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ORIGINAL ARTICLE

Ghost Effect from Boltzmann Theory: Expansion with Remainder

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Abstract

Consider the limit $\varepsilon \to 0$ of the steady Boltzmann problem

$$v \cdot \nabla_x \mathfrak{F} = \varepsilon^{-1} Q[\mathfrak{F}, \mathfrak{F}], \quad \mathfrak{F}\big|_{v\cdot n<0} = M_w \int_{v'\cdot n>0} \mathfrak{F}(v')|v' \cdot n| \mathrm{d}v', \tag{0.1}$$

where $M_w(x_0, v) := \frac{1}{2\pi(T_w(x_0))^2} \exp\left(-\frac{|v|^2}{2T_w(x_0)}\right)$ for $x_0 \in \partial\Omega$ is the wall Maxwellian in the diffuse-reflection boundary condition. We normalize

$$T_w = 1 + O\left(|\nabla T_w|_{L^\infty}\right).$$

In the case of $|\nabla T_w| = O(\varepsilon)$, the Hilbert expansion confirms $\mathfrak{F} \approx (2\pi)^{-\frac{3}{2}} \mathrm{e}^{-\frac{|v|^2}{2}} + \varepsilon(2\pi)^{-\frac{3}{4}} \mathrm{e}^{-\frac{|v|^2}{4}} \left(\rho_1 + T_1 \frac{|v|^2-3}{2}\right)$ where $(2\pi)^{-\frac{3}{2}} \mathrm{e}^{-\frac{|v|^2}{2}}$ is a global Maxwellian and (ρ_1, T_1) satisfies the celebrated Fourier law

$$\Delta_x T_1 = 0.$$

In the natural case of $|\nabla T_w| = O(1)$, for any constant $P > 0$, the Hilbert expansion leads to

$$\mathfrak{F} \approx \mu + \varepsilon \left\{ \mu \left(\rho_1 + u_1 \cdot v + T_1 \frac{|v|^2 - 3T}{2} \right) - \mu^{\frac{1}{2}} \left(\mathscr{A} \cdot \frac{\nabla_x T}{2T^2} \right) \right\},$$

where $\mu(x, v) := \frac{\rho(x)}{(2\pi T(x))^{\frac{3}{2}}} \exp\left(-\frac{|v|^2}{2T(x)}\right)$, and (ρ, u_1, T) is determined by a Navier–Stokes–Fourier system with "ghost" effect

$$\begin{cases} P = \rho T, \\ \rho(u_1 \cdot \nabla_x u_1) + \nabla_x \mathfrak{p} = \nabla_x \cdot \left(\tau^{(1)} - \tau^{(2)}\right), \\ \nabla_x \cdot (\rho u_1) = 0, \\ \nabla_x \cdot \left(\kappa \frac{\nabla_x T}{2T^2}\right) = 5P(\nabla_x \cdot u_1), \end{cases} \tag{0.2}$$

with the boundary condition

$$T\Big|_{\partial\Omega} = T_w, \quad u_1\Big|_{\partial\Omega} := (u_{1,\iota_1}, u_{1,\iota_2}, u_{1,n})\Big|_{\partial\Omega} = (\beta_0 \partial_{\iota_1} T_w, \beta_0 \partial_{\iota_2} T_w, 0). \tag{0.3}$$

Dedication to Professor Carlos Kenig for his 70th birthday.

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Here $\kappa[T] > 0$ is the heat conductivity, (ι_1, ι_2) are two tangential variables and n is the normal variable, $\beta_0 = \beta_0[T_w]$ is a function of T_w, $\tau^{(1)} := \lambda \left(\nabla_x u_1 + (\nabla_x u_1)^t - \frac{2}{3}(\nabla_x \cdot u_1)\mathbf{1}\right)$ and $\tau^{(2)} := \frac{\lambda^2}{P}\left(K_1\left(\nabla_x^2 T - \frac{1}{3}\Delta_x T\mathbf{1}\right) + \frac{K_2}{T}\left(\nabla_x T \otimes \nabla_x T - \frac{1}{3}|\nabla_x T|^2\mathbf{1}\right)\right)$ for some smooth function $\lambda[T] > 0$, the viscosity coefficient, and positive constants K_1 and K_2. Tangential temperature variation creates non-zero first-order velocity u_1 at the boundary (0.3), which plays a surprising "ghost" effect [26, 27] in determining zeroth-order density and temperature field (ρ, T) in (0.2). Such a ghost effect cannot be predicted by the classical fluid theory, while it has been an intriguing outstanding mathematical problem to justify (0.2) from (0.1) due to fundamental analytical challenges. The goal of this paper is to construct $\mathfrak{F}$ in the form of

$$\mathfrak{F}(x, v) = \mu + \mu^{\frac{1}{2}}\left(\varepsilon f_1 + \varepsilon^2 f_2\right) + \mu_w^{\frac{1}{2}}\left(\varepsilon f_1^B\right) + \varepsilon^\alpha \mu^{\frac{1}{2}} R \tag{0.4}$$

for interior solutions f_1, f_2 and boundary layer f_1^B, where μ_w is μ computed for $T = T_w$, and derive equation for the remainder R with some constant $\alpha \geq 1$. To prove the validity of the expansion suitable bounds on R are needed, which are provided in the companion paper (Esposito 2023).

Keywords Boltzmann theory · Hydrodynamic limit · Ghost effect

Mathematics Subject Classification (2010) Primary 35Q20, 82B40 · Secondary 76P05, 35Q35, 35Q70

1 Introduction

The diffusive hydrodynamic limit of the Boltzmann equation in the low Mach number regime is described by the incompressible Navier–Stokes–Fourier equations under the extra assumption that the initial density and temperature profiles differ from constants at most for terms of the order of the Knudsen number. Such behavior has been proved in several papers and an overview is provided in [23] and [13], to which we refer for a partial list of references on the subject. We also stress that a similar result can be obtained starting from the compressible Navier–Stokes equations, which converge, in the low Mach number limit, to the solutions of the incompressible Navier–Stokes equations [18].

When the density and temperature do not satisfy the above mentioned assumptions, the limiting behavior of the Boltzmann equation deviates from the Navier–Stokes–Fourier equations. Such a discrepancy, called "ghost effect" [27], shows up in the macroscopic equations with the presence of some extra terms reminiscent of the limiting procedure such as some heat flow induced by the vanishingly small velocity field. Thus they are genuine kinetic effects which would be never detected in the standard hydrodynamic equations. Y. Sone has given the suggestive name of "ghost effects" to such phenomena. The meaning of the name is that the velocity field u_1 acts like a ghost since it appears at order ε in the expansion and still affects ρ and T at order 1. In [22] the local well-posedness of the time dependent equations is proven.

In this paper we confine our analysis to the stationary Boltzmann equation for a rarefied gas in a bounded domain with diffuse-reflection boundary data describing a non-homogeneous wall temperature with a gradient of order 1. In this situation the gradient of temperature along the boundary wall produces a flow called in literature thermal creep. For relevant physical background and discussion, we refer to [24].

We give a formal derivation of such new equations when the Mach number, proportional to the Knudsen number ε, goes to 0, and prove their well-posedness. In the companion paper [12] we study the much more involved problem of the rigorous proof of such a derivation. Here we construct the formal solution by a truncated expansion in ε plus a remainder, both in the interior and in a boundary layer of size ε. In view of the control of the remainder, we carefully prepare the expansion by truncating at the second order in ε in the bulk and at the first order in the boundary layer. Then a matching procedure allows to determine the boundary conditions for the limiting equations.

The explicit form of the equations for $(\rho, u_1, T, \mathfrak{p})$ is given in (0.2). The main difference between these equations and the incompressible ones is that $\nabla_x \cdot u_1$ is not anymore zero but is related to the gradient of the temperature. This is the analog of the constraint $\nabla_x \cdot u_1 = 0$ in the incompressible Navier–Stokes equations and is compensated by the Lagrangian multiplier $\mathfrak{p}$ in the equation for u_1. Moreover, in the equation for u_1 there are the usual Navier–Stokes terms involving u_1 and also a term $\tau^{(2)}$ depending on the first and second gradient of the temperature. In particular, the "thermal stress" $\tau^{(2)}$ is a new contribution different from the standard fluid theories. It is exactly this term that cannot be obtained from the compressible Navier–Stokes equation. The relevance of these equations, as also noted by Bobylev [5], is that they cannot be derived from the compressible Navier–Stokes equations. Let us notice that the particular solution corresponding to homogeneous initial condition for density and temperature is also solution of the incompressible Navier–Stokes equations.

We give also the proof of the existence of the solution to (0.2) under the assumption of small temperature gradient. The main difficulty in getting a rigorous proof of the hydrodynamic limit is the control of the remainder. This is achieved in [12].

Before stating the main results, we briefly introduce the history of the study of the ghost effect. Sone [25] and [19, 20] pointed out the new thermal effects in stationary situations. In [11], the equations from the Boltzmann equations in the time dependent case were formally derived, but without computing the transport coefficients. These equations were then discussed by Bobylev [5], who analyzed the behavior of the solutions in particular situations. He also showed that the thermodynamic entropy decreases in time. Finally, Sone and the Kyoto group exploited many other kinds of ghost effects in many papers [28, 29], both analytically and numerically and gave computations of the transport coefficients for the hard sphere case and for Maxwellian molecules. A detailed analysis can be found in [26] and [27] and references therein. Rigorous results in deriving the equations where obtained only in one-dimensional stationary cases [7, 8] and [1]. There are no rigorous results in the time dependent case, not even on the torus, but for [16] where the Korteweg theory is derived from the one-dimensional Boltzmann equation on the infinite line. We also refer to [15, 17] and the references therein.

1.1 Formulation of the Problem

We consider the stationary Boltzmann equation in a bounded three-dimensional C^3 domain $\Omega \ni x = (x_1, x_2, x_3)$ with velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The density function $\mathfrak{F}(x, v)$ satisfies

$$\begin{cases} v \cdot \nabla_x \mathfrak{F} = \varepsilon^{-1} Q[\mathfrak{F}, \mathfrak{F}] & \text{in } \Omega \times \mathbb{R}^3, \\ \mathfrak{F}(x_0, v) = \mathfrak{P}_\gamma[\mathfrak{F}] & \text{for } x_0 \in \partial\Omega \text{ and } v \cdot n(x_0) < 0. \end{cases} \tag{1.1}$$

Here Q is the hard-sphere collision operator

$$Q[F, G] := \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} q(\omega, |\mathfrak{u} - v|) \left(F(\mathfrak{u}_*) G(v_*) - F(\mathfrak{u}) G(v)\right) \mathrm{d}\omega \mathrm{d}\mathfrak{u},$$

with $\mathfrak{u}_* := \mathfrak{u} + \omega((v-\mathfrak{u})\cdot\omega)$, $v_* := v - \omega((v-\mathfrak{u})\cdot\omega)$, and the hard-sphere collision kernel $q(\omega, |\mathfrak{u}-v|) := q_0|\omega\cdot(v-\mathfrak{u})|$ for a positive constant q_0.

In the diffuse-reflection boundary condition

$$\mathfrak{P}_\gamma[\mathfrak{F}] := M_w(x_0, v)\int_{v'\cdot n(x_0)>0}\mathfrak{F}(x_0, v')|v'\cdot n(x_0)|\mathrm{d}v',$$

$n(x_0)$ is the unit outward normal vector at x_0, and the Knudsen number ε satisfies $0<\varepsilon\ll 1$. The wall Maxwellian

$$M_w(x_0, v) := \frac{1}{2\pi(T_w(x_0))^2}\exp\left(-\frac{|v|^2}{2T_w(x_0)}\right)$$

for any $T_w(x_0)>0$ satisfies

$$\int_{v\cdot n(x_0)>0} M_w(x_0, v)|v\cdot n(x_0)|\mathrm{d}v = 1.$$

The boundary condition in (1.1) implies that the total max flux across the boundary is zero.

1.2 Notation and Convention

Based on the flow direction, we can divide the boundary $\gamma := \{(x_0, v) : x_0\in\partial\Omega, v\in\mathbb{R}^3\}$ into the incoming boundary γ_-, the outgoing boundary γ_+, and the grazing set γ_0 based on the sign of $v\cdot n(x_0)$. In particular, the boundary condition of (1.1) is only given on γ_-.

Denote the bulk and boundary norms

$$\|f\|_{L^r} := \left(\iint_{\Omega\times\mathbb{R}^3}|f(x,v)|^r\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{r}}, \qquad |f|_{L^r_{\gamma_\pm}} := \left(\int_{\gamma_\pm}|f(x,v)|^r|v\cdot n|\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{r}}.$$

Define the weighted L^∞ norms for $T_M>0$, $0\leq\varrho<\frac{1}{2}$ and $\vartheta\geq 0$ (see (4.7))

$$\|f\|_{L^\infty_{\varrho,\vartheta}} := \operatorname*{ess\,sup}_{(x,v)\in\Omega\times\mathbb{R}^3}\left(\langle v\rangle^\vartheta \mathrm{e}^{\varrho\frac{|v|^2}{2T_M}}|f(x,v)|\right),$$

$$|f|_{L^\infty_{\gamma_\pm,\varrho,\vartheta}} := \operatorname*{ess\,sup}_{(x,v)\in\gamma_\pm}\left(\langle v\rangle^\vartheta \mathrm{e}^{\varrho\frac{|v|^2}{2T_M}}|f(x,v)|\right).$$

Denote the ν-norm

$$\|f\|_{L^2_\nu} := \left(\iint_{\Omega\times\mathbb{R}^3}\nu(x,v)|f(x,v)|^2\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{2}}.$$

Let $\|\cdot\|_{W^{k,p}}$ denote the usual Sobolev norm for $x\in\Omega$ and $|\cdot|_{W^{k,p}}$ for $x\in\partial\Omega$. Let $\|\cdot\|_{W^{k,p}L^q}$ denote $W^{k,p}$ norm for $x\in\Omega$ and L^q norm for $v\in\mathbb{R}^3$. The similar notation also applies when we replace L^q by $L^\infty_{\varrho,\vartheta}$ or L^q_γ.

Define the quantities (where $\mathcal{L}$ is defined in (2.2))

$$\begin{aligned}
\overline{\mathscr{A}} &:= v\cdot\left(|v|^2-5T\right)\mu^{\frac{1}{2}}\in\mathbb{R}^3, \quad \mathscr{A} := \mathcal{L}^{-1}\left[\overline{\mathscr{A}}\right]\in\mathbb{R}^3,\\
\overline{\mathscr{B}} &= \left(v\otimes v-\frac{|v|^2}{3}\mathbf{1}\right)\mu^{\frac{1}{2}}\in\mathbb{R}^{3\times 3}, \quad \mathscr{B} = \mathcal{L}^{-1}\left[\overline{\mathscr{B}}\right]\in\mathbb{R}^{3\times 3},\\
\kappa\mathbf{1} &:= \int_{\mathbb{R}^3}\left(\mathscr{A}\otimes\overline{\mathscr{A}}\right)\mathrm{d}v, \quad \lambda := \frac{1}{T}\int_{\mathbb{R}^3}\mathscr{B}_{ij}\overline{\mathscr{B}}_{ij} \quad \text{for } i\neq j.
\end{aligned}\tag{1.2}$$

Throughout this paper, $C > 0$ denotes a constant that only depends on the domain Ω, but does not depend on the data or ε. It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity z. We write $a \lesssim b$ to denote $a \leq Cb$ and $a \gtrsim b$ to denote $a \geq Cb$.

In this paper, we will use $o(1)$ to denote a sufficiently small constant independent of the data. Also, let o_T be a small constant depending on T_w satisfying

$$o_T = o(1) \to 0 \quad \text{as} \quad |\nabla T_w|_{W^{3,\infty}} \to 0. \tag{1.3}$$

In principle, while o_T is determined by ∇T_w a priori, we are free to choose $o(1)$ depending on the estimate.

1.3 Main Theorem

Throughout this paper, we assume that

$$|\nabla T_w|_{W^{3,\infty}} = o(1). \tag{1.4}$$

Theorem 1.1 *Under the assumption* (1.4)*, for any given* $P > 0$*, there exists a unique solution* $(\rho, u_1, T; \mathfrak{p})$ *(where* $\mathfrak{p}$ *has zero average) to the ghost-effect* (0.2) *and* (0.3) *satisfying for any* $s \in [2, \infty)$

$$\|u_1\|_{W^{3,s}} + \|\mathfrak{p}\|_{W^{2,s}} + \|T - 1\|_{W^{4,s}} \lesssim o_T.$$

Also, we can construct f_1*,* f_2 *and* f_1^B *as in* (2.31)*,* (2.32)*,* (2.48) *such that*

$$\|f_1\|_{W^{3,s}L^\infty_{\varrho,\vartheta}} + |f_1|_{W^{3-\frac{1}{s},s}L^\infty_{\varrho,\vartheta}} \lesssim o_T,$$
$$\|f_2\|_{W^{2,s}L^\infty_{\varrho,\vartheta}} + |f_2|_{W^{2-\frac{1}{s},s}L^\infty_{\varrho,\vartheta}} \lesssim o_T,$$

and for some $K_0 > 0$ *and any* $0 < r \leq 3$

$$\left\| \mathrm{e}^{K_0\eta} f_1^B \right\|_{L^\infty_{\varrho,\vartheta}} + \left\| \mathrm{e}^{K_0\eta} \frac{\partial^r f_1^B}{\partial \iota_1^r} \right\|_{L^\infty_{\varrho,\vartheta}} + \left\| \mathrm{e}^{K_0\eta} \frac{\partial^r f_1^B}{\partial \iota_2^r} \right\|_{L^\infty_{\varrho,\vartheta}} \lesssim o_T.$$

2 Asymptotic Analysis

In this section we construct a solution to (1.1) by a truncated expansion in ε and determine the ghost effect equation in terms of the first terms of the expansion.

