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ORIGINAL ARTICLE

On the Stability of Shear Flows in Bounded Channels, I: Monotonic Shear Flows

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Abstract

We discuss some of our recent work on the linear and nonlinear stability of shear flows as solutions of the 2D Euler equations in the bounded channel $\mathbb{T} \times [0, 1]$. More precisely, we consider shear flows $u = (b(y), 0)$ given by smooth functions $b : [0, 1] \to \mathbb{R}$. We prove linear inviscid damping and linear stability provided that b is strictly increasing and a suitable spectral condition involving the function b is satisfied. Then we show that this can be extended to full nonlinear inviscid damping and asymptotic nonlinear stability, provided that b is linear outside a compact subset of the interval $(0, 1)$ (to avoid boundary contributions which are not compatible with inviscid damping) and the vorticity is smooth in a Gevrey space. In the second article in this series we will discuss the case of non-monotonic shear flows b with non-degenerate critical points (like the classical Poiseuille flow $b : [-1, 1] \to \mathbb{R}$, $b(y) = y^2$). The situation here is different, as nonlinear stability is a major open problem. We will prove a new result in the linear case, involving polynomial decay of the associated stream function.

Keywords Euler equations · Linear inviscid damping · Monotonic shear flows

Mathematics Subject Classification (2010) 35B40 · 35Q31 · 35P25

1 Introduction

The purpose of this series of two articles is twofold. We first review some of our recent work in [22] and [19] on the linear and nonlinear stability of strictly monotonic shear flows in bounded channels. Then we will prove a new linear stability theorem in the case of non-monotonic shear flows with non-degenerate critical points.

Dedicated to Carlos Kenig, on the occasion of his 70th birthday.

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We invest the global dynamics of solutions of the two dimensional incompressible Euler equation in a bounded channel. More precisely we consider solutions $u : [0, \infty) \times \mathbb{T} \times [0, 1] \to \mathbb{R}^2$ of the equation

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \qquad \operatorname{div} u = 0, \tag{1.1}$$

with the boundary condition $u^y|_{y=0,\,1} \equiv 0$. Letting $\omega := -\partial_y u^x + \partial_x u^y$ be the vorticity field, the equation (1.1) can be written in vorticity form as

$$\partial_t \omega + u \cdot \nabla \omega = 0, \qquad u = \nabla^{\perp}\psi = (-\partial_y \psi, \partial_x \psi), \tag{1.2}$$

for $(x, y) \in \mathbb{T} \times [0, 1]$, $t \geq 0$, where the stream function ψ is determined through

$$\Delta\psi = \omega \quad \text{on } \mathbb{T} \times [0, 1], \qquad \psi(x, 0) \equiv 0, \;\; \psi(x, 1) \equiv C_0, \tag{1.3}$$

and C_0 is a constant preserved by the flow. We remark that our domain is a finite, periodic channel: periodicity in x is a key condition for inviscid damping and stability, while compactness in y is a physical choice motivated by finite energy considerations.

The two dimensional incompressible Euler equation is globally well-posed for smooth initial data, by the classical results of Wolibner [40] and Yudovich [34, 35]. The long time behavior of general solutions is however very difficult to understand, due to the lack of an asymptotic relaxation mechanism.

A more realistic goal is to study the global nonlinear dynamics of solutions that are close to steady states of the 2D Euler equation. Coherent structures, such as shear flows and vortices, are particularly important in the study of the 2D Euler equation, since precise numerical simulations and physical experiments, such as those of McWilliams [27, 28], Benzi–Paladin–Patarnello [10], Branchet–Meneguzzi–Politano–Sulem [11], Santanqelo–Benzi–Leqras [32], and Bassom–Gilbert [2, 3], show that coherent structures tend to form and become the dominant feature of incompressible 2D Euler evolutions, for a long time.

In this paper we consider a perturbative regime for the Euler equation (1.1) on the bounded channel $\mathbb{T} \times [0, 1]$, with velocity field given by $(b(y), 0)) + u(x, y)$ and vorticity given by $-b'(y) + \omega$. In vorticity formulation, the system (1.2)–(1.3) is equivalent to the following evolution equation for the vorticity deviation ω,

$$\begin{cases} \partial_t \omega + b(y)\partial_x \omega - b''(y)\partial_x \psi + u \cdot \nabla \omega = 0, \\ u = (u^x, u^y) = (-\partial_y \psi, \partial_x \psi), \quad \Delta\psi = \omega, \quad \psi(t, x, 0) = \psi(t, x, 1) = 0, \\ \omega(0, x, y) = \omega_0(x, y), \quad \int_{\mathbb{T}\times[0,1]} \omega_0(x, y)\, dx dy = 0. \end{cases} \tag{1.4}$$

The normalization condition in the last line may be assumed by modifying (linearly in y) the function b, and is preserved by the flow.

Our main topic in this article is asymptotic stability. The study of stability properties of shear flows and vortices is one of the most important problems in hydrodynamics, and has a long history. Early investigations were started by Kelvin [24], Rayleigh [31], Orr [30], Taylor [33], among many others, with a focus on mode stability. Later, more detailed understanding of the general spectral properties and suitable linear decay estimates were also obtained, see Section 1.1.2 below for more references. In the direction of nonlinear results, Arnold [1] proved a general stability criteria, using the energy Casimir method, but this method does not give asymptotic information on the global dynamics. The full nonlinear asymptotic stability problem has only been investigated in recent years, starting with the remarkable work of Bedrossian–Masmoudi [7], who proved inviscid damping and global nonlinear stability in the simplest case of perturbations of the Couette flow on $\mathbb{T} \times \mathbb{R}$.

1.1 Linear Stability

We start by analyzing the linearized Euler equation associated to the system (1.4), which can be written in the form

$$\begin{cases} \partial_t \omega + b(y)\partial_x \omega - b''(y)\partial_x \psi = 0, \\ \Delta \psi = \omega, \quad \psi(t,x,0) = \psi(t,x,1) = 0, \\ \omega(0,x,y) = \omega_0(x,y), \quad \int_{\mathbb{T}\times[0,1]} \omega_0(x,y)\,dxdy = 0. \end{cases} \tag{1.5}$$

One can gain some intuition by examining a simple explicit case, corresponding to the Couette flow $b(y) = y$. In this case $b''(y) = 0$ and the linearized equation (1.5) becomes

$$\partial_t \omega + y\partial_x \omega = 0,$$

which was studied by Orr in a pioneering work [30]. To simplify the discussion, we assume $x \in \mathbb{T}$, $y \in \mathbb{R}$ (to avoid the boundary issue which is not our main concern here).

By direct calculation we have

$$\omega(t,x,y) = \omega_0(x - yt, y).$$

The stream function is given by $\Delta\psi(t,x,y) = \omega(t,x,y)$ for $(x,y) \in \mathbb{T}\times\mathbb{R}$, so in the Fourier space we have the formulas

$$\widetilde{\omega}(t,k,\xi) = \widetilde{\omega_0}(k,\xi+kt), \qquad \widetilde{\psi}(t,k,\xi) = -\frac{\widetilde{\omega_0}(k,\xi+kt)}{k^2+|\xi|^2}, \tag{1.6}$$

where $\widetilde{g}$ denotes the Fourier transform in x and y.

We make some important observations by examining these explicit formulas. Assume that ω_0 is smooth, so $\widetilde{\omega_0}(k,\xi)$ decays fast in k, ξ. Then:

(1) The main contribution comes from the frequencies $\xi = -kt + O(1)$, therefore $\widetilde{\psi}(t,k,\xi)$ decays like $|k|^{-2}\langle t\rangle^{-2}$ if $k \neq 0$. Similarly, the relations $u^x = -\partial_y\psi$ and $u^y = \partial_x\psi$ show that $\widetilde{u^x}$ decays like $|k|^{-1}\langle t\rangle^{-1}$ and $\widetilde{u^y}$ decays like $|k|^{-1}\langle t\rangle^{-2}$.

(2) It can be seen from (1.6) that the functions $\omega(t,x,y)$ and $\psi(t,x,y)$ are not uniformly smooth as $t \to \infty$, in the original variables x, y. To obtain smooth "profiles" we define

$$F(t,x,y) = \omega(t,x+ty,y), \qquad \phi(t,x,y) = \psi(t,x+ty,y). \tag{1.7}$$

Notice that $F(t,x,y) = \omega_0(x,y)$ (independent of t), while $\phi(t,x,y)$ is uniformly smooth for all t provided that ω_0 is smooth. Taking the Fourier transform in x, y, we have the formula

$$\widetilde{\phi}(t,k,\xi) = -\frac{\widetilde{\omega_0}(k,\xi)}{k^2+|\xi-kt|^2}. \tag{1.8}$$

(3) An important observation of Orr is that for $k \neq 0$ and large ξ, the normalized stream function ϕ (as well as the velocity field) may experience a *transient growth* as t approaches the "critical time" $t_c = \xi/k$ before decaying to zero. This can be seen easily from the formula (1.8). This transient growth on the linearized level turns out to be crucial for the nonlinear analysis as well, and leads to the high regularity assumptions (Gevrey spaces) that are required for the nonlinear perturbation theory.

