - \xi(c_1 + c_2)), \\ \nu \in \left(0, \min\left\{1, \frac{1}{\xi c_2}\right\}\right). \end{cases} \tag{2.9}$$

Then, the solution mapping T is strongly quasi-nonexpansive with constant $(1 - \nu)$.

Proof Let $x^* \in \text{Sol}(C, f)$, i.e., $f(x^*, x) \ge 0$ for all $x \in C$. By replacing $t = x^*$ into Lemma 2.1, we get

$$\begin{aligned}\|Tx - x^*\|^2 \le \|x - x^*\|^2 - (1-\nu)\|x - Tx\|^2 - \nu(1 - 2\gamma - 2\xi c_1)\|x - Sx\|^2 \\ -\nu(1 - 2\xi c_2)\|Tx - Sx\|^2 + 2\nu\xi f(Sx, x^*).\end{aligned} \tag{2.10}$$

Since $Sx \in C$, f is quasimonotone and (2.10), we deduce $f(Sx, x^*) \le 0$ and

$$\begin{aligned}\|Tx - x^*\|^2 &\le \|x - x^*\|^2 - (1-\nu)\|x - Tx\|^2 - \nu(1 - 2\gamma - 2\xi c_1)\|x - Sx\|^2 \\ &\quad -\nu(1 - 2\xi c_2)\|Tx - Sx\|^2 + 2\nu\xi f(Sx, x^*) \\ &\le \|x - x^*\|^2 - (1-\nu)\|x - Tx\|^2 - \nu(1 - 2\gamma - 2\xi c_1)\|x - Sx\|^2 \\ &\quad -\nu(1 - 2\xi c_2)\|Tx - Sx\|^2.\end{aligned} \tag{2.11}$$

Combining this and conditions (2.9), we have

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - (1-\nu)\|x - Tx\|^2, \quad \forall x^* \in \mathrm{Sol}(C, f), x \in \mathcal{H}.$$

By the definition and Lemma 2.2, we get that $\mathrm{Fix}(T) = \mathrm{Sol}(C, f)$ and the solution mapping T is $(1-\nu)$-strongly quasi-nonexpansive. □

Remark 2.4 In [14, Theorem 3.7], the author showed that if $f : C \times C \to \mathcal{R}$ is ζ-strongly monotone and there exist functions $\alpha_i : C \times C \to \mathcal{H}$, $\beta_i : C \to \mathcal{H}$ $(i = 1, \ldots, p)$ such that

$$f(x, y) + f(y, z) \geq f(x, z) + \sum_{i=1}^{p} \langle \alpha_i(x, y), \beta_i(y - z) \rangle, \quad \forall x, y, z \in C,$$

where β_i is K_i-Lipschitz continuous, $\alpha_i(x, y) + \alpha_i(y, x) = 0$ and $|\alpha_i(x, y)| \leq L_i\|x - y\|$ for all $x, y \in C$, $i = 1, \ldots, p$. Under condition $\xi \in (0, \frac{2\zeta}{M^2})$ where $M = \sum_{i=1}^{p} K_i L_i$, the mapping S defined by (2.1) is contractive with constant $\delta = \sqrt{1 - \xi(2\zeta - \xi M^2)} \in (0, 1)$.

Lemma 2.5 [23, Remark 2.1] *Let $T : \mathcal{H} \to \mathcal{H}$ is quasi-nonexpansive and $T_\omega = (1-\omega)\mathrm{Id} + \omega T$ with $\omega \in (0, 1]$ such that $\mathrm{Fix}(T) \neq \emptyset$, where Id is an identify operator. Then, the following statements hold:*

(i) $\mathrm{Fix}(T) = \mathrm{Fix}(T_\omega)$.
(ii) *T_ω is quasi-nonexpansive.*
(iii) $\|T_\omega x - v\|^2 \leq \|x - v\|^2 - \omega(1-\omega)\|Tx - x\|^2$ *for all $x \in \mathcal{H}$ and $v \in \mathrm{Fix}(T)$.*
(iv) $\langle x - T_\omega x, x - u \rangle \geq \frac{\omega}{2}\|x - Tx\|^2$ *for all $x \in \mathcal{H}$ and $u \in \mathrm{Fix}(T)$.*

3 Subgradient Auxiliary Principle Algorithm

Let $G : \mathcal{H} \to \mathcal{H}$ be L_G-Lipschitz continuous and β_G-strongly monotone, and $g : \mathcal{H} \to \mathcal{H}$ is L_g-Lipschitz continuous. In this section, we propose a new algorithm which is called *Subgradient Auxiliary Principle Algorithm* for solving the problem $\mathrm{EPs}(C, f)$ via the mappings G and g. Under certain conditions we obtain the desired convergence for the algorithm. First, we give the restrictions governing the cost bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ and the sequence of parameters below.

(R_1) The solution set $\mathrm{Sol}(C, f)$ of the problem $\mathrm{EPs}(C, f)$ is nonempty.
(R_2) The cost bifunction f is quasimonotone and Lipschitz-type continuous with constants $c_1 > 0$ and $c_2 > 0$. f is jointly weakly continuous on $\mathcal{H} \times C$ in the sense that, if $\{x^k\}$, $\{y^k\}$ converge weakly to $\hat{x}, \hat{y}$, respectively, then $f(x^k, y^k) \to f(\hat{x}, \hat{y})$ as $k \to \infty$.
(R_3) For every integer $k \geq 0$, all the positive parameters satisfy the following restrictions:

$$\begin{cases} \xi_k \in \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right), \quad \lim_{k\to\infty} \xi_k = \xi > 0, \\ \gamma_k \in (0, 1 - \xi_k(c_1 + c_2)), \\ \nu \in \left(0, \min\left\{1, \frac{1}{\xi_k c_2}\right\}\right), \\ \omega \in \left(0, \frac{1}{2}\right), \ \mu \in \left(0, \frac{2\beta_G}{L_G^2}\right), \ \gamma \in \left(0, \frac{\mu}{L_g}\left(\beta_G - \frac{\mu L_G^2}{2}\right)\right), \ \tau \in (\gamma L_g, \mu\beta_G), \\ \alpha_k \in \left(0, \min\left\{1, \frac{2(\mu\beta_G - \tau)}{\mu^2 L_G^2 - \tau^2}, \frac{1}{\tau - \gamma L_g}\right\}\right), \ \sum_{k=0}^{\infty} \alpha_k = \infty, \ \lim_{k\to\infty} \alpha_k = 0. \end{cases}$$

Let the mappings S and T be defined by (2.1)–(2.3). Now we present the Subgradient Auxiliary Principle Algorithm for solving the problem EPs(C, f).

Algorithm 1 Subgradient Auxiliary Principle Algorithm (SAPA).

Choose starting points $x^0 \in \mathcal{H}$, $k = 0$, $\nu > 0$, $\omega > 0$, $\gamma > 0$, $\mu > 0$, three positive sequences $\{\xi_k\}$, $\{\alpha_k\}$ and $\{\gamma_k\}$.

Step 1. Solve the strongly convex auxiliary problem:

$$y^k = \arg\min\left\{\xi_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : x \in C\right\}.$$

If $y^k = x^k$ then Stop. Otherwise, go to Step 2.

Step 2. Calculate $w^k \in \partial f(x^k, \cdot)(y^k)$ and the next iterate

$$z^k = \arg\min\left\{\nu\xi_k f(y^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in H_k\right\},$$

where $H_k = \{w \in \mathcal{H} : \langle x^k - \xi_k w^k - y^k, w - y^k\rangle \le \gamma_k\|x^k - y^k\|^2\}$. If $z^k = x^k$ then Stop. Otherwise, go to Step 3.