We seek a solution in the form

$$\begin{aligned}\mathfrak{F}(x, v) &= f + f^B + \varepsilon^\alpha \mu^{\frac{1}{2}} R \\ &= \mu + \mu^{\frac{1}{2}} \left(\varepsilon f_1 + \varepsilon^2 f_2\right) + \mu_w^{\frac{1}{2}} \left(\varepsilon f_1^B\right) + \varepsilon^\alpha \mu^{\frac{1}{2}} R,\end{aligned}$$

where f is the interior solution

$$f(x, v) := \mu(x, v) + \mu^{\frac{1}{2}}(x, v) \left(\varepsilon f_1(x, v) + \varepsilon^2 f_2(x, v)\right), \tag{2.1}$$

and f^B is the boundary layer term

$$f^B(x, v) := \mu_w^{\frac{1}{2}}(x_0, v) \left(\varepsilon f_1^B(x, v)\right).$$

Here $R(x, v)$ is the remainder, $\mu(x, v)$ denotes a local Maxwellian which will be specified below and $\mu_w(x_0, v) = \mu(x_0, v)$ the boundary Maxwellian. The parameter $\alpha \geq 1$, will be equal to 1 in the companion paper [12].

We start to determine the first terms of the expansion. Inserting (2.1) into (1.1), at the lowest order of ε, we have

$$\textbf{Order } 0\textbf{:} \quad -Q[\mu, \mu] = 0.$$

This equation guarantees that μ is a local Maxwellian. Denote

$$\mu(x, v) := \frac{\rho(x)}{\big(2\pi T(x)\big)^{\frac{3}{2}}} \exp\left(-\frac{|v|^2}{2T(x)}\right),$$

where $\rho(x) > 0$ and $T(x) > 0$ will be determined later in terms of the solutions of the ghost equations. Notice that this local Maxwellian does not contain the velocity field since we are assuming the Mach number of order ε.

Linearized Boltzmann Operator Define the symmetrized version of Q

$$Q^*[F, G] := \frac{1}{2} \iint_{\mathbb{R}^3\times\mathbb{S}^2} q(\omega, |\mathfrak{u} - v|) \times \big(F(\mathfrak{u}_*)G(v_*) + F(v_*)G(\mathfrak{u}_*) - F(\mathfrak{u})G(v) - F(v)G(\mathfrak{u})\big)\, \mathrm{d}\omega \mathrm{d}\mathfrak{u}.$$

Clearly, $Q[F, F] = Q^*[F, F]$. Denote the linearized Boltzmann operator $\mathcal{L}$

$$\mathcal{L}[f] := -2\mu^{-\frac{1}{2}} Q^*\left[\mu, \mu^{\frac{1}{2}} f\right] := \nu(v) f - K[f], \tag{2.2}$$

where for some kernels $k(\mathfrak{u}, v)$ (see [10, 14]),

$$\begin{aligned} \nu(v) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(\omega, |\mathfrak{u} - v|)\mu(\mathfrak{u})\mathrm{d}\omega\mathrm{d}\mathfrak{u}, \\ K[f](v) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(\omega, |\mathfrak{u} - v|)\mu^{\frac{1}{2}}(\mathfrak{u}) \left(\mu^{\frac{1}{2}}(v_*) f(\mathfrak{u}_*) + \mu^{\frac{1}{2}}(\mathfrak{u}_*) f(v_*)\right) \mathrm{d}\omega\mathrm{d}\mathfrak{u} \\ &\quad -\mu^{\frac{1}{2}}(v) \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(\omega, |\mathfrak{u} - v|)\mu^{\frac{1}{2}}(\mathfrak{u}) f(\mathfrak{u})\mathrm{d}\omega\mathrm{d}\mathfrak{u}. \end{aligned}$$

Note that $\mathcal{L}$ is self-adjoint in $L^2_\nu(\mathbb{R}^3)$. Also, the null space $\mathcal{N}$ of $\mathcal{L}$ is a five-dimensional space spanned by the orthogonal basis

$$\mu^{\frac{1}{2}} \left\{1, v, \left(|v|^2 - 3T\right)\right\}.$$

Denote $\mathcal{N}^\perp$ the orthogonal complement of $\mathcal{N}$ in $L^2(\mathbb{R}^3)$, and $\mathcal{L}^{-1} : \mathcal{N}^\perp \to \mathcal{N}^\perp$ the quasi-inverse of $\mathcal{L}$. Define the kernel operator $\mathbf{P}$ as the orthogonal projection onto the null space $\mathcal{N}$ of $\mathcal{L}$, and the non-kernel operator $\mathbf{I} - \mathbf{P}$. Also, denote the nonlinear Boltzmann operator Γ as

$$\Gamma[f, g] := \mu^{-\frac{1}{2}} Q^*\left[\mu^{\frac{1}{2}} f, \mu^{\frac{1}{2}} g\right] \in \mathcal{N}^\perp.$$

2.1 Derivation of Interior Solution

Further inserting (2.1) into (1.1), we have

$$\textbf{Order 1:}\quad v\cdot\nabla_x\mu-2Q^*\left[\mu,\mu^{\frac{1}{2}}f_1\right]=0, \tag{2.3}$$

$$\textbf{Order } \boldsymbol{\varepsilon}\textbf{:}\quad v\cdot\nabla_x\left(\mu^{\frac{1}{2}}f_1\right)-2Q^*\left[\mu,\mu^{\frac{1}{2}}f_2\right]-Q^*\left[\mu^{\frac{1}{2}}f_1,\mu^{\frac{1}{2}}f_1\right]=0. \tag{2.4}$$

Inspired by the continuation of the expansion, we also require an additional condition that

$$\textbf{Order } \boldsymbol{\varepsilon}^{\mathbf{2}}\textbf{:}\quad \mu^{-\frac{1}{2}}\left(v\cdot\nabla_x\left(\mu^{\frac{1}{2}}f_2\right)\right)\perp v\mu^{\frac{1}{2}}. \tag{2.5}$$

Note that we stop the bulk expansion at order ε^2, so we do not need the orthogonality with $\mu^{\frac{1}{2}}$ and $|v|^2\mu^{\frac{1}{2}}$.

2.1.1 Equation (2.3)

Lemma 2.1 *Equation* (2.3) *is equivalent to*

$$\nabla_x P=\nabla_x(\rho T)=0 \tag{2.6}$$

and for some $\rho_1(x)$, $u_1(x)$, $T_1(x)$,

$$f_1=-\mathscr{A}\cdot\frac{\nabla_x T}{2T^2}+\mu^{\frac{1}{2}}\left(\frac{\rho_1}{\rho}+\frac{u_1\cdot v}{T}+\frac{T_1(|v|^2-3T)}{2T^2}\right). \tag{2.7}$$

Proof Equation (2.3) can be rewritten as

$$\mu^{-\frac{1}{2}}\left(v\cdot\nabla_x\mu\right)=-\mathcal{L}[f_1]. \tag{2.8}$$

Then, by the orthogonality of $\mathcal{L}$ to $\mathcal{N}$, to satisfy (2.8) we must have

$$\int_{\mathbb{R}^3}\left(v\cdot\nabla_x\mu\right)\mathrm{d}v=0,\quad \int_{\mathbb{R}^3}v\left(v\cdot\nabla_x\mu\right)\mathrm{d}v=\mathbf{0},\quad \int_{\mathbb{R}^3}|v|^2\left(v\cdot\nabla_x\mu\right)\mathrm{d}v=0. \tag{2.9}$$

Note that

$$v\cdot\nabla_x\mu=\mu\left(v\cdot\frac{\nabla_x\rho}{\rho}+v\cdot\frac{\nabla_x T(|v|^2-3T)}{2T^2}\right). \tag{2.10}$$

Then the first and third conditions in (2.9) are satisfied by oddness. The second condition in (2.9) can be rewritten in the component form for $i\in\{1,2,3\}$ and summation over $j\in\{1,2,3\}$

$$\begin{aligned}\int_{\mathbb{R}^3}v_iv_j\mu\left(\frac{\partial_j\rho}{\rho}+\frac{\partial_jT(|v|^2-3T)}{2T^2}\right)\mathrm{d}v&=\int_{\mathbb{R}^3}\delta_{ij}\frac{|v|^2}{3}\mu\left(\frac{\partial_j\rho}{\rho}+\frac{\partial_jT(|v|^2-3T)}{2T^2}\right)\mathrm{d}v\\&=\delta_{ij}\left(\rho T\cdot\frac{\partial_j\rho}{\rho}+5\rho T^2\cdot\frac{\partial_jT}{2T^2}-\rho T\cdot\frac{3\partial_jT}{2T}\right)\\&=\delta_{ij}\left(T\partial_j\rho+\rho\partial_jT\right)=\delta_{ij}\partial_j(\rho T)=0.\end{aligned} \tag{2.11}$$

Hence, (2.11) is actually (2.6).

Since $T\nabla_x\rho+\rho\nabla_xT=0$, we deduce $\frac{\nabla_x\rho}{\rho}=-\frac{\nabla_xT}{T}$. Thus, inserting this into (2.10), we have

$$v\cdot\nabla_x\mu=\mu\left(v\cdot\nabla_xT\right)\frac{|v|^2-5T}{2T^2}. \tag{2.12}$$

Considering (2.8) and (2.12), we know

$$\mathcal{L}[f_1] = -\overline{\mathscr{A}} \cdot \frac{\nabla_x T}{2T^2},$$

and (2.7) holds. □

2.1.2 Equation (2.4)

Lemma 2.2 *Equation (2.4) is equivalent to*

$$\nabla_x \cdot (\rho u_1) = 0, \tag{2.13}$$

$$\nabla_x P_1 = \nabla_x (T\rho_1 + \rho T_1) = 0, \tag{2.14}$$

$$5P(\nabla_x \cdot u_1) = \nabla_x \cdot \left(\kappa \frac{\nabla_x T}{2T^2}\right), \tag{2.15}$$

and for some $\rho_2(x)$, $u_2(x)$, $T_2(x)$,

$$f_2 = -\mathcal{L}^{-1}\left[\mu^{-\frac{1}{2}} v \cdot \nabla_x \left(\mu^{\frac{1}{2}} f_1\right)\right] + \mathcal{L}^{-1}\left[\Gamma[f_1, f_1]\right] + \mu^{\frac{1}{2}}\left(\frac{\rho_2}{\rho} + \frac{u_2 \cdot v}{T} + \frac{T_2(|v|^2 - 3T)}{2T^2}\right). \tag{2.16}$$

Proof Since the Q^* terms in (2.4) are orthogonal to $\mathcal{N}$, we must have

$$\int_{\mathbb{R}^3} \left(v \cdot \nabla_x \left(\mu^{\frac{1}{2}} f_1\right)\right) \mathrm{d}v = 0, \quad \int_{\mathbb{R}^3} v \left(v \cdot \nabla_x \left(\mu^{\frac{1}{2}} f_1\right)\right) \mathrm{d}v = 0, \quad \int_{\mathbb{R}^3} |v|^2 \left(v \cdot \nabla_x \left(\mu^{\frac{1}{2}} f_1\right)\right) \mathrm{d}v = 0. \tag{2.17}$$

Using (2.7), the first condition in (2.17) can be rewritten as

$$\nabla_x \cdot \left(-\int_{\mathbb{R}^3} v\mu^{\frac{1}{2}} \mathscr{A} \cdot \frac{\nabla_x T}{2T^2} \mathrm{d}v + \int_{\mathbb{R}^3} v\mu \left(\frac{\rho_1}{\rho} + \frac{u_1 \cdot v}{T} + \frac{T_1(|v|^2 - 3T)}{2T^2}\right) \mathrm{d}v\right) = 0. \tag{2.18}$$

Since $\mathscr{A}$ is orthogonal to $\mathcal{N}$, the first term in (2.18) vanishes. Due to oddness, the ρ_1 and T_1 terms in (2.18) vanish. Hence, we are left with (2.13).

Similarly, the second condition in (2.17) can be rewritten as

$$\nabla_x \cdot \left(-\int_{\mathbb{R}^3} v \otimes v\mu^{\frac{1}{2}} \mathscr{A} \cdot \frac{\nabla_x T}{2T^2} \mathrm{d}v + \int_{\mathbb{R}^3} v \otimes v\mu \left(\frac{\rho_1}{\rho} + \frac{u_1 \cdot v}{T} + \frac{T_1(|v|^2 - 3T)}{2T^2}\right) \mathrm{d}v\right) = 0. \tag{2.19}$$

Due to the oddness of $\mathscr{A}$, the first term in (2.19) vanishes. For the same reason, the u_1 term in (2.19) also vanishes. Thus we are left with (2.14).

Finally, the third condition in (2.17) can be rewritten as

$$\nabla_x \cdot \left(-\int_{\mathbb{R}^3} v|v|^2\mu^{\frac{1}{2}} \mathscr{A} \cdot \frac{\nabla_x T}{2T^2} \mathrm{d}v + \int_{\mathbb{R}^3} v|v|^2\mu \left(\frac{\rho_1}{\rho} + \frac{u_1 \cdot v}{T} + \frac{T_1(|v|^2 - 3T)}{2T^2}\right) \mathrm{d}v\right) = 0. \tag{2.20}$$

Using the orthogonality of $\mathscr{A}$ to $\mathcal{N}$, we know

$$\int_{\mathbb{R}^3} v|v|^2\mu^{\frac{1}{2}} \mathscr{A} \cdot \frac{\nabla_x T}{2T^2} \mathrm{d}v = \int_{\mathbb{R}^3} \overline{\mathscr{A}} \mathscr{A} \cdot \frac{\nabla_x T}{2T^2} \mathrm{d}v = \kappa \frac{\nabla_x T}{2T^2},$$

where κ is defined in (1.2).

Due to oddness, the ρ_1 and T_1 terms in (2.20) vanish, so the u_1 term in (2.20) can be computed

$$\int_{\mathbb{R}^3} v|v|^2\mu\frac{u_1\cdot v}{T}\mathrm{d}v = 5\rho T u_1 = 5Pu_1.$$

Hence, (2.20) becomes

$$\nabla_x\cdot\left(-\kappa\frac{\nabla_x T}{2T^2}+5Pu_1\right)=0,$$

which is equivalent to (2.15).