We are now ready to state our first linear stability result, for monotonic shear flows. We assume that the background shear flow $b \in C^4([0,1])$ satisfies the following properties:

(A) For some $\vartheta_0 \in (0, 1/10]$

$$\vartheta_0 \le b'(y) \ \text{ for any } y \in [0,1] \qquad \text{and} \qquad \|b\|_{C^4[0,1]} \le 1/\vartheta_0.$$

(B) For any $k \in \mathbb{Z}\backslash\{0\}$ the associated linearized operator $L_k : L^2(0,1) \to L^2(0,1), k \in \mathbb{Z}\backslash\{0\}$, given by

$$L_k g(y) := b(y)g(y) + b''(y)\int_0^1 G_k(y,z)g(z)dz, \tag{1.9}$$

has no discrete eigenvalues, where G_k is the Green's function for the operator $-\partial_y^2 + k^2$ on $(0, 1)$ with zero Dirichlet boundary conditions, given by

$$G_k(y,z) := \frac{1}{k\sinh k}\begin{cases}\sinh(k(1-z))\sinh(ky) & \text{if } 0 \le y \le z \le 1,\\ \sinh(kz)\sinh(k(1-y)) & \text{if } 0 \le z \le y \le 1.\end{cases} \tag{1.10}$$

For any function $H(x, y)$ we let $\langle H\rangle(y)$ denote the average of H in x. Our first linear result is the following theorem of Wei–Zhang–Zhao [36] and the second author [22].

Theorem 1.1 *Assume that $b \in C^4([0,1])$ satisfies properties (A) and (B) above and $\omega_0 \in H^4(\mathbb{T}\times[0,1])$ satisfies the properties*

$$\|\omega_0\|_{H^4(\mathbb{T}\times[0,1])} \le 1, \qquad \int_{\mathbb{T}\times[0,1]} \omega_0(x,y)\,dxdy = 0.$$

(i) *Then there is a global solution $\omega \in C([0,\infty) : H^4)$ of the linear initial value problem (1.5) and a function $F \in L^\infty(\mathbb{T}\times[0,1])$ such that for all $t \ge 0$,*

$$\|\omega(t, x + tb(y), y) - F(x,y)\|_{L^\infty(\mathbb{T}\times[0,1])} \lesssim \langle t\rangle^{-1}, \tag{1.11}$$

(ii) *The velocity field $u = (u^x, u^y) = (-\partial_y\psi, \partial_x\psi)$ decays as $t \to \infty$, i.e.*

$$\begin{aligned}\|u^x(t,x,y) - \langle u^x\rangle(t,y)\|_{L^\infty(\mathbb{T}\times[0,1])} &\lesssim \langle t\rangle^{-1},\\ \|u^y(t,x,y)\|_{L^\infty(\mathbb{T}\times[0,1])} &\lesssim \langle t\rangle^{-2}.\end{aligned} \tag{1.12}$$

We will discuss some the main ingredients in the proof of Theorem 1.1 in Section 2 below. We conclude this subsection with some remarks of this theorem, in the context of the general problem of linear inviscid damping.

1.1.1 The Main Assumptions (A) and (B)

The assumption (A) that $b(y)$ is strictly monotonic in y is important for our proof to ensure a uniform rate of inviscid damping and sharp pointwise decay of the velocity fields in (1.12).

The spectral assumption (B) is also important, since inviscid damping fails if any of the operators L_k has any eigenvalues. Since L_k is a compact perturbation of the simple multiplication operator $f \to b(y)\cdot f$, by the general theory of Fredholm operators, the spectrum of L_k is purely continuous spectrum $[b(0), b(1)]$ for all $k \in \mathbb{Z}\backslash\{0\}$.

Finally, we note that there is a large class of shear flows b satisfying our assumptions. For instance, if b is strictly convex satisfying $b'' > 0$, or if b satisfies $|b'| \ge 1$ and $|b'''| < 1$ then the spectrum of the operators L_k consist entirely of the continuous spectrum $[b(0), b(1)]$ for $k \in \mathbb{Z}\backslash\{0\}$.

1.1.2 Previous Work

The linear stability problem has a long history, starting with the pioneering work of Kelvin [24], Rayleigh [31], Orr [30], and Taylor [33], among many others, with a focus on mode stability. It has been investigated intensely in the last few years, in particular around general shear flows and vortices, motivated mainly by the potential applications to the full nonlinear stability problem.

Without attempting to be exhaustive, we mention the results of Wei–Zhang–Zhao [36] and the second author [22, 23], who proved optimal decay rates for the linearized problem near monotone shear flows, and the results of Bedrossian–Coti Zelati–Vicol [4] and the authors [20], who proved sharp linear decay estimates for general vortices with decreasing profiles. We also refer the reader to recent work on the linear inviscid damping in the case of non-monotonic shear flows, see Wei–Zhang–Zhao [37, 38] and Grenier–Nguyen–Rousset–Soffer [16] and in the case of circular flows, see Coti Zelati–Zillinger [13].

1.1.3 The Main Conclusions

The main issue is to prove the time decay of the velocity fields claimed in (1.12). These bounds are equivalent to showing that

$$|k|^2\langle t\rangle^2\|\psi_k(t,y)\|_{L^\infty_y}+|k|\langle t\rangle\|\partial_y\psi_k(t,y)\|_{L^\infty_y}\lesssim|k|^4\|\omega_{0k}(y)\|_{L^2_y}+\|\omega_{0k}(y)\|_{H^4_y} \tag{1.13}$$

for any $k\in\mathbb{Z}\setminus\{0\}$ and any $t\geq 0$, where

$$\omega_k(t,y):=\frac{1}{2\pi}\int_{\mathbb{T}}\omega(t,x,y)e^{-ikx}\,dx,\qquad \psi_k(t,y):=\frac{1}{2\pi}\int_{\mathbb{T}}\psi(t,x,y)e^{-ikx}\,dx. \tag{1.14}$$

The bounds (1.11), which show pointwise convergence of the linear profile, follow easily from (1.13) and the main (1.5).

Notice that the conclusions of the theorem match well with the conclusions derived in the case of the Couette flow on $\mathbb{R}$ from the explicit formulas (1.6)–(1.8).

1.2 Asymptotic Nonlinear Stability

Nonlinear asymptotic stability results are difficult for the $2D$ incompressible Euler equation, because the rate of stabilization is slow, the convergence of the vorticity field holds only in the weak sense, the nonlinear effect is strong, and the space of possible final states is very large.

To state our main theorem we define the Gevrey spaces $\mathcal{G}^{\lambda,s}(\mathbb{T}\times\mathbb{R})$ as the spaces of L^2 functions f on $\mathbb{T}\times\mathbb{R}$ defined by the norm

$$\|f\|_{\mathcal{G}^{\lambda,s}(\mathbb{T}\times\mathbb{R})}:=\left\|e^{\lambda\langle k,\xi\rangle^s}\widetilde{f}(k,\xi)\right\|_{L^2_{k,\xi}}<\infty,\qquad s\in(0,1],\ \lambda>0.$$

In the above $(k,\xi)\in\mathbb{Z}\times\mathbb{R}$ and $\widetilde{f}$ denotes the Fourier transform of f in (x,y). More generally, for any interval $I\subseteq\mathbb{R}$ we define the Gevrey spaces $\mathcal{G}^{\lambda,s}(\mathbb{T}\times I)$ by

$$\|f\|_{\mathcal{G}^{\lambda,s}(\mathbb{T}\times I)}:=\|Ef\|_{\mathcal{G}^{\lambda,s}(\mathbb{T}\times\mathbb{R})},$$

where $Ef(x):=f(x)$ if $x\in I$ and $Ef(x):=0$ if $x\notin I$.

