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Characterization of Weakly Compact Multipliers on Left ϕ-Amenable Banach Algebras

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Abstract
Let $\mathcal{A}$ be a left ϕ-amenable Banach algebra, where ϕ denotes a nonzero character on $\mathcal{A}$. In this paper, we show that the existence of a nonzero compact or weakly compact right multiplier on $\mathcal{A}$ is equivalent to the concept of left ϕ-contractibility of $\mathcal{A}$. As an important class of Banach algebras, we employ Lau algebras. Commutative Lau algebras are left ε-amenable, where ε is the identity of $\mathcal{A}^*$; the dual space of $\mathcal{A}$. As an application, we characterize the existence of a nonzero (weakly) compact multiplier on the Fourier algebra $A(H)$ of an ultraspherical hypergroup H, and the Fourier algebra $A(G)$, where G is a locally compact group.

Keywords Banach algebra · Lau algebra · Fourier algebra · Left amenable · Locally compact hypergroup · Weakly compact multiplier

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1 Introduction

Let $\mathcal{A}$ be a Banach algebra. A bounded operator Θ on $\mathcal{A}$ is called a right multiplier (resp. left multiplier), if $\Theta(ab) = a\,\Theta(b)$ (resp. $\Theta(ab) = \Theta(a)\,b$) for all $a, b \in \mathcal{A}$. The study of the existence of nonzero compact and weakly compact multipliers on various kinds of Banach algebras is one of the most frequent subjects in harmonic analysis. Several authors investigated the existence of nonzero compact and weakly compact multipliers on the class of Banach algebras related to a locally compact group G such as the group algebra $L^1(G)$ [1, 9, 24], the Fourier algebra $A(G)$ [17], and the Lebesgue–Fourier algebra $\mathcal{L}A(G)$ [10, 11]. We refer to [2] for some results about compact and weakly compact multipliers on Banach algebras related to locally compact quantum groups and [6] for compact and weakly compact multipliers on Banach algebras related to locally compact hypergroups. See [15] for the general theory of multipliers.

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Let $\mathcal{A}$ be a Banach algebra, $\phi \in \Delta(\mathcal{A})$ be a character on $\mathcal{A}$, that is a nonzero linear multiplicative functional on $\mathcal{A}$, and $\Delta(\mathcal{A})$ be the set of all characters from $\mathcal{A}$ into the complex numbers. Let $\mathcal{A}^{**}$ be the second dual of $\mathcal{A}$. The concepts of ϕ-amenability and ϕ-contractibility of $\mathcal{A}$ are two main concepts in harmonic analysis that imply the existence kinds of topological invariant means in $\mathcal{A}^{**}$ and $\mathcal{A}$, respectively. The Banach algebra $\mathcal{A}$ is called left ϕ-amenable if there is a bounded linear functional $n_0 \in \mathcal{A}^{**}$ satisfying $\langle n_0, \phi\rangle \neq 0$ and $a\, n_0 = \langle \phi, a\rangle\, n_0$ for all $a \in \mathcal{A}$ (cf. [13, 14]). The concept of left ϕ-contractibility is significantly stronger than the left ϕ-amenability of Banach algebras. $\mathcal{A}$ is said to be left ϕ-contractible if there is $a_0 \in \mathcal{A}$ such that $\langle \phi, a_0\rangle \neq 0$ and $a\, a_0 = \langle \phi, a\rangle\, a_0$ for all $a \in \mathcal{A}$ (cf. [12] and [20, Theorem 2.1]). For detailed information see [12, 20, 21]. The concepts of ϕ-amenability and ϕ-contractibility of Banach algebras are connected to other main concepts, such as the fixed point theory [5] and the homology theory of Banach modules [20].

In this paper, we characterize the existence of a nonzero compact or weakly compact right multiplier on a left ϕ-amenable Banach algebra $\mathcal{A}$. We show that this is equivalent to left ϕ-contractibility of $\mathcal{A}$. As an important application, we prove that the existence of a nonzero compact or weakly compact positive right multiplier on a left ε-amenable Lau algebra $\mathcal{A}$ is equivalent to left ε-contractibility of $\mathcal{A}$. It was known from [16] that any commutative Lau algebra is left ε-amenable. Thus all results about left ε-amenable Lau algebras also satisfy for commutative Lau algebras. As an application of commutative Lau algebras, we study the existence of a nonzero (weakly) compact multiplier on the Fourier algebra $A(H)$ of an ultraspherical hypergroup H. This result generalizes Corollary 4.6 of [6]. Moreover, we present equivalent conditions with the existence of a nonzero (weakly) compact multiplier on the Fourier algebra $A(G)$, where G is a locally compact group.