Equation (2.4) can be rewritten as

$$\mu^{-\frac{1}{2}}v\cdot\nabla_x\left(\mu^{\frac{1}{2}}f_1\right)-\Gamma[f_1,f_1]=-\mathcal{L}[f_2],$$

and thus (2.16) holds. □

2.1.3 Equation (2.5)

Lemma 2.3 *We have the identity*

$$\int_{\mathbb{R}^3}\mathscr{B}\Gamma\left[(u_1\cdot v)\mu^{\frac{1}{2}},\mathscr{A}\right]=-\int_{\mathbb{R}^3}\mathscr{B}\left(\frac{u_1\cdot v}{2}\right)\overline{\mathscr{A}}+T\int_{\mathbb{R}^3}\mathscr{B}(u_1\cdot\overline{\mathscr{B}}). \tag{2.21}$$

Proof We follow the idea in [4]. Denote the translated quantities

$$\mu_s(x,v):=\frac{\rho(x)}{(2\pi T(x))^{\frac{3}{2}}}\exp\left(-\frac{|v-su_1|^2}{2T(x)}\right),\quad \mathcal{L}_s[f]:=-2\mu_s^{-\frac{1}{2}}Q^*\left[\mu_s,\mu_s^{\frac{1}{2}}f\right],$$

and

$$\overline{\mathscr{A}}_s=\overline{\mathscr{A}}(v-su_1),\quad \mathscr{A}_s=\mathcal{L}_s^{-1}[\overline{\mathscr{A}}_s],\quad \overline{\mathscr{B}}_s=\overline{\mathscr{B}}(v-su_1),\quad \mathscr{B}_s=\mathcal{L}_s^{-1}[\overline{\mathscr{B}}_s].$$

Note that translation will not change the orthogonality, i.e. for any $s\in\mathbb{R}$

$$\int_{\mathbb{R}^3}\mathscr{B}_s\overline{\mathscr{A}}_s=\int_{\mathbb{R}^3}\overline{\mathscr{B}}_s\mathscr{A}_s=0.$$

Taking s derivative, we know

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^3}\overline{\mathscr{B}}_s\mathscr{A}_s=0,$$

which is equivalent to

$$\int_{\mathbb{R}^3}\frac{\mathrm{d}\overline{\mathscr{B}}_s}{\mathrm{d}s}\mathcal{L}_s^{-1}\left[\overline{\mathscr{A}}_s\right]+\int_{\mathbb{R}^3}\overline{\mathscr{B}}_s\frac{\mathrm{d}\mathcal{L}_s^{-1}}{\mathrm{d}s}\left[\overline{\mathscr{A}}_s\right]+\int_{\mathbb{R}^3}\overline{\mathscr{B}}_s\mathcal{L}_s^{-1}\left[\frac{\mathrm{d}\overline{\mathscr{A}}_s}{\mathrm{d}s}\right]=0. \tag{2.22}$$

For the first term in (2.22), due to oddness and orthogonality, we can directly verify that

$$\lim_{s\to 0}\int_{\mathbb{R}^3}\frac{\mathrm{d}\overline{\mathscr{B}}_s}{\mathrm{d}s}\mathcal{L}_s^{-1}\left[\overline{\mathscr{A}}_s\right]=\int_{\mathbb{R}^3}\overline{\mathscr{B}}\left(\frac{u_1\cdot v}{2T}\right)\mathscr{A}. \tag{2.23}$$

For the second term in (2.22), we have

$$\int_{\mathbb{R}^3}\overline{\mathscr{B}}_s\frac{\mathrm{d}\mathcal{L}_s^{-1}}{\mathrm{d}s}\left[\overline{\mathscr{A}}_s\right]=-\int_{\mathbb{R}^3}\overline{\mathscr{B}}_s\mathcal{L}_s^{-1}\frac{\mathrm{d}\mathcal{L}_s}{\mathrm{d}s}\mathcal{L}_s^{-1}\left[\overline{\mathscr{A}}_s\right]=-\int_{\mathbb{R}^3}\mathscr{B}_s\frac{\mathrm{d}\mathcal{L}_s}{\mathrm{d}s}[\mathscr{A}_s].$$

Notice that for any $g(v)$

$$\lim_{s\to 0}\frac{\mathrm{d}\mathcal{L}_s}{\mathrm{d}s}[g] = -2\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\left(\mu_s^{-\frac{1}{2}}Q^*\left[\mu_s,\mu_s^{\frac{1}{2}}g\right]\right)$$
$$= -2\left\{-\frac{u_1\cdot v}{2T}\mu^{-\frac{1}{2}}Q^*\left[\mu,\mu^{\frac{1}{2}}g\right]+\mu^{-\frac{1}{2}}Q^*\left[\frac{u_1\cdot v}{T}\mu,\mu^{\frac{1}{2}}g\right]\right.$$
$$\left.+\mu^{-\frac{1}{2}}Q^*\left[\mu,\frac{u_1\cdot v}{2T}\mu^{\frac{1}{2}}g\right]\right\}$$
$$= -\frac{u_1\cdot v}{2T}\mathcal{L}[g]-2\mu^{-\frac{1}{2}}Q^*\left[\frac{u_1\cdot v}{T}\mu,\mu^{\frac{1}{2}}g\right]+\mathcal{L}\left[\frac{u_1\cdot v}{2T}g\right].$$

Hence, we have

$$\lim_{s\to 0}\int_{\mathbb{R}^3}\overline{\mathscr{B}}_s\frac{\mathrm{d}\mathcal{L}_s^{-1}}{\mathrm{d}s}\left[\overline{\mathscr{A}}_s\right] = -\lim_{s\to 0}\int_{\mathbb{R}^3}\mathscr{B}_s\frac{\mathrm{d}\mathcal{L}_s}{\mathrm{d}s}\left[\mathscr{A}_s\right] \tag{2.24}$$
$$= -\int_{\mathbb{R}^3}\mathscr{B}\left(-\frac{u_1\cdot v}{2T}\right)\overline{\mathscr{A}}+2\int_{\mathbb{R}^3}\mathscr{B}\mu^{-\frac{1}{2}}Q^*\left[\frac{u_1\cdot v}{T}\mu,\mu^{\frac{1}{2}}\mathscr{A}\right]-\int_{\mathbb{R}^3}\mathscr{B}\mathcal{L}\left[\frac{u_1\cdot v}{2T}\mathscr{A}\right]$$
$$= \int_{\mathbb{R}^3}\mathscr{B}\left(\frac{u_1\cdot v}{2T}\right)\overline{\mathscr{A}}+2\int_{\mathbb{R}^3}\mathscr{B}\Gamma\left[\frac{u_1\cdot v}{T}\mu^{\frac{1}{2}},\mathscr{A}\right]-\int_{\mathbb{R}^3}\overline{\mathscr{B}}\left(\frac{u_1\cdot v}{2T}\right)\mathscr{A}.$$

For the third term in (2.22), we have

$$\lim_{s\to 0}\int_{\mathbb{R}^3}\overline{\mathscr{B}}_s\mathcal{L}_s^{-1}\left[\frac{\mathrm{d}\overline{\mathscr{A}}_s}{\mathrm{d}s}\right] = \lim_{s\to 0}\int_{\mathbb{R}^3}\mathscr{B}_s\frac{\mathrm{d}\overline{\mathscr{A}}_s}{\mathrm{d}s} = \int_{\mathbb{R}^3}\mathscr{B}\left(\frac{u_1\cdot v}{2T}\right)\overline{\mathscr{A}}-2\int_{\mathbb{R}^3}\mathscr{B}(u_1\cdot\overline{\mathscr{B}}). \tag{2.25}$$

Inserting (2.23), (2.24) and (2.25) into (2.22), we have

$$\int_{\mathbb{R}^3}\overline{\mathscr{B}}\left(\frac{u_1\cdot v}{2T}\right)\mathscr{A}+\int_{\mathbb{R}^3}\mathscr{B}\left(\frac{u_1\cdot v}{2T}\right)\overline{\mathscr{A}}+2\int_{\mathbb{R}^3}\mathscr{B}\Gamma\left[\frac{u_1\cdot v}{T}\mu^{\frac{1}{2}},\mathscr{A}\right]$$
$$-\int_{\mathbb{R}^3}\overline{\mathscr{B}}\left(\frac{u_1\cdot v}{2T}\right)\mathscr{A}+\int_{\mathbb{R}^3}\mathscr{B}\left(\frac{u_1\cdot v}{2T}\right)\overline{\mathscr{A}}-2\int_{\mathbb{R}^3}\mathscr{B}(u_1\cdot\overline{\mathscr{B}})=0.$$

Hence, we know that

$$\int_{\mathbb{R}^3}\mathscr{B}\Gamma\left[\frac{u_1\cdot v}{T}\mu^{\frac{1}{2}},\mathscr{A}\right] = -\int_{\mathbb{R}^3}\mathscr{B}\left(\frac{u_1\cdot v}{2T}\right)\overline{\mathscr{A}}+\int_{\mathbb{R}^3}\mathscr{B}(u_1\cdot\overline{\mathscr{B}}).$$

This verifies (2.21). □

Lemma 2.4 *We have the identity*

$$\Gamma\left[\mu^{\frac{1}{2}}\left(\frac{\rho_1}{\rho}+\frac{u_1\cdot v}{T}+\frac{T_1(|v|^2-3T)}{2T^2}\right),\mu^{\frac{1}{2}}\left(\frac{\rho_1}{\rho}+\frac{u_1\cdot v}{T}+\frac{T_1(|v|^2-3T)}{2T^2}\right)\right]$$
$$= -\mathcal{L}\left[\mu\left(\frac{\rho_1}{\rho}+\frac{u_1\cdot v}{T}+\frac{T_1(|v|^2-3T)}{2T^2}\right)^2\right]. \tag{2.26}$$

Proof The proof can be found in [3, (60)]. A different derivation can be achieved by considering the expansion with respect to ε in $Q[\mu_F,\mu_F]=0$ where

$$\mu_F = \frac{\rho_F}{(2\pi T_F)^{\frac{3}{2}}}\exp\left(-\frac{|v-u_F|^2}{2T_F}\right)$$
$$= \frac{(\rho+\varepsilon\rho_1+\varepsilon^2\rho_2)}{\left(2\pi(T_0+\varepsilon T_1+\varepsilon^2T_2)\right)^{\frac{3}{2}}}\exp\left(-\frac{|v-(\varepsilon u_1+\varepsilon^2u_2)|^2}{2(T_0+\varepsilon T_1+\varepsilon^2T_2)}\right).$$

□

Lemma 2.5 *Equation (2.5) is equivalent to*

$$-\frac{P}{T}\nabla_x\cdot\left(-\frac{2}{3}|u_1|^2\mathbf{1}+2(u_1\otimes u_1)\right)+\nabla_x\mathfrak{p}+\nabla_x\cdot\left(\tau^{(1)}-\tau^{(2)}\right)=0,$$

where

$$\begin{aligned}\tau^{(1)} &:= \int_{\mathbb{R}^3}\mathscr{B}\left\{\mu^{\frac{1}{2}}\left(v\cdot\frac{\nabla_x u_1}{T}\cdot v\right)\right\},\\ \tau^{(2)} &:= \int_{\mathbb{R}^3}\mathscr{B}\left\{v\cdot\nabla_x^2 T\cdot\frac{\mathscr{A}}{2T^2}\right\}+\int_{\mathbb{R}^3}\mathscr{B}\left\{\mu^{-\frac{1}{2}}v\cdot\nabla_x\left(\mu^{\frac{1}{2}}\frac{\mathscr{A}}{2T^2}\right)\cdot\nabla_x T\right\}\\ &\quad+\int_{\mathbb{R}^3}\mathscr{B}\Gamma\left[\mathscr{A}\cdot\frac{\nabla_x T}{2T^2},\mathscr{A}\cdot\frac{\nabla_x T}{2T^2}\right].\end{aligned}$$

Proof Equation (2.5) is equivalent to

$$\int_{\mathbb{R}^3}v\left(v\cdot\nabla_x\left(\mu^{\frac{1}{2}}f_2\right)\right)\mathrm{d}v=0. \tag{2.27}$$

Using (2.16), (2.27) can be rewritten as

$$\begin{aligned}&\nabla_x\cdot\left(-\int_{\mathbb{R}^3}v\otimes v\mu^{\frac{1}{2}}\mathcal{L}^{-1}\left[\mu^{-\frac{1}{2}}\left(v\cdot\nabla_x\left(\mu^{\frac{1}{2}}f_1\right)\right)\right]+\int_{\mathbb{R}^3}v\otimes v\mu^{\frac{1}{2}}\mathcal{L}^{-1}[\Gamma[f_1,f_1]]\right.\\ &\quad\left.+\int_{\mathbb{R}^3}v\otimes v\mu\left(\frac{\rho_2}{\rho}+\frac{u_2\cdot v}{T}+\frac{T_2(|v|^2-3T)}{2T^2}\right)\right)=0.\end{aligned} \tag{2.28}$$

<u>*First Term in* (2.28)</u> For the first term in (2.28), by orthogonality, since $\mathcal{L}^{-1}$ is self-adjoint, using (2.7), we have

$$\begin{aligned}&-\int_{\mathbb{R}^3}v\otimes v\mu^{\frac{1}{2}}\mathcal{L}^{-1}\left[\mu^{-\frac{1}{2}}\left(v\cdot\nabla_x\left(\mu^{\frac{1}{2}}f_1\right)\right)\right]\\ &=-\int_{\mathbb{R}^3}\left(v\otimes v-\frac{|v|^2}{3}\mathbf{1}\right)\mu^{\frac{1}{2}}\mathcal{L}^{-1}\left[\mu^{-\frac{1}{2}}\left(v\cdot\nabla_x\left(\mu^{\frac{1}{2}}f_1\right)\right)\right]\\ &=-\int_{\mathbb{R}^3}\mathcal{L}^{-1}\left[\overline{\mathscr{B}}\right]\left(\mu^{-\frac{1}{2}}\left(v\cdot\nabla_x\left(\mu^{\frac{1}{2}}f_1\right)\right)\right)\\ &=-\int_{\mathbb{R}^3}\mathscr{B}\left(\mu^{-\frac{1}{2}}\left(v\cdot\nabla_x\left(\mu^{\frac{1}{2}}f_1\right)\right)\right),\\ &=-\int_{\mathbb{R}^3}\mathscr{B}\left\{\mu^{-\frac{1}{2}}v\cdot\nabla_x\left(-\mu^{\frac{1}{2}}\mathscr{A}\cdot\frac{\nabla_x T}{2T^2}+\mu\left(\frac{\rho_1}{\rho}+\frac{u_1\cdot v}{T}+\frac{T_1(|v|^2-3T)}{2T^2}\right)\right)\right\}.\end{aligned} \tag{2.29}$$

Due to oddness, the ρ_1 and T_1 terms in (2.29) vanish. Hence, the first term in (2.28) is actually

$$-\int_{\mathbb{R}^3}\mathscr{B}\left\{\mu^{-\frac{1}{2}}v\cdot\nabla_x\left(-\mu^{\frac{1}{2}}\mathscr{A}\cdot\frac{\nabla_x T}{2T^2}+\mu\frac{u_1\cdot v}{T}\right)\right\}=-\tau^{(1)}+\widetilde{\tau}^{(2)}+\widetilde{\varsigma},$$

where

$$\begin{aligned}\tau^{(1)} &:= \int_{\mathbb{R}^3}\mathscr{B}\left\{\mu^{\frac{1}{2}}\left(v\cdot\frac{\nabla_x u_1}{T}\cdot v\right)\right\},\\ \widetilde{\tau}^{(2)} &:= \int_{\mathbb{R}^3}\mathscr{B}\left\{v\cdot\nabla_x^2 T\cdot\frac{\mathscr{A}}{2T^2}\right\}+\int_{\mathbb{R}^3}\mathscr{B}\left\{\mu^{-\frac{1}{2}}v\cdot\nabla_x\left(\mu^{\frac{1}{2}}\frac{\mathscr{A}}{2T^2}\right)\cdot\nabla_x T\right\},\end{aligned}$$

and

$$\begin{aligned}\widetilde{\varsigma} &:= \int_{\mathbb{R}^3} \mathscr{B}\left\{\mu^{\frac{1}{2}}\left(v \cdot \frac{\nabla_x T}{T^2}\right)(u_1 \cdot v)\right\} - \int_{\mathbb{R}^3} \mathscr{B}\left\{\mu^{-\frac{1}{2}}(v \cdot \nabla_x \mu)\left(\frac{u_1 \cdot v}{T}\right)\right\} \\ &= \int_{\mathbb{R}^3} \mathscr{B}\left\{\mu^{\frac{1}{2}}\left(v \cdot \frac{\nabla_x T}{T^2}\right)(u_1 \cdot v)\right\} - \int_{\mathbb{R}^3} \mathscr{B}\left\{\mu^{\frac{1}{2}}\left(v \cdot \frac{\nabla_x T}{2T^3}\right)\left(|v|^2 - 5T\right)(u_1 \cdot v)\right\} \\ &= \frac{\nabla_x T}{T^2} \cdot \int_{\mathbb{R}^3} \mathscr{B} \cdot \left\{(u_1 \cdot v) v \mu^{\frac{1}{2}}\right\} - \frac{\nabla_x T}{2T^3} \cdot \int_{\mathbb{R}^3} \mathscr{B} \cdot \left\{\overline{\mathscr{A}}(u_1 \cdot v)\right\}.\end{aligned}$$

Second Term of (2.28) For the second term of (2.28), we have

$$\begin{aligned}&\int_{\mathbb{R}^3} v \otimes v \mu^{\frac{1}{2}} \mathcal{L}^{-1}\left[\Gamma[f_1, f_1]\right] = \int_{\mathbb{R}^3} \bar{\mathscr{B}} \mathcal{L}^{-1}\left[\Gamma[f_1, f_1]\right] = \int_{\mathbb{R}^3} \mathscr{B}\Gamma[f_1, f_1] \qquad (2.30)\\ &= \int_{\mathbb{R}^3} \mathscr{B}\Gamma\left[-\mathscr{A} \cdot \frac{\nabla_x T}{2T^2}, -\mathscr{A} \cdot \frac{\nabla_x T}{2T^2}\right] \\ &\quad + 2\int_{\mathbb{R}^3} \mathscr{B}\Gamma\left[-\mathscr{A} \cdot \frac{\nabla_x T}{2T^2}, \mu^{\frac{1}{2}}\left(\frac{\rho_1}{\rho} + \frac{u_1 \cdot v}{T} + \frac{T_1|v|^2 - 3T)}{2T^2}\right)\right] \\ &\quad + \int_{\mathbb{R}^3} \mathscr{B}\Gamma\left[\mu^{\frac{1}{2}}\left(\frac{\rho_1}{\rho} + \frac{u_1 \cdot v}{T} + \frac{T_1(|v|^2 - 3T)}{2T^2}\right), \mu^{\frac{1}{2}}\left(\frac{\rho_1}{\rho} + \frac{u_1 \cdot v}{T} + \frac{T_1(|v|^2 - 3T)}{2T^2}\right)\right].\end{aligned}$$

For the first term in (2.30), denote

$$\overline{\tau}^{(2)} := \int_{\mathbb{R}^3} \mathscr{B}\Gamma\left[-\mathscr{A} \cdot \frac{\nabla_x T}{2T^2}, -\mathscr{A} \cdot \frac{\nabla_x T}{2T^2}\right].$$