Concerning the background shear flow $b\in C^\infty(\mathbb{R})$, we replace the assumption (A) with the following stronger assumption:

(A') For some $\vartheta_0\in(0,1/10]$ and $\beta_0>0$

$$\vartheta_0\leq b'(y)\leq 1/\vartheta_0\ \text{ for } y\in[0,1]\quad\text{and}\quad b''(y)\equiv 0\ \text{ for } y\notin[2\vartheta_0,1-2\vartheta_0], \tag{1.15}$$

and

$$\|b\|_{L^\infty(0,1)}+\|b''\|_{\mathcal{G}^{\beta_0,1/2}}\leq 1/\vartheta_0. \tag{1.16}$$

Our main nonlinear stability result in the article is the following:

Theorem 1.2 *Assume that the function b satisfies the properties* (*A'*) *and* (*B*) *above, with constants* $\beta_0, \vartheta_0, \kappa > 0$ *as defined in* (1.15)–(1.16) *and* (2.8)*. Then there are constants* $\beta_1 = \beta_1(\beta_0, \vartheta_0, \kappa) > 0$ *and* $\overline{\varepsilon} = \overline{\varepsilon}(\beta_0, \vartheta_0, \kappa) > 0$ *such that the following statement is true:*
Assume that the initial data ω_0 *has compact support in* $\mathbb{T} \times [2\vartheta_0, 1 - 2\vartheta_0]$*, and satisfies*

$$\|\omega_0\|_{\mathcal{G}^{\beta_0,1/2}(\mathbb{T}\times\mathbb{R})} = \varepsilon \leq \overline{\varepsilon}, \qquad \int_{\mathbb{T}} \omega_0(x, y)\, dx = 0 \text{ for any } y \in [0, 1]. \tag{1.17}$$

Then there is a unique smooth global solution $\omega : [0, \infty) \times \mathbb{T} \times [0, 1] \to \mathbb{R}$ *of the Euler* (1.4) *with the following properties:*

(i) *For all* $t \geq 0$, $\operatorname{supp} \omega(t) \subseteq \mathbb{T} \times [\vartheta_0, 1 - \vartheta_0]$.
(ii) *There exists* $F_\infty(x, y) \in \mathcal{G}^{\beta_1,1/2}$ *with* $\operatorname{supp} F_\infty \subseteq \mathbb{T} \times [\vartheta_0, 1 - \vartheta_0]$ *such that for all* $t \geq 0$,

$$\|\omega(t, x + tb(y) + \Phi(t, y), y) - F_\infty(x, y)\|_{\mathcal{G}^{\beta_1,1/2}(\mathbb{T}\times[0,1])} \lesssim_{\beta_0,\vartheta_0,\kappa} \varepsilon\langle t\rangle^{-1}, \tag{1.18}$$

where

$$\Phi(t, y) := \int_0^t \langle u^x\rangle(\tau, y)\, d\tau.$$

(iii) *We define the smooth functions* $\psi_\infty, u_\infty : [0, 1] \to \mathbb{R}$ *by*

$$\partial_y^2 \psi_\infty = \langle F_\infty\rangle, \qquad \psi_\infty(0) = \psi_\infty(1) = 1, \qquad u_\infty(y) := -\partial_y \psi_\infty.$$

Then the velocity field $u = (u^x, u^y)$ *satisfies*

$$\left\|\langle u^x\rangle(t, y) - u_\infty(y)\right\|_{\mathcal{G}^{\beta_1,1/2}(\mathbb{T}\times[0,1])} \lesssim_{\beta_0,\vartheta_0,\kappa} \varepsilon\langle t\rangle^{-2}, \tag{1.19}$$

$$\left\|u^x(t, x, y) - \langle u^x\rangle(t, y)\right\|_{L^\infty(\mathbb{T}\times[0,1])} \lesssim_{\beta_0,\vartheta_0,\kappa} \varepsilon\langle t\rangle^{-1}, \tag{1.20}$$

$$\left\|u^y(t, x, y)\right\|_{L^\infty(\mathbb{T}\times[0,1])} \lesssim_{\beta_0,\vartheta_0,\kappa} \varepsilon\langle t\rangle^{-2}. \tag{1.21}$$

This theorem was proved by the authors in [19], and we will present the main ideas of the proof in Section 3 below. A similar theorem was proved slightly later and independently by Masmoudi–Zhao [26]. In the rest of this subsection we discuss some of the assumptions and the conclusions of Theorem 1.2, and provide more references and remarks.

1.2.1 Nonlinear Asymptotic Stability of Euler Equations in 2D

The first nonlinear asymptotic stability result was proved by Bedrossian–Masmoudi [7], who showed that small perturbations of the Couette flow on the infinite cylinder $\mathbb{T} \times \mathbb{R}$ converge weakly to nearby shear flows. This result was extended by the authors [17] to the finite channel $\mathbb{T} \times [0, 1]$, in order to be able to consider solutions with finite energy. In [18] the authors also proved asymptotic stability of point vortex solutions in $\mathbb{R}^2$, showing that small and Gevrey smooth perturbations converge to a smooth radial profile, and the position of the point vortex stabilizes rapidly and forms the center of the final radial profile. Finally, the authors [19] and Masmoudi–Zhao [26] independently proved asymptotic stability of monotonic shear flows in bounded channels, which is Theorem 1.2 above. These results are the only known results on nonlinear asymptotic stability of stationary solutions for the Euler equations.

We remark that inviscid damping is much better understood in the linear case, in a variety of settings, as summarized in Section 1.1.2 above. The reason for this is that there is a very large gap between linear and nonlinear theory. In fact, even in the simplest case of the Couette flow, to prove nonlinear stability one needs to bound the contribution of the so-called "resonant times", which can only be detected by working in the Fourier space, in a specific coordinate system. This requires refined Fourier analysis techniques, including energy functionals with suitable weights (in the Fourier space), which are not compatible with the natural spectral theory of the variable-coefficient linearized problems associated to general shear flows and vortices. In addition, the final state of the flow is determined dynamically by the global evolution and cannot be described in terms of the initial data and nonlinear decay comes at the expense of loss of regularity.

Overall, proving nonlinear inviscid damping appears to be a very challenging problem even in the simplest cases not covered so far, for example for the Poisseuille flow $b(y) = y^2$. We hope that the general framework we develop here can be adapted to establish nonlinear asymptotic stability in other outstanding open problems involving 2D or 3D Euler and Navier-Stokes equations, such as the stability of smooth radially decreasing vortices in 2D.

1.2.2 The Support Assumptions

The assumption on the compact support of ω_0 is likely necessary to prove scattering in Gevrey spaces. Indeed, Zillinger [41] showed that scattering does not hold in high Sobolev spaces unless one assumes that the vorticity vanishes at high order at the boundary. This is due to what is called "boundary effect", which is not consistent with inviscid damping. This boundary effect can also be seen clearly in [22] as the main asymptotic term for the stream function. Understanding quantitatively the boundary effect in the context of asymptotic stability of Euler or Navier–Stokes equations is a very interesting topic by itself, but we will not address it here.

The assumption on the support of b'' is necessary to preserve the compact support of $\omega(t)$ in $\mathbb{T} \times [\vartheta_0, 1 - \vartheta_0]$, due to the nonlocal term $b''(y)\partial_x\psi$ in (1.4). In principle, one could hope to remove this assumption (and replace it with a milder decay assumption) by working in the infinite cylinder $\mathbb{T} \times \mathbb{R}$ domain instead of the finite channel $\mathbb{T} \times [0, 1]$, but this would be at the expense of considering solutions of infinite energy.

1.2.3 Gevrey Regularity

The use of Gevrey spaces is necessary in the context of inviscid damping, mainly due to loss of regularity during the flow. In contrast, Sobolev spaces provide control only on finitely many derivatives, which is not sufficient in our case. Analytic functions have also been used in certain cases, but analyticity is a very rigid condition which is not compatible with the type of localization arguments we need in our problem (the point is that one can work in the class of compactly supported Gevrey functions, but there are no non-trivial compactly supported analytic functions).

The Gevrey regularity assumption (1.17) on the initial data ω_0 is likely sharp. See the recent construction of nonlinear instability of Deng–Masmoudi [14] for the Couette flow in slightly larger Gevrey spaces, and the more definitive counter-examples to inviscid damping in low Sobolev spaces by Lin–Zeng [25].

1.2.4 The Main Conclusions

The most important statement in Theorem 1.2 is the bound (1.18), which provides strong control on the "profile" of the vorticity and from which the other statements follow easily. We note that the convergence (1.18) of the profile for vorticity holds in a slightly weaker Gevrey space ($\beta_1 < \beta_0$). This is connected with the use of energy functionals with decreasing time-dependent weights to control the profile, and is a reflection of the phenomenon that "decay costs regularity" in inviscid damping.

We also remark that the pointwise decay of the velocity fields in (1.20)–(1.21) is sharp and matches the linear pointwise decay in (1.12) (and the decay in the case of the Couette flow, which follows from the explicit formulas (1.6)).

At the qualitative level, our main conclusion (1.18) shows that the vorticity ω converges weakly to the function $\langle F_\infty\rangle(y)$. This is consistent with a far-reaching conjecture regarding the long time behavior of the $2D$ Euler equation, which predicts that for general generic solutions the vorticity field converges, as $t \to \infty$, *weakly but not strongly* in L^2_{loc} to a steady state. Proving such a conjecture for general solutions is, of course, well beyond the current PDE techniques, but the nonlinear asymptotic stability results we have so far in [7, 17–19, 26] are consistent with this conjecture.

