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# Partially Symmetric Tensors and the Non-defectivity of Secant Varieties of Products with a Projective Line as a Factor

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**Abstract**
We prove (with a mild restriction on the multidegrees) that all secant varieties of Segre–Veronese varieties with $k > 2$ factors, $k - 2$ of them being $\mathbb{P}^1$, have the expected dimension. This is equivalent to compute the dimension of the set of all partially symmetric tensors with a fixed rank and the same format. The proof uses the case $k = 2$ proved by Galuppi and Oneto. Our theorem is an easy consequence of a theorem proved here for arbitrary projective varieties with a projective line as a factor and with respect to complete linear systems.

**Keywords** Secant variety · Segre–veronese varieties · Partially symmetric tensors

**Mathematics Subject Classification (2010)** 14N05 · 14N07 · 15A69

## 1 Introduction

For any positive integer $z$ and for any integral and non-degenerate variety $W \subset \mathbb{P}^r$ let $\sigma_z(W)$ denote the $z$-secant variety of $W$, i.e. the closure of the union of all linear spaces spanned by $z$ points of $W$. Many practical linear algebra problems and applications use secant varieties or, at least, their dimensions [18]. For instance, if $\sigma_{z-1}(W) \neq \mathbb{P}^r$, then the integer $\dim \sigma_z(W)$ is the dimension of the set of all $q \in \mathbb{P}^r$ with $W$-rank $z$. If we take as $W$ a multiprojective space, then the $W$-rank decompositions are the partially symmetric tensor decompositions with the minimal number of addenda. If we take $W = \mathbb{P}^n$, then the $W$-ranks correspond to the additive decompositions of forms in $n + 1$ variables with a minimal number of addenda.

Segre–Veronese varieties are related to partially symmetric tensors and hence their secant varieties (or at least their dimensions) are an active topic of research with several papers devoted to the study of the dimensions of their secant varieties [1, 2, 11, 15–17]. As a corollary of our results we prove the following result in which we only use the secant non-defectivity of almost all Segre–Veronese varieties with 2-factors [15].

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**Theorem 1** *Take $X := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{k-2}$, $k \geq 3$, embedded by the complete linear system $\mathcal{O}_X(d_1, \dots, d_k)$ with $d_i \geq 2$ for all $i$, $d_1 \geq 3$ and $d_2 \geq 3$. Then this embedding of $X$ is not secant defective.*

We prove a more interesting result, which applies to all varieties with a projective line as a factor and with respect to complete linear systems (Theorem 2).

Let $Y$ be an integral projective variety. Set $X := Y \times \mathbb{P}^1$. Let $\pi_1 : X \to Y$ and $\pi_2 : X \to \mathbb{P}^1$ denote the 2 projections. For any $\mathcal{L} \in \mathrm{Pic}(Y)$ and any integer $t$ set $\mathcal{L}[t] := \pi_1^*(\mathcal{L}) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(t))$.

**Theorem 2** *Let $Y$ be an integral projective variety. Fix an integer $t \geq 2$. Set $X := Y \times \mathbb{P}^1$. Let $\mathcal{L}$ be a very ample line bundle on $Y$ with $h^1(\mathcal{L}) = 0$. Set $n := \dim X$, $\alpha := h^0(\mathcal{L})$ and $e := \lfloor \alpha/n \rfloor$. Assume $n \geq 3$, $\alpha > n^2$ and that the $e$-th secant variety of $(Y, \mathcal{L})$ has the expected dimension. Then the pair $(X, \mathcal{L}[t])$ is not secant defective.*

We assume $n \geq 3$ in Theorem 2 because if $n = t = 2$ no lower bound on $\alpha$ may work, as shown by Example 1. The integer $e$ is the maximal integer such that the $e$-secant variety may have dimension $ne - 1$. We assume that it has dimension $ne - 1$, the expected dimension, but if $ne < \alpha$ we do not assume that $(Y, \mathcal{L})$ is not secant defective, i.e. we do not assume that $\sigma_{e+1}(Y)$ is the entire projective space. We do not assume it, because we do not need it and assuming it will not help for the proof. There are plenty of defective pairs $(Y, \mathcal{L})$ and $t$ such that $(Y \times \mathbb{P}^1, \mathcal{L}[t])$ is not defective. For instance, take $Y = \mathbb{P}^n$ with $\mathcal{L} := \mathcal{O}_{\mathbb{P}^n}(a)$ defective and either $t$ odd or $a > 2$ [7, Theorem 3.1]. Conversely, $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ is not defective, but if we take $t = 1$ we get a defective variety [7, Theorem 3.1]. This is not very surprising. For any $o \in \mathbb{P}^1$, the subvariety $Y \times \{o\}$ is only a tiny part of $X = Y \times \mathbb{P}^1$, while the secant varieties only depend on a Zariski open subset of $X$. Having a covering family of defective varieties does not (a priori) exclude the non-defectivity of $X$. Assuming that $\sigma_{e+1}(Y)$ is the full projective space would not shorten our proof.

Theorem 1 is an easy consequence of Theorem 2.

Our tools work even if $(Y, \mathcal{L})$ is secant defective, adding conditions on $t$ and/or $\alpha$. As an example we prove one case in which we only assume that $\sigma_{\lfloor \alpha/n \rfloor - 1}(Y)$ has the expected dimension (Theorem 3). We also see that even for $t = 1$ we may get non-trivial results (Propositions 1 and 2).

We often use the Differential Horace Lemma [5, 6] and an inductive procedure (the Horace Method) but from top to bottom with smaller and smaller zero-dimensional schemes to be handled. This approach may be considered as a controlled asymptotic tool which does not require the low cases to start the inductive procedure [8, 9], [12, Lemma 3]. A long and detailed explanation of this method is contained in [8, pp. 1005–1008]; see in particular the diagram of logical implications in [8, p. 1008]. See Lemma 1 and Remark 2 for a key part of this approach. Then sometimes the low cases may be proved, e.g. with a computer assisted proof [9, 13]. However, a standard use of this tool would only give a very weak result (e.g. Theorem 2 only for $t \geq \dim Y + 2$ and hence the inductive proof to get Theorem 1 for $k \geq 4$ would require very large $t$).