Then denote

$$\tau^{(2)} := \widetilde{\tau}^{(2)} + \overline{\tau}^{(2)}.$$

For the second term in (2.30), using identity (2.21), we obtain

$$\begin{aligned}\overline{\varsigma} &:= 2\int_{\mathbb{R}^3} \mathscr{B}\Gamma\left[-\mathscr{A} \cdot \frac{\nabla_x T}{2T^2}, \mu^{\frac{1}{2}}\left(\frac{\rho_1}{\rho} + \frac{u_1 \cdot v}{T} + \frac{T_1(|v|^2 - 3T)}{2T^2}\right)\right] \\ &= -\int_{\mathbb{R}^3} \mathscr{B}\Gamma\left[\mathscr{A} \cdot \frac{\nabla_x T}{T^2}, \mu^{\frac{1}{2}}\left(\frac{u_1 \cdot v}{T}\right)\right] = -\frac{\nabla_x T}{T^3} \cdot \int_{\mathbb{R}^3} \mathscr{B}\Gamma\left[\mathscr{A}, \mu^{\frac{1}{2}}(u_1 \cdot v)\right] \\ &= \frac{\nabla_x T}{2T^3} \cdot \int_{\mathbb{R}^3} \mathscr{B} \cdot \overline{\mathscr{A}}(u_1 \cdot v) - \frac{\nabla_x T}{T^2} \cdot \int_{\mathbb{R}^3} \mathscr{B} \cdot (u_1 \cdot \overline{\mathscr{B}}).\end{aligned}$$

Then we have

$$\widetilde{\varsigma} + \overline{\varsigma} = 0.$$

For the third term in (2.30), direct computation using (2.26) and oddness reveals that

$$\begin{aligned}&-\int_{\mathbb{R}^3} \mathscr{B}\Gamma\left[\mu^{\frac{1}{2}}\left(\frac{\rho_1}{\rho} + \frac{u_1 \cdot v}{T} + \frac{T_1(|v|^2 - 3T)}{2T^2}\right), \mu^{\frac{1}{2}}\left(\frac{\rho_1}{\rho} + \frac{u_1 \cdot v}{T} + \frac{T_1(|v|^2 - 3T)}{2T^2}\right)\right] \\ &= \int_{\mathbb{R}^3} \overline{\mathscr{B}} \mathcal{L}^{-1}\left[\mathcal{L}\left[\mu^{\frac{1}{2}} \frac{(u_1 \cdot v)^2}{T^2}\right]\right] = -\frac{2P}{3T}|u_1|^2 \mathbf{1} + \frac{2P}{T}(u_1 \otimes u_1).\end{aligned}$$

Third Term of (2.28) For the third term of (2.28), due to oddness, u_2 terms vanish, and thus we have

$$\int_{\mathbb{R}^3} v \otimes v \mu \left(\frac{\rho_2}{\rho} + \frac{u_2 \cdot v}{T} + \frac{T_2(|v|^2 - 3T)}{2T^2}\right) = (T\rho_2 + \rho T_2)\mathbf{1}.$$

□

2.1.4 Ghost-Effect Equations

Collecting all above and rearranging the terms, we have

$$\mu(x, v) = \frac{\rho(x)}{\big(2\pi T(x)\big)^{\frac{3}{2}}} \exp\left(-\frac{|v|^2}{2T(x)}\right)$$

and

$$f_1 = -\mathscr{A} \cdot \frac{\nabla_x T}{2T^2} + \mu^{\frac{1}{2}} \left(\frac{\rho_1}{\rho} + \frac{u_1 \cdot v}{T} + \frac{T_1(|v|^2 - 3T)}{2T^2}\right), \tag{2.31}$$

$$f_2 = -\mathcal{L}^{-1}\left[\mu^{-\frac{1}{2}}\left(v \cdot \nabla_x \left(\mu^{\frac{1}{2}} f_1\right)\right)\right] + \mathcal{L}^{-1}\left[\Gamma[f_1, f_1]\right] + \mu^{\frac{1}{2}}\left(\frac{\rho_2}{\rho} + \frac{u_2 \cdot v}{T} + \frac{T_2\big(|v|^2 - 3T\big)}{2T^2}\right), \tag{2.32}$$

where $(\rho, 0, T)$, (ρ_1, u_1, T_1) and (ρ_2, u_2, T_2) satisfy

- Order 1 equation:
$$\nabla_x P = \nabla_x(\rho T) = 0. \tag{2.33}$$

- Order ε system:
$$\nabla_x \cdot (\rho u_1) = 0, \tag{2.34}$$
$$\nabla_x P_1 = \nabla_x(T\rho_1 + \rho T_1) = 0,$$
$$\nabla_x \cdot \left(\kappa \frac{\nabla_x T}{2T^2}\right) = 5P(\nabla_x \cdot u_1). \tag{2.35}$$

- Order ε^2 system:
$$\rho(u_1 \cdot \nabla_x u_1) + \nabla_x \mathfrak{p} = \nabla_x \cdot \left(\tau^{(1)} - \tau^{(2)}\right). \tag{2.36}$$

Here $u_k = (u_{k,1}, u_{k,2}, u_{k,3})$,

$$P := \rho T, \quad P_1 := T\rho_1 + \rho T_1, \quad \mathfrak{p} := T\rho_2 + \rho T_2, \tag{2.37}$$
$$\overline{\mathscr{A}} := v \cdot \big(|v|^2 - 5T\big)\mu^{\frac{1}{2}}, \quad \mathscr{A} := \mathcal{L}^{-1}[\overline{\mathscr{A}}] = \mathcal{L}^{-1}\left[v \cdot \big(|v|^2 - 5T\big)\mu^{\frac{1}{2}}\right],$$
$$\kappa\mathbf{1} := \int_{\mathbb{R}^3} \mathscr{A} \otimes \overline{\mathscr{A}} \mathrm{d}v,$$

and

$$\tau^{(1)} := \lambda\left(\nabla_x u_1 + (\nabla_x u_1)^t - \frac{2}{3}(\nabla_x \cdot u_1)\mathbf{1}\right),$$
$$\tau^{(2)} := \frac{\lambda^2}{P}\left(K_1\left(\nabla_x^2 T - \frac{1}{3}\Delta_x T\mathbf{1}\right) + \frac{K_2}{T}\left(\nabla_x T \otimes \nabla_x T - \frac{1}{3}|\nabla_x T|^2\mathbf{1}\right)\right)$$

for smooth functions $\lambda[T] > 0$, and positive constants K_1 and K_2 [5, 20, 26].

We observe that (2.33), (2.34), (2.35) and (2.36) are a set of equations sufficient to determine $(\rho, u_1, T, \nabla_x \mathfrak{p})$ uniquely once suitable boundary conditions are specified:

$$\begin{cases} \nabla_x P = \nabla_x(\rho T) = 0, \\ \rho(u_1 \cdot \nabla_x u_1) + \nabla_x \mathfrak{p} = \nabla_x \cdot \left(\tau^{(1)} - \tau^{(2)}\right), \\ \nabla_x \cdot (\rho u_1) = 0, \\ \nabla_x \cdot \left(\kappa \dfrac{\nabla_x T}{2T^2}\right) = 5P(\nabla_x \cdot u_1). \end{cases}$$

Notice that $\mathfrak{p}$ enters in the equations only through its gradient so we are free to choose a definite value by imposing $\int_\Omega \mathfrak{p} = 0$.

Also, we are left with an additional requirement:

$$\nabla_x P_1 = \nabla_x(T\rho_1 + \rho T_1) = 0. \tag{2.38}$$

The higher-order terms of the expansion will be discussed in Section 3.

2.2 Normal Chart Near Boundary

In order to define the boundary layer correction, we need to design a coordinate system based on the normal and tangential directions on the boundary surface. Our main goal is to rewrite the three-dimensional transport operator $v \cdot \nabla_x$ in this new coordinate system. This is basically textbook-level differential geometry, so we omit the details.

Substitution 1: Spatial Substitution: For a smooth manifold $\partial\Omega$, there exists an orthogonal curvilinear coordinates system (ι_1, ι_2) such that the coordinate lines coincide with the principal directions at any $x_0 \in \partial\Omega$ (at least locally).

Assume $\partial\Omega$ is parameterized by $\mathbf{r} = \mathbf{r}(\iota_1, \iota_2)$. Let $|\cdot|$ denote the length. Hence, $\partial_{\iota_1}\mathbf{r}$ and $\partial_{\iota_2}\mathbf{r}$ represent two orthogonal tangential vectors. Denote $L_i = |\partial_{\iota_i}\mathbf{r}|$ for $i = 1, 2$. Then define the two orthogonal unit tangential vectors

$$\varsigma_1 := \frac{\partial_{\iota_1}\mathbf{r}}{L_1}, \qquad \varsigma_2 := \frac{\partial_{\iota_2}\mathbf{r}}{L_2}.$$

Also, the outward unit normal vector is

$$n := \frac{\partial_{\iota_1}\mathbf{r} \times \partial_{\iota_2}\mathbf{r}}{|\partial_{\iota_1}\mathbf{r} \times \partial_{\iota_2}\mathbf{r}|} = \varsigma_1 \times \varsigma_2.$$

Obviously, $(\varsigma_1, \varsigma_2, n)$ forms a new orthogonal frame. Hence, consider the corresponding new coordinate system $(\iota_1, \iota_2, \mathfrak{n})$, where $\mathfrak{n}$ denotes the normal distance to boundary surface $\partial\Omega$, i.e.

$$x = \mathbf{r} - \mathfrak{n} n.$$

Note that $\mathfrak{n} = 0$ means $x \in \partial\Omega$ and $\mathfrak{n} > 0$ means $x \in \Omega$ (before reaching the other side of $\partial\Omega$). Using this new coordinate system and denoting κ_i the principal curvatures, the transport operator becomes

$$v \cdot \nabla_x = -(v \cdot n)\frac{\partial}{\partial \mathfrak{n}} - \frac{v \cdot \varsigma_1}{L_1(\kappa_1 \mathfrak{n} - 1)}\frac{\partial}{\partial \iota_1} - \frac{v \cdot \varsigma_2}{L_2(\kappa_2 \mathfrak{n} - 1)}\frac{\partial}{\partial \iota_2}.$$

Substitution 2: Velocity Substitution: Define the orthogonal velocity substitution for $\mathfrak{v} := (v_\eta, v_\phi, v_\psi)$ as

$$\begin{cases} -v \cdot n := v_\eta, \\ -v \cdot \varsigma_1 := v_\phi, \\ -v \cdot \varsigma_2 := v_\psi. \end{cases}$$

Then the transport operator becomes

$$\begin{aligned} v \cdot \nabla_x = {} & v_\eta \frac{\partial}{\partial \mathfrak{n}} - \frac{1}{R_1 - \mathfrak{n}} \left(v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) - \frac{1}{R_2 - \mathfrak{n}} \left(v_\psi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\psi \frac{\partial}{\partial v_\psi} \right) \\ & + \frac{1}{L_1 L_2} \left(\frac{R_1 \partial_{\iota_1 \iota_1} \mathbf{r} \cdot \partial_{\iota_2} \mathbf{r}}{L_1 (R_1 - \mathfrak{n})} v_\phi v_\psi + \frac{R_2 \partial_{\iota_1 \iota_2} \mathbf{r} \cdot \partial_{\iota_2} \mathbf{r}}{L_2 (R_2 - \mathfrak{n})} v_\psi^2 \right) \frac{\partial}{\partial v_\phi} \\ & + \frac{1}{L_1 L_2} \left(R_2 \frac{\partial_{\iota_2 \iota_2} \mathbf{r} \cdot \partial_{\iota_1} \mathbf{r}}{L_2 (R_2 - \mathfrak{n})} v_\phi v_\psi + \frac{R_1 \partial_{\iota_1 \iota_2} \mathbf{r} \cdot \partial_{\iota_1} \mathbf{r}}{L_1 (R_1 - \mathfrak{n})} v_\phi^2 \right) \frac{\partial}{\partial v_\psi} \\ & + \left(\frac{R_1 v_\phi}{L_1 (R_1 - \mathfrak{n})} \frac{\partial}{\partial \iota_1} + \frac{R_2 v_\psi}{L_2 (R_2 - \mathfrak{n})} \frac{\partial}{\partial \iota_2} \right), \end{aligned}$$

where $R_i = \kappa_i^{-1}$ represent the radii of principal curvature.

Substitution 3: Scaling Substitution: Finally, we define the scaled variable $\eta = \frac{\mathfrak{n}}{\varepsilon}$, which implies $\frac{\partial}{\partial \mathfrak{n}} = \frac{1}{\varepsilon} \frac{\partial}{\partial \eta}$. Then the transport operator becomes

$$\begin{aligned} v \cdot \nabla_x = {} & \frac{1}{\varepsilon} v_\eta \frac{\partial}{\partial \eta} - \frac{1}{R_1 - \varepsilon\eta} \left(v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) - \frac{1}{R_2 - \varepsilon\eta} \left(v_\psi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\psi \frac{\partial}{\partial v_\psi} \right) \\ & + \frac{1}{L_1 L_2} \left(\frac{R_1 \partial_{\iota_1 \iota_1} \mathbf{r} \cdot \partial_{\iota_2} \mathbf{r}}{L_1 (R_1 - \varepsilon\eta)} v_\phi v_\psi + \frac{R_2 \partial_{\iota_1 \iota_2} \mathbf{r} \cdot \partial_{\iota_2} \mathbf{r}}{L_2 (R_2 - \varepsilon\eta)} v_\psi^2 \right) \frac{\partial}{\partial v_\phi} \\ & + \frac{1}{L_1 L_2} \left(R_2 \frac{\partial_{\iota_2 \iota_2} \mathbf{r} \cdot \partial_{\iota_1} \mathbf{r}}{L_2 (R_2 - \varepsilon\eta)} v_\phi v_\psi + \frac{R_1 \partial_{\iota_1 \iota_2} \mathbf{r} \cdot \partial_{\iota_1} \mathbf{r}}{L_1 (R_1 - \varepsilon\eta)} v_\phi^2 \right) \frac{\partial}{\partial v_\psi} \\ & + \left(\frac{R_1 v_\phi}{L_1 (R_1 - \varepsilon\eta)} \frac{\partial}{\partial \iota_1} + \frac{R_2 v_\psi}{L_2 (R_2 - \varepsilon\eta)} \frac{\partial}{\partial \iota_2} \right). \end{aligned}$$

2.3 Milne Problem with Tangential Dependence

To construct the Hilbert expansion in a general domain, it is important to study the Milne problem depending on the tangential variable (ι_1, ι_2). Notice that, in the new variables, $\mu_w = \mu_w(\iota_1, \iota_2, \mathfrak{v})$. Set

$$\mathcal{L}_w[f] := -2\mu_w^{-\frac{1}{2}} Q^* \left[\mu_w, \mu_w^{\frac{1}{2}} f \right] = \nu_w f - K_w[f].$$

Let $\Phi(\eta, \iota_1, \iota_2, \mathfrak{v})$ be solution to the Milne problem

$$v_\eta \frac{\partial \Phi}{\partial \eta} + \nu_w \Phi - K_w[\Phi] = 0, \tag{2.39}$$

with in-flow boundary condition at $\eta = 0$

$$\Phi(0, \iota_1, \iota_2, \mathfrak{v}) = \mathscr{A} \cdot \frac{\nabla_x T}{2T^2} \quad \text{for } v_\eta > 0, \tag{2.40}$$

and the zero mass-flux condition

$$\int_{\mathbb{R}^3} v_\eta \mu_w^{\frac{1}{2}}(\iota_1,\iota_2,\mathfrak{v})\Phi(0,\iota_1,\iota_2,\mathfrak{v})\mathrm{d}\mathfrak{v}=0. \tag{2.41}$$

Theorem 2.6 *Assume that $\nabla_x T\in W^{k,\infty}(\partial\Omega)$ for some $k\in\mathbb{N}$ and $|T|_{L^\infty_{\partial\Omega}}\lesssim 1$. Then there exists*

$$\Phi_\infty(\iota_1,\iota_2,\mathfrak{v}):=\Phi_\infty(\iota_1,\iota_2,v):=\mu_w^{\frac{1}{2}}\left(\frac{\rho^B}{\rho_w}+\frac{u^B\cdot v}{T_w}+\frac{T^B(|v|^2-3T_w)}{2T_w^2}\right)\in\mathcal{N}, \tag{2.42}$$

for $\rho_w:=PT_w^{-1}$ and some $(\rho^B(\iota_1,\iota_2),u^B(\iota_1,\iota_2),T^B(\iota_1,\iota_2))$ such that

$$\int_{\mathbb{R}^3} v_\eta\mu_w^{\frac{1}{2}}(\mathfrak{v})\Phi_\infty(\mathfrak{v})\mathrm{d}\mathfrak{v}=0,$$

and a unique solution $\Phi(\eta,\iota_1,\iota_2,\mathfrak{v})$ to (2.39) such that $\widetilde{\Phi}:=\Phi-\Phi_\infty$ satisfies

$$\begin{cases} v_\eta\dfrac{\partial\widetilde{\Phi}}{\partial\eta}+\nu_w\widetilde{\Phi}-K_w\left[\widetilde{\Phi}\right]=0,\\ \widetilde{\Phi}(0,\iota_1,\iota_2,\mathfrak{v})=\Phi(0,\iota_1,\iota_2,\mathfrak{v})-\Phi_\infty(\iota_1,\iota_2,\mathfrak{v})\ \textit{for}\ v_\eta>0,\\ \int_{\mathbb{R}^3} v_\eta\mu_w^{\frac{1}{2}}(\iota_1,\iota_2,\mathfrak{v})\widetilde{\Phi}(0,\iota_1,\iota_2,\mathfrak{v})\mathrm{d}\mathfrak{v}=0\end{cases} \tag{2.43}$$

and for some $K_0>0$ and any $0<r\le k$

$$|\Phi_\infty|+\left\|\mathrm{e}^{K_0\eta}\widetilde{\Phi}\right\|_{L^\infty_{\varrho,\vartheta}}\lesssim|\nabla_x T|_{L^\infty_{\partial\Omega}}, \tag{2.44}$$

$$\left\|\mathrm{e}^{K_0\eta}v_\eta\partial_\eta\widetilde{\Phi}\right\|_{L^\infty_{\varrho,\vartheta}}+\left\|\mathrm{e}^{K_0\eta}v_\eta\partial_{v_\eta}\widetilde{\Phi}\right\|_{L^\infty_{\varrho,\vartheta}}\lesssim|\nabla_x T|_{L^\infty_{\partial\Omega}}, \tag{2.45}$$

$$\left\|\mathrm{e}^{K_0\eta}\partial_{v_\phi}\widetilde{\Phi}\right\|_{L^\infty_{\varrho,\vartheta}}+\left\|\mathrm{e}^{K_0\eta}\partial_{v_\psi}\widetilde{\Phi}\right\|_{L^\infty_{\varrho,\vartheta}}\lesssim|\nabla_x T|_{L^\infty_{\partial\Omega}}, \tag{2.46}$$