Step 3. Calculate

$$x^{k+1} = \alpha_k\gamma g(x^k) + (\mathrm{Id} - \alpha_k\mu G)(h^k),$$

where $h^k = (1-\omega)x^k + \omega z^k$. Let $k := k + 1$ and go to Step 1.

For each $k \ge 0$ and $x \in \mathcal{H}$, set

$$S_k x = \arg\min\left\{\xi_k f(x, y) + \frac{1}{2}\|y - x\|^2 : x \in C\right\}, \tag{3.1}$$

$$T_k x = \arg\min\left\{\nu\xi_k f(S_k x, y) + \frac{1}{2}\|y - x\|^2 : x \in H_k\right\}, \tag{3.2}$$

where $H_k = \{w \in \mathcal{H} : \langle x - \xi_k w_x - S_k x, w - S_k x\rangle \le \gamma_k\|x - S_k x\|^2\}$ and $w_x \in \partial f(x, \cdot)(S_k x)$.

Remark 3.1 (i) Since x^* is a solution of the problem EPs(C, f) if and only if it is a fixed point of the mapping S_k defined by (3.1). Therefore, if $y^k = x^k$ in Algorithm 1, i.e., $x^k = S_k x^k$ under the assumption $\xi_k > 0$, then x^k is a solution of the problem EPs(C, f). The stopping criterion in Step 1 is valid.

(ii) By Lemma 2.2, x^k is a solution of the problem EPs(C, f) if and only if it is a fixed point of the solution mapping T_k defined by (3.2) under the assumptions (R_1)–(R_3). Thus, if $z^k = x^k$ in Algorithm 1, i.e., $x^k = T_k x^k$, then x^k is a solution of the problem EPs(C, f). The stopping criterion in Step 2 is valid.

(iii) As usual, for each $\varepsilon > 0$, an iteration point x^k defined in Algorithm 1 is ε-solution of the problem EPs(C, f), if $\|y^k - x^k\| \le \varepsilon$ or $\|z^k - x^k\| \le \varepsilon$. Equivalently, $\max\{\|y^k - x^k\|, \|z^k - x^k\|\} \le \varepsilon$.

The next lemma is crucial for the proof of our convergent theorem.

Lemma 3.2 *Let $\{x^k\}$ and $\{y^k\}$ be the two sequences generated by Algorithm 1 and let $x^* \in$ Sol(C, f). Under assumptions (R_2) and (R_3), the following claim holds*

$$\|z^k - x^*\|^2 \le \|x^k - x^*\|^2 - (1-\nu)\|x^k - z^k\|^2 - \nu(1 - 2\gamma_k - 2\xi_k c_1)\|x^k - y^k\|^2$$
$$-\nu(1 - 2\xi_k c_2)\|z^k - y^k\|^2.$$

Proof From (2.1) and Step 1, it follows $y^k = S_k x^k$, where the mapping S_k is defined in (3.1). Combining (3.2) and Step 2, it yields that $x^{k+1} = T_k x^k$. By using the $(1-\nu)$-strongly quasi-nonexpansive property of the mapping T_k in Lemma 2.3, (2.11), assumptions (R_2) and (R_3), we obtain

$$\begin{aligned}\|z^k - x^*\|^2 &= \|T_k x^k - x^*\|^2\\ &\le \|x^k - x^*\|^2 - (1-\nu)\|x^k - T_k x^k\|^2 - \nu(1-2\gamma_k - 2\xi_k c_1)\|x^k - S_k x^k\|^2\\ &\quad -\nu(1-2\xi_k c_2)\|T_k x^k - S_k x^k\|^2.\end{aligned}$$

which completes the proof. □

Theorem 3.3 *Let the cost bifunction f and the parameters satisfy assumptions (R_1)–(R_3). Then, two iteration sequences $\{x^k\}$ and $\{y^k\}$ generated by Algorithm 1 converge strongly to the unique solution $x^* \in \mathrm{Sol}(C, f)$ of the following variational inequality:*

$$\langle(\mu G - \gamma g)(x^*), x - x^*\rangle \ge 0, \quad \forall x \in \mathrm{Sol}(C, f). \tag{3.3}$$

Proof Let x^* satisfy the inequality (3.3). Then, $x^* \in \mathrm{Sol}(C, f)$. From Lemma 2.2, it follows that x^* is a fixed point of T_k defined by (3.2). By Lemma 2.3, the mapping T_k is strongly quasi-nonexpansive on $\mathcal{H}$. Since (3.2) and Step 2, we have $z^k = T_k x^k$. For every $x \in \mathcal{H}$, since G is β_G-strongly monotone and L_G-Lipschitz continuous, we have

$$\begin{aligned}&\|(\mathrm{Id} - \alpha_k\mu G)(x) - (\mathrm{Id} - \alpha_k\mu G)(y)\|^2\\ &\quad= \|x-y\|^2 - 2\alpha_k\mu\langle x-y, G(x) - G(y)\rangle + \alpha_k^2\mu^2\|G(x) - G(y)\|^2\\ &\quad\le (1 - 2\alpha_k\mu\beta_G + \alpha_k^2\mu^2 L_G^2)\|x-y\|^2\\ &\quad\le (1-\alpha_k\tau)^2\|x-y\|^2, \quad \forall x, y \in \mathcal{H},\end{aligned} \tag{3.4}$$

where the last inequality is deduced from the condition $\alpha_k \in \left(0, \min\left\{1, \frac{2(\mu\beta_G - \tau)}{\mu^2 L_G^2 - \tau^2}\right\}\right)$ and $\tau \in (0, \mu\beta_G)$ of (R_3). Since g is L_g-Lipschitz continuous, (3.4) and Lemma 2.5(iii), we obtain

$$\begin{aligned}\|x^{k+1} - x^*\| &= \|\alpha_k\gamma g(x^k) + (\mathrm{Id} - \alpha_k\mu G)(h^k) - x^*\|\\ &= \|\alpha_k\gamma[g(x^k) - g(x^*)] + \alpha_k[\gamma g(x^*) - \mu G(x^*)]\\ &\quad +(\mathrm{Id} - \alpha_k\mu G)(h^k) - (\mathrm{Id} - \alpha_k\mu G)(x^*)\|\\ &\le \alpha_k\gamma\|g(x^k) - g(x^*)\| + \alpha_k\|\gamma g(x^*) - \mu G(x^*)\|\\ &\quad +\|(\mathrm{Id} - \alpha_k\mu G)(h^k) - (\mathrm{Id} - \alpha_k\mu G)(x^*)\|\\ &\le \alpha_k\gamma L_g\|x^k - x^*\| + \alpha_k\|\gamma g(x^*) - \mu G(x^*)\| + (1-\alpha_k\tau)\|h^k - x^*\|\\ &= \alpha_k\gamma L_g\|x^k - x^*\| + \alpha_k\|\gamma g(x^*) - \mu G(x^*)\|\\ &\quad +(1-\alpha_k\tau)\|(1-\omega)x^k + \omega T_k x^k - x^*\| \qquad (3.5)\\ &\le \alpha_k\gamma L_g\|x^k - x^*\| + \alpha_k\|\gamma g(x^*) - \mu G(x^*)\| + (1-\alpha_k\tau)\|x^k - x^*\|\\ &= [1 - \alpha_k(\tau - \gamma L_g)]\|x^k - x^*\| + \alpha_k(\tau - \gamma L_g)\frac{\|\gamma g(x^*) - \mu G(x^*)\|}{\tau - \gamma L_g}\\ &\le \max\left\{\|x^k - x^*\|, \frac{\|\gamma g(x^*) - \mu G(x^*)\|}{\tau - \gamma L_g}\right\}\\ &\le \cdots\\ &\le \max\left\{\|x^0 - x^*\|, \frac{\|\gamma g(x^*) - \mu G(x^*)\|}{\tau - \gamma L_g}\right\},\end{aligned}$$