For our proof of Theorem 2 the key part is the proof of the case $t = 2$. Then the cases $t > 2$ have a short inductive proof using Lemmas 2, 3 and 4.

Our method works for some non-complete linear systems (see Remark 5).

## 2 Preliminaries

Let $W$ a projective variety and $D$ an effective Cartier divisor of $W$. For any $p \in W_{\mathrm{reg}}$ let $(2p, W)$ denote the closed subscheme of $W$ with $(\mathcal{I}_{p,W})^2$ as its ideal sheaf. We have $\deg((2p, W)) = \dim W + 1$ and $(2p, W)_{\mathrm{red}} = \{p\}$. For any finite set $S \subset W_{\mathrm{reg}}$ set $(2S, W) := \cup_{p \in S}(2p, W)$. We often write $2p$ and $2S$ instead of $(2p, X)$ and $(2S, X)$. For any zero-dimensional scheme $Z \subset W$ let $\mathrm{Res}_D(Z)$ denote the residual scheme of $Z$ with respect to $D$, i.e. the closed subscheme of $W$ with $\mathcal{I}_Z : \mathcal{I}_D$ has its ideal sheaf. We have $\deg(Z) = \deg(\mathrm{Res}_D(Z)) + \deg(Z \cap D)$, $\mathrm{Res}_D(Z) \subseteq Z$, $\mathrm{Res}_D(Z) = Z$ if $Z \cap D = \emptyset$, $\mathrm{Res}_D(Z) = \emptyset$ if $Z \subset D$ and $\mathrm{Res}_D(Z) = \mathrm{Res}_D(A) \cup \mathrm{Res}_D(B)$ if $Z = A \cup B$ and $A \cap B = \emptyset$. If $p \in D_{\mathrm{reg}} \cap W_{\mathrm{reg}}$, then $\mathrm{Res}_D((2p, W)) = \{p\}$ and $(2p, W) \cap D = (2p, D)$. For any line bundle $\mathcal{R}$ on $W$ there is an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{R}(-D) \to \mathcal{I}_Z \otimes \mathcal{R} \to \mathcal{I}_{Z \cap D, D} \otimes \mathcal{R}_{|D} \to 0 \tag{1}$$

of coherent sheaves on $W$ which we call the *residual sequence of* $D$. Fix a positive integer $z$. By the Terracini Lemma [3, Corollary 1.11], [18, 5.3.1.1] the integer $\dim \sigma_z(W)$ is the codimension of the linear span of the zero-dimensional scheme $(2S, W)$, where $S$ is a general subset of $W$ of cardinality $z$. Thus if the embedding of $W$ is induced by the complete linear system $|\mathcal{R}|$, then $\dim \sigma_z(W) = h^0(\mathcal{R}) - 1 - h^0(\mathcal{I}_{(2S,W)} \otimes \mathcal{R})$. Hence $\dim \sigma_z(W) = z(\dim W + 1) - 1$ if and only if $h^1(\mathcal{I}_{(2S,W)} \otimes \mathcal{R}) = h^1(\mathcal{R})$.

Now we describe the so-called Differential Horace Lemma [5, 6].

**Remark 1** Let $E \subset W$ be a zero-dimensional scheme. Fix $i \in \{0, 1\}$ and an integer $g > 0$. Let $D$ be an integral divisor of $W$. Let $F \subset D$ be a general subset of $D$ with $\#F = g$. Suppose you want to prove that $h^i(\mathcal{I}_{E \cup (2S,W)} \otimes \mathcal{R}) = 0$ for a general $S \subset W$ such that $\#S = g$. It is sufficient to prove that $h^i(H, \mathcal{I}_{(E \cap D) \cup F} \otimes \mathcal{R}_{|D}) = 0$ and $h^i(W, \mathcal{I}_{\mathrm{Res}_D(E) \cup (2F,D)} \otimes \mathcal{R}(-D)) = 0$ [5, 6].

Take $X := Y \times \mathbb{P}^1$. Every line bundle on $X$ is of the form $\mathcal{L}[t]$ for a uniquely determined $\mathcal{L} \in \mathrm{Pic}(Y)$ and a unique $t \in \mathbb{Z}$ [14, Proposition 3]. Now assume $t \geq 0$ and $\alpha := h^0(\mathcal{L}) > 0$. The Künneth formula gives $h^0(\mathcal{L}[t]) = (t + 1)\alpha$ and $h^1(\mathcal{L}[t]) = (t + 1)h^1(\mathcal{L})$. From now on we also assume $h^1(\mathcal{L}) = 0$ and hence $h^1(\mathcal{L}[t]) = 0$. Note that $\mathcal{L}[t]_{|H} = \mathcal{L}[0]_{|H}$ for all $t$ and that a double point $2p$ of $X$ gives the same number of condition of a linear subspace of $H^0(X, \mathcal{L}[0])$ as the double point $2p_1(p)$ of $Y$ to the corresponding linear subspace of $H^0(Y, \mathcal{L})$.

As in [8, 9] we often use the following observation, which explain we using Remark 1 in the step $\mathcal{L}[t] \Longrightarrow \mathcal{L}[t - 1]$ of a descending induction we also need to handle $\mathcal{L}[t - 2]$.

**Lemma 1** *Let $E \subset W$ be a zero-dimensional scheme. Let $D$ be an integral divisor of $W$ and assume $h^1(\mathcal{I}_E \otimes \mathcal{R}) = 0$. Set $b := h^0(\mathcal{I}_E \otimes \mathcal{R})$. Fix $a \in \mathbb{N}$ and a general $A \subset D$ such that $\#A = a$. We have $h^1(\mathcal{I}_{E \cup A} \otimes \mathcal{R}) = \max\{0, b - a\}$ and $h^0(\mathcal{I}_{E \cup A} \otimes \mathcal{R}) = \max\{0, b - a\}$ if and only if $h^0(\mathcal{I}_{\mathrm{Res}_D(E)} \otimes \mathcal{R}(-D)) \leq \max\{0, b - a\}$.*