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}\partial_{\iota_1}^r\widetilde{\Phi}\right\|_{L^\infty_{\varrho,\vartheta}}+\left\|\mathrm{e}^{K_0\eta}\partial_{\iota_2}^r\widetilde{\Phi}\right\|\lesssim|\nabla_x T|_{L^\infty_{\partial\Omega}}&+\sum_{j=1}^r\left|\partial_{\iota_1}^j\nabla_x T\right|_{L^\infty_{\partial\Omega}}\\ &+\sum_{j=1}^r\left|\partial_{\iota_2}^j\nabla_x T\right|_{L^\infty_{\partial\Omega}}.\end{aligned} \tag{2.47}$$

Proof Based on [2] and [30], we have the well-posedness of (2.39). Also, estimates (2.44), (2.45) and (2.46) follow. Hence, we will focus on (2.47). Let $W:=\frac{\partial\widetilde{\Phi}}{\partial\iota_i}$ for $i=1,2$. Then W satisfies

$$\begin{cases} v_\eta\dfrac{\partial W}{\partial\eta}+\nu_w W-K_w[W]=-\dfrac{\partial\nu_w}{\partial\iota_i}\widetilde{\Phi}+\dfrac{\partial K_w}{\partial\iota_i}[\widetilde{\Phi}],\\ W(0,\iota_1,\iota_2,\mathfrak{v})=-\dfrac{\partial}{\partial\iota_i}\left(\mathscr{A}\cdot\dfrac{\nabla_x T}{2T^2}\right)-\dfrac{\partial\widetilde{\Phi}_\infty}{\partial\iota_i}(\iota_1,\iota_2,\mathfrak{v})\quad\text{for}\ \ v_\eta>0,\\ \displaystyle\int_{\mathbb{R}^3} v_\eta\mu_w^{\frac{1}{2}}(\mathfrak{v})W(0,\iota_1,\iota_2,\mathfrak{v})\mathrm{d}\mathfrak{v}=-\int_{\mathbb{R}^3} v_\eta\frac{\partial\mu_w^{\frac{1}{2}}}{\partial\iota_i}(\iota_1,\iota_2,\mathfrak{v})\widetilde{\Phi}(0,\iota_1,\iota_2,\mathfrak{v})\mathrm{d}\mathfrak{v}.\end{cases}$$

Multiplying $|\mathfrak{v}|^2\mu_w^{\frac{1}{2}}$ on both sides of (2.43) and integrating over $\mathbb{R}^3$ yield

$$\int_{\mathbb{R}^3} v_\eta|\mathfrak{v}|^2\mu_w^{\frac{1}{2}}(\iota_1,\iota_2,\mathfrak{v})\widetilde{\Phi}(0,\iota_1,\iota_2,\mathfrak{v})\mathrm{d}\mathfrak{v}=\int_{\mathbb{R}^3} v_\eta|\mathfrak{v}|^2\mu_w^{\frac{1}{2}}(\iota_1,\iota_2,\mathfrak{v})\widetilde{\Phi}(\infty,\iota_1,\iota_2,\mathfrak{v})\mathrm{d}\mathfrak{v}=0,$$

which, combined with the zero mass-flux of $\widetilde{\Phi}$, further implies

$$\int_{\mathbb{R}^3} v_\eta \frac{\partial \mu_w^{\frac{1}{2}}}{\partial \iota_i}(\iota_1, \iota_2, \mathfrak{v})\widetilde{\Phi}(0, \iota_1, \iota_2, \mathfrak{v})\mathrm{d}\mathfrak{v} = 0.$$

Hence, W still satisfies the zero mass-flux condition. Also, notice that

$$\left\| \mathrm{e}^{K_0\eta}\left(-\frac{\partial \nu_w}{\partial \iota_i}\widetilde{\Phi} + \frac{\partial K_w}{\partial \iota_i}[\widetilde{\Phi}]\right)\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim |\nabla_x T|_{L^\infty_{\partial\Omega}}.$$

Therefore, based on [2], there exists a unique $W_\infty \in \mathcal{N}$ such that

$$|W_\infty| + \left\| \mathrm{e}^{K_0\eta}(W - W_\infty)\right\|_{L^\infty_{\varrho,\vartheta}} \lesssim |\nabla_x T|_{L^\infty_{\partial\Omega}} + |\partial_{\iota_i}\nabla_x T|_{L^\infty_{\partial\Omega}}.$$

In particular, since $\widetilde{\Phi} \to 0$ as $\eta \to \infty$, we must have $W_\infty = 0$. Hence, (2.47) is verified for $r = 1$. The $r > 1$ cases follow inductively. □

Let $\chi(y) \in C^\infty(\mathbb{R})$ and $\overline{\chi}(y) = 1 - \chi(y)$ be smooth cut-off functions satisfying

$$\chi(y) = \begin{cases} 1 & \text{if } |y| \le 1, \\ 0 & \text{if } |y| \ge 2, \end{cases}$$

In view of the later regularity estimates (see the companion paper [12]), we define a cutoff boundary layer f_1^B:

$$f_1^B(\eta, \iota_1, \iota_2, \mathfrak{v}) := \overline{\chi}(\varepsilon^{-1}v_\eta)\chi(\varepsilon\eta)\widetilde{\Phi}(\eta, \iota_1, \iota_2, \mathfrak{v}). \tag{2.48}$$

We can verify that f_1^B satisfies

$$\begin{aligned} &v_\eta \frac{\partial f_1^B}{\partial \eta} + \nu_w f_1^B - K_w\left[f_1^B\right] \\ &= v_\eta \overline{\chi}(\varepsilon^{-1}v_\eta)\frac{\partial \chi(\varepsilon\eta)}{\partial \eta}\widetilde{\Phi} + \chi(\varepsilon\eta)\left(\overline{\chi}\left(\varepsilon^{-1}v_\eta\right)K_w[\widetilde{\Phi}] - K_w[\overline{\chi}(\varepsilon^{-1}v_\eta)\widetilde{\Phi}]\right), \end{aligned}$$

with

$$f_1^B(0, \iota_1, \iota_2, \mathfrak{v}) = \overline{\chi}(\varepsilon^{-1}v_\eta)\left(-\mathscr{A}\cdot\frac{\nabla_x T}{2T^2} - \Phi_\infty(\iota_1, \iota_2, \mathfrak{v})\right) \quad \text{for } v_\eta > 0.$$

Due to the cutoff $\overline{\chi}$, f_1^B cannot preserve the zero mass-flux condition, i.e.

$$\begin{aligned} &\int_{\mathbb{R}^3} v_\eta \mu_w^{\frac{1}{2}}(\iota_1, \iota_2, \mathfrak{v}) f_1^B(0, \iota_1, \iota_2, \mathfrak{v})\mathrm{d}\mathfrak{v} \\ &= \int_{\mathbb{R}^3} v_\eta \mu_w^{\frac{1}{2}}(\iota_1, \iota_2, \mathfrak{v})\overline{\chi}(\varepsilon^{-1}v_\eta)\widetilde{\Phi}(0, \iota_1, \iota_2, \mathfrak{v})\mathrm{d}\mathfrak{v} \\ &= \int_{\mathbb{R}^3} v_\eta \mu_w^{\frac{1}{2}}(\iota_1, \iota_2, \mathfrak{v})\chi(\varepsilon^{-1}v_\eta)\widetilde{\Phi}(0, \iota_1, \iota_2, \mathfrak{v})\mathrm{d}\mathfrak{v} \lesssim o_T\varepsilon. \end{aligned} \tag{2.49}$$

The zero mass-flux condition will be restored with the help of f_2 in (3.11).

2.4 Analysis of Boundary Matching

Considering the boundary condition in (1.1) and the expansion (0.4), we require the matching condition for $x_0 \in \partial\Omega$ and $v \cdot n < 0$:

$$\mu_w\big|_{v\cdot n<0} = M_w \int_{v'\cdot n>0} \mu_w |v' \cdot n| \mathrm{d}v', \tag{2.50}$$

$$\mu_w^{\frac{1}{2}}\left(f_1 + f_1^B\right)\big|_{v\cdot n<0} = M_w \int_{v'\cdot n>0} \mu_w^{\frac{1}{2}}\left(f_1 + f_1^B\right)|v' \cdot n| \mathrm{d}v' + O(\varepsilon). \tag{2.51}$$

In order to guarantee (2.50), we deduce that

$$T(x_0) = T_w(x_0). \tag{2.52}$$

This determines the boundary conditions for T.

In order to guarantee (2.51), due to (2.49), it suffices to require that at $\eta = 0$

$$\mu_w^{\frac{1}{2}}(f_1 + \Phi - \Phi_\infty)\big|_{v\cdot n<0} = M_w \int_{v'\cdot n>0} \mu_w^{\frac{1}{2}}(f_1 + \Phi - \Phi_\infty)|v' \cdot n| \mathrm{d}v'. \tag{2.53}$$

Lemma 2.7 *With the boundary condition* (2.40) *for* (2.39), *and for* $x_0 \in \partial\Omega$

$$u_1(x_0) = u^B, \quad T_1(x_0) = T^B, \tag{2.54}$$

(2.53) *is valid.*

Proof Using (2.31) and (2.42), we have for $x_0 \in \partial\Omega$

$$f_1 + \Phi - \Phi_\infty = \mathscr{A} \cdot \frac{\nabla_x T}{2T^2} + \mu^{\frac{1}{2}}\left(\frac{\rho_1}{\rho} + \frac{u_1 \cdot v}{T} + \frac{T_1(|v|^2 - 3T)}{2T^2}\right)$$
$$+\Phi - \mu^{\frac{1}{2}}\left(\frac{\rho^B}{\rho} + \frac{u^B \cdot v}{T} + \frac{T^B(|v|^2 - 3T_w)}{2T^2}\right).$$

With (2.54), we have

$$f_1 + \Phi - \Phi_\infty = \left(\Phi + \mathscr{A} \cdot \frac{\nabla_x T}{2T^2}\right) + \mu^{\frac{1}{2}}\left(\frac{\rho_1}{\rho} - \frac{\rho^B}{\rho}\right).$$

Since direct computation reveals that

$$\mu^{\frac{1}{2}}\left(\frac{\rho_1}{\rho} - \frac{\rho^B}{\rho}\right)\bigg|_{v\cdot n<0} = M_w \int_{v'\cdot n>0} \mu^{\frac{1}{2}}\left(\frac{\rho_1}{\rho} - \frac{\rho^B}{\rho}\right)|v' \cdot n| \mathrm{d}v',$$

in order to verify (2.53), it suffices to require

$$\left(\Phi + \mathscr{A} \cdot \frac{\nabla_x T}{2T^2}\right)\bigg|_{v\cdot n<0} = M_w \int_{v'\cdot n>0} \left(\Phi + \mathscr{A} \cdot \frac{\nabla_x T}{2T^2}\right)|v' \cdot n| \mathrm{d}v'. \tag{2.55}$$

When (2.40) is valid, we know that

$$\left(\Phi + \mathscr{A} \cdot \frac{\nabla_x T}{2T^2}\right)\bigg|_{v\cdot n<0} = 0. \tag{2.56}$$

Also, due to (2.41) and orthogonality of $\mathscr{A}$, we have

$$M_w \int_{\mathbb{R}^3} \mu^{\frac{1}{2}}\Phi(v' \cdot n)\mathrm{d}v' = M_w \int_{\mathbb{R}^3} \mu^{\frac{1}{2}}\left(\mathscr{A} \cdot \frac{\nabla_x T}{2T^2}\right)(v' \cdot n)\mathrm{d}v' = 0,$$

which, combined with (2.56), yields

$$M_w \int_{v' \cdot n > 0} \mu^{\frac{1}{2}} \left(\Phi + \mathscr{A} \cdot \frac{\nabla_x T}{2T^2} \right) |v' \cdot n| \mathrm{d}v' = 0.$$

Then clearly (2.55) is true. □

3 Construction of Expansion

In this section, we will present the detailed construction of ghost-effect solution, f_1, f_2 and f_1^B based on the analysis in Section 2.4. Since the boundary conditions are tangled together, we divide the construction into several stages.

3.1 Construction of Boundary Layer f_1^B-Stage I

Since (2.40) involves $\nabla_n T$, which is not fully provided by T_w, we will have to split the tangential and normal parts of the boundary layer

$$f_1^B = f_{1,\iota_1}^B + f_{1,\iota_2}^B + f_{1,n}^B, \quad \Phi = \Phi_{\iota_1} + \Phi_{\iota_2} + \Phi_n.$$

Define

$$\Phi_{\iota_i} := \left(\partial_{\iota_i} T_w\right) \mathcal{H}^{(i)},$$

where $\mathcal{H}^{(i)}$ for $i = 1, 2$ solves the Milne problem

$$\begin{cases} v_\eta \dfrac{\partial \mathcal{H}^{(i)}}{\partial \eta} + \mathcal{L}_w\left[\mathcal{H}^{(i)}\right] = 0, \\ \mathcal{H}^{(i)}(0, \mathfrak{v}) = -\dfrac{\mathscr{A} \cdot \varsigma_i}{2T^2} \quad \text{for } v_\eta > 0, \\ \lim_{\eta \to \infty} \mathcal{H}^{(i)}(\eta, \mathfrak{v}) = \mathcal{H}_\infty^{(i)} \in \mathcal{N}, \end{cases} \tag{3.1}$$

with the zero mass-flux condition

$$\int_{\mathbb{R}^3} v_\eta \mu_w^{\frac{1}{2}}(\mathfrak{v}) \mathcal{H}^{(i)} = 0.$$

Denote

$$\Phi_{\iota_i,\infty} := (\partial_{\iota_i} T_w) \mathcal{H}_\infty^{(i)}.$$

Since we lack the information of Φ_n at this stage, we are not able to determine the boundary condition $T_1 = T^B$ yet. However, we can fully determine the boundary condition $u_1 = u^B$. Denote $u_1 = (u_{1,\iota_1}, u_{1,\iota_2}, u_{1,n})$ for the two tangential components $(u_{1,\iota_1}, u_{1,\iota_2})$ and one normal component $u_{1,n}$. Due to (2.41), we have

$$u_{1,n}(x_0) = 0. \tag{3.2}$$

Due to oddness, the projection of $\mathcal{H}^{(i)}$ and $\mathcal{H}_\infty^{(i)}$ on $v\mu_w^{\frac{1}{2}}$ only has contribution on $(v \cdot \varsigma_i)\mu^{\frac{1}{2}}$. Hence, from (2.54), we deduce

$$u_{1,\iota_1}(x_0) = \beta_1[T_w] \partial_{\iota_1} T_w, \quad u_{1,\iota_2}(x_0) = \beta_2[T_w] \partial_{\iota_2} T_w,$$

where β_i are functions depending on T_w. Due to isotropy, we know that $\beta_1 = \beta_2$, and we denote it β_0. Hence, we arrive at

$$u_{1,\iota_1}(x_0) = \beta_0[T_w] \partial_{\iota_1} T_w, \quad u_{1,\iota_2}(x_0) = \beta_0[T_w] \partial_{\iota_2} T_w. \tag{3.3}$$

Lemma 3.1 *Under the assumption* (1.4)*, for any* $s \geq 1$*, the boundary data of* u_1 *satisfies*

$$|u_1|_{W^{3,\infty}} \lesssim o_T.$$

Proof Taking ι_i derivatives for $i = 1, 2$ on both sides of (3.1), using (1.4) and (2.44), we conclude that

$$|u_{1,\iota_1}|_{W^{3,\infty}} + |u_{1,\iota_2}|_{W^{3,\infty}} \lesssim o_T.$$

Then using (3.2), we obtain the desired estimates. □

Remark 3.2 Note that the boundary condition of u_1 only depends on T_w and ∇T_w directly without referring to T in the bulk.