2 Weakly Compact Multipliers on Left ϕ-Amenable Banach Algebras

Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. In the following theorem, we prove that the existence of a nonzero compact or weakly compact right multiplier on a left ϕ-amenable Banach algebra $\mathcal{A}$ is equivalent to left ϕ-contractibility of $\mathcal{A}$. Note that for an operator Θ on $\mathcal{A}$, the adjoint of Θ is an operator on $\mathcal{A}^*$ and is denoted by Θ^*. Also, for any $a \in \mathcal{A}$, the operator $R_a : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by $R_a(b) = b\, a$ for all $b \in \mathcal{A}$.

Theorem 2.1 *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \Delta(\mathcal{A})$. If $\mathcal{A}$ is left ϕ-amenable, then the following statements are equivalent.*

(i) *There exists a (weakly) compact right multiplier Θ on $\mathcal{A}$ such that $\Theta^*(\phi) \neq 0$.*
(ii) *There exists $a_0 \in \mathcal{A}$ such that $\langle \phi, a_0\rangle \neq 0$ and R_{a_0} is (weakly) compact.*
(iii) *$\mathcal{A}$ is left ϕ-contractible.*

Proof (i) $\Rightarrow$ (ii): Let $\Theta : \mathcal{A} \longrightarrow \mathcal{A}$ be a weakly compact right multiplier such that $\Theta^*(\phi) \neq 0$. So there exists $b_0 \in \mathcal{A}$ such that

$$\langle \phi, \Theta(b_0)\rangle = \langle \Theta^*(\phi), b_0\rangle \neq 0.$$

Put $a_0 := \Theta(b_0)$. So $\langle \phi, a_0\rangle \neq 0$. Consider $R_{a_0} = R_{\Theta(b_0)} : \mathcal{A} \longrightarrow \mathcal{A}$. Since Θ is weakly compact, it follows that $R_{a_0} = R_{\Theta(b_0)}$ is also weakly compact.

(ii) $\Rightarrow$ (iii): Let there be $a_0 \in \mathcal{A}$ such that $\langle \phi, a_0\rangle \neq 0$ and R_{a_0} is weakly compact. Since R_{a_0} is weakly compact, it follows that $R^{**}_{a_0}(\mathcal{A}^{**}) \subseteq \mathcal{A}$. On the other hand, $\mathcal{A}$ is left

ϕ-amenable. So there exists $m_0 \in \mathcal{A}^{**}$ such that $\langle m_0, \phi\rangle \neq 0$ and $am_0 = \langle \phi, a\rangle\, m_0$ for all $a \in \mathcal{A}$. Let $u_0 := R_{a_0}^{**}(m_0) \in \mathcal{A}$. Then

$$\langle \phi, u_0\rangle = \langle \phi, m_0 a_0\rangle = \underbrace{\langle m_0, \phi\rangle}_{\neq 0}\ \underbrace{\langle \phi, a_0\rangle}_{\neq 0} \neq 0. \tag{2.1}$$

Also, for every $a \in \mathcal{A}$ we have

$$au_0 = a(m_0 a_0) = (am_0)a_0 = (\langle \phi, a\rangle m_0)a_0 = \langle \phi, a\rangle (m_0 a_0) = \langle \phi, a\rangle u_0. \tag{2.2}$$

By (2.1) and (2.2), $\mathcal{A}$ is left ϕ-contractible.