***Proof*** First assume $a = 0$. We have the “only if” part (i.e. the part we will not use in the proofs of the theorems) by the $H^0$-part of the cohomology sequence of (1). Thus we may assume $a > 0$ and use induction on the integer $a$. We order the points $p_1, \ldots, p_a$ of $A$. Set $A' := A \setminus \{p_a\}$. By the inductive assumption we have $h^1(\mathcal{I}_{E \cup A'} \otimes \mathcal{R}) = \max\{0, b - a + 1\}$ and $h^0(\mathcal{I}_{E \cup A'} \otimes \mathcal{R}) = \max\{0, b - a + 1\}$ if and only if $h^0(\mathcal{I}_{\mathrm{Res}_D(E)} \otimes \mathcal{R}(-D)) \leq \max\{0, b - a + 1\}$. If $h^0(\mathcal{I}_{E \cup A'} \otimes \mathcal{R}) = 0$, i.e. if $a > b$, adding the point $p_a \notin E_{\mathrm{red}}$ does not change any of the 3 max. Now assume $h^0(\mathcal{I}_{E \cup A'} \otimes \mathcal{R}) > 0$ and hence $h^1(\mathcal{I}_{E \cup A'} \otimes \mathcal{R}) = 0$. Adding $p_a$

does not change $\mathrm{Res}_D(E)$. Since $p_a$ is general in $D$, we have $h^1(\mathcal{I}_{E\cup A}\otimes\mathcal{R})=0$ if and only if $D$ is not in the base locus of $|\mathcal{I}_{E\cup A'}\otimes\mathcal{R}|$, i.e. if and only if $h^0(\mathcal{I}_{\mathrm{Res}_D(E)}\otimes\mathcal{R}(-D))<h^0(\mathcal{I}_{E\cup A'}\otimes\mathcal{R})=b-a+1$. □

**Remark 2** In the applications of Lemma 1 to prove Theorem 2 and similar results we have $E=Z'\cup\Delta$, where $Z'$ is a general union of double points of $W$ and $\Delta$ is a general union of double points of $D$. Note that $\mathrm{Res}_D(E)=Z'$.

Suppose there is a line bundle $\mathcal{R}$ on $W$ such that the embedding $W\subset\mathbb{P}^r$ is induced by the complete linear system $|\mathcal{R}|$. We call the secant varieties of the embedded variety $W\subset\mathbb{P}^r$ the secant varieties of the pair $(W,\mathcal{R})$.

**Remark 3** Let $W\subset\mathbb{P}^r$ be an integral and non-degenerate variety. Set $n:=\dim W$, $z_1:=\lfloor(r+1)/(n+1)\rfloor$ and $z_2:=\lceil(r+1)/(n+1)\rceil$. Suppose that $\sigma_{z_1}(W)$ and $\sigma_{z_2}(W)$ have the expected dimension. Since $(n+1)z_2\ge r+1$, $\sigma_x(W)=\mathbb{P}^r$ for all $x\ge z_2$. Either $z_1=z_2$ or $z_1=z_2-1$. Let $S\subset W_{\mathrm{reg}}$ be a general subset such that $\#S=z_1$. Since $\dim\sigma_{z_1}(W)=(n+1)z_1-1$, the Terracini Lemma gives $\dim(2S,W)=(n+1)z_1-1$, i.e. the scheme $(2S,W)$ is linearly independent. Hence, $(2A,W)$ is linearly independent for any $A\subset S$. The Terracini Lemma gives $\dim\sigma_y(W)=(n+1)y-1$ for all $1\le y<z_1$ (a similar statement is proved in [4, Proposition 2.1(i)]). Thus $W$ is not secant defective. Note that $z_2$ is the minimal integer $z$ such that $(\dim W+1)z\ge r+1$. Thus to prove that $W$ is not secant defective it is sufficient prove that $\sigma_z(W)$ has the expecting dimension for $z=z_2$ and for all $z\le z_1$, i.e. for all positive integers $z$ such that $(\dim W+1)z\le r+1+\dim W$. In the set-up of Theorem 2 it is sufficient to test all positive integers $z$ such that $(n+1)z\le(t+1)\alpha+n$.

## 3 The Proofs and Related Results

Set $X:=Y\times\mathbb{P}^1$ and $n:=\dim X$. We fix $o\in\mathbb{P}^1$ and set $H:=Y\times\{o\}$. The set $H$ is an effective Cartier divisor of $X$ and $H\cong Y$. We first give 2 cases in which the proof is short (Propositions 1 and 2). Then we discuss the main idea of the proofs.

**Proposition 1** *Fix a positive integer $z$ such that $n(z-e)+e\le\alpha$ and assume that the $e$-secant variety of $(Y,\mathcal{L})$ has dimension $ne-1$. Then the $z$-secant variety of the pair $(X,\mathcal{L}[1])$ has dimension $z(n+1)-1$.*

***Proof*** It is sufficient to do the cases with $z\ge e$. Take a general $(A,B)\subset H\times H$ such that $\#A=z-e$ and $\#B=e$. By assumption $\#A+\#B\le\alpha$. Since the $e$-secant variety of $(Y,\mathcal{L})$ has dimension $ne-1$ and $e$ is a non-negative integer, $h^1(H,\mathcal{I}_{(2A,H)\cup B,H}\otimes\mathcal{L}[1]_{|H})=0$. By the Differential Horace Lemma (Remark 1) to prove the proposition it is sufficient to prove that $h^1(\mathcal{I}_{(2B,H)}\otimes\mathcal{L})=0$. Since $(2B,H)\subset H$ and $h^1(\mathcal{L}[-1])=0$ by the Künneth formula, we have $h^1(\mathcal{I}_{(2B,H)}\otimes\mathcal{L})=h^1(H,\mathcal{I}_{(2B,H),H}\otimes\mathcal{L}_{|H})=0$ (because the $e$-secant variety of $(Y,\mathcal{L})$ has dimension $ne-1$). □

**Proposition 2** *Assume $\alpha\equiv 0\pmod{n+1}$ and that the $\alpha/(n+1)$-secant variety of $(Y,\mathcal{L})$ has the expected dimension. Then $(X,\mathcal{L}[1])$ is not defective*