where $\alpha_k(\tau-\gamma L_g)\in(0,1)$ is deduced from the conditions (R_3). Therefore, $\{x^k\}$ is bounded. Applying Lemma 2.5(iv) for $T_\omega:=(1-\omega)\mathrm{Id}+\omega T_k$, using the Cauchy–Schwarz inequality and (3.4), we have

$$\begin{aligned}
&\langle x^{k+1}-x^k+\alpha_k[\mu G(x^k)-\gamma g(x^k)], x^k-x^*\rangle\\
&\quad=\langle(\mathrm{Id}-\alpha_k\mu G)(h^k)-(\mathrm{Id}-\alpha_k\mu G)(x^k), x^k-x^*\rangle\\
&\quad=\langle(\mathrm{Id}-\alpha_k\mu G)(T_\omega x^k)-(\mathrm{Id}-\alpha_k\mu G)(x^k), x^k-x^*\rangle\\
&\quad=(1-\alpha_k)\langle T_\omega x^k-x^k, x^k-x^*\rangle+\alpha_k\langle(\mathrm{Id}-\mu G)T_\omega x^k-(\mathrm{Id}-\mu G)x^k, x^k-x^*\rangle\\
&\quad\le(1-\alpha_k)\langle T_\omega x^k-x^k, x^k-x^*\rangle+\alpha_k\langle(\mathrm{Id}-\mu G)T_\omega x^k-(\mathrm{Id}-\mu G)x^k, x^k-x^*\rangle\\
&\quad\le-(1-\alpha_k)\frac{\omega}{2}\|T_kx^k-x^k\|^2+\alpha_k(1-\alpha_k\tau)\|T_\omega x^k-x^k\|\|x^k-x^*\|\\
&\quad=-\frac{(1-\alpha_k)\omega}{2}\|T_kx^k-x^k\|^2+\omega\alpha_k(1-\alpha_k\tau)\|T_kx^k-x^k\|\|x^k-x^*\|.
\end{aligned}$$

Then, using the relation

$$2\langle x,y\rangle=\|x+y\|^2-\|x\|^2-\|y\|^2,\quad\forall x,y\in\mathcal{H},$$

we get

$$\begin{aligned}
2\langle x^{k+1}-x^k, x^k-x^*\rangle&=\|x^{k+1}-x^*\|^2-\|x^{k+1}-x^k\|^2-\|x^k-x^*\|^2\\
&\le 2\alpha_k\langle\mu G(x^k)-\gamma g(x^k), x^*-x^k\rangle-(1-\alpha_k)\omega\|T_kx^k-x^k\|^2\\
&\quad+2\omega\alpha_k(1-\alpha_k\tau)\|T_kx^k-x^k\|\|x^k-x^*\|.
\end{aligned}\tag{3.6}$$

From Step 3, it follows that

$$\begin{aligned}
\|x^{k+1}-x^k\|^2&=\|\alpha_k\gamma g(x^k)+(\mathrm{Id}-\alpha_k\mu G)(h^k)-x^k\|^2\\
&=\|\alpha_k\gamma g(x^k)+(\mathrm{Id}-\alpha_k\mu G)(T_\omega x^k)-x^k\|^2\\
&=\|\alpha_k[\gamma g(x^k)-\mu G(x^k)]+(\mathrm{Id}-\alpha_k\mu F)T_\omega x^k-(\mathrm{Id}-\alpha_k\mu G)(x^k)\|^2\\
&\le 2\alpha_k^2\|\gamma g(x^k)-\mu G(x^k)\|^2+2\|(\mathrm{Id}-\alpha_k\mu F)T_\omega x^k-(\mathrm{Id}-\alpha_k\mu G)(x^k)\|^2\\
&\le 2\alpha_k^2\|\gamma g(x^k)-\mu G(x^k)\|^2+2(1-\alpha_k\tau)^2\|T_\omega x^k-x^k\|^2\\
&=2\alpha_k^2\|\gamma g(x^k)-\mu G(x^k)\|^2+2(1-\alpha_k\tau)^2\omega^2\|T_kx^k-x^k\|^2.
\end{aligned}\tag{3.7}$$

Set $a_k:=\|x^k-x^*\|$. Combining (3.6) and (3.7), it yields

$$\begin{aligned}
a_{k+1}&\le a_k+\|x^{k+1}-x^k\|^2+2\alpha_k[\mu G(x^k)-\gamma g(x^k)], x^*-x^k\rangle\\
&\quad-(1-\alpha_k)\omega\|T_kx^k-x^k\|^2+2\omega\alpha_k(1-\alpha_k\tau)\|T_kx^k-x^k\|\|x^k-x^*\|\\
&\le a_k+2\alpha_k\langle\mu G(x^k)-\gamma g(x^k), x^*-x^k\rangle-(1-\alpha_k)\omega\|T_kx^k-x^k\|^2\\
&\quad+2\alpha_k^2\|\gamma g(x^k)-\mu G(x^k)\|^2+2(1-\alpha_k\tau)^2\omega^2\|T_kx^k-x^k\|^2\\
&\quad+2\omega\alpha_k(1-\alpha_k\tau)\|T_kx^k-x^k\|\|x^k-x^*\|\\
&=a_k+2\alpha_k\langle\mu G(x^k)-\gamma g(x^k), x^*-x^k\rangle-\omega[1-\alpha_k-2\omega(1-\alpha_k\tau)^2]\|T_kx^k-x^k\|^2\\
&\quad+2\alpha_k^2\|\gamma g(x^k)-\mu G(x^k)\|^2+2\omega\alpha_k(1-\alpha_k\tau)\|T_kx^k-x^k\|\|x^k-x^*\|.
\end{aligned}\tag{3.8}$$

Let us consider two following cases:

Case 1. There exists a positive integer k_0 such that $a_{k+1}\le a_k$ for all $k\ge k_0$. Then, the limit $\lim_{k\to\infty}a_k=A<\infty$ exists. Passing to the limit into (3.8) as $k\to\infty$, using the boundedness of $\{x^k\}$ and $\lim_{k\to\infty}\alpha_k=0$, we obtain $\lim_{k\to\infty}\|T_kx^k-x^k\|=0$. From (3.8), it follows

$$\alpha_k\Gamma_k\le a_k-a_{k+1},\quad\forall k\ge k_0,$$

where $\Gamma_k := -2\langle \mu G(x^k) - \gamma g(x^k), x^* - x^k\rangle - 2\alpha_k\|\gamma g(x^k) - \mu G(x^k)\|^2 - 2\omega\alpha_k(1 - \alpha_k\tau)\|T_k x^k - x^k\|\|x^k - x^*\|$. By the condition $\sum_{k=0}^{\infty}\alpha_k = \infty$ in (R_3), we deduce $\liminf_{k\to\infty}\Gamma_k \le 0$ and hence

$$\liminf_{k\to\infty}\langle \mu G(x^k) - \gamma g(x^k), x^k - x^*\rangle \le 0.$$

Combining this with the relation

$$\langle (\mu G - \gamma g)(x) - (\mu G - \gamma g)(y), x - y\rangle \ge (\mu\beta_G - \gamma L_g)\|x - y\|^2, \quad \forall x, y \in \mathcal{H},$$

yields

$$\begin{aligned}0 &\ge \liminf_{k\to\infty}\langle \mu G(x^k) - \gamma g(x^k), x^k - x^*\rangle \\ &\ge \liminf_{k\to\infty}\Big[\langle \mu G(x^*) - \gamma g(x^*), x^k - x^*\rangle + (\mu\beta_G - \gamma L_g)\|x^k - x^*\|^2\Big].\end{aligned} \tag{3.9}$$