(iii) $\Rightarrow$ (i): Let $\mathcal{A}$ be left ϕ-contractible. So there exists $u_0 \in \mathcal{A}$ such that $\langle \phi, u_0\rangle \neq 0$ and $au_0 = \langle \phi, a\rangle u_0$ for all $a \in \mathcal{A}$. Consider $\Theta = R_{u_0} : \mathcal{A} \longrightarrow \mathcal{A}$. It is clear that Θ is a right multiplier. Also, $\dim(\mathrm{Im}(R_{u_0})) = 1$, so it is weakly compact. In addition,

$$\langle \Theta^*(\phi), u_0\rangle = \langle \phi, \Theta(u_0)\rangle = \langle \phi, R_{u_0}(u_0)\rangle = \langle \phi, u_0 u_0\rangle = \underbrace{\langle \phi, u_0\rangle}_{\neq 0}\ \underbrace{\langle \phi, u_0\rangle}_{\neq 0} \neq 0,$$

as required. □

In continue, we focus on Lau algebras. It was known that a large class of Banach algebras is the class of Lau algebras. A Banach algebra $\mathcal{A}$ is called a Lau algebra or an F-algebra if $\mathcal{A}$ is a unique predual of a von Neumann algebra $\mathcal{M}$ such that the identity of $\mathcal{M}$—that will be denoted by ε—is a character on $\mathcal{A}$ [16]. Let $\mathcal{A}$ be a Lau algebra. The concepts of left ε-amenability and left ε-contractibility of $\mathcal{A}$ can be found in [16] and [12, 20, 21].

For a Lau algebra $\mathcal{A}$, the set $P(\mathcal{A})$ denotes the set of all elements $a \in \mathcal{A}$ which induces positive linear functionals on the von Neumann algebra $\mathcal{A}^*$; the dual space of $\mathcal{A}$. It is known from [23, Propositions 1.5.1 and 1.5.2] that

$$P(\mathcal{A}) = \{a \in \mathcal{A} : \langle \varepsilon, a\rangle = \|a\|\}$$

and $\mathcal{A}$ is generated by the set $P(\mathcal{A})$. It is known that $P(\mathcal{A})$ is a convex cone (cf. [23, Theorem 1.4.2]). Moreover,

$$P_1(\mathcal{A}) = \{a \in \mathcal{A} : \langle \varepsilon, a\rangle = \|a\| = 1\}$$

denotes the set of all elements $a \in \mathcal{A}$ that induces states on $\mathcal{A}^*$. It is derived from [23, Proposition 1.17.1] that $P_1(\mathcal{A})$ also generates $\mathcal{A}$.

Examples of Lau algebras include the group algebra $L^1(G)$, the measure algebra $M(G)$, the Fourier algebra $A(G)$, and the Fourier–Stieltjes algebra $B(G)$ of a locally compact group G. Moreover, the measure algebra $M(S)$ of a locally compact semi-topological semigroup S is a Lau algebra. The class of Lau algebras also includes the predual of a Hopf von Neumann algebra, and the algebras $L^1(H)$ in the case where H has a left Haar measure, $M(H)$, and Fourier hypergroup $A(H)$ of a locally compact hypergroup H.

In the following, we prove that the existence of a nonzero compact or weakly compact positive right multiplier on a left ε-amenable Lau algebra $\mathcal{A}$ is equivalent to left ε-contractibility of $\mathcal{A}$.

Theorem 2.2 *Let $\mathcal{A}$ be a Lau algebra. If $\mathcal{A}$ is left ε-amenable, then the following statements are equivalent.*

(i) *There exists a (weakly) compact positive right multiplier Θ on $\mathcal{A}$ such that $\Theta^*(\varepsilon) \neq 0$.*
(ii) *There exists a nonzero (weakly) compact positive right multiplier on $\mathcal{A}$.*
(iii) *There exists $a_0 \in P_1(\mathcal{A})$ such that R_{a_0} is (weakly) compact.*

(iv) *$\mathcal{A}$ is left ε-contractible.*

Proof (i) $\Rightarrow$ (ii): Suppose that Θ is a weakly compact positive right multiplier on $\mathcal{A}$ such that $\Theta^*(\varepsilon) \neq 0$. So there exists $a_0 \in \mathcal{A}$ such that

$$\langle \varepsilon, \Theta(a_0)\rangle = \langle \Theta^*(\varepsilon), a_0\rangle \neq 0.$$

Thus $\Theta(a_0) \neq 0$, and therefore Θ is nonzero.