***Proof*** Set $z:=2\alpha/(n+1)$. We have $h^0(\mathcal{L}[1])=2\alpha$. Since $\alpha\equiv 0\pmod{n+1}$, it is sufficient to prove that $h^i(\mathcal{I}_{2S}\otimes\mathcal{L}[1])=0$, $i=0,1$, where $S$ is a general subset of $X$. Take a general $S'\cup S''\subset H$ such that $\#S'=\#S''=z/2$ and $S'\cap S''=\emptyset$. Since $\dim\sigma_{z/2}(Y)=nz/2$ and $\alpha=(n+1)z/2=nz/2+z/2$, we have $h^i(H,\mathcal{I}_{(2S',H)\cup S''}\otimes\mathcal{L}[1]_{|H})=0$, $i=0,1$ and

$h^i(\mathcal{I}_{S'\cup(2S'',H)}\otimes\mathcal{L}[0]) = h^i(H, \mathcal{I}_{S'\cup(2S'',H)}\mathcal{L}[0]_{|H}) = 0$, $i = 0, 1$. We specialize $z/2$ of the points of $S$ to $S'$, while use Differential Horace at each point of $S''$. □

In the statement of Proposition 2 we often have $\alpha/(n+1) < e$.

**Discussion of the Main Idea** Suppose you want to prove the first step in the proof that the $z$-secant variety of $(X, \mathcal{L}[t])$ has the expected dimension and that $(n+1)z \le (t+1)\alpha$. Let $S \subset X$ be a general union of $z$ double points of $X$. We need to prove that $h^1(\mathcal{I}_Z\otimes\mathcal{L}[t]) = 0$. We want to find a zero-dimensional scheme $W$ such that $\deg(W) = \deg(Z)-\alpha$, $h^1(\mathcal{I}_W\otimes\mathcal{L}[t-1]) = 0$ and such that $h^1(\mathcal{I}_W \otimes \mathcal{L}[t-1]) = 0$ implies $h^1(\mathcal{I}_{2S}\otimes\mathcal{L}[t]) = 0$. Take $(x, y) \in \mathbb{N}^2$ and assume $z \ge x + y$. Let $Z' \subset X$ be a general union of $z - x - y$ double points of $X$. One can try to specialize $2S$ to the union of $Z'$, $x$ general double points of $X$ with reduction contained in $H$ and apply $y$ times Differential Horace with respect to $y$ general points of $H$. On $H$ we get a scheme $\Delta$ which is a general union of $x$ double points of $H$ and $y$ points of $H$. To get $h^i(H, \mathcal{I}_\Delta \otimes \mathcal{L}[t]_{|H}) = 0$, $i = 0, 1$, we need to assume $nx + y = \alpha$. Since $y \ge 0$, we have $x \le e$ and hence the assumption $nx + y = \alpha$ gives $h^i(H, \mathcal{I}_\Delta \otimes \mathcal{L}[t]_{|H}) = 0$, $i = 0, 1$. By the Differential Horace Lemma it is sufficient to prove that $h^1(\mathcal{I}_{Z'\cup S'\cup(2S'',H)} \otimes \mathcal{L}[t-1]) = 0$. Since $S' \cup (2S'', H)$ has degree $x + ny$, we certainly need $x + ny \le \alpha$, i.e. $y \le x$. But we also have to handle $Z'$. We first check that, for the integers $x$ and $y$ we carefully took we have $h^1(\mathcal{I}_{Z'\cup(2S'',H} \otimes \mathcal{L}[t-1]) = 0$. This is often possible even if we do not know that $(X, \mathcal{L}[t-1])$ is not defective, because $h^0(\mathcal{L}[t]) - h^0(\mathcal{L}[(t-1)]) = \alpha$, $\deg(Z' \cup (2S'', H)) = \deg(Z) - \alpha - x$ and $x$ is “large”. Then we use that $S'$ is general in $H$ to reduce the proof that $h^1(\mathcal{I}_{Z'\cup S'\cup(2S'',H}\otimes\mathcal{L}[t-1]) = 0$ to $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[t-2]) \le \max\{0, t\alpha - \deg(Z' \cup S' \cup (2S'', H)\}$ (Lemma 1 and Remark 2). Note that $h^0(\mathcal{L}[t-2]) = (t-1)\alpha$, while we require $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[t-2]) \le \max\{0, t\alpha - \deg(Z' \cup S' \cup (2S'', H)\}$. Lemma 1 explains the use of $\mathcal{L}[t-2]$. For $t = 2$ we use that either $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[0]) = \alpha - n(z - x - y)$ (case $z - x - y \le e$) or $h^0(\mathcal{I}_{Z'} \otimes \mathcal{L}[0]) \le n - 2$ (case $z - x - y > e$). For $t > 2$ the proof is easier, because we may use that $\mathcal{L}[t-1]$ is not defective and, if $t \ge 4$, even that $\mathcal{L}[t-2]$ is not defective. For $t > 2$ we always take $x := e$ and $y := f$, which is the usual choice to apply the Differential Horace Lemma. We use different $(x, y)$ for $t = 2$, because we cannot assume that $\mathcal{L}[1]$ is not defective.

For all positive integers $t$ and $z$ call $A(t, z)$ the following statement:

$A(t, z)$: We have $h^0(\mathcal{I}_{2S} \otimes \mathcal{L}[t]) = \max\{\alpha(t+1) - (n+1)z, 0\}$ for a general $S \subset X$ such that $\#S = z$.

By the Terracini Lemma $A(t, z)$ is true if and only if the $z$-secant variety of the pair $(X, \mathcal{L}[t])$ has the expected dimension. Since $h^1(\mathcal{L}[t]) = 0$, $A(t, z)$ is equivalent to $h^1(\mathcal{I}_{2S} \otimes \mathcal{L}[t]) = \max\{0, (n+1)z - \alpha(t+1)\}$ for a general $S \subset X$ such that $\#S = z$.

We say that $A(t)$ is true if $A(t, z)$ are true for all $z \in \{\lfloor(t+1)\alpha/(n+1)\rfloor, \lceil(t+1)\alpha/(n+1)\rceil\}$. Remark 3 shows that $A(t)$ is true if and only if $A(t, z)$ is true for all positive integers $z$.

Write $\alpha = ne + f$ with $e, f$ integers and $0 \le f < n$, i.e. set $e := \lfloor\alpha/n\rfloor$ and $f := \alpha - ne$.

The following example is well-known [17, p. 1457].