3.2 Well-Posedness of Ghost-Effect Equation

Based on our analysis above, the ghost-effect equation (0.2) will be accompanied with the boundary conditions (2.52), (3.2) and (3.3).

$$T(x_0) = T_w, \quad u_{1,\iota_1}(x_0) = \beta_0[T_w]\partial_{\iota_1}T_w, \quad u_{1,\iota_2}(x_0) = \beta_0[T_w]\partial_{\iota_2}T_w, \quad u_{1,n}(x_0) = 0. \tag{3.4}$$

Theorem 3.3 *Under the assumption* (1.4)*, for any given* $P > 0$*, there exists a unique solution* $(\rho, u_1, T; \mathfrak{p})$ *(*$\mathfrak{p}$ *has zero average) to the ghost-effect* (0.2) *with the boundary condition* (3.4) *satisfying for any* $s \in [2, \infty)$

$$\|u_1\|_{W^{3,s}} + \|\mathfrak{p}\|_{W^{2,s}} + \|T - 1\|_{W^{4,s}} \lesssim o_T.$$

Proof

Simplified Equations Denote $\overline{u} := \rho u_1$. From the first and third equations in (0.2)

$$\nabla_x \cdot \overline{u} = \nabla_x \cdot (\rho u_1) = P\nabla_x \cdot \left(\frac{u_1}{T}\right) = 0,$$

we have

$$\nabla_x \cdot u_1 = u_1 \cdot \frac{\nabla_x T}{T}. \tag{3.5}$$

From the second equation in (0.2) and (3.5), we have

$$\begin{aligned} -\frac{5}{3}\lambda[1]\Delta_x u_1 + \nabla_x \mathfrak{p} &= -\frac{5}{3}\left(\lambda[1] - \lambda[T]\right)\Delta_x u_1 \\ &-\nabla_x \cdot \left(\frac{\lambda^2[T]}{P}\left(K_1[T]\left(\nabla_x^2 T - \frac{1}{3}\Delta_x T\mathbf{1}\right) + \frac{K_2[T]}{T}\left(\nabla_x T \otimes \nabla_x T - \frac{1}{3}|\nabla_x T|^2\mathbf{1}\right)\right)\right) \\ &+\nabla_x\lambda[T] \cdot \left(\nabla_x u_1 + (\nabla_x u_1)^t - \frac{2}{3}(\nabla_x \cdot u_1)\mathbf{1}\right) + \lambda[T]\nabla_x\left(u_1 \cdot \frac{\nabla_x T}{T}\right) - \frac{P}{T}u_1 \cdot \nabla_x u_1. \end{aligned} \tag{3.6}$$

Hence, we know

$$\begin{aligned} -\frac{5}{3P}\lambda[1]\Delta_x\overline{u} + \nabla_x\mathfrak{p} &= -\frac{5}{3P}\big(\lambda[1] - \lambda[T]\big)\Delta_x\overline{u} + \frac{5}{3P}\lambda[T]\Delta_x\big((T-1)\overline{u}\big) \\ &-\nabla_x \cdot \left(\frac{\lambda^2[T]}{P}\left(K_1[T]\left(\nabla_x^2 T - \frac{1}{3}\Delta_x T\mathbf{1}\right) + \frac{K_2[T]}{T}\left(\nabla_x T \otimes \nabla_x T - \frac{1}{3}|\nabla_x T|^2\mathbf{1}\right)\right)\right) \\ &+\nabla_x\lambda[T] \cdot \left(\nabla_x\left(P^{-1}T\overline{u}\right) + \left(\nabla_x(P^{-1}T\overline{u})\right)^t - \frac{2}{3}\left(\nabla_x \cdot (P^{-1}T\overline{u})\right)\mathbf{1}\right) \\ &+\lambda[T]\nabla_x\left(P^{-1}\overline{u} \cdot \nabla_x T\right) - \overline{u} \cdot \nabla_x\left(P^{-1}T\overline{u}\right). \end{aligned}$$

Furthermore, from the fourth equations in (0.2)

$$\nabla_x \cdot u_1 = \frac{1}{5P}\nabla_x \cdot \left(\kappa \frac{\nabla_x T}{2T^2}\right) = \frac{1}{5P}\frac{\kappa}{2T^2}\Delta_x T + \frac{1}{5P}\nabla_x\left(\frac{\kappa}{2T^2}\right)\cdot\nabla_x T,$$

we have

$$\frac{1}{5P}\frac{\kappa}{2T^2}\Delta_x T = P^{-1}\overline{u}\cdot\nabla_x T - \frac{1}{5P}\nabla_x\left(\frac{\kappa}{2T^2}\right)\cdot\nabla_x T.$$

Then we know

$$\Delta_x T = \frac{10T^2}{\kappa[T]}(\overline{u}\cdot\nabla_x T) - \frac{2T^2}{\kappa[T]}\nabla_x\left(\frac{\kappa[T]}{2T^2}\right)\cdot\nabla_x T. \tag{3.7}$$

Setup of Contraction Mapping Collecting (3.5), (3.6) and (3.7), this is a system for the pair $(\overline{u}, T)$. Then we can design a mapping $W^{3,s}\times W^{4,s}\to W^{3,s}\times W^{4,s} : (\widetilde{u},\widetilde{T})\to(\overline{u},T)$

$$\begin{cases} -\frac{5}{3P}\lambda[1]\Delta_x\overline{u} + \nabla_x\mathfrak{p} = Z_1,\\ \nabla_x\cdot\overline{u} = 0,\\ \Delta_x T = Z_3,\end{cases}$$

where

$$\begin{aligned} Z_1 :=& -\frac{5}{3P}\left(\lambda[1]-\lambda[\widetilde{T}]\right)\Delta_x\widetilde{u} + \frac{5}{3P}\lambda[\widetilde{T}]\Delta_x\big((\widetilde{T}-1)\widetilde{u}\big)\\ &-\nabla_x\cdot\left(\frac{\lambda^2[\widetilde{T}]}{P}\left(K_1[\widetilde{T}]\left(\nabla_x^2\widetilde{T}-\frac{1}{3}\Delta_x\widetilde{T}\mathbf{1}\right)+\frac{K_2}{\widetilde{T}}[\widetilde{T}]\left(\nabla_x\widetilde{T}\otimes\nabla_x\widetilde{T}-\frac{1}{3}|\nabla_x\widetilde{T}|^2\mathbf{1}\right)\right)\right)\\ &+\nabla_x\lambda[\widetilde{T}]\cdot\left(\nabla_x\big(P^{-1}\widetilde{T}\widetilde{u}\big)+\big(\nabla_x\big(P^{-1}\widetilde{T}\widetilde{u}\big)\big)^t-\frac{2}{3}\left(\nabla_x\cdot\big(P^{-1}\widetilde{T}\widetilde{u}\big)\right)\mathbf{1}\right)\\ &+\lambda[\widetilde{T}]\nabla_x\left(P^{-1}\widetilde{u}\cdot\nabla_x\widetilde{T}\right)-\widetilde{u}\cdot\nabla_x\big(P^{-1}\widetilde{T}\widetilde{u}\big),\\ Z_3 :=& \frac{10\widetilde{T}^2}{\kappa[\widetilde{T}]}\big(\widetilde{u}\cdot\nabla_x\widetilde{T}\big)-\frac{2\widetilde{T}^2}{\kappa[\widetilde{T}]}\nabla_x\left(\frac{\kappa[\widetilde{T}]}{2\widetilde{T}^2}\right)\cdot\nabla_x\widetilde{T}.\end{aligned}$$

Boundedness and Contraction Based on [9] and [6, Theorem IV.5.8], noticing the compatibility condition

$$\int_{\partial\Omega}\overline{u}\cdot n = \int_\Omega(\nabla_x\cdot\overline{u}) = 0,$$

we know that

$$\|\overline{u}\|_{W^{3,s}} + \|\mathfrak{p}\|_{W^{2,s}} \lesssim \|Z_1\|_{W^{1,s}} + |\overline{u}|_{W^{3-\frac{1}{s},s}}.$$

Based on standard elliptic estimates [21], we have

$$\|T-1\|_{W^{4,s}} \lesssim \|Z_3\|_{W^{2,s}} + |T|_{W^{4-\frac{1}{s},s}}.$$

Under the assumption

$$\|\widetilde{u}_1\|_{W^{3,s}} + \|\widetilde{T}-1\|_{W^{4,s}} \lesssim 2o_T,$$

we directly obtain

$$\begin{aligned}\|Z_1\|_{W^{1,s}} &\lesssim o_T\left(\|\widetilde{u}\|_{W^{3,s}} + \|\nabla_x\widetilde{T}\|_{W^{3,s}}\right),\\ \|Z_3\|_{W^{2,s}} &\lesssim o_T\left(\|\widetilde{u}\|_{W^{3,s}} + \|\nabla_x\widetilde{T}\|_{W^{3,s}}\right).\end{aligned}$$

Hence, we know that

$$\|\overline{u}\|_{W^{3,s}} + \|\mathfrak{p}\|_{W^{2,s}} + \|T-1\|_{W^{4,s}} \lesssim o_T\left(\|\widetilde{u}\|_{W^{3,s}} + \|\nabla_x\widetilde{T}\|_{W^{3,s}}\right) + |\nabla T_w|_{W^{3,\infty}} \le 2o_T.$$

Hence, this mapping is bounded.

By a similar argument, for $(\widetilde{u}^{[k]}, \widetilde{T}^{[k]}) \to (\overline{u}^{[k]}, T^{[k]})$ with $k = 1, 2$, we can show that

$$\begin{aligned} &\left\| \overline{u}^{[1]} - \overline{u}^{[2]} \right\|_{W^{3,s}} + \left\| \mathfrak{p}^{[1]} - \mathfrak{p}^{[2]} \right\|_{W^{2,s}} + \left\| T^{[1]} - T^{[2]} \right\|_{W^{4,s}} \\ &\lesssim \left(\|\widetilde{u}\|_{W^{3,s}} + \|\nabla_x \widetilde{T}\|_{W^{3,s}} \right) \left(\left\| \widetilde{u}^{[1]} - \widetilde{u}^{[2]} \right\|_{W^{3,s}} + \left\| \nabla_x \widetilde{T}^{[1]} - \nabla_x \widetilde{T}^{[2]} \right\|_{W^{3,s}} \right), \end{aligned}$$

which yields

$$\begin{aligned} &\left\| \overline{u}^{[1]} - \overline{u}^{[2]} \right\|_{W^{3,s}} + \left\| \mathfrak{p}^{[1]} - \mathfrak{p}^{[2]} \right\|_{W^{2,s}} + \left\| T^{[1]} - T^{[2]} \right\|_{W^{4,s}} \\ &\lesssim o_T \left(\left\| \widetilde{u}^{[1]} - \widetilde{u}^{[2]} \right\|_{W^{3,s}} + \left\| \nabla_x \widetilde{T}^{[1]} - \nabla_x \widetilde{T}^{[2]} \right\|_{W^{3,s}} \right). \end{aligned}$$

Hence, this is a contraction mapping.

In summary, we know that there exists a unique solution to (0.2) satisfying

$$\|\overline{u}\|_{W^{3,s}} + \|\mathfrak{p}\|_{W^{2,s}} + \|T - 1\|_{W^{4,s}} \lesssim o_T,$$

and further

$$\|u_1\|_{W^{3,s}} + \|\mathfrak{p}\|_{W^{2,s}} + \|T - 1\|_{W^{4,s}} \lesssim o_T.$$

□

Remark 3.4 Based on the first equation in (2.37), we have

$$\rho = PT^{-1} \in W^{4,s}.$$

Then we have

$$P|\Omega| = \int_\Omega \rho(x)T(x)\mathrm{d}x = \frac{1}{3}\iint_{\Omega\times\mathbb{R}^3} |v|^2 \mu(x, v)\mathrm{d}v\mathrm{d}x. \tag{3.8}$$

3.3 Construction of Boundary Layer f_1^B-Stage II

Now we can define the full boundary layer. Define

$$\Phi_n := (\partial_n T)\mathcal{H}^{(n)},$$

where $\mathcal{H}^{(n)}$ solves the Milne problem

$$\begin{cases} v_\eta \dfrac{\partial \mathcal{H}^{(n)}}{\partial \eta} + \mathcal{L}_w \left[\mathcal{H}^{(n)}\right] = 0, \\ \mathcal{H}^{(n)}(0, \mathfrak{v}) = -\dfrac{\mathscr{A} \cdot n}{2T^2} \quad \text{for } v_\eta > 0, \\ \lim_{\eta\to\infty} \mathcal{H}^{(n)}(\eta, \mathfrak{v}) = \mathcal{H}^{(n)}_\infty \in \mathcal{N}, \end{cases}$$

with the zero mass-flux condition

$$\int_{\mathbb{R}^3} v_\eta \mu_w^{\frac{1}{2}}(\mathfrak{v})\mathcal{H}^{(n)} = 0.$$

Denote

$$\Phi_{n,\infty} := (\partial_n T)\mathcal{H}^{(n)}_\infty.$$

Here $\partial_n T$ comes from the ghost-effect equation (0.2) and is well-defined due to Theorem 3.3.

Finally, we have the full boundary layer from (2.48):

$$
\begin{aligned}
f_1^B(\eta,\mathfrak{v}) =& \overline{\chi}(\varepsilon^{-1}v_\eta)\chi(\varepsilon\eta)\left(\Phi_{\iota_1}(\eta,\mathfrak{v})+\Phi_{\iota_2}(\eta,\mathfrak{v})+\Phi_n(\eta,\mathfrak{v})-\Phi_{\iota_1,\infty}-\Phi_{\iota_2,\infty}-\Phi_{n,\infty}\right)\\
=& \overline{\chi}(\varepsilon^{-1}v_\eta)\chi(\varepsilon\eta)\left(\widetilde{\Phi}_{\iota_1}(\eta,\mathfrak{v})+\widetilde{\Phi}_{\iota_2}(\eta,\mathfrak{v})+\widetilde{\Phi}_n(\eta,\mathfrak{v})\right).
\end{aligned}
$$

Since the cutoff in f_1^B is only defined in the normal direction, we can deduce tangential regularity estimates from Theorem 2.6:

Theorem 3.5 *Under the assumption* (1.4)*, we can construct f_1^B such that for $i=1,2$, some $K_0>0$ and any $0<r\le 3$*

$$
\left\|\mathrm{e}^{K_0\eta}f_1^B\right\|+\left\|\mathrm{e}^{K_0\eta}\frac{\partial^r f_1^B}{\partial\iota_i^r}\right\|\lesssim o_T. \tag{3.9}
$$

From (2.54) and (2.42), this fully determines the boundary condition of T_1:

$$
T_1(x_0)=T^B.
$$

3.4 Construction of (ρ_1, T_1)

Theorem 3.6 *Under the assumption* (1.4)*, we can construct (ρ_1,T_1) such that for any $s\in[2,\infty)$*

$$
\|f_1\|_{W^{3,s}L^\infty_{\varrho,\vartheta}}+|f_1|_{W^{3-\frac{1}{s},s}L^\infty_{\varrho,\vartheta}}\lesssim o_T.
$$

Proof The boundary condition in (2.54) and Theorem 3.5 imply that

$$
|T_1|_{W^{3,s}}\lesssim o_T.
$$

Then we can freely define a Sobolev extension for T_1 such that

$$
\|T_1\|_{W^{3+\frac{1}{s},s}}\lesssim o_T.
$$

We choose the constant

$$
P_1=0.
$$

Then we can deduce that

$$
\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^3}|v|^2\left(\mu^{\frac12}f_1+\mu_w^{\frac12}f_1^B+\varepsilon\mu^{\frac12}f_2(x,v)\right)\mathrm{d}x\mathrm{d}v \\
&=\int_\Omega\big(3\rho_1(x)T(x)+3T_1(x)\rho(x)+3\varepsilon\rho_2(x)T(x)+3\varepsilon\rho(x)T_2(x)\big)\mathrm{d}x\\
&\quad+\iint_{\Omega\times\mathbb{R}^3}|v|^2\mu_w^{\frac12}f_1^B\mathrm{d}x\mathrm{d}v\\
&=\int_\Omega(3\rho_1(x)T(x)+3T_1(x)\rho(x))\mathrm{d}x+\iint_{\Omega\times\mathbb{R}^3}|v|^2\mu_w^{\frac12}f_1^B\mathrm{d}x\mathrm{d}v\\
&=\int_\Omega 3P_1\mathrm{d}x+\iint_{\Omega\times\mathbb{R}^3}|v|^2\mu_w^{\frac12}f_1^B\mathrm{d}x\mathrm{d}v=\iint_{\Omega\times\mathbb{R}^3}|v|^2\mu_w^{\frac12}f_1^B\mathrm{d}x\mathrm{d}v,
\end{aligned} \tag{3.10}
$$

where we have used $\int_\Omega\mathfrak{p}=\int_\Omega(T\rho_2+\rho T_2)=0$.

Then based on (2.38), we have

$$
\rho_1=-T^{-1}(\rho T_1),
$$

and thus

$$\|\rho_1\|_{W^{3+\frac{1}{s},s}} \lesssim o_T.$$

Note that ρ_1 is not necessarily equal to ρ^B on $\partial\Omega$. However, (2.53) can still hold due to (2.50).