(ii) $\Rightarrow$ (iii): Let $\Theta : \mathcal{A} \longrightarrow \mathcal{A}$ be a nonzero weakly compact positive right multiplier. So there exists $b_0 \in \mathcal{A}$ such that $\Theta(b_0) \neq 0$. Since $\mathcal{A}$ is a Lau algebra, it follows that there exist $b_1^+, b_1^-, b_2^+, b_2^- \in P(\mathcal{A})$ such that $b_0 = (b_1^+ - b_1^-) + i(b_2^+ - b_2^-)$, where $b_1 := b_1^+ - b_1^-$ and $b_2 := b_2^+ - b_2^-$ are self-adjoint elements of $\mathcal{A}$. Thus

$$\Theta(b_0) = \Theta\left((b_1^+ - b_1^-) + i(b_2^+ - b_2^-)\right) = \left(\Theta(b_1^+) - \Theta(b_1^-)\right) + i\left(\Theta(b_2^+) - \Theta(b_2^-)\right).$$

It is clear that at least one of the elements $\Theta(b_1^+)$, $\Theta(b_1^-)$, $\Theta(b_2^+)$, or $\Theta(b_2^-)$ is nonzero. Let $\Theta(b_1^+) \neq 0$. Take $a_0 := \frac{\Theta(b_1^+)}{\|\Theta(b_1^+)\|} \neq 0$. Since Θ is positive, it follows that $\langle \varepsilon, \Theta(b_1^+)\rangle = \|\Theta(b_1^+)\|$. So

$$\|a_0\| = \frac{\|\Theta(b_1^+)\|}{\|\Theta(b_1^+)\|} = 1 = \frac{\langle \varepsilon, \Theta(b_1^+)\rangle}{\|\Theta(b_1^+)\|} = \left\langle \varepsilon, \frac{\Theta(b_1^+)}{\|\Theta(b_1^+)\|} \right\rangle = \langle \varepsilon, a_0\rangle.$$

Thus $a_0 \in P_1(\mathcal{A})$. Consider $R_{a_0} : \mathcal{A} \longrightarrow \mathcal{A}$. Since Θ is weakly compact, it follows that R_{a_0} is also weakly compact.

(iii) $\Rightarrow$ (iv): This implication holds by a similar argument to the implication (ii) $\Rightarrow$ (iii) of Theorem 2.1.

(iv) $\Rightarrow$ (i): Suppose that $\mathcal{A}$ is a left ε-contractible Lau algebra. So there exists $b_0 \in \mathcal{A}$ such that

$$\langle \varepsilon, b_0\rangle \neq 0, \quad ab_0 = \langle \varepsilon, a\rangle b_0 \qquad (a \in \mathcal{A}).$$

By a similar argument as the proof of [16, Theorem 4.1], $b_0 \in \mathcal{A}$ induces a nonzero element $a_0 \in P(\mathcal{A})$ such that

$$aa_0 = \langle \varepsilon, a\rangle a_0 \qquad (a \in \mathcal{A}).$$

Now, we easily can define the operator Θ as follows

$$\begin{cases} \Theta : \mathcal{A} \longrightarrow \mathcal{A} \\ \qquad a \longmapsto \Theta(a) := aa_0 \qquad (a \in \mathcal{A}). \end{cases}$$

An easy calculation shows that Θ is a bounded linear right multiplier. Furthermore,

$$\langle \Theta^*(\varepsilon), a_0\rangle = \langle \varepsilon, \Theta(a_0)\rangle = \langle \varepsilon, a_0^2\rangle = \|a_0\|^2 \neq 0.$$

Moreover, since $a_0 \in P(\mathcal{A})$, it follows that $\Theta(P(\mathcal{A})) \subseteq P(\mathcal{A})$, and therefore Θ is positive. It is also straightforward that $\dim(\mathrm{Im}(\Theta)) = 1$, so it is weakly compact. □

It is derived from [16] that any commutative Lau algebra is left ε-amenable. So Theorem 2.2 is also valid for commutative Lau algebras. Also, by combining Theorems 2.1 and 2.2, we get the following result for left ε-amenable, and especially, for commutative Lau algebras.

Corollary 2.3 *Let $\mathcal{A}$ be a commutative Lau algebra. Then the following statements are equivalent.*

(i) *There exists a (weakly) compact multiplier Θ on $\mathcal{A}$ such that $\Theta^*(\varepsilon) \neq 0$.*
(ii) *There exists a nonzero (weakly) compact positive multiplier on $\mathcal{A}$.*
(iii) *There exists $a_0 \in \mathcal{A}$ with $\langle \varepsilon, a_0 \rangle \neq 0$ such that R_{a_0} is (weakly) compact.*
(iv) *$\mathcal{A}$ is left ε-contractible.*

3 Application to Fourier Algebras of Ultraspherical Hypergroups

As an application of the previous section, we focus on the Fourier algebra of an ultraspherical hypergroup. For more information, the reader can refer to [3, 18, 19]. It is appropriate to note that there are new researches in the area of hypergroups. See for example [6] and [22].