**Example 1** Fix a positive integer $a$, $t = 2$ and $(Y, \mathcal{L}) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a))$. We have $n = 2$, $\alpha = 2a + 1$, $(X, \mathcal{L}[2])$ is secant defective with only defective the $(2a + 1)$-secant variety. Indeed, a general $S \subset \mathbb{P}^1 \times \mathbb{P}^1$ with $\#S = 2a + 1$ is contained in the singular locus of a unique $D \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2a, 2)|$, the double curve $2T$ with $T$ the unique element of $|\mathcal{I}_S(a, 1)|$.

**Remark 4** It is very important for our proof that $f \le e$. Since $f \le n - 1$, it is sufficient to assume $e \ge n - 1$. If $\dim \sigma_e(Y) = ne - 1$, it is sufficient to assume $\alpha \ge n(n - 1)$, which is quite mild.

For all positive integers $t$ and $z$ we call $B(t, z)$ the following statement:

$B(t, z)$: Either $z < e + f$ or

$$h^0(\mathcal{I}_E \otimes \mathcal{L}[t-1]) = \max\{0, t\alpha - (n+1)(z-e-f) - ne - f\},$$

where $E \subset X$ is a general union of $z-e-f$ double points of $X$, $f$ double points of $H$ and $e$ points of $H$.

Note that $B(t, z)$ is always true if $z < e + f$. In the proof of Theorem 2 we will separate the 2 cases, $z < e + f$ and $z \geq e + f$.

For all $t \geq 2$ we call $C(t, z)$ the following statement:

$C(t, z)$: We have $h^0(\mathcal{I}_W \otimes \mathcal{L}[t-2]) \leq \max\{0, (t-1)\alpha - \deg(W)\}$, where $W$ is a general union of $\max\{0, z-e-f\}$ double points of $X$.

Note that $C(2, z)$ is true if and only if $z \leq e + f$.

We say that $B(t)$ (resp. $C(t)$) is true if $B(t, z)$ (resp. $C(t, z)$) is true for all $z$. As in [8] the assertions $B(t, z)$ and $C(t, z)$ are motivated by Lemma 1 and Remark 2.

Using the Differential Horace Lemma it is quite easy to get $A(t)$ if we know $B(t)$. A key step is to get $B(t)$ knowing that $C(t)$ is true. Indeed, for $z \leq \lceil h^0(\mathcal{L}[t])/(n+1)\rceil$ the integer $z-e-f$ usually is much smaller than $\lfloor h^0(\mathcal{L}[t-2])/(n+1)\rfloor$ and hence it should be "easy" to prove that $h^1(\mathcal{I}_W \otimes \mathcal{L}[t-2]) = 0$. But of course, $t-2 < t$ and so to use this strategy we need to prove that $\lfloor h^0(\mathcal{L}[t-2])/(n+1)\rfloor - (z-e-f)$ is very large (depending on $t$). For $t = 2$ we also need another trick. Then we prove $C(3)$. Then Lemmas 2, 3, and 4 give the case $t \geq 4$ by induction on $t$.

**Lemma 2** *Assume that the e-secant variety of $(Y, \mathcal{L})$ has dimension $ne-1$. If $t \geq 2$ and $B(t, z)$ is true, then $A(t, z)$ is true.*

***Proof*** Let $Z \subset X$ be a general union of $z$ double points.

First assume $z \geq e + f$. Let $Z' \subset X$ be a general union of $z-e-f$ double points. Fix a general $S \subset H$ such that $\#S = e + f$ and write $S = S' \cup S''$ with $\#S' = e$ and $\#S'' = 2$. Set $A := (2S', H) \cup S''$ and $B := S' \cup (2S'', H)$. Note that $\deg(A) = \alpha$. Since $\dim \sigma_e(Y) = en - 1$, $h^1(H, \mathcal{I}_{(2S',H)} \otimes \mathcal{L}[t]_{|H}) = 0$. Since $S''$ is general in $H \cong Y$ and $\deg(A) = \alpha$, $h^i(H, \mathcal{I}_{A,H} \otimes \mathcal{L}[t]_{|H}) = 0$, $i = 0, 1$. The Differential Horace Lemma (Remark 1) gives $h^i(\mathcal{I}_Z \otimes \mathcal{L}[t]) = h^i(\mathcal{I}_{Z' \cup S' \cup (2S'',H)} \otimes \mathcal{L}[t-1])$ for $i = 0, 1$. Thus $B(t, z)$ implies $A(t, z)$ if $z \geq e + f$.

Now assume $z \leq e + f - 1$. The proof of the case $z \geq e + f$ works taking $Z' = \emptyset$, $\#S' = \min\{e, z\}$ and $\#S'' = z - \#S'$. □

**Lemma 3** *Assume $\alpha > n^2$, $t \geq 3$ and that the e-secant variety of $(Y, \mathcal{L})$ has dimension $ne-1$. Take $z \leq \lceil (t+1)\alpha/(n+1)\rceil$. If $C(t, z)$ is true and either $z \leq e + f$ or $A(t-2, z-e-f)$ is true, then $B(t, z)$ and $A(t, z)$ are true.*

***Proof*** By Remark 3 it is sufficient to check all positive integers $z$ such that $(n+1)z \leq (t+1)\alpha + n$. By Lemma 2 it is sufficient to prove $B(t, z)$. By the definition of $B(t, z)$ we may assume $z \geq e + f$. Let $W \subset X$ be a general union of $z-e-f$ double points. Take a general $S \subset H$ such that $\#S = e + f$ and write $S = S' \cup S''$ with $\#S' = e$ and $\#S'' = f$. By assumption $h^0(\mathcal{I}_W \otimes \mathcal{L}[t-2]) \leq \max\{0, (t-1)\alpha - \deg(W)\}$, i.e. either $h^0(\mathcal{I}_W \otimes \mathcal{L}[t-2]) = 0$ or $h^1(\mathcal{I}_W \otimes \mathcal{L}[t-2]) = 0$. Assume that $B(t, z)$ fails. Hence $h^1(\mathcal{I}_{W \cup (2S'',H) \cup S'} \otimes \mathcal{L}[t-1]) > 0$.