1.2.5 Some Technical Remarks

The equation (1.4) for the vorticity deviation is equivalent to the original Euler equations (1.1)–(1.3). The condition $\int_{\mathbb{T}} \omega_0(x, y)\,dx = 0$ can be imposed without loss of generality, because we may replace the shear flow $b(y)$ by the nearby shear flow $b(y) + \langle u_0^x\rangle(y)$. In fact, since $\partial_y\langle\partial_y\psi\rangle = \langle\omega\rangle$, this condition is equivalent to

$$\langle u_0^x\rangle(y) = 0 \qquad \text{for any } y \in [0, 1].$$

These identities only hold for the initial data, and are not propagated by the flow (1.4). However, it is not hard to see that

$$\langle u^x\rangle(t, y) \equiv 0 \qquad \text{for } y \in [0, 1] \setminus [\vartheta_0, 1 - \vartheta_0] \text{ and } t \in [0, T],$$

as long as the vorticity ω is supported in $[0, T] \times \mathbb{T} \times [\vartheta_0, 1 - \vartheta_0]$.

There are several parameters in our proof, and we summarize their roles here. The parameters $\beta_0, \vartheta_0, \kappa > 0$ (the structural constants of the problem) are assumed fixed, and implicit constants in inequalities like $A \lesssim B$ are allowed to depend on these parameters. We will later fix a constant $\delta_0 > 0$ sufficiently small depending on these parameters, as part of the construction of our main weights defined in (3.9)–(3.10).

These weights will also depend on a small parameter $\delta > 0$, much smaller than δ_0, which is needed at many places, such as in commutator estimates using inequalities like (3.18). We will use the general notation $A \lesssim_\delta B$ to indicate inequalities where the implicit constants may depend on δ. Finally, the parameters ε and $\varepsilon_1 = \varepsilon^{2/3}$, which bound the size of the perturbation, are assumed to be much smaller than δ.

1.2.6 Further Remarks and References

(1) The problem of nonlinear inviscid damping we consider here is connected to the well-known Landau damping effect for the Vlasov–Poisson equations, and we refer to the celebrated work of Mouhot–Villani [29] for the physical background and more references.

(2) Inviscid damping is a very subtle mechanism of stability, which has only been proved rigorously in 2D for Euler-type equations. In fact, inviscid damping does not seem to hold at the nonlinear level even for small variations of the 2D Euler equations, such

as the generalized SQG equations with slightly less regular stream functions (see [21, Section 3]).

(3) The Euler equations can be viewed as the limiting case of the Navier–Stokes equations with small viscosity $\nu > 0$. In the presence of viscosity, one can have more robust stability results for initial data that is sufficiently small relative to ν, which exploit the enhanced dissipation due to the mixing of the fluid. See [5, 8, 9, 15, 39] and the references therein. Moreover, in the limit $\nu \to 0$ and if there is boundary then the boundary layer becomes an important issue, and there are significant additional difficulties. We refer the interested reader to [6, 12] for more details and further references.

1.3 Non-monotonic Shear Flows

Nonlinear asymptotic stability in the case of shear flows that are not monotonic is a wide open and important problem, even in the simplest case of the Poiseuille flow $b(y) = y^2$ on the infinite channel $\mathbb{T} \times \mathbb{R}$. However, some linear stability are known, see for example the recent work of Wei–Zhang–Zhao [37].

In the second part of this series of two papers we will prove a new linear stability result for a certain class of shear flows with one critical point. We remark that the conclusions are weaker than in the case of monotonic shear flows, but we are still able to prove quantitative decay for the associated stream function, thus improving on the results of [37].

1.4 Organization

The rest of the paper is organized as follows: in Section 2 we present the main ideas in the proof of Theorem 1.1, while in Section 3 we present the main ideas in the proof of Theorem 1.2.

2 Linear Stability

In this section we outline the proof of Theorem 1.1, following the paper [22].

2.1 The Main Formulas

Taking Fourier transform in x in the (1.5) for ω, we obtain that

$$\partial_t \omega_k + ikb(y)\omega_k - ikb''(y)\psi_k = 0,$$

for $k \in \mathbb{Z}$, $t \geq 0$, $y \in [0, 1]$, where ω_k and ψ_k are defined as in (1.14). This is equivalent to

$$\partial_t \omega_k + ikL_k\omega_k = 0, \tag{2.1}$$

see (1.9)–(1.10). The spectrum of the operator L_k is included in the interval $[-A, A]$ for A sufficiently large, due to the spectral assumption (B). By the standard theory of spectral projections, it follows from (2.1) that for any $y \in [0, 1]$,

$$\omega_k(t, y) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0+} \int_{[-A,A]} e^{i\lambda kt} \left\{ \left[(\lambda + L_k - i\varepsilon)^{-1} - (\lambda + L_k + i\varepsilon)^{-1} \right] \omega_{0k} \right\} (y)\, d\lambda,$$

where $(\lambda + L_k \pm i\varepsilon)^{-1}$ denote resolvent operators. Thus, for any $y \in [0,1]$,

$$\psi_k(t,y) = -\frac{1}{2\pi i}\lim_{\varepsilon\to 0+}\int_{[-A,A]} e^{-ik\lambda t}\left[\psi^-_{k,\varepsilon}(y,\lambda) - \psi^+_{k,\varepsilon}(y,\lambda)\right] d\lambda, \tag{2.2}$$

where, for $y \in [0,1]$ and $\lambda \in [-A, A]$,

$$\psi^{\pm}_{k,\varepsilon}(y,\lambda) := \int_0^1 G_k(y,z)\left[(-\lambda + L_k \pm i\varepsilon)^{-1}\omega_{0k}\right](z)\, dz. \tag{2.3}$$

The definition (2.3) shows that, for any $\lambda \in \mathbb{R}$ and $\varepsilon \in [-1,1] \setminus \{0\}$,

$$-k^2\psi^{\pm}_{k,\varepsilon}(y,\lambda) + \frac{d^2}{dy^2}\psi^{\pm}_{k,\varepsilon}(y,\lambda) - \frac{b''(y)}{b(y)-\lambda\pm i\varepsilon}\psi^{\iota}_{k,\varepsilon}(y,\lambda) = \frac{-\omega_{0k}(y)}{b(y)-\lambda\pm i\varepsilon}.$$

Therefore

$$\psi^{\pm}_{k,\varepsilon}(y,\lambda) + (T^1_{k,\pm\varepsilon,\lambda}\psi^{\pm}_{k,\varepsilon})(y,\lambda) = (T^0_{k,\pm\varepsilon,\lambda}\omega_{0k})(y,\lambda) \tag{2.4}$$

where, for any $\lambda \in \mathbb{R}$ and $\rho \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} T^0_{k,\rho,\lambda} f(y,\lambda) &:= \int_0^1 G_k(y,z)\frac{f(z)}{b(z)-\lambda+i\rho}\, dz, \\ T^1_{k,\rho,\lambda} f(y,\lambda) &:= \int_0^1 G_k(y,z)\frac{b''(z)f(z)}{b(z)-\lambda+i\rho}\, dz. \end{aligned} \tag{2.5}$$

The main issue is to prove decay of the stream functions ψ_k and their y-derivatives,

$$|k|^2\langle t\rangle^2\|\psi_k(t,y)\|_{L^\infty_y} + |k|\langle t\rangle\|\partial_y\psi_k(t,y)\|_{L^\infty_y} \lesssim |k|^4\|\omega_{0k}(y)\|_{L^2_y} + \|\omega_{0k}(y)\|_{H^4_y}. \tag{2.6}$$

The idea is to use the identity (2.2) and integrate by parts in the spectral parameter λ. For this we need good bounds on the generalized eigenfunctions $\psi^{\pm}_{k,\varepsilon}$ and their first and second order derivatives in λ; we prove such bounds by analyzing the identity (2.4).

2.2 The Operators $T^l_{k,\rho,\lambda}$, $l \in \{0, 1\}$

To implement this strategy we need to understand well the operators $T^0_{k,\rho,\lambda}$ and $T^1_{k,\rho,\lambda}$ defined in (2.5). Here we start using the lower bound $b'(z) \gtrsim 1$ in assumption (A); to gain some intuition we integrate by parts in z, to eliminate the singular factor $b(z)-\lambda+i\rho$, and calculate

$$T^0_{k,\rho,\lambda} f(y,\lambda) = -\int_0^1 \log[b(z)-\lambda+i\rho]\frac{d}{dz}\left\{G_k(y,z)\frac{f(z)}{b'(z)}\right\} dz, \tag{2.7}$$

and then

$$\begin{aligned} (\partial_y T^0_{k,\rho,\lambda} f)(y,\lambda) = &-\log[b(y)-\lambda+i\rho]\frac{f(y)}{b'(y)} - \int_0^1 \log[b(z)-\lambda+i\rho]\, G'_k(y,z)\frac{f(z)}{b'(z)}\, dz \\ &- \int_0^1 \log[b(z)-\lambda+i\rho](\partial_y G_k)(y,z)\frac{d}{dz}\frac{f(z)}{b'(z)}\, dz, \end{aligned}$$

where

$$\begin{aligned} \partial_y\partial_z G_k(y,z) &= \delta_0(y-z) + G'_k(y,z), \\ G'_k(y,z) &:= \frac{1}{\sinh k}\begin{cases} -k\cosh(k(1-z))\cosh(ky) & \text{if } 0 \le y \le z \le 1, \\ -k\cosh(kz)\cosh(k(1-y)) & \text{if } 0 \le z \le y \le 1. \end{cases} \end{aligned}$$

These formulas show that the operators $T^0_{k,\rho,\lambda}$ have a smoothing effect. For example, if $f \in H^1(0,1)$ then

$$\|T^0_{k,\rho,\lambda} f\|_{L^\infty_y} \lesssim |k| \|f\|_{H^1},$$
$$\left\| (\partial_y T^0_{k,\rho,\lambda} f)(y,\lambda) + \log[b(y) - \lambda + i\rho] \frac{f(y)}{b'(y)} \right\|_{L^\infty_y} \lesssim |k| \|f\|_{H^1},$$
$$\left\| \partial_y T^0_{k,\rho,\lambda} f \right\|_{L^p_y} \lesssim_p |k| \|f\|_{H^1}, \qquad p \in [2,\infty),$$

uniformly for $\lambda \in [-A, A]$ and $\delta \in [-1, 1] \setminus \{0\}$. Similar bounds hold for the operators $T^1_{k,\rho,\lambda}$ as well. The $|k|$ dependence in the right-hand sides of these inequalities is not optimal, but this is not an issue here.