Let H be a hypergroup. Note that the hypergroups in this paper are all assumed to possess left Haar measure. Let $\widetilde{H}$ denote the equivalence classes of all representations of H and λ denote the left regular representation of H on $L^2(H)$ as follows

$$\begin{cases} \lambda : H \longrightarrow \mathcal{B}(L^2(H)) \\ \quad x \longmapsto \begin{cases} \lambda(x) : L^2(H) \longrightarrow L^2(H) \\ \quad g \longmapsto \begin{cases} \lambda(x)(g) : H \longrightarrow \mathbb{C} \\ \quad y \longmapsto \lambda(x)(g)(y) = g(\check{x} * y), \end{cases} \end{cases} \end{cases}$$

where $x, y \in H$ and $g \in L^2(H)$. Let $C^\star(H)$ denote the full $C^\star$-algebra of H and $C^\star_\lambda(H)$ denote the reduced $C^\star$-algebra of H. The von Neumann algebra associated to λ of H, that is $[\lambda(L^1(H))]''$ (the bicommutant of $\lambda(L^1(H))$) in $\mathcal{B}(L^2(H))$ is called the von Neumann algebra of H and is denoted by $VN(H)$. The Banach space dual of $C^\star(H)$ is called the Fourier–Stieltjes space and is denoted by $B(H)$ and the Banach space dual of the reduced $C^\star$-algebra $C^\star_\lambda(H)$ is denoted by $B_\lambda(H)$. It is known that $B_\lambda(H)$ is a closed subspace of $B(H)$. The closed subspace spanned by $\{f * \tilde{f} : f \in C_c(H)\}$ in $B_\lambda(H)$ is called the Fourier space of H and is denoted by $A(H)$ (cf. [18, 19]). The Banach space dual of $A(H)$ is the von Neumann algebra $VN(H)$ by [18, Theorem 2.19]. The duality between $A(H)$ and $VN(H)$ is denoted by $\langle T, \psi \rangle$ for all $T \in VN(H)$ and $\psi \in A(H)$. In particular,

$$\langle \lambda(x), \psi \rangle = \psi(x) \qquad (x \in H,\ \psi \in A(H)). \tag{3.1}$$

It is appropriate to note that in general, $A(H)$ might be not an algebra under the pointwise multiplication. A hypergroup H is called a Fourier hypergroup if the Fourier space $A(H)$ forms an algebra with the pointwise multiplication, and there exists a norm on $A(H)$ which is equivalent to the original norm with respect to which $A(H)$ forms a Banach algebra. H is called a regular Fourier hypergroup if $A(H)$ is a Banach algebra with its original norm and pointwise multiplication. In other words, H is a regular Fourier hypergroup if it is a Fourier hypergroup with its regular Fourier norm.

In continue, we focus on an important regular Fourier hypergroup. Let G be a locally compact group, $\pi : C_c(G) \longrightarrow C_c(G)$ be a spherical projector defined in [19], and $\pi^* : M(G) \longrightarrow M(G)$ be the transpose of π. Also, let $\mathcal{O}_x = \text{supp}(\pi(\delta_x)) = \text{supp}(\pi^*(\delta_x))$, where δ_x is the Dirac measure at x. A function $f \in C_c(G)$ is called π-radial if $\pi(f) = f$. Similarly, a measure $\mu \in M(G)$ is called π radial if $\pi^*(\mu) = \mu$. Let $H = \{\mathcal{O}_x : x \in G\}$ with the natural quotient topology under the quotient map $p : G \longrightarrow H$. A general element of H is denoted by $\dot{x}$ instead of $\mathcal{O}_x$. It is shown in [19, Theorem 2.12] that the space $H = \{\mathcal{O}_x : x \in G\} = \{\dot{x} : x \in G\}$ forms a hypergroup, which is called a spherical hypergroup. A spherical hypergroup H is an ultraspherical hypergroup if the modular function of the group G is π-radial. The Fourier space $A(H)$ of an ultraspherical hypergroup H is a Banach algebra under the pointwise multiplication by [19, Corollary 3.11].