(a) Assume $h^1(\mathcal{I}_W \otimes \mathcal{L}[t-2])) = 0$. Since $f \leq e$, Remark 3 gives $\dim \sigma_f(Y) = nf - 1$. Thus $h^1(H, \mathcal{I}_{(2S'',H),H} \otimes \mathcal{L}[t-1]_{|H}) = 0$. Thus the residual exact sequence of $H$ gives $h^1(\mathcal{I}_{W \cup (2S'',H)} \otimes \mathcal{L}[t-1]) = 0$. Let $a$ be the maximal integer such that $h^1(\mathcal{I}_{W \cup (2S'',H) \cup A} \otimes$

$\mathcal{L}[t-1]) = 0$, where $A$ is a general subset of $H$ with cardinality $a$. By assumption $a < e$. Take a general $p \in H$. The definition of $a$ gives $h^1(\mathcal{I}_{W\cup(2S'',H)\cup A\cup\{p\}} \otimes \mathcal{L}[t-1]) > 0$. Since $h^1(\mathcal{I}_{W\cup(2S'',H)\cup A} \otimes \mathcal{L}[t-1]) = 0$, we see that $H$ is in the base locus of $|\mathcal{I}_{W\cup(2S'',H)\cup A} \otimes \mathcal{L}[t-1]|$, i.e. (since $\mathrm{Res}_H(W\cup(2S'',H)\cup A) = W$ and $h^1(\mathcal{I}_W \otimes \mathcal{L}[t-2]) = 0$), $h^0(\mathcal{I}_{W\cup(2S'',H)\cup A} \otimes \mathcal{L}[t-1]) = (t-1)\alpha - \deg(W)$. We get $a + nf = \alpha$, which is false because $a < e$, $e \ge f$ (Remark 4) and $\alpha = ne + f$.

(b) Assume $h^0(\mathcal{I}_W \otimes \mathcal{L}[t-2]) = 0$. We get $(n+1)(z-e-f) \ge (t-1)\alpha$. By assumption $(n+1)z \le (t+1)\alpha + n$. Hence $2\alpha \le (n+1)e + (n+1)f + n$. We have $ne + f = \alpha$. Thus $\alpha \le nf + n \le n^2$, a contradiction. □

**Lemma 4** *Fix an integer $t \ge 3$ and assume $A(t-2)$. Then $C(t)$ is true.*

***Proof*** Fix a positive integer $z$ for which we want to prove $C(t, z)$. If $z < e + f$, then $C(t, z)$ is true. Now assume $z \ge e + f$ and let $W \subset X$ be a general union of $z - e - f$ double points of $X$. If $\deg(W) \ge h^0(\mathcal{L}[t-2])$, we get $h^0(\mathcal{I}_W \otimes \mathcal{L}[t-2]) = 0$ by $A(t-2)$. If $\deg(W) < h^0(\mathcal{L}[t-2])$, then $h^1(\mathcal{I}_W \otimes \mathcal{L}[t-2]) = 0$, i.e. $h^0(\mathcal{I}_W \otimes \mathcal{L}[t-2]) = (t-1)\alpha - \deg(W)$. Thus $C(t, z)$ is true. □

***Proof of Theorem 2*** By Remark 3 it is sufficient to test all positive integers $z$ such that $(n+1)z \le (t+1)\alpha + n$.

**Outline of the Proof** We start with the proof of some numerical inequalities. Recall that $B(t, z)$ implies $A(t, z)$ for $t \ge 3$ (Lemma 3). We first prove the theorem for $t = 2$ without using Lemma 3 (step (a)). Then we take $t = 3$ and prove $C(3)$ and $B(3)$ (step (b)). Then for $t \ge 4$ we prove the theorem by induction on $t$ using that $A(t-2)$ is proved.

Let $S \subset H$ (resp. $S' \subset H$, resp. $S''$) be a general subset of $H$ with cardinality $e$ (resp. $f$, resp. $e - f$). Let $Z \subset X$ be a general union of $z$ double points of $X$. By the Terracini Lemma it is sufficient to prove that either $h^0(\mathcal{I}_Z \otimes \mathcal{L}[t]) = 0$ or $h^1(\mathcal{I}_Z \otimes \mathcal{L}[t]) = 0$. Since $f \le n-1$, $f + ne = \alpha$ and $\alpha \ge n^2 - 1$, we have $f \le e$. Thus the $f$-secant variety of the pair $(Y, \mathcal{L})$ has dimension $fn - 1$ (Remark 3). Note that $h^i(Y, \mathcal{I}_{(2A,Y)\cup B} \otimes \mathcal{L}) = 0$, $i = 0, 1$, where $A$ is a general subset of $Y$ with cardinality $e$ and $B$ is a general subset of $Y$ with cardinality $f$. Set $\Delta := (t+1)\alpha + n - z(n+1)$ and $w := z - (t-1)e - f$.

***Claim 1*** We have

$$\Delta + z \ge tf + e + n \tag{2}$$

***Proof of Claim 1*** We have $\Delta + (n+1)z = (t+1)\alpha + n$. Thus $(n+1)\Delta + (n+1)z = n\Delta + (t+1)\alpha + n$. Since $\alpha = ne + f$, to prove Claim 1 it is sufficient to prove that $(t+1)ne + (t+1)f - n \ge (n+1)tf + (n+1)e + n(n+1)$, i.e. $\varphi_n(t) \ge 0$, where $\varphi_n(t) := (t+1)ne + (t+1)f - (n+1)tf - (n+1)e - n(n+2)$. We have $\varphi_n(2) = (2n-1)ne + (1-2n)f - n(n+2)$. Since $f \le e$, $\varphi_n(2) \ge (2n-1)(n-1)e - n(n+2)$. Since $\alpha \ge n^2$, $e \ge n$. Since $n \ge 3$, $\varphi_n(2) \ge 0$. The derivative $\varphi_n'(t)$ of $\varphi_n(t)$ is $ne - (n+1)f$. Since $f \le n-1$, $\varphi_n'(t) \ge 0$ if $e \ge n$, i.e. if $\alpha \ge n^2$. □

***Claim 2*** We have $nf + n(w-e) + e \le \alpha$.

***Proof of Claim 2*** By Remark 4 we may assume $w \ge e$, i.e. $z \ge te + f$. Since $w = z - (t-1)e - f$, $w - e = z - te - f$. Thus $nf + n(w-e) + e = (n+1)z - z - t\alpha + tf + 1$. Since $(n+1)z = (t+1)\alpha + n - \Delta$, we have $nf + n(w-e) + e \le \alpha$ by (2). □