2.3 The Limiting Absorption Principle

The spectral condition (B) is a qualitative condition, and we need to make it quantitative in order to link it to the perturbation theory. For this we define, for any $k \in \mathbb{Z} \setminus \{0\}$,

$$\|f\|_{H^1_k(0,1)} := \|f\|_{L^2(0,1)} + |k|^{-1} \|f'\|_{L^2(0,1)}.$$

The following lemma provides the critical quantitative bounds.

Lemma 2.1 *Then there is a constant $\kappa > 0$ such that, for any $f \in H^1_k(0,1)$,*

$$\|f + T^1_{k,\varepsilon,\lambda} f\|_{H^1_k(0,1)} \geq \kappa \|f\|_{H^1_k(0,1)}, \tag{2.8}$$

uniformly in $\lambda \in [-A, A]$, $k \in \mathbb{Z}\setminus\{0\}$, and $\varepsilon \in [-\kappa, \kappa] \setminus \{0\}$.

2.4 Smoothness of the Generalized Eigenfunctions $\psi^{\pm}_{k,\varepsilon}$

To use the formula (2.2) and prove $\langle t\rangle^{-2}$ decay of the stream function (in the form (2.6)) we need to integrate by parts twice in λ. For this we need to calculate λ-derivatives of the the operators $T^l_{k,\delta,\lambda}$, $l \in \{0, 1\}$. Taking λ-derivatives leads to more singular factors in the integrals representing these operators, which require additional integrations by parts in z. For example, starting from the formula (2.7) and integrating by parts once more,

$$\begin{aligned}
(\partial_\lambda T^0_{k,\rho,\lambda} f)(y,\lambda) &= \int_0^1 \frac{1}{b(z) - \lambda + i\rho} \frac{d}{dz} \left\{ G_k(y,z) \frac{f(z)}{b'(z)} \right\} dz \\
&= \left\{ \log[b(z) - \lambda + i\rho] \partial_z G_k(y,z) \frac{f(z)}{b'(z)} \right\} \Big|_0^1 \\
&\quad - \int_0^1 \log[b(z) - \lambda + i\rho] \frac{d}{dz} \left\{ \frac{1}{b'(z)} \frac{d}{dz} \left\{ G_k(y,z) \frac{f(z)}{b'(z)} \right\} \right\} dz.
\end{aligned}$$

Since $\partial_z^2 G_k(y,z) = k^2 G_k(y,z) - \delta_0(y - z)$, this can be written in the form

$$(\partial_\lambda T^0_{k,\rho,\lambda} f)(y,\lambda) = \log[b(1) - \lambda + i\rho] \partial_z G_k(y,1) \frac{f(1)}{b'(1)} - \log[b(0) - \lambda + i\rho] \partial_z G_k(y,0) \frac{f(0)}{b'(0)}$$

$$+\log[b(y)-\lambda+i\rho]\frac{f(y)}{[b'(y)]^2}-\int_0^1\log[b(z)-\lambda+i\rho]\Big\{\partial_z G_k(y,z)\frac{2f'(z)b'(z)-3f(z)b''(z)}{[b'(z)]^3}$$
$$+G_k(y,z)\frac{b'(z)[f''(z)b'(z)-f(z)b'''(z)]-3b''(z)[f'(z)b'(z)-f(z)b''(z)]+k^2f(z)[b'(z)]^2}{[b'(z)]^4}\Big\}\,dz.$$

Similar calculations give explicit formulas for $\partial_y\partial_\lambda T^0_{k,\rho,\lambda}f$ and $\partial^2_\lambda T^0_{k,\rho,\lambda}f$, involving the singular factors $1/(b(p)-\lambda-i\rho)$, $p\in\{0,1,y\}$. One can then use limiting absorption principles, similar to Lemma 2.1, and the identities (2.4) to prove bounds on the functions $\partial_\lambda\psi^{\pm}_{k,\varepsilon}$ and $\partial^2_\lambda\psi^{\pm}_{k,\varepsilon}$. Then we use the formula (2.2) and integrate by parts in λ twice to prove the desired pointwise decay estimates (2.6).

3 Nonlinear Inviscid Damping and Asymptotic Stability

We describe now some of the main ideas involved in the proof of Theorem 1.2, following the paper [19].

3.1 Renormalization and the Main Equations

To obtain uniform control in time we need to unwind the transportation in x. As in [7, 17, 18], we make the nonlinear change of variables

$$v=b(y)+\frac{1}{t}\int_0^t\langle u^x\rangle(\tau,y)\,d\tau,\qquad z=x-tv. \tag{3.1}$$

The point of this change of variables is to eliminate two of the non-decaying terms in the evolution equation in (1.4), namely the terms $b(y)\partial_x\omega$ and $\langle u^x\rangle\partial_x\omega$. The change of variable $y\to v$ is crucial for our analysis, and it allows us to link the renormalized stream function ϕ to the profile F using the elliptic equation (3.8). The point is that this equation has constant coefficients at the top linear level (in a suitable sense), so it is compatible with Fourier analysis.

This change of variables leads to new functions satisfying new equations. We summarize our main conclusions in the following proposition:

Proposition 3.1 *Assume $\omega:[0,T]\times\mathbb{T}\times[0,1]\to\mathbb{R}$ is a sufficiently smooth solution of the system (1.4) on some time interval $[0,T]$, with initial data ω_0 satisfying (1.17). Assume that $\omega(t)$ is supported in $\mathbb{T}\times[\vartheta_0,1-\vartheta_0]$ and that $\|\langle\omega\rangle(t)\|_{H^{10}}\ll 1$ for all $t\in[0,T]$. Then*

$$\langle u^x\rangle(t,y)=0\qquad \textit{for any } t\in[0,T] \textit{ and } y\in[0,\vartheta_0]\cup[1-\vartheta_0,1].$$

We let $(z,v):[0,T]\times\mathbb{T}\times[0,1]\to[0,T]\times\mathbb{T}\times[b(0),b(1)]$ denote the change of variables (3.1), and define the new variables $F,\phi:[0,T]\times\mathbb{T}\times[b(0),b(1)]\to\mathbb{R}$ and $V',V'',\dot V,B',B'',\mathcal{H}:[0,T]\times[b(0),b(1)]\to\mathbb{R}$ by the formulas

$$F(t,z,v):=\omega(t,x,y),\qquad \phi(t,z,v):=\psi(t,x,y), \tag{3.2}$$
$$V'(t,v):=\partial_y v(t,y),\qquad V''(t,v)=\partial_{yy}v(t,y),\qquad \dot V(t,v)=\partial_t v(t,y), \tag{3.3}$$
$$B'(t,v):=\partial_y b(y),\qquad B''(t,v):=\partial_{yy}b(y), \tag{3.4}$$
$$\mathcal{H}(t,v):=tV'(t,v)\partial_v\dot V(t,v)=B'(t,v)-V'(t,v)-\langle F\rangle(t,v). \tag{3.5}$$

Then $V'(t,v) \geq \vartheta_0/2$ and the new variables F, $V' - B'$, $\dot{V}$, and $\mathcal{H}$ are supported in $[0,T] \times \mathbb{T} \times [b(\vartheta_0), b(1-\vartheta_0)]$ *and satisfy the evolution equations*

$$\partial_t F - B''\partial_z\phi = V'\partial_v P_{\neq 0}\phi\, \partial_z F - (\dot{V} + V'\partial_z\phi)\, \partial_v F, \tag{3.6}$$
$$\partial_t B'(t,v) + \dot{V}\partial_v B'(t,v) = \partial_t B''(t,v) + \dot{V}\partial_v B''(t,v) = 0,$$
$$\partial_t (V' - B') + \dot{V}\partial_v (V' - B') = \mathcal{H}/t,$$
$$\partial_t \mathcal{H} + \dot{V}\partial_v \mathcal{H} = -\mathcal{H}/t - V'\langle \partial_v P_{\neq 0}\phi\, \partial_z F\rangle + V'\langle \partial_z\phi\, \partial_v F\rangle. \tag{3.7}$$