From (3.5), it follows

$$\begin{aligned}\|x^{k+1} - x^*\| &\le \alpha_k\gamma L_g\|x^k - x^*\| + \alpha_k\|\gamma g(x^*) - \mu G(x^*)\| \\ &\quad + (1 - \alpha_k\tau)\|(1-\omega)x^k + \omega T_k x^k - x^*\| \\ &\le \alpha_k\gamma L_g\|x^k - x^*\| + \alpha_k\|\gamma g(x^*) - \mu G(x^*)\| \\ &\quad + (1 - \alpha_k\tau)(1-\omega)\|x^k - x^*\| + (1 - \alpha_k\tau)\omega\|z^k - x^*\|.\end{aligned}$$

Passing to the limit as $k \to \infty$, we have

$$A \le \lim_{k\to\infty}\|z^k - x^*\|.$$

By Lemma 3.2, we also have

$$\lim_{k\to\infty}\|z^k - x^*\| \le A.$$

Thus,

$$\lim_{k\to\infty}\|z^k - x^*\| = A.$$

Using Lemma 3.2 yields

$$\lim_{k\to\infty}\|y^k - x^k\| = 0.$$

Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_j}\}$ such that $x^{k_j} \rightharpoonup \bar{x}$ as $j \to \infty$ and

$$\liminf_{k\to\infty}\langle \mu G(x^*) - \gamma g(x^*), x^k - x^*\rangle = \lim_{j\to\infty}\langle \mu G(x^*) - \gamma g(x^*), x^{k_j} - x^*\rangle.$$

Then, $y^{k_j} \rightharpoonup \bar{x}$. Since C is closed and convex, C is weakly closed. Thus, from $\{y^k\} \subset C$, we obtain $\bar{x} \in C$. By the proof of [1, Lemma 3.1],

$$\xi_k[f(x^k, y) - f(x^k, y^k)] \ge \langle x^k - y^k, y - y^k\rangle, \quad \forall y \in C.$$

Passing to the limit in the last inequality as $k \to \infty$ and using the assumption (R_2) and $\lim_{k\to\infty}\xi_k = \xi > 0$, we get $f(\bar{x}, y) \ge 0$ for all $y \in C$. Thus, $\bar{x} \in \mathrm{Sol}(C, f) = \mathrm{Fix}(T_k)$.

Since x^* is the solution of (3.3) and (3.9), we have

$$\begin{aligned}(\mu\beta_G-\gamma L_g)\lim_{k\to\infty}a_k&=(\mu\beta_G-\gamma L_g)\lim_{k\to\infty}\|x^k-x^*\|^2\\&\le-\liminf_{k\to\infty}\langle\mu G(x^*)-\gamma g(x^*),x^k-x^*\rangle\\&=-\lim_{j\to\infty}\langle\mu G(x^*)-\gamma g(x^*),x^{k_j}-x^*\rangle\\&=-\langle\mu G(x^*)-\gamma g(x^*),\bar{x}-x^*\rangle\\&\le 0.\end{aligned}$$

Using $\gamma\in\left(0,\frac{\mu}{L_g}\left(\beta_G-\frac{\mu L_G^2}{2}\right)\right)$, it implies $\lim_{k\to\infty}a_k=0$. Thus, both $\{x^k\}$ and $\{y^k\}$ converge strongly to x^*.

Case 2. There does not exist any integer k_0 such that $a_{k+1}\le a_k$ for all $k\ge k_0$. Then, consider the sequence of integers as follows:

$$\phi(k)=\max\{j\le k: a_j<a_{j+1}\},\quad\forall k\ge k_0.$$

By [23], $\{\phi(k)\}$ is a nondecreasing sequence verifying

$$\lim_{k\to\infty}\phi(k)=\infty,\quad a_{\phi(k)}\le a_{\phi(k)+1},\quad a_k\le a_{\phi(k)+1},\qquad\forall k\ge k_0.\tag{3.10}$$

Replacing k by $\phi(k)$ into (3.8), it follows that

$$\begin{aligned}&\omega[1-\alpha_{\phi(k)}-2\omega(1-\alpha_{\phi(k)}\tau)^2]\|T_{\phi(k)}x^{\phi(k)}-x^{\phi(k)}\|^2\\&\le a_{\phi(k)}-a_{\phi(k)+1}+2\alpha_{\phi(k)}\langle\mu G(x^{\phi(k)})-\gamma g(x^{\phi(k)}),x^*-x^{\phi(k)}\rangle\\&\quad+2\alpha_{\phi(k)}^2\|\gamma g(x^{\phi(k)})-\mu G(x^{\phi(k)})\|^2\\&\quad+2\omega\alpha_{\phi(k)}(1-\alpha_{\phi(k)}\tau)\|T_{\phi(k)}x^{\phi(k)}-x^{\phi(k)}\|\|x^{\phi(k)}-x^*\|\\&\le 2\alpha_{\phi(k)}\langle\mu G(x^{\phi(k)})-\gamma g(x^{\phi(k)}),x^*-x^{\phi(k)}\rangle+2\alpha_{\phi(k)}^2\|\gamma g(x^{\phi(k)})-\mu G(x^{\phi(k)})\|^2\\&\quad+2\omega\alpha_{\phi(k)}(1-\alpha_{\phi(k)}\tau)\|T_{\phi(k)}x^{\phi(k)}-x^{\phi(k)}\|\|x^{\phi(k)}-x^*\|.\end{aligned}\tag{3.11}$$

Taking the limit as $k\to\infty$ in (3.11) and using the boundedness of $\{x^k\}$, we obtain

$$\|T_{\phi(k)}x^{\phi(k)}-x^{\phi(k)}\|\to 0\quad\text{as }\ k\to\infty.\tag{3.12}$$

From (3.11), it implies

$$\begin{aligned}&\langle\mu G(x^{\phi(k)})-\gamma g(x^{\phi(k)}),x^{\phi(k)}-x^*\rangle\\&\le\alpha_{\phi(k)}\|\gamma g(x^{\phi(k)})-\mu G(x^{\phi(k)})\|^2+2\omega(1-\alpha_{\phi(k)}\tau)\|T_{\phi(k)}x^{\phi(k)}-x^{\phi(k)}\|\|x^{\phi(k)}-x^*\|.\end{aligned}\tag{3.13}$$

Consider (3.9) again, we have

$$\langle\mu G(x^{\phi(k)})-\gamma g(x^{\phi(k)}),x^{\phi(k)}-x^*\rangle\ge\langle\mu G(x^*)-\gamma g(x^*),x^{\phi(k)}-x^*\rangle+(\mu\beta_G-\gamma L_g)a_{\phi(k)}.$$

Combining this and (3.13), it leads

$$\begin{aligned}&\langle\mu G(x^*)-\gamma g(x^*),x^{\phi(k)}-x^*\rangle+(\mu\beta_G-\gamma L_g)a_{\phi(k)}\\&\le\alpha_{\phi(k)}\|\gamma g(x^{\phi(k)})-\mu G(x^{\phi(k)})\|^2+2\omega(1-\alpha_{\phi(k)}\tau)\|T_{\phi(k)}x^{\phi(k)}-x^{\phi(k)}\|\|x^{\phi(k)}-x^*\|.\end{aligned}$$