Hence, we have shown that

$$\|f_1\|_{W^{3,s}L^\infty_{\varrho,\vartheta}} + |f_1|_{W^{3-\frac{1}{s},s}L^\infty_{\varrho,\vartheta}} \lesssim \|\rho_1\|_{W^{3,s}} + \|u_1\|_{W^{3,s}} + \|T_1\|_{W^{3,s}} + \|T\|_{W^{3,s}} \lesssim o_T.$$

□

Remark 3.7 We assume that the remainder R satisfies

$$\iint_{\Omega\times\mathbb{R}^3} |v|^2\mu^{\frac{1}{2}} R(x,v)\mathrm{d}x\mathrm{d}v = 0.$$

Hence, combining (3.8), (3.10) and (0.4), we know

$$\iint_{\Omega\times\mathbb{R}^3} |v|^2\mathfrak{F}(x,v)\mathrm{d}v\mathrm{d}x = \iint_{\Omega\times\mathbb{R}^3} |v|^2\mu(x,v)\mathrm{d}v\mathrm{d}x$$
$$= 3P|\Omega| + \varepsilon \iint_{\Omega\times\mathbb{R}^3} |v|^2\mu_w^{\frac{1}{2}} f_1^B\mathrm{d}x\mathrm{d}v.$$

3.5 Construction of (ρ_2, u_2, T_2)

Theorem 3.8 *Under the assumption (1.4), we can construct (ρ_2, u_2, T_2) such that for any $s\in[2,\infty)$*

$$\|f_2\|_{W^{2,s}L^\infty_{\varrho,\vartheta}} + |f_2|_{W^{2-\frac{1}{s},s}L^\infty_{\varrho,\vartheta}} \lesssim o_T.$$

Proof Denote

$$Y(\iota_1,\iota_2) := -\varepsilon^{-1}P^{-1}\int_{\mathbb{R}^3} v_\eta \mu_w^{\frac{1}{2}}(\mathfrak{v}) f_1^B(0,\mathfrak{v})\mathrm{d}\mathfrak{v}.$$

Due to (2.49), we have $|Y| \lesssim o_T$. Then we define u_2 via $u_2 = \nabla_x\psi$ where ψ solves

$$\begin{cases} -\Delta_x\psi = -|\Omega|^{-1}\displaystyle\int_{\partial\Omega} Y(\iota_1,\iota_2)\mathrm{d}s & \text{in } \Omega,\\ \dfrac{\partial\psi}{\partial n} = Y & \text{on } \partial\Omega. \end{cases}$$

Due to classical elliptic theory, we know that this equation is well-posed. In particular, due to (3.9), we know $Y\in W^{3,\infty}(\partial\Omega)$. Then we have $\psi\in W^{4,s}$ and thus $u_2\in W^{3,s}$ satisfying

$$\|u_2\|_{W^{3,s}} \lesssim o_T.$$

From Theorem 3.3 and the third equation in (2.37), we know that

$$T\rho_2 + \rho T_2 \in W^{2,s}.$$

We are free to take $\rho_2 = 0$ in Ω, and thus T_2 is determined and satisfies

$$\|T_2\|_{W^{2,s}} \lesssim o_T.$$

Hence, we have shown that

$$\|f_2\|_{W^{2,s}L^\infty_{\varrho,\vartheta}} + |f_2|_{W^{2-\frac{1}{s},s}L^\infty_{\varrho,\vartheta}} \lesssim \|f_1\|_{W^{2,s}L^\infty_{\varrho,\vartheta}} + \|\rho_2\|_{W^{2,s}} + \|u_2\|_{W^{2,s}} + \|T_2\|_{W^{2,s}} \lesssim o_T.$$

□

Remark 3.9 Such choice of u_2 implies that on the boundary $\partial\Omega$

$$u_2 \cdot n = Y.$$

Hence, we know

$$\int_{\mathbb{R}^3} \left(\varepsilon^2 f_2 + \varepsilon f_1^B\right) \mu_w(v\cdot n) = \varepsilon^2 P(u_2\cdot n) + \varepsilon \int_{\mathbb{R}^3} v_\eta \mu_w^{\frac{1}{2}}(\mathfrak{v}) f_1^B(0,\mathfrak{v})\mathrm{d}\mathfrak{v} = 0, \quad (3.11)$$

and thus

$$\int_{\mathbb{R}^3} \left(\mu^{\frac{1}{2}} + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon f_1^B\right) \mu_w^{\frac{1}{2}}(v\cdot n) = 0.$$

We restore the zero mass-flux condition of $\mu^{\frac{1}{2}} + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon f_1^B$.

4 Remainder Equation

For sake of completeness, in this section we will present the remainder equation for R and report the main result in [12].

Now we begin to derive the remainder equation for R in (0.4), or equivalently the nonlinear Boltzmann equation (1.1). Denote

$$\begin{aligned} Q[F,F] &= Q_{\mathrm{gain}}[F,F] - Q_{\mathrm{loss}}[F,F] \\ &:= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(\omega, |\mathfrak{u}-v|)F(\mathfrak{u}_*)F(v_*)\mathrm{d}\omega\mathrm{d}\mathfrak{u} \\ &\quad - F(v)\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} q(\omega,|\mathfrak{u}-v|)F(\mathfrak{u})\mathrm{d}\omega\mathrm{d}\mathfrak{u} = \nu(F)F. \end{aligned}$$

Denote $\mathfrak{F} = \mathfrak{F}_{\mathrm{a}} + \varepsilon^\alpha \mu^{\frac{1}{2}} R$, where

$$\mathfrak{F}_{\mathrm{a}} := \mu + \mu^{\frac{1}{2}}(\varepsilon f_1 + \varepsilon^2 f_2) + \mu_w^{\frac{1}{2}}(\varepsilon f_1^B).$$

We can split $\mathfrak{F} = \mathfrak{F}_+ - \mathfrak{F}_-$ where $\mathfrak{F}_+ = \max\{\mathfrak{F}, 0\}$ and $\mathfrak{F}_- = \max\{-\mathfrak{F}, 0\}$ denote the positive and negative parts, and the similar notation also applies to $\mathfrak{F}_{\mathrm{a}}$ and R.

In order to study (1.1), we first consider an auxiliary equation (which is equivalent to (1.1) when $\mathfrak{F} \geq 0$)

$$v\cdot\nabla_x\mathfrak{F} + \varepsilon^{-1}\left(Q_{\mathrm{loss}}[\mathfrak{F},\mathfrak{F}] - Q_{\mathrm{gain}}[\mathfrak{F}_+,\mathfrak{F}_+]\right) = \mathfrak{z}\iint_{\Omega\times\mathbb{R}^3} \varepsilon^{-1}\left(Q_{\mathrm{loss}}[\mathfrak{F},\mathfrak{F}] - Q_{\mathrm{gain}}[\mathfrak{F}_+,\mathfrak{F}_+]\right), \quad (4.1)$$

with diffuse-reflection boundary condition

$$\mathfrak{F}(x_0,v) = M_w(x_0,v)\int_{v'\cdot n(x_0)>0} \mathfrak{F}(x_0,v')|v'\cdot n(x_0)|\mathrm{d}v' \quad \text{for } x_0\in\partial\Omega \text{ and } v\cdot n(x_0)<0.$$

Here $\mathfrak{z} = \mathfrak{z}(v) > 0$ is a smooth function with support contained in $\{|v| \leq 1\}$ such that $\iint_{\Omega\times\mathbb{R}^3} \mathfrak{z} = 1$.

The auxiliary system (4.1) is equivalent to

$$\begin{aligned} v\cdot\nabla_x\mathfrak{F} - \varepsilon^{-1}Q[\mathfrak{F},\mathfrak{F}] &= -\varepsilon^{-1}\left(Q_{\mathrm{gain}}[\mathfrak{F},\mathfrak{F}] - Q_{\mathrm{gain}}[\mathfrak{F}_+,\mathfrak{F}_+]\right) \\ &\quad + \mathfrak{z}\iint_{\Omega\times\mathbb{R}^3} \varepsilon^{-1}\left(Q_{\mathrm{loss}}[\mathfrak{F},\mathfrak{F}] - Q_{\mathrm{gain}}[\mathfrak{F}_+,\mathfrak{F}_+]\right), \end{aligned}$$

and due to orthogonality of Q, is further equivalent to

$$v \cdot \nabla_x \mathfrak{F} - \varepsilon^{-1} Q[\mathfrak{F}, \mathfrak{F}] = -\varepsilon^{-1} \left(Q_{\text{gain}}[\mathfrak{F}, \mathfrak{F}] - Q_{\text{gain}}[\mathfrak{F}_+, \mathfrak{F}_+] \right) + \mathfrak{z} \iint_{\Omega \times \mathbb{R}^3} \varepsilon^{-1} \left(Q_{\text{gain}}[\mathfrak{F}, \mathfrak{F}] - Q_{\text{gain}}[\mathfrak{F}_+, \mathfrak{F}_+] \right). \tag{4.2}$$

Remark 4.1 The extra terms

$$-\varepsilon^{-1} \left(Q_{\text{gain}}[\mathfrak{F}, \mathfrak{F}] - Q_{\text{gain}}[\mathfrak{F}_+, \mathfrak{F}_+] \right) + \mathfrak{z} \iint_{\Omega \times \mathbb{R}^3} \varepsilon^{-1} \left(Q_{\text{gain}}[\mathfrak{F}, \mathfrak{F}] - Q_{\text{gain}}[\mathfrak{F}_+, \mathfrak{F}_+] \right).$$

on the right hand side of (4.2) plays a significant role in justifying the positivity of $\mathfrak{F}$ (see [12]). Clearly, when $\mathfrak{F} \geq 0$, i.e. $\mathfrak{F} = \mathfrak{F}_+$, the above extra terms vanish and the auxiliary equation (4.2) reduces to (1.1).

Inserting $\mathfrak{F} = \mathfrak{F}_{\text{a}} + \varepsilon^{\alpha} \mu^{\frac{1}{2}} R := \mu + \widetilde{\mathfrak{F}}_{\text{a}} + \varepsilon^{\alpha} \mu^{\frac{1}{2}} R$ into (4.2), we have

$$\begin{aligned}
&v \cdot \nabla_x \left(\mu^{\frac{1}{2}} R \right) + \varepsilon^{-1} \mu^{\frac{1}{2}} \mathcal{L}[R] \\
&= \mathscr{S} + \varepsilon^{-1} \left(2Q^* \left[\widetilde{\mathfrak{F}}_{\text{a}}, \mu^{\frac{1}{2}} R \right] + \varepsilon^{\alpha} Q^* \left[\mu^{\frac{1}{2}} R, \mu^{\frac{1}{2}} R \right] \right) \\
&\quad -\varepsilon^{-\alpha} \left(\varepsilon^{-1} Q_{\text{gain}} \left[\mathfrak{F}_{\text{a}} + \varepsilon^{\alpha} \mu^{\frac{1}{2}} R, \mathfrak{F}_{\text{a}} + \varepsilon^{\alpha} \mu^{\frac{1}{2}} R \right] \right. \\
&\quad \left. -\varepsilon^{-1} Q_{\text{gain}} \left[\left(\mathfrak{F}_{\text{a}} + \varepsilon^{\alpha} \mu^{\frac{1}{2}} R \right)_+, \left(\mathfrak{F}_{\text{a}} + \varepsilon^{\alpha} \mu^{\frac{1}{2}} R \right)_+ \right] \right) \\
&\quad +\varepsilon^{-\alpha} \mathfrak{z} \iint_{\Omega \times \mathbb{R}^3} \left(\varepsilon^{-1} Q_{\text{gain}} \left[\mathfrak{F}_{\text{a}} + \varepsilon^{\alpha} \mu^{\frac{1}{2}} R, \mathfrak{F}_{\text{a}} + \varepsilon^{\alpha} \mu^{\frac{1}{2}} R \right] \right. \\
&\quad \left. -\varepsilon^{-1} Q_{\text{gain}} \left[\left(\mathfrak{F}_{\text{a}} + \varepsilon^{\alpha} \mu^{\frac{1}{2}} R \right)_+, \left(\mathfrak{F}_{\text{a}} + \varepsilon^{\alpha} \mu^{\frac{1}{2}} R \right)_+ \right] \right),
\end{aligned} \tag{4.3}$$

where

$$\mathscr{S} := -\varepsilon^{-\alpha} v \cdot \nabla_x \mathfrak{F}_{\text{a}} + \varepsilon^{-\alpha-1} Q^* \left[\mathfrak{F}_{\text{a}}, \mathfrak{F}_{\text{a}} \right]. \tag{4.4}$$

Hence, we know that the equation for the remainder R is

$$\begin{cases} v \cdot \nabla_x \left(\mu^{\frac{1}{2}} R \right) + \varepsilon^{-1} \mu^{\frac{1}{2}} \mathcal{L}[R] = \mu^{\frac{1}{2}} S & \text{in } \Omega \times \mathbb{R}^3, \\ R(x_0, v) = \mathcal{P}_{\gamma}[R](x_0, v) + h(x_0, v) & \text{for } x_0 \in \partial\Omega \text{ and } v \cdot n(x_0) < 0. \end{cases} \tag{4.5}$$

where

$$\mathcal{P}_{\gamma}[R](x_0, v) := m_w(x_0, v) \int_{v' \cdot n(x_0) > 0} \mu^{\frac{1}{2}}(x_0, v') R(x_0, v') |v' \cdot n(x_0)| \mathrm{d}v',$$

with

$$m_w(x_0, v) := M_w \mu_w^{-\frac{1}{2}},$$

satisfying the normalization condition

$$\mu_w^{\frac{1}{2}}(x_0, v) = m_w(x_0, v) \int_{v' \cdot n(x_0) > 0} \mu_w(x_0, v') |v' \cdot n(x_0)| \mathrm{d}v' = \frac{P}{\left(2\pi T_w(x_0) \right)^{\frac{1}{2}}} m_w(x_0, v).$$

The source term S includes the nonlinear terms and the terms of the expansion coming from higher orders and h is a correction on the boundary condition.

Lemma 4.2 *We have*

$$\begin{aligned}h &:= \varepsilon^{-\alpha}\left(\mathcal{P}_\gamma\left[\mu^{-\frac{1}{2}}\widetilde{\mathfrak{F}}_a\right]-\mu^{-\frac{1}{2}}\widetilde{\mathfrak{F}}_a\right)\\ &= \varepsilon^{2-\alpha}\left(m_w\int_{v'\cdot n>0}\mu_w^{\frac{1}{2}}(v')f_2(v')|v'\cdot n|\mathrm{d}v'-f_2\big|_{v\cdot n<0}\right)\\ &\quad -\varepsilon^{1-\alpha}\left(m_w\int_{v'\cdot n>0}\mu_w^{\frac{1}{2}}\chi(\varepsilon^{-1}v_\eta)\widetilde{\Phi}|v'\cdot n|\mathrm{d}v'-\mu_w^{\frac{1}{2}}\chi(\varepsilon^{-1}v_\eta)\widetilde{\Phi}\big|_{v\cdot n<0}\right).\end{aligned}$$

Proof From (0.4), we know

$$h := \varepsilon^{-\alpha}\left(\mathcal{P}_\gamma\left[\mu^{-\frac{1}{2}}\widetilde{\mathfrak{F}}_{\mathrm{a}}\right]-\mu^{-\frac{1}{2}}\widetilde{\mathfrak{F}}_{\mathrm{a}}\right).$$

Then due to (2.53) and (2.54), we know

$$\begin{aligned}&\varepsilon^{-\alpha}\left(\mathcal{P}_\gamma\left[\mu^{-\frac{1}{2}}(\varepsilon f_1+\varepsilon f_1^B)\right]-\mu^{-\frac{1}{2}}(\varepsilon f_1+\varepsilon f_1^B)\right)\\ &= \varepsilon^{1-\alpha}\left(\mathcal{P}_\gamma\left[\mu^{-\frac{1}{2}}\left(f_1+\widetilde{\Phi}-\chi(\varepsilon^{-1}v_\eta)\widetilde{\Phi}\right)\right]-\mu^{-\frac{1}{2}}\left(f_1+\widetilde{\Phi}-\chi(\varepsilon^{-1}v_\eta)\widetilde{\Phi}\right)\right)\\ &= -\varepsilon^{1-\alpha}\left(m_w\int_{v'\cdot n>0}\mu_w^{\frac{1}{2}}\chi(\varepsilon^{-1}v_\eta)\widetilde{\Phi}|v'\cdot n|\mathrm{d}v'-\mu_w^{\frac{1}{2}}\chi(\varepsilon^{-1}v_\eta)\widetilde{\Phi}\Big|_{v\cdot n<0}\right).\end{aligned}$$

