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ORIGINAL ARTICLE

On Well-Posedness of Nonlocal Evolution Equations

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Abstract
This work studies questions of existence, uniqueness, dependence on initial data, and regularity of solutions to the Cauchy problem for nonlocal evolution equations with data in Sobolev spaces. The focus is on integrable Camassa–Holm type equations and in particular the Novikov equation and its dispersive modification. These equations apart from being interesting on their own right, also they can serve as "toy" models for the Euler equations.

Keywords Camassa–Holm equation · Modified Novikov equation · Nonlocal equations · Initial value problem · Well-posedness · Sobolev spaces · Nonuniform dependence · Trilinear estimates · Bourgain spaces

Mathematics Subject Classification (2010) Primary 35Q55 · 35G31 · 35G16 · 37K10

1 Introduction and Results

In this work, we investigate questions of existence, uniqueness, dependence on initial data, and regularity of solutions to the initial value problem of nonlocal evolution equations. Our focus is on Camassa–Holm type equations with initial data in Sobolev spaces. We begin with four integrable equations, which Vladimir Novikov [69] derived in a unified way by examining the question of integrability for Camassa–Holm type equations of the form

$$(1-\partial_x^2)u_t = P(u, u_x, u_{xx}, u_{xxx}, \dots),$$

where P is a polynomial of u and its x-derivatives. Using as definition of integrability the existence of an infinite hierarchy of (quasi-) local higher symmetries, he produced about

Dedicated to Professor Carlos Kenig

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20 integrable equations with quadratic nonlinearities that include the Camassa–Holm (CH) equation (see [14])

$$\partial_t u + \frac{1}{2}\partial_x(u^2) + (1-\partial_x^2)^{-1}\partial_x\left[u^2 + \frac{1}{2}(\partial_x u)^2\right] = 0$$

and the Degasperis–Procesi (DP) equation (see [24])

$$\partial_t u + u\partial_x u + (1-\partial_x^2)^{-1}\partial_x\left[\frac{3}{2}u^2\right] = 0,$$

both written in their nonlocal evolutionary form. Also, he produced about 10 integrable equations with cubic nonlinearities that include the following one

$$\partial_t u + \frac{1}{3}\partial_x(u^3) + (1-\partial_x^2)^{-1}\partial_x\left[u^3 + \frac{3}{2}u(\partial_x u)^2\right] + (1-\partial_x^2)^{-1}\left[\frac{1}{2}(\partial_x u)^3\right] = 0,$$

which is called the Novikov equation (NE) and the Fokas–Olver–Rosenau–Qiao (FORQ) equation

$$\partial_t u + \frac{1}{3}\partial_x(u^3) - \frac{1}{3}(\partial_x u)^3 + (1-\partial_x^2)^{-1}\partial_x\left[\frac{2}{3}u^3 + uu_x^2\right] + (1-\partial_x^2)^{-1}\partial_x\left[\frac{1}{3}u_x^3\right] = 0,$$

which was derived earlier in [28, 70] and [72], and which is the only non-quasilinear integrable CH type equation considered here.

The CH equation arose initially in the context of hereditary symmetries studied by Fuchssteiner and Fokas [29]. However, it was written explicitly as a water wave equation by Camassa and Holm [14], who derived it from the Euler equations of hydrodynamics using asymptotic expansions. Also, they derived its peakon solutions. The existence of peakon solutions is a common phenomenon of CH, DP, NE and FORQ. On the line, these are of the form

$$\begin{aligned}&\text{CH, DP: } u(x,t) = ce^{-|x-ct|}, \quad \text{NE: } u(x,t) = \sqrt{c}e^{-|x-ct|},\\ &\text{FORQ: } u(x,t) = \sqrt{\frac{3c}{2}}e^{-|x-ct|}.\end{aligned} \tag{1.1}$$

In fact, the discovery of CH by Camassa and Holm in 1993 was partly driven by the desire to find a water wave equation which has traveling wave solutions that "break" (see [76]). Recall, that the celebrated Korteweg–de Vries (KdV) equation [10, 57]

$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0,$$

which is a model of long water waves propagating in a channel, has only smooth solitons like

$$u(x,t) = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right).$$

Also, CH, DP and NE have multipeakon solutions of the form

$$u(x,t) = \sum_{j=1}^{n} p_j(t)e^{-|x-q_j(t)|},$$

which will be discussed further later since they play an important role in the well-posedness theory of these equations. For more about peakon and multipeakon solutions of Camassa–Holm type equations we refer the reader to [2, 4, 5, 20, 62] and the references therein.

Next, we provide a brief survey on the well-posedness results for these equations. We begin by recalling the definition of well-posedness in Sobolev spaces on the line or the circle

$H^s = H^s$ ($\mathbb{R}$ or $\mathbb{T}$) according to Hadamard, which is expressed by the following three properties:

(i) For any initial data $u(0) \in H^s$ there exists (a lifespan) $T = T_{u(0)} > 0$ and a solution $u \in C([0, T]; H^s)$ to the CH (DP, NE or FORQ) Cauchy problem.
(ii) This solution u is unique in the Hadamard space $C([0, T]; H^s)$.
(iii) The data-to-solution map $u(0) \mapsto u(t)$ is continuous from H^s into $C([0, T]; H^s)$.