The variables ϕ, V'', and $\dot{V}$ satisfy the elliptic-type identities

$$\partial_z^2\phi + (V')^2(\partial_v - t\partial_z)^2\phi + V''(\partial_v - t\partial_z)\phi = F, \tag{3.8}$$
$$\partial_v \dot{V} = \mathcal{H}/(tV'), \qquad \dot{V}(t, b(0)) = \dot{V}(t, b(1)) = 0, \qquad V'' = V'\partial_v V'.$$

We explain briefly the roles of our new variables (see also the more precise discussion after the statement of Proposition 3.2):

(1) The main variable we need to control is F, which is the profile for the vorticity ω. The second important variable is the renormalized stream function ϕ, which is linked to F through the elliptic equation (3.8).
(2) The functions V', V'', B', B'' are connected to the change of variables $y \to v$. These functions appear in many of the nonlinear terms in the equations, so it is important to control their smoothness as well, as part of a combined bootstrap argument, in a way that is consistent with the smoothness of the functions F and ϕ.
(3) The variables $\dot{V}$ and $\mathcal{H}$ play a different role: they encode the convergence of the coordinate system as $t \to \infty$. The function $\mathcal{H}$ satisfies the more favorable (3.7), and we use this equation to prove asymptotic decay. The identity in (3.5) can be proved using the definitions.

3.2 Energy Functionals and the Bootstrap Proposition

The main idea of the proof is to estimate the increment of suitable energy functionals, which are defined using special weights. These weights are defined by

$$A_{NR}(t,\xi) := \frac{e^{\lambda(t)\langle\xi\rangle^{1/2}}}{b_{NR}(t,\xi)} e^{\sqrt{\delta}\langle\xi\rangle^{1/2}}, \qquad A_R(t,\xi) := \frac{e^{\lambda(t)\langle\xi\rangle^{1/2}}}{b_R(t,\xi)} e^{\sqrt{\delta}\langle\xi\rangle^{1/2}}, \tag{3.9}$$

and

$$A_k(t,\xi) := e^{\lambda(t)\langle k,\xi\rangle^{1/2}} \left(\frac{e^{\sqrt{\delta}\langle\xi\rangle^{1/2}}}{b_k(t,\xi)} + e^{\sqrt{\delta}|k|^{1/2}} \right), \tag{3.10}$$

where $k \in \mathbb{Z}$, $t \in [0,\infty)$, $\xi \in \mathbb{R}$. The function $\lambda : [0,\infty) \to [\delta_0, 3\delta_0/2]$ is defined by

$$\lambda(0) = \frac{3}{2}\delta_0, \quad \lambda'(t) = -\frac{\delta_0\sigma_0^2}{\langle t\rangle^{1+\sigma_0}},$$

where $\delta_0 > 0$ is a fixed parameter and $\sigma_0 = 0.01$. In particular, λ is decreasing on $[0,\infty)$, and the functions A_{NR}, A_R, A_k are also decreasing in t. The parameter $\delta > 0$, which appears also in the weights b_R, b_{NR}, b_k, is to be taken sufficiently small, depending only on the structural parameters δ_0, ϑ_0, and κ.

The precise definitions of the weights b_{NR}, b_R, b_k are very important; we will discuss some of the basic requirements in Section 3.2.1 below. For now we note that these functions are essentially increasing in t and satisfy

$$e^{-\delta\sqrt{|\xi|}} \le b_R(t,\xi) \le b_k(t,\xi) \le b_{NR}(t,\xi) \le 1 \quad \text{for any } t, \xi, k.$$

In other words, the weights $1/b_{NR}$, $1/b_R$, $1/b_k$ are small when compared to the main factors $e^{\lambda(t)\langle\xi\rangle^{1/2}}$ and $e^{\lambda(t)\langle k,\xi\rangle^{1/2}}$ in (3.9)–(3.10). However, their relative contributions are important as they are used to distinguish between resonant and non-resonant times.

Assume that $\omega : [0,T]\times\mathbb{T}\times[0,1]\to\mathbb{R}$ is as in Proposition 3.1 and define the functions $F, \phi, V', V'', \dot{V}, B', B'', \mathcal{H}$ as in (3.2)–(3.5). To construct useful energy functionals we need to modify the functions V', B', B'' which are not "small", so we define the new variables

$$B_0'(v) := B'(0,v) = (\partial_y b)(b^{-1}(v)), \qquad B_0''(v) := B''(0,v) = (\partial_y^2 b)(b^{-1}(v)),$$
$$V_*' := V' - B_0', \qquad B_*' := B' - B_0', \qquad B_*'' := B'' - B_0''.$$

Our main goal is to control the functions F and ϕ. For this we need to consider two auxiliary functions F^* and ϕ'. We define first the function $\phi'(t,z,v) : [0,T]\times\mathbb{T}\times[b(0),b(1)]\to\mathbb{R}$ as the unique solution to the equation

$$\partial_z^2\phi' + (B_0')^2(\partial_v - t\partial_z)^2\phi' + B_0''(\partial_v - t\partial_z)\phi' = F, \quad \phi'(t,z,b(0)) = \phi'(t,z,b(1)) = 0, \tag{3.11}$$

on $\mathbb{T}\times[b(0),b(1)]$. Then we define the modified profile

$$F^*(t,z,v) := F(t,z,v) - B_0''(v)\int_0^t \partial_z\phi'(\tau,z,v)\,d\tau, \tag{3.12}$$

and the renormalized elliptic profiles

$$\Theta(t,z,v) := (\partial_z^2 + (\partial_v - t\partial_z)^2)\,(\Psi(v)\,\phi(t,z,v))\,,$$
$$\Theta^*(t,z,v) := (\partial_z^2 + (\partial_v - t\partial_z)^2)\,\big(\Psi(v)\,(\phi(t,z,v) - \phi'(t,z,v))\big)\,,$$

where $\Psi : \mathbb{R}\to[0,1]$ is a Gevrey class cut-off function, satisfying

$$\left\| e^{\langle\xi\rangle^{3/4}}\widetilde{\Psi}(\xi)\right\|_{L^\infty} \lesssim 1,$$
$$\operatorname{supp}\Psi \subseteq [b(\vartheta_0/4), b(1-\vartheta_0/4)]\,, \quad \Psi \equiv 1 \text{ in } [b(\vartheta_0/3), b(1-\vartheta_0/3)]\,.$$

Our bootstrap argument is based on controlling simultaneously energy functionals and space-time integrals. We define these quantities in the Fourier space, since one of our main concerns is to capture accurately the contributions of the resonances $(t,k,\xi)\in[0,T\times\mathbb{Z}\times\mathbb{R}$ satisfying $\xi - tk = 0$. Let $\dot{A}_Y(t,\xi) := (\partial_t A_Y)(t,\xi) \le 0$, $Y\in\{NR, R, k\}$ and let $\widetilde{f}$ denote the Fourier transform of f, either on $\mathbb{T}\times\mathbb{R}$ or on $\mathbb{R}$. We define

$$\mathcal{E}_f(t) := \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} A_k^2(t,\xi)\left|\widetilde{f}(t,k,\xi)\right|^2 d\xi, \qquad f\in\{F,F^*\},$$
$$\mathcal{B}_f(t) := \int_1^t\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} |\dot{A}_k(s,\xi)|A_k(s,\xi)\left|\widetilde{f}(s,k,\xi)\right|^2 d\xi ds,$$

$$\mathcal{E}_{F-F^*}(t) := \sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}}(1+\langle k,\xi\rangle/\langle t\rangle)A_k^2(t,\xi)\left|(\widetilde{F-F^*})(t,k,\xi)\right|^2\,d\xi, \quad (3.13)$$

$$\mathcal{B}_{F-F^*}(t) := \int_1^t\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}}(1+\langle k,\xi\rangle/\langle s\rangle)|\dot{A}_k(s,\xi)|A_k(s,\xi)\left|(\widetilde{F-F^*})(s,k,\xi)\right|^2\,d\xi ds, \quad (3.14)$$

$$\mathcal{E}_{\Phi}(t) := \sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}}A_k^2(t,\xi)\frac{|k|^2\langle t\rangle^2}{|\xi|^2+|k|^2\langle t\rangle^2}\left|\widetilde{\Phi}(t,k,\xi)\right|^2\,d\xi, \qquad \Phi\in\{\Theta,\Theta^*\},$$

$$\mathcal{B}_{\Phi}(t) := \int_1^t\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}}|\dot{A}_k(s,\xi)|A_k(s,\xi)\frac{|k|^2\langle s\rangle^2}{|\xi|^2+|k|^2\langle s\rangle^2}\left|\widetilde{\Phi}(s,k,\xi)\right|^2\,d\xi ds,$$