Let H be an ultraspherical hypergroup and $\dot{x} \in H$. The functional

$$\begin{cases} \omega_{\dot{x}} : A(H) \longrightarrow \mathbb{C} \\ \qquad\quad \psi \longmapsto \omega_{\dot{x}}(\psi) = \psi(\dot{x}) \end{cases}$$

is a nonzero homomorphism, and the map

$$\begin{cases} H \longrightarrow \Delta(A(H)) \\ \dot{x} \longmapsto \omega_{\dot{x}} \end{cases}$$

is a homeomorphism from H onto $\Delta(A(H))$; the set of all characters from $A(H)$ into the complex numbers, when $\Delta(A(H))$ is equipped with the Gelfand topology. Moreover, the Banach algebra $A(H)$ is regular, semisimple, and Tauberian by [19, Theorem 3.13]. It is obvious that the regular Fourier hypergroup $A(H)$ is a commutative Banach algebra under the pointwise multiplication. Note that by (3.1) we can write

$$\langle \lambda(\dot{x}), \psi \rangle = \psi(\dot{x}) = \langle \omega_{\dot{x}}, \psi \rangle \qquad (\dot{x} \in H,\ \psi \in A(H)).$$

So we get

$$\langle \omega_{\dot{e}}, \psi \rangle = \psi(\dot{e}) = \langle \lambda(\dot{e}), \psi \rangle = \langle I, \psi \rangle \qquad (\psi \in A(H)).$$

In other words, $\omega_{\dot{e}} = I$, where $I \in VN(H)$ is the identity operator. Now, let $\psi, \Psi \in A(H)$ be two arbitrary elements. So

$$\langle I, \psi\Psi \rangle = \langle \omega_{\dot{e}}, \psi\Psi \rangle = (\psi\Psi)(\dot{e}) = \psi(\dot{e})\,\Psi(\dot{e}) = \langle \omega_{\dot{e}}, \psi \rangle\, \langle \omega_{\dot{e}}, \Psi \rangle = \langle I, \psi \rangle\, \langle I, \Psi \rangle.$$

Therefore $A(H)$ is a Lau algebra. In other words, $A(H)$ is a commutative Lau algebra. So by [16] $A(H)$ is left $\omega_{\dot{e}}$-amenable.

In the following, we characterize the existence of a nonzero (weakly) compact multiplier on the Fourier algebra $A(H)$ of an ultraspherical hypergroup H.

Theorem 3.1 *Let H be an ultraspherical hypergroup. Then the following statements are equivalent.*

(i) *There exists a (weakly) compact multiplier Θ on $A(H)$ such that $\Theta^*(\omega_{\dot{e}}) \neq 0$.*
(ii) *There exists a nonzero (weakly) compact positive multiplier on $A(H)$.*
(iii) *There exists $\psi_0 \in A(H)$ with $\langle \psi_0, \dot{e} \rangle \neq 0$ such that R_{ψ_0} is (weakly) compact.*
(iv) *$A(H)$ is left $\omega_{\dot{e}}$-contractible.*
(v) *H is discrete.*

Proof $A(H)$ is a commutative and left $\omega_{\dot{e}}$-amenable Lau algebra. Therefore by Corollary 2.3 we obtain that the statements (i), (ii), (iii), and (iv) are equivalent.

(iv) $\Rightarrow$ (v): Let $A(H)$ be left $\omega_{\dot{e}}$-contractible. So there exists $\psi_0 \in A(H)$ such that

$$\langle I, \psi_0 \rangle = \langle \omega_{\dot{e}}, \psi_0 \rangle \neq 0, \qquad \psi\psi_0 = \langle I, \psi \rangle\, \psi_0 = \langle \omega_{\dot{e}}, \psi \rangle\, \psi_0$$

for all $\psi \in A(H)$. Similar to the proof of Theorem 2.2, $\psi_0 \in A(H)$ induces a nonzero element $\Psi_0 \in P(A(H))$ satisfying $\psi\Psi_0 = \langle I, \psi \rangle\, \Psi_0 = \langle \omega_{\dot{e}}, \psi \rangle\, \Psi_0 = \psi(\dot{e})\, \Psi_0$ for all $\psi \in A(H)$. Therefore it is clear that

$$\psi \frac{\Psi_0}{\|\Psi_0\|} = \psi(\dot{e}) \frac{\Psi_0}{\|\Psi_0\|} \qquad (\psi \in A(H))$$

and

$$\left\langle I, \frac{\Psi_0}{\|\Psi_0\|} \right\rangle = \left\langle \omega_{\dot{e}}, \frac{\Psi_0}{\|\Psi_0\|} \right\rangle = \left\| \frac{\Psi_0}{\|\Psi_0\|} \right\| = 1.$$