(a) Assume $t = 2$. We assume $z \geq e + f$, since the case $z < e + f$ only requires a small modification and it is never a critical case $z \in \{\lfloor 3\alpha/(n+1) \rfloor, \lceil 3\alpha/(n+1) \rceil\}$. Set $w := z - e - f$. We write $E \cup Z'$ with $E \cap Z' = \emptyset$, $E$ union of $w - e - f$ general double points and $Z'$ a general union of $e + f$ double points. We use Differential Horace Lemma (Remark 1) $e + f$ times to the connected of $Z'$ with respect to the general subset $S'$ of $H$. To prove that $h^0(\mathcal{I}_Z \otimes \mathcal{L}[2]) = \max\{0, 3\alpha - (n+1)z\}$ it is sufficient to prove that $h^0(\mathcal{I}_{E\cup S \cup (2S',H)} \otimes \mathcal{L}[1]) = \max\{0, 2\alpha - (n+1)(z - e - f) - nf - e)\}$. Since $e \geq n \geq (n+1)z - 3\alpha$ and $S$ is general in $H$, it is sufficient to prove that $h^1(\mathcal{I}_{E\cup(2S',H)} \otimes \mathcal{L}[1]) = 0$ and that $h^0(\mathcal{I}_E \otimes \mathcal{L}[0]) \leq \max\{0, h^0(\mathcal{I}_{E\cup(2S',H)} \otimes \mathcal{L}[1]) - e\}$. The last inequality is critical for the proof (see Claim 4) and it fails in Example 1 with $n = 2$, $z = 2a + 1$, $e = a$, $f = 0$, because $h^0(\mathcal{I}_E \otimes \mathcal{L}[0]) = 1$ and (assuming $h^1(\mathcal{I}_{E\cup(2S',H)} \otimes \mathcal{L}[1]) = 0$) $h^0(\mathcal{I}_{E\cup(2S',H)} \otimes \mathcal{L}[1]) - e = 4a + 2 - 3a - 2 - a = 0$.

***Claim 3*** Either $h^1(\mathcal{I}_{E\cup(2S',H)} \otimes \mathcal{L}[1]) = 0$ or $h^0(\mathcal{I}_{E\cup(2S',H)} \otimes \mathcal{L}[1]) = 0$.

***Proof of Claim 3*** Take a general $S_3 \subset H$ such that $\#S_3 = w - e$ and a general $S_4 \subset H \setminus S$ such that $\#S_4 = e$. By the Differential Horace Lemma to prove Claim 3 it is sufficient to prove that $h^1(\mathcal{I}_{2S_4,H} \otimes \mathcal{L}[0]) = 0$ and either $h^1(H, \mathcal{I}_{(2S',H)\cup(2S_3,H)\cup S_4} \otimes \mathcal{L}[1]_{|H}) = 0$ or $h^0(H, \mathcal{I}_{(2S',H)\cup(2S_3,H)\cup S_4} \otimes \mathcal{L}[1]_{|H}) = 0$. We have $h^1(\mathcal{I}_{2S_4,H} \otimes \mathcal{L}[0]) = h^1(H, \mathcal{I}_{(2S_4,H)} \otimes \mathcal{L}) = 0$ (Remark 4). We have $h^1(H, \mathcal{I}_{(2S',H)\cup(2S_3,H)} \otimes \mathcal{L}[1]_{|H}) = 0$, because $\#S' + \#S_3 = w - e + f \leq e$ by Claim 2. Since $S_4$ is general in $H$, either $h^1(H, \mathcal{I}_{(2S',H)\cup(2S_3,H)\cup S_4} \otimes \mathcal{L}[1]_{|H}) = 0$ or $h^0(H, \mathcal{I}_{(2S',H)\cup(2S_3,H)\cup S_4} \otimes \mathcal{L}[1]_{|H}) = 0$. □

If $h^0(\mathcal{I}_{E\cup(2S',H)} \otimes \mathcal{L}[1]) = 0$, then $h^0(\mathcal{I}_Z \otimes \mathcal{L}[2]) = 0$, proving the theorem in this case. Thus we may assume $h^1(\mathcal{I}_{E\cup(2S',H)} \otimes \mathcal{L}[1]) = 0$, i.e. $h^0(\mathcal{I}_{E\cup(2S',H)} \otimes \mathcal{L}[1]) = 2\alpha - \deg(E) - nf$.

***Claim 4*** $h^0(\mathcal{I}_E \otimes \mathcal{L}[0]) \leq \max\{0, h^0(\mathcal{I}_{E\cup(2S',H)} \otimes \mathcal{L}[1]) - e\}$.

***Proof of Claim 4*** We saw that we may assume $h^0(\mathcal{I}_{E\cup(2S',H)} \otimes \mathcal{L}[1]) = 2\alpha - \deg(E) - nf$. Since $E$ is a general union of some, $\gamma$, double points of $X$, $h^0(\mathcal{I}_E \otimes \mathcal{L}[0]) = h^0(Y, \mathcal{I}_U \otimes \mathcal{L})$, where $U$ is a general union of $\gamma$ double points of $Y$. First assume $\gamma \leq e$. The assumption on $(Y, \mathcal{L})$ gives $h^0(Y, \mathcal{I}_U \otimes \mathcal{L}) = \alpha - n\gamma$ (Remark 3). We have $\deg(E) = (n+1)\gamma$. Thus in this case it is sufficient to check that $\alpha \geq \gamma + nf$. Since $\gamma \leq e$, it is sufficient to use that $ne + f \geq e + nf$ (Remark 4). Now assume $\gamma > e$. A general union $\Gamma$ of $e$ double points of $Y$ and $f$ points of $Y$ satisfies $h^i(Y, \mathcal{I}_\Gamma \otimes \mathcal{L}) = 0$, $i = 0, 1$. Thus $h^0(Y, \mathcal{I}_U \otimes \mathcal{L}) = 0$ if $\gamma \geq e + f$, while if $e < \gamma < \gamma + f$, then $h^0(Y, \mathcal{I}_U \otimes \mathcal{L}) \leq f - (\gamma - e)$. If $e < \gamma < \gamma + f$ we have $h^0(\mathcal{I}_{E\cup(2S',H)} \otimes \mathcal{L}[1]) = 2\alpha - (n+1)\gamma - nf = \alpha - \gamma - (n-1)f \geq \alpha - e - nf - 1$. It is sufficient to have $\alpha \geq e + (n+1)f - 2$. Since $\alpha = ne + f$, it is sufficient to have $(n-1)e + 2 \geq nf$, i.e. $n(n-1)e + 2n \geq n^2 f$. Since $f \leq (n-1)$, it is sufficient to have $ne + 2n/(n-1) \geq n^2$, which is true for $\alpha > n^2$. □

Claims 3 and 4 prove the case $t = 2$.