$$\mathcal{E}_g(t) := \int_{\mathbb{R}}A_R^2(t,\xi)\,|\widetilde{g}(t,\xi)|^2\,d\xi, \qquad g\in\{V'_*, B'_*, B''_*\},$$

$$\mathcal{B}_g(t) := \int_1^t\int_{\mathbb{R}}|\dot{A}_R(s,\xi)|A_R(s,\xi)\,|\widetilde{g}(s,\xi)|^2\,d\xi ds,$$

$$\mathcal{E}_{\mathcal{H}}(t) := \mathcal{K}^2\int_{\mathbb{R}}A_{NR}^2(t,\xi)\,(\langle t\rangle/\langle\xi\rangle)^{3/2}\left|\widetilde{\mathcal{H}}(t,\xi)\right|^2\,d\xi,$$

$$\mathcal{B}_{\mathcal{H}}(t) := \mathcal{K}^2\int_1^t\int_{\mathbb{R}}|\dot{A}_{NR}(s,\xi)|A_{NR}(s,\xi)\,(\langle s\rangle/\langle\xi\rangle)^{3/2}\left|\widetilde{\mathcal{H}}(s,\xi)\right|^2\,d\xi ds,$$

for any $t\in[0,T]$, where $\mathbb{Z}^* := \mathbb{Z}\setminus\{0\}$ and $\mathcal{K}\geq 1$ is a large constant that depends only on δ.

Our main bootstrap proposition is the following:

Proposition 3.2 *Assume $T\geq 1$ and $\omega\in C([0,T]:\mathcal{G}^{2\delta_0,1/2})$ is a sufficiently smooth solution of the system (1.4) on the time interval $[0,T]$, with initial data ω_0 satisfying (1.17). Assume that $\omega(t)$ is supported in $\mathbb{T}\times[\vartheta_0,1-\vartheta_0]$ and that $\|\langle\omega\rangle(t)\|_{H^{10}}\ll 1$ for all $t\in[0,T]$. Define F, F^*, Θ, Θ^* B'_*, B''_*, V'_*, $\mathcal{H}$ as above. Assume that ε_1 is sufficiently small (depending on δ),*

$$\sum_{g\in\{F,F^*,F-F^*,\Theta,\Theta^*,V'_*,B'_*,B''_*,\mathcal{H}\}}\mathcal{E}_g(t)\leq\varepsilon_1^3 \qquad \text{for any } t\in[0,1],$$

and

$$\sum_{g\in\{F,F^*,F-F^*,\Theta,\Theta^*,V'_*,B'_*,B''_*,\mathcal{H}\}}\left[\mathcal{E}_g(t)+\mathcal{B}_g(t)\right]\leq\varepsilon_1^2 \qquad \text{for any } t\in[1,T].$$

Then for any $t\in[1,T]$ we have the improved bounds

$$\sum_{g\in\{F,F^*,F-F^*,\Theta,\Theta^*,V'_*,B'_*,B''_*,\mathcal{H}\}}\left[\mathcal{E}_g(t)+\mathcal{B}_g(t)\right]\leq\varepsilon_1^2/2. \quad (3.15)$$

Moreover, we also have the stronger bounds for $t\in[1,T]$

$$\sum_{g\in\{F,\Theta\}}\left[\mathcal{E}_g(t)+\mathcal{B}_g(t)\right]\lesssim_\delta\varepsilon_1^3.$$

Proposition 3.2 is the main ingredient in the proof of Theorem 1.2. Our argument involves proving simultaneous control of nine main quantities $F, F^*, F-F^*, \Theta, \Theta^*, V'_*, B'_*, B''_*, \mathcal{H}$. We summarize briefly the roles of these quantities:

(1) The main variables are the vorticity profile F and the renormalized elliptic profile Θ. Our primary goal is to prove global bounds on these quantities.

(2) The functions F^* and Θ^* are auxiliary variables, and we analyze them as an intermediate step to controlling the main variables F and Θ. The function F^* satisfies a better transport equation than F, without any other linear terms, while the function Θ^* satisfies a better elliptic equation than Θ, again without linear terms in the right-hand side.
(3) A significant component of the proof is to control the function $F - F^*$, which allows us to pass from the modified profile F^* to the true profile F. This is based on the theory of the linearized equation in Gevrey spaces, as developed in [23], and requires the spectral assumption (B) on the shear flow. We remark that the bootstrap control on the variable $F - F^*$ is slightly stronger than on the variables F and F^* separately, which is needed to compensate for the lack of symmetry in some of the transport terms.
(4) The functions V'_*, B'_*, and B''_* are present due to the change of variables $y \to v$, and appear in many of the nonlinear terms in the equations. To close the entire argument it is important to control these functions as well, as part of a combined bootstrap argument, in a way that is consistent with the control on the main functions F and Θ.
(5) Finally, the function $\mathcal{H}$, which decays in time, captures the convergence of the system as $t \to \infty$. This function decays at a rate of $\langle t\rangle^{-3/4}$, in a weaker topology, which shows that the function $\partial_v \dot{V}$ decays fast at an integrable rate of $\langle t\rangle^{-7/4}$, again in a weaker topology.

3.2.1 The Weights A_k, A_{NR}, and A_R

To make the bootstrap argument work the weights A_k, A_{NR}, and A_R in Proposition 3.2 have to be defined very carefully. These weights are required to satisfy several strong properties, and can only be used if the initial data is in Gevrey spaces.

These weights have been defined and analyzed in [17–19] (as refinements of the weights introduced in [7], with an additional smoothing procedure to make them compatible with commutator estimates). We will not provide the precise definitions here; instead, we will state and explain three critical properties that these weights need to satisfy.

(1) Assume first that F and ϕ satisfy the simplified closed system

$$\partial_t F - \partial_v P_{\neq 0}\phi\, \partial_z F = 0, \qquad \partial_z^2\phi + (\partial_v - t\partial_z)^2\phi = F,$$

for $(z, v, t) \in \mathbb{T} \times \mathbb{R} \times [0, \infty)$. Compared to the original equation (3.6), we assume that $b'' \equiv 0$ (the Couette flow) and keep only one nonlinear term, the "reaction term" $\partial_v \mathbb{P}_{\neq 0}\phi \cdot \partial_z F$. We would like to control, uniformly in time, an energy functional of the form

$$\mathcal{E}(t) := \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} A_k^2(t, \xi) \left|\widetilde{F}(t, k, \xi)\right|^2 d\xi, \tag{3.16}$$

as well as a similar energy functional for the function ϕ, for a suitable weight $A_k(t, \xi)$ which decreases in t. The main observation is that

$$\widetilde{\partial_v P_{\neq 0}\phi}(t, k, \xi) = -\frac{i\xi}{k^2}\frac{\widetilde{F}(t, k, \xi)}{1 + |t - \xi/k|^2}\mathbf{1}_{k\neq 0}. \tag{3.17}$$

When $|\xi| \gg k^2$, the factor ξ/k^2 in (3.17) indicates a loss of one full derivative in v, which occurs in the resonant region $\{(t, k, \xi) : |t - \xi/k| \ll |\xi|/k^2,\ k^2 + 1 \ll |\xi|\}$. This is a major obstruction to proving stability, which cannot be removed by standard symmetrization techniques.

The key idea is to use *imbalanced weights* $A_k(t, \xi)$ to absorb this derivative loss, taking advantage of the favorable structure of the nonlinearity that does not allow for contributions to the resonant region to come from bilinear interactions of small frequencies and frequencies in the resonant region (due to the factor $\partial_z F$ in the reaction term). More precisely, the weights A_k are constructed to satisfy the property

$$\frac{A_\ell(t,\eta)}{A_k(t,\xi)} \approx \left|\frac{\eta}{\ell^2}\right| \frac{1}{1+|t-\eta/\ell|},$$

when $k \neq \ell$, $\ell \neq 0$, $\xi = \eta + O(1)$, $k = \ell + O(1)$, and $1 + |t-\eta/\ell| \ll |\eta|/\ell^2$.

(2) The weights $A_k(t,\xi)$ have to decrease in time, in the quantitative form,

$$-\frac{\partial_t A_k(t,\xi)}{A_k(t,\xi)} \approx \frac{1}{\langle t-\xi/k\rangle},$$

if $k \in \mathbb{Z}\backslash\{0\}$, $k^2 \lesssim |\xi|$, $|t-\xi/k| \lesssim |\xi|/k^2$, which is needed in order to be able to control some of the nonlinear terms using the Cauchy–Kowalevski terms coming from time differentiation of the energy functional $\mathcal{E}$ in (3.16). This leads to loss of regularity of the profile F during the evolution, which is the price to pay to prove nonlinear decay of the stream function ϕ.