Then, by using $\lim_{k\to\infty}\alpha_k=0$ and (3.12), we have

$$(\mu\beta_G-\gamma L_g)\limsup_{k\to\infty}a_{\phi(k)}\le-\liminf_{k\to\infty}\langle\mu G(x^*)-\gamma g(x^*),x^{\phi(k)}-x^*\rangle.\tag{3.14}$$

We recall the fact that $\{x^{\phi(k)}\}$ is bounded, we can choose a subsequence $\{x^{\phi(k_j)}\}$ such that

$$x^{\phi(k_j)} \rightharpoonup \hat{x} \quad \text{as} \quad j \to \infty$$

and

$$\liminf_{k\to\infty} \langle \mu G(x^*) - \gamma g(x^*), x^{\phi(k)} - x^* \rangle = \langle \mu G(x^*) - \gamma g(x^*), \hat{x} - x^* \rangle.$$

By a similar way as in Case 1, we also have $\hat{x} \in \mathrm{Sol}(C, f)$. It means $\langle \mu G(x^*) - \gamma g(x^*), \hat{x} - x^* \rangle \geq 0$. Then, using (3.14) and $\mu\beta_G - \gamma L_g > 0$ in (R_3), we deduce

$$(\mu\beta_G - \gamma L_g) \limsup_{k\to\infty} a_{\phi(k)} \leq -\langle \mu G(x^*) - \gamma g(x^*), \hat{x} - x^* \rangle \leq 0,$$

and hence $\limsup_{k\to\infty} a_{\phi(k)} = 0$. However, from (3.8) and (3.12), it follows

$$\limsup_{k\to\infty} a_{\phi(k)+1} \leq \limsup_{k\to\infty} a_{\phi(k)}.$$

Consequently

$$\limsup_{k\to\infty} a_{\phi(k)+1} = 0.$$

Recalling $a_k \leq a_{\phi(k)}$ for all $k \geq k_0$ in (3.10), we immediately obtain $\lim_{k\to\infty} a_k = 0$. Thus, both $\{x^k\}$ and $\{y^k\}$ converge strongly to a unique solution x^* of the variational inequality problem (3.3). Which completes the proof. □

Remark 3.4 Theorem 3.3 showed that the strongly cluster point of the sequences $\{x^k\}$ and $\{y^k\}$ constructed by the algorithm $(SAPA)$ is a unique solution of the variational inequality problem (3.3). This result is a fundamental difference of our algorithm with respect to existing algorithms. However, the set $\mathrm{Sol}(C, f)$ is not given explicit. So, the problem (3.3) is not easy to solve.

4 Numerical Experiments

An important application of the problem $\mathrm{EPs}(C, f)$ is the noncooperative n-person games. The problem is to find $x^* \in C$ such that

$$f_i(x^*[y^i]) \to \max, \quad \forall y^i \in C_i,$$

where

- The ith player's strategy set is a closed convex set C_i of the Euclidean space $\mathcal{R}^{s_i}$ for all $i \in I := \{1, 2, \ldots, n\}$.
- The $f_i : C := C_1 \times C_2 \times \cdots \times C_n \to \mathbb{R}$ is the loss function of player i.
- The $x[y^i]$ stands for the vector obtained from $x = (x^1, \ldots, x^n) \in C$ by replacing x^i with y^i.

By [21], a point $x^* \in C$ is said to be a Nash equilibrium point on C if and only if

$$f_i(x^*) \leq f_i(x^*[y^i]), \quad \forall y^i \in C_i, i \in I.$$

Then, we set

$$f(x, y) = \sum_{i=1}^{n} [f_i(x[y^i]) - f_i(x)].$$

We can see that the problem of finding a Nash equilibrium point of f on C can be formulated equivalently to the problem EPs(C, f).

Now we provide some computational results for solving the problem EPs(C, f) to illustrate the effectiveness of Subgradient Auxiliary Principle Algorithm (SAPA), and also to compare this algorithm with two well-known algorithms using the solution mapping S defined in the form (2.1): Extragradient Algorithm (EA) introduced by Quoc et al. [32, Algorithm 1] with the auxiliary bifunction $L(x, y) = \frac{1}{2}\|y-x\|^2$ for all $x, y \in \mathcal{H}$ and Halpern Subgradient Extragradient Algorithm (HSEA) proposed by Hieu [15, Algorithm 3.2]. As we know, the iteration point x^k defined by S is a solution of the problem EPs(C, f) if and only if $y^k = x^k$. Therefore, we have used the sequence $\{S_k = \|x^k - y^k\| : k = 0, 1, \dots\}$ to consider the convergent rate of all above algorithms. And, we can say that x^k is an ε-solution to the problem EPs(C, f) where $\varepsilon > 0$, if $S_k \le \varepsilon$.

To test all above algorithms, the parameters are chosen as follows.

– Subgradient Auxiliary Principle Algorithm (SAPA):

$$\xi_k = \frac{1}{4c_1} + \frac{1}{5k+400}, \quad \gamma_k = \frac{1}{2}(1 - \xi_k(c_1 + c_2)), \quad \nu = \frac{1}{2}\min\left\{1, \frac{1}{\xi_k c_2}\right\},$$
$$\omega = \frac{1}{4}, \quad \mu = \frac{\beta_G}{L_G}, \quad \gamma = \frac{\mu}{2L_g}\left(\beta_G - \frac{\mu L_G^2}{2}\right), \quad \tau = \frac{1}{2}\min(\gamma L_g, \mu\beta_G),$$
$$\alpha_k = \frac{a}{k+1} \quad \text{where } a = \min\left\{1, \frac{2(\mu\beta_G - \tau)}{\mu^2 L_G^2 - \tau^2}, \frac{1}{\tau - \gamma L_g}\right\}.$$

– Extragradient Algorithm (EA):

$$\beta := \frac{1}{2}, \quad \rho := \frac{1}{2\|z\|(h|e_1| + g|p_1|)} \in \left(0, \min\left\{\frac{\beta}{2c_1}, \frac{\beta}{2c_2}\right\}\right).$$

– Halpern Subgradient Extragradient Algorithm (HSEA):

$$\lambda := \frac{1}{4c_1} \in \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right), \quad \alpha_n := \frac{1}{5n+10}, \quad \forall n \ge 0.$$

Auxiliary convex problems in the algorithms are computed effectively by the function *fmincon* in Matlab 2018a Optimization Toolbox. All the programs are performed on a PC Desktop Intel(R) Core(TM) i7-12700F CPU @ 2.10 GHz 2.50 GHz, RAM 32.00 GB.

Let $\mathcal{H}$ be a real Hilbert space. We introduce a new cost bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ and the constraint C are given in the forms

$$C = \{x \in \mathcal{H} : \|x\|^2 \le R^2, \ \langle r, x\rangle \le l\}, \tag{4.1}$$
$$f(x, y) = \langle [g\sin(p_1\|x\| + p_2) + h\cos(e_1\|y\| + e_2) + m]z, y - x\rangle, \tag{4.2}$$

where $x, y \in \mathcal{H}, R, p_1, p_2, e_1, e_2 \in \mathcal{R}, l > 0, g > 0, h > 0, m \in (g+h, \infty), (z, r) \in \mathcal{H}\times\mathcal{H}$. Then, the C is nonempty closed convex, and the f has the following properties.