Then the result follows by adding the f_2 contribution. □

Lemma 4.3 *We have*

$$\begin{aligned}S &:= \mu^{-\frac{1}{2}}\mathscr{S}+\varepsilon^{-1}\mu^{-\frac{1}{2}}\left(2Q^*\left[\widetilde{\mathfrak{F}}_a,\mu^{\frac{1}{2}}R\right]+\varepsilon^\alpha Q^*\left[\mu^{\frac{1}{2}}R,\mu^{\frac{1}{2}}R\right]\right)\\ &\quad -\varepsilon^{-\alpha}\mu^{-\frac{1}{2}}\left(\varepsilon^{-1}Q_{gain}\left[\mathfrak{F}_a+\varepsilon^\alpha\mu^{\frac{1}{2}}R,\mathfrak{F}_a+\varepsilon^\alpha\mu^{\frac{1}{2}}R\right]\right.\\ &\quad \left.-\varepsilon^{-1}Q_{gain}\left[\left(\mathfrak{F}_a+\varepsilon^\alpha\mu^{\frac{1}{2}}R\right)_+,\left(\mathfrak{F}_a+\varepsilon^\alpha\mu^{\frac{1}{2}}R\right)_+\right]\right)\\ &\quad +\varepsilon^{-\alpha}\mathfrak{z}\mu^{-\frac{1}{2}}\iint_{\Omega\times\mathbb{R}^3}\left(\varepsilon^{-1}Q_{gain}\left[\mathfrak{F}_a+\varepsilon^\alpha\mu^{\frac{1}{2}}R,\mathfrak{F}_a+\varepsilon^\alpha\mu^{\frac{1}{2}}R\right]\right.\\ &\quad \left.-\varepsilon^{-1}Q_{gain}\left[\left(\mathfrak{F}_a+\varepsilon^\alpha\mu^{\frac{1}{2}}R\right)_+,\left(\mathfrak{F}_a+\varepsilon^\alpha\mu^{\frac{1}{2}}R\right)_+\right]\right),\end{aligned}$$

where $\mathscr{S}$ *is defined in* (4.4). *The detailed expression is*

$$S=-\mathcal{L}^1[R]+\overline{S},$$

where

$$\begin{aligned}\mathcal{L}^1[R] &:= -2\varepsilon^{-1}\mu^{-\frac{1}{2}}Q^*\left[\mu^{\frac{1}{2}}(\varepsilon f_1),\mu^{\frac{1}{2}}R\right]=-2\Gamma[f_1,R],\\ \overline{S} &:= S_0+S_1+S_2+S_3+S_4+S_5+S_6,\end{aligned}$$

for

$$
\begin{aligned}
S_0 &:= 2\varepsilon^{-1}\mu^{-\frac{1}{2}} Q^*\left[\mu^{\frac{1}{2}}(\varepsilon^2 f_2), \mu^{\frac{1}{2}} R\right] = 2\varepsilon\Gamma[f_2, R],\\
S_1 &:= 2\varepsilon^{-1}\mu^{-\frac{1}{2}} Q^*\left[\mu_w^{\frac{1}{2}}(\varepsilon f_1^B), \mu^{\frac{1}{2}} R\right] = 2\Gamma\left[\mu^{-\frac{1}{2}}\mu_w^{\frac{1}{2}} f_1^B, R\right],\\
S_2 &:= \varepsilon^{\alpha-1}\mu^{-\frac{1}{2}} Q^*\left[\mu^{\frac{1}{2}} R, \mu^{\frac{1}{2}} R\right] = \varepsilon^{\alpha-1}\Gamma[R, R],\\
S_3 &:= \varepsilon^{1-\alpha}\mu^{-\frac{1}{2}}\frac{1}{R_1-\varepsilon\eta}\left(v_\phi^2\frac{\partial}{\partial v_\eta} - v_\eta v_\phi\frac{\partial}{\partial v_\phi}\right)\left(\mu_w^{\frac{1}{2}} f_1^B\right)\\
&\quad +\varepsilon^{1-\alpha}\mu^{-\frac{1}{2}}\frac{1}{R_2-\varepsilon\eta}\left(v_\psi^2\frac{\partial}{\partial v_\eta} - v_\eta v_\psi\frac{\partial}{\partial v_\psi}\right)\left(\mu_w^{\frac{1}{2}} f_1^B\right)\\
&\quad -\varepsilon^{1-\alpha}\mu^{-\frac{1}{2}}\frac{1}{L_1L_2}\left(\frac{R_1\partial_{\iota_1\iota_1}\mathbf{r}\cdot\partial_{\iota_2}\mathbf{r}}{L_1(R_1-\varepsilon\eta)}v_\phi v_\psi + \frac{R_2\partial_{\iota_1\iota_2}\mathbf{r}\cdot\partial_{\iota_2}\mathbf{r}}{L_2(R_2-\varepsilon\eta)}v_\psi^2\right)\frac{\partial}{\partial v_\phi}\left(\mu_w^{\frac{1}{2}} f_1^B\right)\\
&\quad -\varepsilon^{1-\alpha}\mu^{-\frac{1}{2}}\frac{1}{L_1L_2}\left(\frac{R_2\partial_{\iota_2\iota_2}\mathbf{r}\cdot\partial_{\iota_1}\mathbf{r}}{L_2(R_2-\varepsilon\eta)}v_\phi v_\psi + \frac{R_1\partial_{\iota_1\iota_2}\mathbf{r}\cdot\partial_{\iota_1}\mathbf{r}}{L_1(R_1-\varepsilon\eta)}v_\phi^2\right)\frac{\partial}{\partial v_\psi}\left(\mu_w^{\frac{1}{2}} f_1^B\right)\\
&\quad -\varepsilon^{1-\alpha}\mu^{-\frac{1}{2}}\left(\frac{R_1 v_\phi}{L_1(R_1-\varepsilon\eta)}\frac{\partial}{\partial\iota_1} + \frac{R_2 v_\psi}{L_2(R_2-\varepsilon\eta)}\frac{\partial}{\partial\iota_2}\right)\left(\mu_w^{\frac{1}{2}} f_1^B\right)\\
&\quad +\varepsilon^{-\alpha}\mu^{-\frac{1}{2}} v_\eta\overline{\chi}(\varepsilon^{-1}v_\eta)\frac{\partial\chi(\varepsilon\eta)}{\partial\eta}\left(\mu_w^{\frac{1}{2}}\widetilde{\Phi}\right)\\
&\quad +\varepsilon^{-\alpha}\mu^{-\frac{1}{2}}\mu_w^{\frac{1}{2}}\chi(\varepsilon\eta)\left(\overline{\chi}(\varepsilon^{-1}v_\eta)K_w[\widetilde{\Phi}] - K_w[\overline{\chi}(\varepsilon^{-1}v_\eta)\widetilde{\Phi}]\right),\\
S_4 &:= -\varepsilon^{-\alpha}\mu^{-\frac{1}{2}}\left(v\cdot\nabla_x\left(\mu^{\frac{1}{2}}\left(\varepsilon^2 f_2\right)\right)\right) = -\varepsilon^{2-\alpha}\mu^{-\frac{1}{2}}\left(v\cdot\nabla_x\left(\mu^{\frac{1}{2}} f_2\right)\right),\\
S_5 &:= \varepsilon^{3-\alpha}\mu^{-\frac{1}{2}} Q^*\left[\mu^{\frac{1}{2}} f_2, \mu^{\frac{1}{2}} f_2\right] + 2\varepsilon^{2-\alpha}\mu^{-\frac{1}{2}} Q^*\left[\mu^{\frac{1}{2}} f_2, \mu^{\frac{1}{2}} f_1\right]\\
&\quad +2\varepsilon^{2-\alpha}\mu^{-\frac{1}{2}} Q^*\left[\mu^{\frac{1}{2}} f_2, \mu_w^{\frac{1}{2}} f_1^B\right] + 2\varepsilon^{1-\alpha}\mu^{-\frac{1}{2}} Q^*\left[\mu_w^{\frac{1}{2}} f_1^B, \mu^{\frac{1}{2}} f_1\right]\\
&\quad +\varepsilon^{1-\alpha}\mu^{-\frac{1}{2}} Q^*\left[\mu_w^{\frac{1}{2}} f_1^B, \mu_w^{\frac{1}{2}} f_1^B\right] + \varepsilon^{-\alpha}\mu^{-\frac{1}{2}} Q^*\left[\mu-\mu_w, \mu_w^{\frac{1}{2}} f_1^B\right]\\
&= \varepsilon^{3-\alpha}\Gamma[f_2, f_2] + 2\varepsilon^{2-\alpha}\Gamma[f_2, f_1] + 2\varepsilon^{2-\alpha}\Gamma\left[f_2, \mu^{-\frac{1}{2}}\mu_w^{\frac{1}{2}} f_1^B\right]\\
&\quad +2\varepsilon^{1-\alpha}\Gamma\left[\mu^{-\frac{1}{2}}\mu_w^{\frac{1}{2}} f_1^B, f_1\right] + \varepsilon^{1-\alpha}\Gamma\left[\mu^{-\frac{1}{2}}\mu_w^{\frac{1}{2}} f_1^B, \mu^{-\frac{1}{2}}\mu_w^{\frac{1}{2}} f_1^B\right]\\
&\quad +\varepsilon^{-\alpha}\Gamma\left[\mu^{-\frac{1}{2}}(\mu-\mu_w), \mu^{-\frac{1}{2}}\mu_w^{\frac{1}{2}} f_1^B\right],
\end{aligned}
$$

and

$$
\begin{aligned}
S_6 := -\varepsilon^{-\alpha}\mu^{-\frac{1}{2}}&\Big(\varepsilon^{-1} Q_{gain}\left[\mathfrak{F}_a + \varepsilon^\alpha\mu^{\frac{1}{2}} R, \mathfrak{F}_a + \varepsilon^\alpha\mu^{\frac{1}{2}} R\right]\\
&-\varepsilon^{-1} Q_{gain}\left[\left(\mathfrak{F}_a + \varepsilon^\alpha\mu^{\frac{1}{2}} R\right)_+, \left(\mathfrak{F}_a + \varepsilon^\alpha\mu^{\frac{1}{2}} R\right)_+\right]\Big)\\
&+\varepsilon^{-\alpha}\mathfrak{z}\mu^{-\frac{1}{2}}\iint_{\Omega\times\mathbb{R}^3}\Big(\varepsilon^{-1} Q_{gain}\left[\mathfrak{F}_a + \varepsilon^\alpha\mu^{\frac{1}{2}} R, \mathfrak{F}_a + \varepsilon^\alpha\mu^{\frac{1}{2}} R\right]
\end{aligned}
$$

$$-\varepsilon^{-1} Q_{gain}\left[\left(\mathfrak{F}_a + \varepsilon^{\alpha}\mu^{\frac{1}{2}}R\right)_+, \left(\mathfrak{F}_a + \varepsilon^{\alpha}\mu^{\frac{1}{2}}R\right)_+\right]\Big).$$

Proof This follows directly from (4.3). □

We decompose

$$R = \mathbf{P}[R] + (\mathbf{I}-\mathbf{P})[R] := \mu^{\frac{1}{2}}(v)\left(p_R(x) + v\cdot\mathbf{b}_R(x) + (|v|^2 - 5T)c_R(x)\right) + (\mathbf{I}-\mathbf{P})[R],$$

We further define the orthogonal split

$$(\mathbf{I}-\mathbf{P})[R] = \mathscr{A}\cdot\mathbf{d}_R(x) + (\mathbf{I}-\overline{\mathbf{P}})[R],$$

where $(\mathbf{I}-\overline{\mathbf{P}})[R]$ is the orthogonal complement to $\mathscr{A}\cdot\mathbf{d}_R(x)$ in $\mathcal{N}^{\perp}$ with respect to $(\cdot,\cdot)_{\mathcal{L}} = (\cdot, \mathcal{L}[\cdot])_{\mathcal{L}}$, i.e.

$$(\mathscr{A}, (\mathbf{I}-\overline{\mathbf{P}})[R])_{\mathcal{L}} = \langle\overline{\mathscr{A}}, (\mathbf{I}-\overline{\mathbf{P}})[R]\rangle = 0.$$

In summary, we decompose the remainder as (4.6),

$$R = \left(p + \mathbf{b}\cdot v + c(|v|^2 - 5T)\right)\mu^{\frac{1}{2}} + \mathbf{d}\cdot\mathscr{A} + (\mathbf{I}-\overline{\mathbf{P}})[R]. \tag{4.6}$$

We can further define the Hodge decomposition $\mathbf{d} = \nabla_x\xi + \mathbf{e}$ with ξ solving the Poisson equation

$$\begin{cases} \nabla_x\cdot(\kappa\nabla_x\xi) = \nabla_x\cdot(\kappa\mathbf{d}) & \text{in } \Omega,\\ \xi = 0 & \text{on } \partial\Omega. \end{cases}$$

We reformulate the remainder equation with a global Maxwellian in order to obtain L^∞ estimates. Considering $\|\nabla_x T\| \lesssim o_T$ for o_T defined in (1.3), choose a constant T_M such that

$$T_M < \min_{x\in\Omega} T < \max_{x\in\Omega} T < 2T_M \quad \text{and} \quad \max_{x\in\Omega} T - T_M = o_T. \tag{4.7}$$

Define a global Maxwellian

$$\mu_M := \frac{P}{(2\pi)^{\frac{3}{2}} T_M^{\frac{5}{2}}}\exp\left(-\frac{|v|^2}{2T_M}\right).$$

We can rewrite (4.5) as

$$\begin{cases} v\cdot\nabla_x R_M + \varepsilon^{-1}\mathcal{L}_M[R] = S_M & \text{in } \Omega\times\mathbb{R}^3,\\ R_M(x_0,v) = \mathcal{P}_M[R_M](x_0,v) + h_M(x_0,v) & \text{for } x_0\in\partial\Omega \text{ and } v\cdot n(x_0) < 0, \end{cases}$$

where $R_M = \mu_M^{-\frac{1}{2}}\mu^{\frac{1}{2}}R$, $S_M = \mu_M^{-\frac{1}{2}}\mu^{\frac{1}{2}}S$, $h_M = \mu_M^{-\frac{1}{2}}\mu^{\frac{1}{2}}h$ and for $m_{M,w} := \mu_M^{-\frac{1}{2}}\mu^{\frac{1}{2}}(x_0,v)m_w(x_0,v) = M_w\mu_M^{-\frac{1}{2}}$

$$\mathcal{L}_M[R_M] := -2\mu_M^{-\frac{1}{2}} Q\left[\mu, \mu_M^{\frac{1}{2}}R_M\right] := \nu_M R_M - K_M[R_M],$$

$$\mathcal{P}_M[R_M](x_0,v) := m_{M,w}(x_0,v)\int_{v'\cdot n(x_0)>0}\mu_M^{\frac{1}{2}}R_M(x_0,v')|v'\cdot n(x_0)|\mathrm{d}v'.$$

Denote the working space X via the norm

$$\begin{aligned}\|R\|_X :=& \varepsilon^{-1}\|p\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|\mathbf{b}\|_{L^2} + \|c\|_{L^2} + \varepsilon^{-1}\|\xi\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|\xi\|_{H^2} + \varepsilon^{-1}\|\mathbf{e}\|_{L^2} \\ &+\varepsilon^{-1}\|(\mathbf{I}-\overline{\mathbf{P}})[R]\|_{L^2_\nu} + \|p\|_{L^6} + \|\mathbf{b}\|_{L^6} + \|c\|_{L^6} + \varepsilon^{-1}\|\xi\|_{L^6} + \|\xi\|_{W^{2,6}} \\ &+\|\mathbf{e}\|_{L^6} + \|(\mathbf{I}-\overline{\mathbf{P}})[R]\|_{L^6} + |\mathcal{P}_\gamma[R]|_{L^2_\gamma} + \varepsilon^{-\frac{1}{2}}|(1-\mathcal{P}_\gamma)[R]|_{L^2_{\gamma+}} \\ &+ \left|\mu^{\frac{1}{4}}(1-\mathcal{P}_\gamma)[R]\right|_{L^4_{\gamma+}} + \varepsilon^{-\frac{1}{2}}|\nabla_x\xi|_{L^2_{\partial\Omega}} + \varepsilon^{\frac{1}{2}}\|R_M\|_{L^\infty_{\varrho,\vartheta}} + \varepsilon^{\frac{1}{2}}|R_M|_{L^\infty_{\gamma,\varrho,\vartheta}}.\end{aligned} \tag{4.8}$$

In the companion paper [12], we prove the following:

Theorem 4.1 *Assume that Ω is a bounded C^3 domain and (1.4) holds. Then for any given $P > 0$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a nonnegative solution $\mathfrak{F}$ to the equation (0.1) represented by (0.4) with $\alpha = 1$ satisfying*

$$\int_\Omega p(x)\mathrm{d}x = 0 \tag{4.9}$$

and

$$\|R\|_X \lesssim o_T, \tag{4.10}$$

where the X norm is defined in (4.8). Such a solution is unique among all solutions satisfying (4.9) and (4.10). This further yields that in the expansion (0.4), $\mu + \varepsilon\mu(u_1 \cdot v)$ is the leading-order terms in the sense of

$$\left\|\mu^{-\frac{1}{2}}[\mathfrak{F}-\mu]\right\|_{L^2_{x,v}} \lesssim \varepsilon$$

and

$$\left\|\int_{\mathbb{R}^3}[\mathfrak{F}-\mu-\varepsilon\mu(u_1\cdot v)]v\right\|_{L^2_x} \lesssim \varepsilon^{\frac{3}{2}},$$

where (ρ, u_1, T) is determined by the ghost-effect equations (0.2) and (0.3).

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