(3) Finally, to prove commutator estimates in the context of our problem, we need to know that the weights vary sufficiently slowly in ξ, ideally something like $|A_k(t,\xi) - A_k(t,\eta)| \lesssim \langle k,\xi\rangle^{-1/2}[A_k(t,\xi)+A_k(t,\eta)]$ if $\langle \xi-\eta\rangle \lesssim 1$. This is not possible, however, in the framework of imbalanced weights as defined above. Our solution to this problem is to allow the weights to depend on another parameter $\delta \ll 1$, and prove weaker estimates of the form

$$|A_k(t,\xi) - A_k(t,\eta)| \lesssim \left[\frac{C(\delta)}{\langle k,\xi\rangle^{1/2}} + \sqrt{\delta}\right] \max\{A_k(t,\xi), A_k(t,\eta)\} \qquad (3.18)$$

if $\langle \xi-\eta\rangle \lesssim 1 \ll \min\{\langle k,\xi\rangle, \langle k,\eta\rangle\}$. Such bounds are still suitable to control the commutators, due to the gain of $\sqrt{\delta}$ for large frequencies.

3.3 The Auxiliary Nonlinear Profile

In the case of general shear flows, an essential new difficulty that is not present in the Couette case, is the additional linear term $B''(t,v)\partial_z\phi$ in (3.6). This extra linear term cannot be treated as a perturbation if b'' is not assumed to be small. On the linearized level, one can understand the evolution by using spectral analysis, especially the regularity analysis of generalized eigenfunctions corresponding to the continuous spectrum. However, it is still a challenge to combine the linear spectral analysis with the more sophisticated Fourier analysis tools needed for controlling the nonlinearity. We deal with this basic issue in two steps: first we define an auxiliary nonlinear profile $F^*(t)$ given by

$$F^*(t,z,v) = F(t,z,v) - \int_0^t B''(0,v)\partial_z\phi'(s,z,v)\,ds. \qquad (3.19)$$

Thus F^* takes into account the linear effect accumulated up to time t and can be bounded perturbatively, using weighted energy estimates. The function ϕ' (not to be confused with the derivative of ϕ) is a small but crucial modification of ϕ, obtained by freezing the coefficients of the elliptic equation defining stream functions at time $t = 0$, in order to keep these coefficients very smooth. See (3.12) for the precise definition.

On a heuristic level, we expect that the full evolution of F consists of two contributions: the main, linear evolution that changes the size of the profile most significantly, and a small but rough (compared with the linear evolution) nonlinear correction. We can view (3.19) as a bounded linear transformation in both space and time from F to F^* which takes into account the bulk linear evolution. Remarkably, the transformation (3.19) can be chosen independently of the nonlinear evolution, once the nonlinear change of coordinates is fixed, and can be studied using just linear analysis. The key point is that this transformation can be inverted to get bounds on the full profile F from bounds on F^*, see Section 3.4 below for an outline of this construction.

The modified profile F^* now evolves in a perturbative fashion, and can be bounded using the method in [17]. However, this construction leads to loss of symmetry in the transport terms $V'\partial_v P_{\neq 0}\phi\, \partial_z F$ and $(\dot{V} + V'\partial_z\phi)\, \partial_v F$, since the main perturbative variable is now F^*. This loss of symmetry causes a derivative loss, so we need to prove stronger bounds on $F - F^*$ than on the variables F, F^*, as described in (3.13)–(3.14).

3.4 Control of the Full Profile

We still need to recover the bounds on F and the improved bounds on $F - F^*$. Since the bounds on F^* are already proved, it suffices to prove the improved bounds (3.15) for $F - F^*$.

This is a critical step where we need to use our main spectral assumption and the precise estimates on the linearized flow. To link $F - F^*$ with the linearized flow, we define an auxiliary function $\phi^* : [0, T] \times \mathbb{T} \times [0, 1] \to \mathbb{R}$ (heuristically a stream function associated with F^*), as a solution of the elliptic equation

$$\partial_z^2\phi^* + (B_0')^2(\partial_v - t\partial_z)^2\phi^* + B_0''(\partial_v - t\partial_z)\phi^* = F^*, \quad \phi^*(t, z, b(0)) = \phi^*(t, z, b(1)) = 0$$

(compare with the definition (3.11)). Now setting $h := F - F^*$, $\psi := \phi' - \phi^*$, the functions h and ψ satisfy the inhomogeneous linear system with trivial initial data

$$\begin{aligned} &\partial_t h - B_0''(v)\partial_z\varphi = H, \qquad h(0, z, v) = 0, \\ &B_0'(v)^2(\partial_v - t\partial_z)^2\psi + B_0''(v)(\partial_v - t\partial_z)\psi + \partial_z^2\psi = h(t, z, v), \end{aligned} \tag{3.20}$$

where $(t, z, v) \in [0, \infty) \times \mathbb{T} \times [b(0), b(1)]$. The functions $B_0'(v) = B'(0, v)$ and $B_0''(v) = B''(0, v)$ are time-independent, very smooth, and can be expressed in terms of the original shear flow b. The source term H is given by $H = B_0''(v)\partial_z\phi^*$.

The function ϕ^* is determined by the auxiliary profile F^*. Since we have already proved quadratic bounds on the profile F^*, we can use elliptic estimates to prove quadratic bounds on ϕ^*, and then on the source term H.

Therefore, we can think of (3.20) as a linear inhomogeneous system with trivial initial data, and attempt to adapt the linear theory to our situation. Decomposing in modes, conjugating by e^{-ikvt}, and using Duhamel's formula, we can further reduce to the study of the homogeneous initial-value problem

$$\begin{aligned} &\partial_t g_k + ikvg_k - ikB_0''\varphi_k = 0, \qquad g_k(0, v) = X_k(v)e^{-ikav}, \\ &(B_0')^2\partial_v^2\varphi_k + B_0''(v)\partial_v\varphi_k - k^2\varphi_k = g_k, \qquad \varphi_k(b(0)) = \varphi_k(b(1)) = 0 \end{aligned} \tag{3.21}$$

for $(t, v) \in [0, \infty) \times [b(0), b(1)]$, where $k \in \mathbb{Z} \setminus \{0\}$ and $a \in \mathbb{R}$.

3.4.1 Analysis of the Linearized Flow

The equation (3.21) was analyzed, at least when $a = 0$, in [36] and [23]. We follow the approach in [23]. The main idea is to use the spectral representation formula and reduce the analysis of the linearized flow to the analysis of generalized eigenfunctions corresponding to the continuous spectrum. More precisely, using general spectral theory, we can express the stream function as an oscillatory integral of the spectral density function (which depends both on the physical and the spectral variables), as in the formula (2.2). An important new feature in the analysis of the linearized equation here is that we have to consider initial data with an oscillatory factor, see (3.21), and the norms we use to measure the spectral density function are adapted to the oscillatory factor. It is well known that the generalized eigenfunctions contain singularities. To obtain precise characterization of these singularities, we make suitable re-normalizations and estimate the resulting functions in Gevrey spaces.

As a result, given data X_k smooth and satisfying $\operatorname{supp} X_k \subseteq [b(\vartheta_0), b(1-\vartheta_0)]$ we find a representation formula

$$\widetilde{g_k}(t,\xi) = \widetilde{X_k}(\xi + kt + ka) + ik \int_0^t \int_{\mathbb{R}} \widetilde{B_0''}(\zeta)\widetilde{\Pi_k'}(\xi + kt - \zeta - k\tau, \xi + kt - \zeta, a)\, d\zeta\, d\tau$$

for the solution g_k of the linear evolution equation (3.21), where $\Pi_k'(\xi,\eta,a)$ can be expressed in terms of the generalized eigenfunctions. These eigenfunctions cannot be calculated explicitly, but can be estimated very precisely in the Fourier space,

$$\left\| (|k| + |\xi|) W_k(\eta + ka) \widetilde{\Pi_k'}(\xi,\eta,a) \right\|_{L^2_{\xi,\eta}} \lesssim_\delta \left\| W_k(\eta)\widetilde{X_k}(\eta) \right\|_{L^2_\eta}$$

for any $a \in \mathbb{R}$, using the limiting absorption principle Lemma 2.1, provided that the weights W_k satisfy smoothness properties of the type

$$|W_k(\xi) - W_k(\eta)| \lesssim e^{2\delta_0 \langle \xi - \eta \rangle^{1/2}} W_k(\eta) \left[\frac{C(\delta)}{\langle k, \eta \rangle^{1/8}} + \sqrt{\delta} \right] \quad \text{for any } \xi, \eta \in \mathbb{R}. \tag{3.22}$$

The inequality (3.22) holds for standard weights, like polynomial weights $W_k(\xi) = (1 + |\xi|^2)^{N/2}$, which correspond to Sobolev spaces, or exponential weights $W_k(\xi) = e^{\lambda \langle \xi \rangle^s}$, $s < 1/2$, which correspond to Gevrey spaces. More importantly, it also holds for our carefully designed weights $A_k(t,\xi)$, as we have already discussed in (3.18). This allows us to adapt and incorporate the linear theory, and close the argument.

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