Proposition 4.1 *Let the bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ be defined by (4.2). Then,*

(i) *f is pseudomonotone;*
(ii) *f is Lipschitz-type continuous with constants $c_1 = c_2 = \frac{\|z\|(h|e_1|+g|p_1|)}{2}$.*

Proof Assume that $f(x, y) \ge 0$ for each $x, y \in \mathcal{H}$. Then, we have

$$\begin{aligned} 0 \le f(x, y) &= \langle [g\sin(p_1\|x\| + p_2) + h\cos(e_1\|y\| + e_2) + m]z, y - x\rangle \\ &= [g(\sin(p_1\|x\| + p_2) + 1) + h(\cos(e_1\|y\| + e_2) + 1) + (m - g - h)]\langle z, y - x\rangle. \end{aligned}$$

From $m \in (g+h, +\infty)$, it yields that $\langle z, y-x\rangle \geq 0$ and

$$\begin{aligned} f(y,x) &= \langle [g\sin(p_1\|y\|+p_2)+h\cos(e_1\|x\|+e_2)+m]z, x-y\rangle \\ &= [g(1+\sin(p_1\|y\|+p_2))+h(1+\cos(e_1\|x\|+e_2))+(m-g-h)]\langle z, x-y\rangle \\ &\leq 0. \end{aligned}$$

By definition, the bifunction f is pseudomonotone on $\mathcal{H}\times\mathcal{H}$.
Since (4.2), it follows that, for every $x, y, t \in \mathcal{H}$,

$$\begin{aligned} & f(x,y)+f(y,t)-f(x,t) \\ &= \langle [g\sin(p_1\|x\|+p_2)+h\cos(e_1\|y\|+e_2)+m]z, y-x\rangle \\ &\quad +\langle [g\sin(p_1\|y\|+p_2)+h\cos(e_1\|t\|+e_2)+m]z, t-y\rangle \\ &\quad -\langle [g\sin(p_1\|x\|+p_2)+h\cos(e_1\|t\|+e_2)+m]z, t-x\rangle \\ &= \langle [g\sin(p_1\|x\|+p_2)+h\cos(e_1\|y\|+e_2)+m]z, y-x\rangle \\ &\quad +\langle [g\sin(p_1\|y\|+p_2)+h\cos(e_1\|t\|+e_2)+m]z, t-y\rangle \\ &\quad -\langle [g\sin(p_1\|x\|+p_2)+h\cos(e_1\|t\|+e_2)+m]z, t-y\rangle \\ &\quad -\langle [g\sin(p_1\|x\|+p_2)+h\cos(e_1\|t\|+e_2)+m]z, y-x\rangle \\ &= h[\cos(e_1\|y\|+e_2)-\cos(e_1\|t\|+e_2)]\langle z, y-x\rangle \\ &\quad +g[\sin(p_1\|y\|+p_2)-\sin(p_1\|x\|+p_2)]\langle z, t-y\rangle \\ &= -2h\sin\left(e_1\frac{\|y\|+\|t\|}{2}+e_2\right)\sin\left(e_1\frac{\|y\|-\|t\|}{2}\right)\langle z, y-x\rangle \\ &\quad +2g\cos\left(p_1\frac{\|y\|+\|x\|}{2}+p_2\right)\sin\left(p_1\frac{\|y\|-\|x\|}{2}\right)\langle z, t-y\rangle \\ &\geq -2h\left|\sin\left(e_1\frac{\|y\|+\|t\|}{2}+e_2\right)\right|\left|\sin\left(e_1\frac{\|y\|-\|t\|}{2}\right)\right|\|z\|\|y-x\| \\ &\quad -2g\left|\cos\left(p_1\frac{\|y\|+\|x\|}{2}+p_2\right)\right|\left|\sin\left(p_1\frac{\|y\|-\|x\|}{2}\right)\right|\|z\|\|t-y\| \\ &\geq -2h\left|\sin\left(e_1\frac{\|y\|-\|t\|}{2}\right)\right|\|z\|\|y-x\|-2g\left|\sin\left(p_1\frac{\|y\|-\|x\|}{2}\right)\right|\|z\|\|t-y\| \\ &\geq -h|e_1|\,|\|y\|-\|t\||\,\|z\|\|y-x\|-g|p_1|\,|\|y\|-\|x\||\,\|z\|\|t-y\|. \end{aligned}$$

where the last inequality is deduced from the relation

$$|\sin\theta| \leq |\theta|, \quad \forall\theta\in\mathcal{R}.$$

By using the relation

$$|\|a_1\|-\|a_2\|| \leq \|a_1-a_2\|, \quad \forall a_1, a_2 \in \mathcal{H},$$

we obtain

$$\begin{aligned} & f(x,y)+f(y,t)-f(x,t) \\ &\geq -h|e_1|\big|\|y\|-\|t\|\big|\|z\|\|y-x\|-g|p_1|\big|\|y\|-\|x\|\big|\|z\|\|t-y\| \\ &\geq -h|e_1|\|y-t\|\|z\|\|y-x\|-g|p_1|\|y-x\|\|z\|\|t-y\| \end{aligned}$$

$$\geq -\frac{h|e_1|\|z\|}{2}\|y-t\|^2 - \frac{h|e_1|\|z\|}{2}\|y-x\|^2 - \frac{g|p_1|\|z\|}{2}\|y-x\|^2 - \frac{g|p_1|\|z\|}{2}\|t-y\|^2$$
$$= -\frac{\|z\|(h|e_1|+g|p_1|)}{2}\|y-x\|^2 - \frac{\|z\|(h|e_1|+g|p_1|)}{2}\|y-t\|^2.$$

Thus, the f is Lipschitz-type continuous with $c_1 = c_2 = \frac{\|z\|(h|e_1|+g|p_1|)}{2}$. The proof is complete. □

Test 1 First, let us run the algorithm (SAPA) in $\mathcal{R}^s$ with $s = 5$. The starting point is $x^0 = (1, 0, 1, 100, 25)^\top$. The parameters $R, g, p, q, h, e, f, m, l$ and the vectors r, z are randomly chosen as follows:

$$R = 5,\ \ l = 2,\ \ g = 3,\ \ p_1 = -5,\ \ p_2 = 7,\ \ h = 3,\ \ e_1 = 8,\ \ e_2 = 2,\ \ m = g + h + 5,$$
$$r = (2, -3, 5, 8, 4)^\top,\quad z = (10, 5, 3, -7, 12)^\top.$$

Consider the mappings $G : \mathcal{R}^s \to \mathcal{R}^s$, $g : \mathcal{R}^s \to \mathcal{R}^s$:

$$g(x) = 10x,\quad G(x) = Qx + q,\qquad \forall x \in \mathcal{R}^s,$$

where $q \in \mathcal{R}^s$, $Q = AA^\top + B + D$, A is a $s \times s$ matrix, B is a $s \times s$ skew-symmetric matrix, and D is a $s \times s$ diagonal matrix with its nonnegative diagonal entries (so Q is positive semidefinite). It is obviously that G is β_G-strongly monotone and L_G-Lipschitz continuous, where $\beta_G = \min\{t : t \in \text{eig}(Q)\}$ is the smallest eigenvalue of Q and $L_G = \|Q\|$. The matrics A, B, D of the mapping G are chosen randomly as follows:

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 & 0 \\ 2 & 1 & 0 & 5 & -3 \\ 4 & 0 & 7 & 9 & 1 \\ 2 & 5 & 0 & -5 & 3 \\ -1 & 9 & 4 & 2 & 3 \end{bmatrix}_{(5\times5)},\quad B = \begin{bmatrix} 0 & 2 & 3 & 4 & 5 \\ -2 & 0 & -5 & 7 & 9 \\ -3 & 5 & 0 & 6 & -8 \\ -4 & -7 & -6 & 0 & 1 \\ -5 & -9 & 8 & 1 & 0 \end{bmatrix}_{(5\times5)},$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}_{(5\times5)},\quad q = (1, 7, -3, 22, 6)^\top.$$

Then, the eigenvalue and the norm of Q are evaluated as follows:

$$\text{eig}(Q) = \{221.2357, 144.1649, 3.3983, 24.9611, 22.2399\},\quad \|Q\| = 222.3145.$$

This implies that the strongly monotone constant of G is $\beta_G = 3.3983$ and the Lipschitz continuous constant of G is given in $L_G = 222.3145$.