In other words, $\frac{\Psi_0}{\|\Psi_0\|}$ is a topological invariant mean on $A(H)^*$ belonging to $A(H)$. Consequently, H is discrete by [25, Theorem 4.4].

(v) $\Rightarrow$ (iv): Suppose that H is discrete. By [25, Theorem 4.4] the set of all topological invariant means on $A(H)^*$ intersects with $A(H)$. Take $\Upsilon \in A(H)$ that is also a topological invariant mean on $A(H)^*$. Thus

$$\langle \omega_{\dot{e}}, \Upsilon \rangle = \langle I, \Upsilon \rangle = \|\Upsilon\| = 1, \qquad \psi\Upsilon = \psi(\dot{e})\Upsilon = \langle \omega_{\dot{e}}, \psi \rangle \Upsilon$$

for all $\psi \in A(H)$. Hence, $A(H)$ is left $\omega_{\dot{e}}$-contractible. □

Note that Theorem 3.1 generalizes [6, Corollary 4.6].

At the end of this section, we examine nonzero (weakly) compact multipliers on the Fourier algebra $A(G)$ with G a locally compact group.

Let G be a locally compact group. Consider the Fourier algebra $A(G)$ introduced in [7]. The Fourier algebra $A(G)$ with the pointwise multiplication is a commutative Banach algebra. Furthermore, the dual of $A(G)$ is the group von Neumann algebra $VN(G)$. In fact, $A(G)$ is a Lau algebra. As a well-known fact, we have

$$\Delta(A(G)) = \{\omega_x : x \in G\},$$

where $\langle \omega_x, \psi \rangle = \psi(x)$ for all $\psi \in A(G)$. Let $e \in G$ be the identity element. It is straightforward that $\omega_e = \varepsilon \in \Delta(A(G))$ is exactly the identity $I \in VN(G)$.

As a corollary of Theorem 3.1, we have:

Corollary 3.2 *Let G be a locally compact group. Then the following statements are equivalent.*

(i) *There exists a (weakly) compact multiplier Θ on $A(G)$ such that $\Theta^*(\omega_e) \neq 0$.*
(ii) *There exists a nonzero (weakly) compact positive multiplier on $A(G)$.*
(iii) *There exists $\psi_0 \in A(G)$ with $\langle \psi_0, e \rangle \neq 0$ such that R_{ψ_0} is (weakly) compact.*
(iv) *$A(G)$ is left ω_e-contractible.*
(v) *G is discrete.*

It is derived from [17] that there exists a nonzero (weakly) compact multiplier on the Fourier algebra $A(G)$ if and only if G is discrete. The reader could compare Corollary 3.2 with this fact from [17].

4 Some Remarks and Open Problems

In this section, we focus on the condition of left ϕ-amenability of $\mathcal{A}$ in Theorem 2.1.

Let X be a Banach space and $\phi \in X^*\backslash\{0\}$. Define the following product on X:

$$ab = a\langle \phi, b \rangle \qquad (a, b \in X). \tag{4.1}$$

X equipped with this product is a Banach algebra denoted by $\mathcal{A}_\phi(X)$. We have the following proposition.

Proposition 4.1 *Let X be a Banach space and $\phi \in X^*\backslash\{0\}$. Then the following statements are equivalent.*

(i) $\mathcal{A}_\phi(X)$ *is left* ϕ*-contractible.*
(ii) $\mathcal{A}_\phi(X)$ *is left* ϕ*-amenable.*
(iii) $\dim(\mathcal{A}_\phi(X)) = 1$.