(b) In this step we prove $C(3, z)$. We may assume $z \geq e + f$. Let $W \subset X$ be a general union of $w' := z - e - f$ double points. We need to prove that $h^0(\mathcal{I}_W \otimes \mathcal{L}[1]) \leq \max\{0, 3\alpha - \deg(W)\}$. If $w' \geq 2e + f$, using twice the Differential Horace Lemma with respect to $H$ we get $h^0(\mathcal{I}_W \otimes \mathcal{L}[1])) = 0$. If $w' \leq e$ using once the usual Horace Lemma we get $h^1(\mathcal{I}_W \otimes \mathcal{L}[1]) = 0$. If $e \leq w' \leq 2e$, using twice the usual Horace Lemma we get $h^1(\mathcal{I}_W \otimes \mathcal{L}[1]) \leq 2e - w'$, which is enough to get $C(3, z)$. If $w' > 2e$ we get $C(3, z)$ in the following way. We first degenerate $W$ to $W' \cup W''$ with $W' \cap W'' = \emptyset$, $W'$ a general union of $e$ double points of $X$ and

$W''$ a general union of $w' - 2e$ double points of $X$ with $S_5 := W''_{\mathrm{red}} \subset H$. Since the $e$-secant variety of $(Y, \mathcal{L})$ has the expected dimension, $h^0(H, \mathcal{I}_{W'' \cap H, H} \otimes \mathcal{L}[1]_{|H}) \le \max\{0, f + 2e - w'\}$. Thus the residual exact sequence of $H$ gives $h^0(\mathcal{I}_W \otimes \mathcal{L}[1]) \le \max\{0, f + 2e - w'\} + h^0(\mathcal{I}_{W' \cup S_5} \otimes \mathcal{L}[0])$. We have $h^0(\mathcal{I}_{W' \cup S_5} \otimes \mathcal{L}[0]) = h^0(H, \mathcal{I}_{W' \cap H \cup S_5} \otimes \mathcal{L}[0]_{|H}) = 0$, because $e \ge f$.

(c) Step (b) and Lemma 3 proves $A(3)$. Thus, we may assume $t \ge 4$ and that $A(x)$ is true for $2 \le x < t$. By Lemma 3 it is sufficient to prove $C(t)$. Since $A(t-2)$ is true, Lemma 4 gives $C(t)$. □

Sometimes we get $(X, \mathcal{L}[t])$ not secant defective even if $\dim \sigma_e(Y) \le ne - 2$, but we need to add other conditions. We give the following example in which we only assume that the $(e-1)$-secant variety of the pair $(Y, \mathcal{L})$ has dimension $n(e-1) - 1$.

**Theorem 3** *Fix integer $t \ge 2$ and $z > 0$. Assume $n \ge 3$, $\alpha \ge 2n^2 + 4n$, $t \ge 2$, $\dim \sigma_{e-1}(Y) = n(e-1) - 1$. Then the $z$-secant variety of $(X, \mathcal{L}[t])$ has the expected dimension.*

***Proof*** The proof is very similar to the one of Theorem 2, just using the integers $g := e - 1$ and $h := f + n$ instead of $e$ and $f$. The stronger assumption on $\alpha$ comes from the inequality $h \le 2n - 1$ instead of the inequality $f \le n - 1$. □

***Proof of Theorem 1*** We have $n := \dim X = n_1 + n_2 + k$.

First assume $k = 3$. We apply Theorem 2 to $Y := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ with $\alpha = \binom{n_1+d_1}{n_1}\binom{n_2+d_2}{n_3}$. Since $\alpha$ is an increasing function of $d_1$ and $d_2$, it is sufficient to check that $\alpha > n^2$ if $d_1 = d_2 = 3$. In this case $\alpha - n^2 - 1 = \binom{n_1+3}{3}\binom{n_2+3}{3} - (n_1 + n_2 + 1) - 1$. Taking the derivatives with respect to $n_1$ and $n_2$ we see that it is sufficient to check the case $n_1 = n_2 = 1$. In this case $n = 3$ and $\alpha = 16$.

Now assume $k > 3$ and that $Y := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{k-1}$ is not secant defective. We apply Theorem 2 to $Y$ with $\alpha = \binom{n_1+d_1}{n_1}\binom{n_2+d_2}{n_3}(d_3+1)\cdots(d_{k-1}+1) \ge 3^{k-1}\binom{n_1+d_1}{n_1}\binom{n_2+d_2}{n_3}$. We immediately get $\alpha > n^2$. □

**Remark 5** Fix a non-degenerate embedding $Y \subset \mathbb{P}^r$ for which we know that many secant varieties have the expect dimension. Set $\mathcal{L} := \mathcal{O}_Y(1)$ and $X := Y \times \mathbb{P}^1$. Let $V \subset H^0(\mathcal{O}_Y(1))$ denote the image of the restriction map $H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(\mathcal{L})$. For any integer $t > 0$ let $V[t]$ denote the linear subspace $\pi_1^*(V) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(t))$ of $\mathcal{L}[t]$. We have $\dim V[t] = (r+1)(t+1)$ and we may use the proofs of this paper with $r + 1$ instead of $\alpha$. Hence, under the assumptions of one of the results we get the non-defectivity of the secant varieties with respect to the embedding induced by a general linear subspace of dimension $(r+1)(t+1)$ of $H^0(\mathcal{L}[t])$.

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## Declarations

**Conflicts of interest** The author has no conflict of interests.

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