It is easy to evaluate that

$$c_1 = c_2 = \frac{1}{2}(g|p_1| + h|e_1|)\|z\| \approx 352.6213,$$

and

$$\partial f(x^k, \cdot)(y^k) = \begin{cases} \Big\{[g\sin(p_1\|x^k\| + p_2) + h\cos(e_1\|y^k\| + e_2) + m]z \\ \quad -\frac{he_1\sin(e_1\|y^k\|+e_2)}{\|y^k\|}\langle y^k, z\rangle(y^k - x^k)\Big\} & \text{if } y^k \neq 0, \\ \Big\{[g\sin(p_1\|x^k\| + p_2) + h\cos(e_2) + m]z \\ \quad +he_1\sin(e_2)\langle u^k, z\rangle x^k : \|u^k\| \leq 1\Big\}, & \text{otherwise.} \end{cases}$$

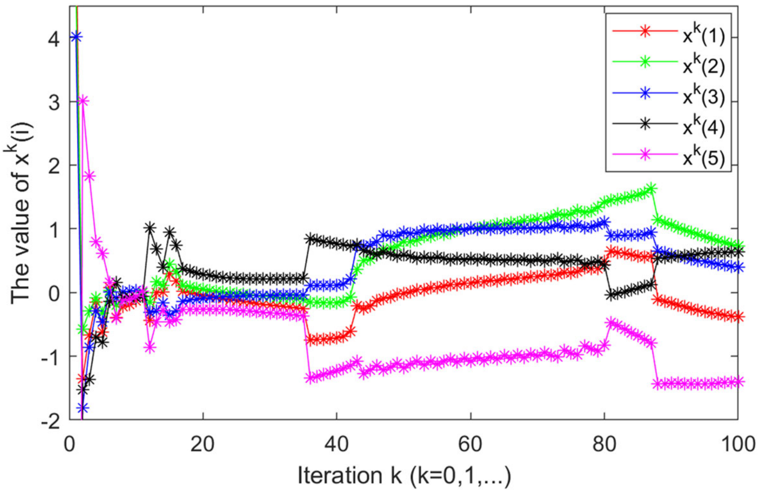

Fig. 1 Performance of the sequence $\{x^k\}$ in the algorithm (SAPA) with the tolerance $\varepsilon = 10^{-3}$. The approximate solution is $x^{196} = (-1.5086, -0.5649, -0.1058, 1.0853, -1.8301)^\top$

With the tolerance $\max\{\|y^k - x^k\|, \|x^{k+1} - x^k\|\} \le \varepsilon = 10^{-3}$, the computational results of the algorithm (SAPA) are showed in Fig. 1 and Table 1.

Test 2 Consider in an infinite-dimensional Hilbert space $\mathcal{H} = L^2[0, 1]$ with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt, \quad \forall x, y \in \mathcal{H},$$

Table 1 Iterations (Iter.) and CPU times (Times) with randomly different parameters, where $a = \min\left\{1, \frac{2(\mu\beta_G-\tau)}{\mu^2 L_G^2-\tau^2}, \frac{1}{\tau-\gamma L_g}\right\}$, $\bar{\gamma_k} = 1 - \xi_k(c_1 + c_2)$, $\bar{\nu_k} = \min\left\{1, \frac{1}{\xi_k c_2}\right\}$ and $\bar{\gamma} = \frac{\mu}{L_g}\left(\beta_G - \frac{\mu L_G^2}{2}\right)$

Test	ξ_k	ν	α_k	γ	μ	ω	Iter.	Times
T_1	$\frac{1}{4c_1} + \frac{1}{5k+400}$	$\frac{1}{2}\bar{\nu_k}$	$\frac{a}{k+1}$	$\frac{1}{2}\bar{\gamma}$	$\frac{\beta_G}{L_G^2}$	0.5	434	0.5313
T_2	$0.001 + \frac{1}{k+1000}$	$\frac{1}{2}\bar{\nu_k}$	$\frac{a}{k+1}$	$\frac{1}{2}\bar{\gamma}$	$\frac{\beta_G}{L_G^2}$	0.5	1605	3.8594
T_3	$\frac{1}{4c_1} + \frac{1}{5k+400}$	$0.2\bar{\nu_k}$	$\frac{a}{k+1}$	$\frac{1}{2}\bar{\gamma}$	$\frac{\beta_G}{L_G^2}$	0.5	333	0.6094
T_4	$\frac{1}{4c_1} + \frac{1}{5k+400}$	$\frac{1}{2}\bar{\nu_k}$	$\frac{a}{10k+100}$	$\frac{1}{2}\bar{\gamma}$	$\frac{\beta_G}{L_G^2}$	0.5	439	0.6250
T_5	$\frac{1}{4c_1} + \frac{1}{5k+400}$	$\frac{1}{2}\bar{\nu_k}$	$\frac{a}{k+1}$	$0.7\bar{\gamma}$	$\frac{\beta_G}{L_G^2}$	0.5	196	4.7500
T_6	$\frac{1}{4c_1} + \frac{1}{5k+400}$	$\frac{1}{2}\bar{\nu_k}$	$\frac{a}{k+1}$	$\frac{1}{2}\bar{\gamma}$	$\frac{\beta_G}{L_G^2}$	0.5	297	2.8906
T_7	$\frac{1}{4c_1} + \frac{1}{5k+400}$	$\frac{1}{2}\bar{\nu_k}$	$\frac{a}{k+1}$	$\frac{1}{2}\bar{\gamma}$	$\frac{\beta_G}{4L_G^2}$	0.5	504	6.4921
T_8	$\frac{1}{4c_1} + \frac{1}{5k+400}$	$\frac{1}{2}\bar{\nu_k}$	$\frac{a}{k+1}$	$\frac{1}{2}\bar{\gamma}$	$\frac{\beta_G}{L_G^2}$	0.9	351	1.9557

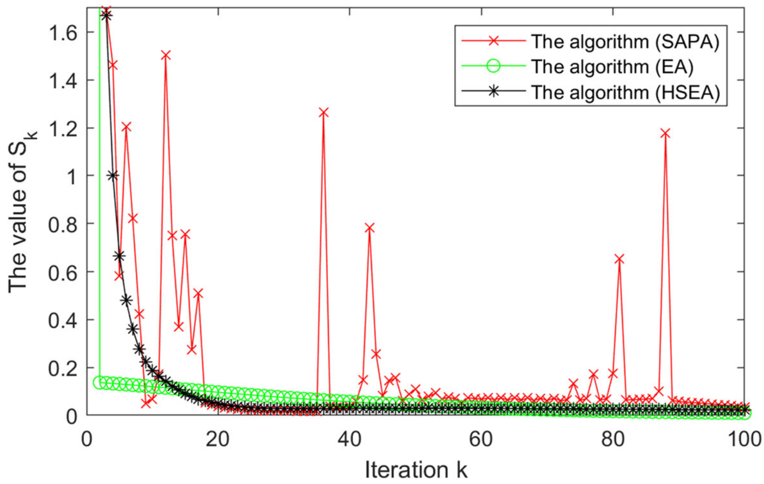

Fig. 2 Comparison results of the algorithm (SAPA) with two algorithms (EA) and (HSEA), where $x^0 = (1, 0, 1, 100, 25)^\top$

and its induced norm

$$\|x\| = \sqrt{\int_0^1 x^2(t)dt}, \quad \forall x \in \mathcal{H}.$$

The constraint C and the cost bifunction f are defined in the forms (4.1) and (4.2). We compare the algorithm (SAPA) with three above algorithm with different starting points x_0. The numerical results are showed in Table 2.