Proof The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are obvious. It is sufficient to show that (ii) implies (iii). Let $\mathcal{A}_\phi(X)$ be left ϕ-amenable. So there exists $n_0 \in \mathcal{A}_\phi(X)^{**}$ such that $\langle n_0, \phi\rangle := 1$ and $an_0 = \langle \phi, a\rangle n_0$ for all $a \in \mathcal{A}_\phi(X)$. Therefore

$$\langle \phi, a\rangle\langle n_0, f\rangle = \langle an_0, f\rangle = \langle n_0, f \cdot a\rangle \qquad (f \in \mathcal{A}_\phi(X)^*). \tag{4.2}$$

In addition, by an easy calculation, we have $f \cdot a = \langle f, a\rangle\phi$. So by (4.2) for every $a \in \mathcal{A}_\phi(X)$ and $f \in \mathcal{A}_\phi(X)^*$ we obtain

$$\langle \phi, a\rangle\langle n_0, f\rangle = \langle n_0, \langle f, a\rangle\phi\rangle = \langle f, a\rangle \underbrace{\langle n_0, \phi\rangle}_{1} = \langle f, a\rangle.$$

Consequently, $\langle \phi, a\rangle = 0\,(a \in \mathcal{A}_\phi(X))$ if and only if $\langle f, a\rangle = 0\,(a \in \mathcal{A}_\phi(X),\ f \in \mathcal{A}_\phi(X)^*)$. In other words,

$$\ker(\phi) = \bigcap_{f \in \mathcal{A}_\phi(X)^*} \ker(f). \tag{4.3}$$

Also, it is clear that $\{0\}^\perp = \{f \in \mathcal{A}_\phi(X)^* : f(0) = 0\} = \mathcal{A}_\phi(X)^*$. Thus by employing [4, Theorem 6.13] we have

$$\{0\} = \overline{\{0\}} = \bigcap_{f \in \{0\}^\perp} \ker(f) = \bigcap_{f \in \mathcal{A}_\phi(X)^*} \ker(f). \tag{4.4}$$

By (4.3) and (4.4) we get $\ker(\phi) = \{0\}$. Thus ϕ is one-to-one and therefore $\mathcal{A}_\phi(X) \cong \mathbb{C}$. Hence, $\dim(\mathcal{A}_\phi(X)) = 1$. □

Remark 4.2 The condition of left ϕ-amenability of $\mathcal{A}$ in Theorem 2.1 can not be omitted. Indeed, if $\mathcal{A}$ is not left ϕ-amenable, then the implication (ii) $\Rightarrow$ (iii) in Theorem 2.1 does not necessarily hold.

For example, consider the Banach algebra $\mathcal{A}_\phi(X)$ equipped with the product (4.1), where X is a Banach space and $\phi \in X^*\backslash\{0\}$. Also, let $\mathcal{A}_\phi(X)$ be a reflexive Banach space with $\dim(\mathcal{A}_\phi(X)) \geq 2$. Then by Proposition 4.1, $\mathcal{A}_\phi(X)$ is not left ϕ-amenable. Now, since $\mathcal{A}_\phi(X)$ is a reflexive Banach space, it follows that the closed unit ball of $\mathcal{A}_\phi(X)$ is weakly compact (cf. [8, Theorem 3.31]). Therefore the identity operator on $\mathcal{A}_\phi(X)$, that we denote by $\mathrm{id}_{\mathcal{A}_\phi(X)}$, is also weakly compact. Choose an element $a_0 \in \mathcal{A}_\phi(X)$ such that $\langle \phi, a_0\rangle = 1$ and consider the operator $R_{a_0}(a) = aa_0\ (a \in \mathcal{A}_\phi(X))$ on $\mathcal{A}_\phi(X)$. So for every $a \in \mathcal{A}_\phi(X)$ we have

$$R_{a_0}(a) = aa_0 = a\,\underbrace{\langle \phi, a_0\rangle}_{1} = a = \mathrm{id}_{\mathcal{A}_\phi(X)}(a).$$

Thus R_{a_0} is a weakly compact right multiplier. Therefore (ii) in Theorem 2.1 holds. But since $\dim(\mathcal{A}_\phi(X)) \geq 2$, it follows by Proposition 4.1 that $\mathcal{A}_\phi(X)$ is not left ϕ-contractible. Thus (iii) in Theorem 2.1 does not hold.

Question 1 Can the left ϕ-amenability of $\mathcal{A}$ in Theorem 2.1 be replaced with another condition?

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