From the comparative results in Fig. 2 and Table 2 of the Subgradient Auxiliary Principle Algorithm (SAPA) with two other agorithms: the Extragradient Algorithm (EA) and the Halpern Subgradient Extragradient Algorithm (HSEA), and the preliminary numerical results reported in Table 1 and Fig. 1, we observe that

- The convergence speed of our algorithm (SAPA) is the most sensitive to all the parameters. The CPU time and iteration number depend very much on the parameter sequence $\{\xi_k\}$.
- The CPU time (second) and the number of iterations of our algorithm are less than those of the algorithms (EA) and (HSEA).

Conclusions In this paper, we introduce a new solution mapping to equilibrium problems in a real Hilbert space. We show that this mapping is strongly quasi-nonexpansiveness under quasimonotone and Lischitz continuous assumptions of the cost bifunction. Then, the Sub-

Table 2 Comparative results with different starting points in $L^2[0, 1]$

Algorithm	$x_0 = \cos t$		$x_0 = \sin t$		$x_0 = 2t^2 + 7t$		$x_0 = 2^t + 5t$	
	S_k	Times	S_k	Times	S_k	Times	S_k	Times
(SAPA)	3.7e-19	19.5	3.5e-17	19.4	2.7e-18	21.6	7.6e-15	23.1
(EA)	2.8e-15	23.6	4.5e-16	21.2	7.31e-18	37.1	9.5e-12	38.5
(HSEA)	6.6e-25	13.5	1.9e-25	13.7	5.0e-17	18.9	4.7e-14	20.8

gradient Auxiliary Principle Algorithm is constructed by the solution mapping and classical auxiliary principle. Finally, the stated theoretical results are verified by several preliminary numerical experiments.

Acknowledgements The authors would like to thank the anonymous referees for their really helpful and constructive comments and suggestions that helped us very much in improving the paper. This research is funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant Number 101.02-2023.01.

References

1. Anh, P.N.: A hybrid extragradient method for pseudomonotone equilibrium problems and fixed point problems. Bull. Malays. Math. Sci. Soc. **36**, 107–116 (2013)
2. Anh, P.N., Anh, T.T.H., Hien, N.D.: Modified basic projection methods for a class of equilibrium problems. Numer. Algor. **79**, 139–152 (2018)
3. Anh, P.N., Le Thi, H.A.: An Armijo-type method for pseudomonotone equilibrium problems and its applications. J. Glob. Optim. **57**, 803–820 (2013)
4. Anh, P.N., Hai, T.N., Tuan, P.M.: On ergodic algorithms for equilibrium problems. J. Glob. Optim. **64**, 179–195 (2016)
5. Anh, P.N., Hieu, D.V.: Multi-step algorithms for solving EPs. Math. Model. Anal. **23**, 453–472 (2018)
6. Anh, P.N., Ansari, Q.H., Tu, H.P.: DC auxiliary principle methods for solving lexicographic equilibrium problems. J. Glob. Optim. **85**, 129–153 (2023)
7. Ansari, Q.H., Balooee, J.: Auxiliary principle technique for solving regularized nonconvex mixed equilibrium problems. Fixed Point Theory **20**, 431–450 (2019)
8. Bianchi, M., Schaible, S.: Generalized monotone bifunctions and equilibrium problems. J. Optim. Theory Appl. **90**, 31–43 (1996)
9. Blum, E., Oettli, W.: From optimization and variational inequality to equilibrium problems. Math. Stud. **63**, 127–149 (1994)
10. Combettes, L.P., Hirstoaga, S.A.: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117–136 (2005)
11. Eskandani, G.Z., Raeisi, M., Rassias, T.M.: A hybrid extragradient method for solving pseudomonotone equilibrium problems using Bregman distance. J. Fixed Point Theory Appl. **20**, 132 (2018)
12. Fan, K.: A minimax inequality and applications. In: Shisha, O. (ed.) Inequality III, pp. 103–113. Academic Press, New York (1972)
13. Giannessi, F., Maugeri, A., Pardalos, P.M. (eds.): Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models. Kluwer Academic Publishers, Dordrecht (2004)
14. Hai, T.N.: Contraction of the proximal mapping and applications to the equilibrium problem. Optimization **66**, 381–396 (2017)
15. Hieu, D.V.: Halpern subgradient extragradient method extended to equilibrium problems. RACSAM **111**, 823–840 (2017)
16. Iiduka, H., Yamada, I.: A subgradient-type method for the equilibrium problem over the fixed point set and its applications. Optimization **58**, 251–261 (2009)
17. Iusem, A.N., Sosa, W.: Iterative algorithms for equilibrium problems. Optimization **52**, 301–316 (2003)
18. Iusem, A.N., Nasri, M.: Inexact proximal point methods for equilibrium problems in Banach spaces. Numer. Funct. Anal. Optim. **28**, 1279–1308 (2007)
19. Khatibzadeh, H., Mohebbi, V.: Proximal point algorithm for infinite pseudo-monotone bifunctions. Optimization **65**, 1629–1639 (2016)
20. Khatibzadeh, H., Mohebbi, V., Ranjbar, S.: Convergence analysis of the proximal point algorithm for pseudo-monotone equilibrium problems. Optim. Methods Softw. **30**, 1146–1163 (2015)
21. Konnov, I.V.: Combined Relaxation Methods for Variational Inequalities. Springer, Berlin (2000)
22. Kraikaew, R., Saejung, S.: Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. J. Optim. Theory Appl. **163**, 399–412 (2014)
23. Maingé, P.E.: The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces. Comput. Math. Appl. **59**, 74–79 (2010)
24. Maingé, P.-E., Moudafi, A.: Coupling viscosity methods with the extragradient algorithm for solving equilibrium problems. J. Nonlinear Convex Anal. **9**, 283–294 (2008)

25. Mastroeni, G.: Gap function for equilibrium problems. J. Glob. Optim. **27**, 411–426 (2003)
26. Mastroeni, G.: On auxiliary principle for equilibrium problems. Publ. Dipart. Math. Univ. Pisa **3**, 1244–1258 (2000)
27. Moudafi, A.: Proximal methods for a class of bilevel monotone equilibrium problems. J. Glob. Optim. **47**, 287–292 (2010)
28. Moreau, J.J.: Proximité et dualité dans un espace hilbertien. Bull. Soc. Math. Fr. **93**, 273–299 (1965)
29. Nikaidô, H., Isoda, K.: Note on noncooperative convex games. Pac. J. Math. **5**, 807–815 (1955)
30. Noor, M.A.: Auxiliary principle technique for equilibrium problems. J. Optim. Theory Appl. **122**, 371–386 (2004)
31. Quoc, T.D., Anh, P.N., Muu, L.D.: Dual extragradient algorithms extended to equilibrium problems. J. Glob. Optim. **52**, 139–159 (2012)
32. Quoc, T.D., Muu, L.D., Hien, N.V.: Extragradient algorithms extended to equilibrium problems. Optimization **57**, 749–776 (2008)
33. Santos, P., Scheimberg, S.: An inexact subgradient algorithm for equilibrium problems. Comput. Appl. Math. **30**, 91–107 (2011)
34. Yordsorn, P., Kumam, P., Ur Rehman, H.: Modified two-step extragradient method for solving the pseudomonotone equilibrium programming in a real Hilbert space. Carpathian J. Math. **36**, 313–330 (2020)