The CH and DP equations Next result states that for $s > 3/2$ CH and DP are well-posed in the Sobolev spaces H^s in the sense of Hadamard and the dependence of solutions on initial data is sharp, that is, it is not better than continuous.

Theorem 1.1 (CH and DP well-posedness) *The Cauchy problem for the CH and DP equations is well-posed in H^s for $s > 3/2$. More precisely, if $s > 3/2$ and $u_0 \in H^s$ then there exist $T > 0$ and a unique solution $u \in C([0, T]; H^s)$ of the initial value problem for CH and DP, which depends continuously on the initial data u_0. Furthermore, we have the estimate*

$$\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s} \quad \text{for } t \geq 1/(2c_s\|u_0\|_{H^s}) \doteq T, \tag{1.2}$$

where $c_s > 0$ is a constant depending on s.

(Nonuniform Dependence) Furthermore, the data-to-solution map for these equations is not uniformly continuous from any bounded subset in H^s into $C([0, T]; H^s)$.

Well-posedness of CH on the circle was proved in [44] by writing it as an ODE in a Banach space where one can prove existence and uniqueness of solutions and their continuous dependence on the initial data when these belong to a Sobolev space $H^s(\mathbb{T})$ with $s > 3/2$. This method follows Arnold's approach [1] for the study of the Euler equations (see [25]). On the real line, well-posedness of CH for $s > 3/2$ was proved by Li and Olver [63] using a regularization technique like in Bona and Smith [7]. The well-posedness of CH was also studied by Danchin [22], Misiolek [68] and Rodriguez-Blanco [73] using various approaches. Other works about CH can be found in [11, 12, 16–19, 23, 45, 47, 49, 50, 52, 66, 67], and [3]. For the well-posedness of DP we refer to [77] and in [34]. At this point we note that the peakon solutions $u(t)$ defined in (1.1) belong in $H^{\frac{3}{2}^-} - H^{\frac{3}{2}}$. Therefore, these solutions are not covered by the (strong) local well posedness results mentioned in this paper. However, they are Lipchitz functions, i.e. they belong in $W^{1,\infty}$. But, the translation operator is not continuous in $W^{1,\infty}(\mathbb{R})$. So, for initial data $X \equiv H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ (containing the peakons) one can only expect to get a unique local solution in $u \in C([-T, T]; H^1(\mathbb{R})) \cap L^\infty([-T, T]; W^{1,\infty}(\mathbb{R}))$, with the data-to-solution map continuous from X to $C([-T, T]; H^1(\mathbb{R}))$. For CH on the circle, this result was first established by de Lellis, Kappeler and Toplalov [23]. For CH on the line, it was proved more recently by Linares, Ponce and Sideris in [64], where they also extended the decay results obtained in [48] to the class of solutions containing peakons.

Nonuniform dependence for CH on $\mathbb{R}$ was proved in [39] by using the method of approximate solutions together with the well-posedness estimate (1.2) derived there. The periodic case was done in [40] using delicate commutator and multiplier estimates in addition to approximate solutions. For the case of DP Theorem 1.1 was proved in [34]. It is interesting that the approximate solutions used for both the CH and DP are the same. Next, we provide a brief description of the method of approximate solutions, which in the context of the Benjamin–Ono equation was first used Koch and Tzvetkov [53]. We prove that there exist two sequences of CH or DP solutions $u_n(t)$ and $v_n(t)$ in $C([0, T]; H^s(\mathbb{R}))$ satisfying the following three conditions:

(1) $\sup_n \|u_n(t)\|_{H^s} + \sup_n \|v_n(t)\|_{H^s} \lesssim 1$,

(2) $\lim_{n\to\infty} \|u_n(0) - v_n(0)\|_{H^s} = 0$,
(3) $\liminf_n \|u_n(t) - v_n(t)\|_{H^s} \gtrsim \sin t$, $0 \le t < T \le 1$.

We do this in the periodic case. For both the CH and DP the approximate solutions

$$u^{\omega,n}(x,t) = \omega n^{-1} + n^{-s}\cos(nx - \omega t) \quad \text{for } \omega = -1, 1,$$

where $n \in \mathbb{Z}^+$, satisfy conditions (1)–(3) for nonuniform dependence **but they are not solutions**. However, the error $E = CH(u^{\omega,n})$ or $E = DP(u^{\omega,n})$ is small in the H^s-norm. Then, solving the Cauchy problem with initial data $u^{\omega,n}(x,0)$ gives actual solutions, which one proves that they satisfy the three conditions of nonuniform continuity.
Ill-posedness and norm inflation The following result shows that $s = 3/2$ is critical for the well-posedness for CH and DP in Sobolev spaces.

Theorem 1.2 (Ill-posedness) *For $s < 3/2$, the Cauchy problem for CH and DP is not well-posed in H^s in the sense of Hadamard.*

For DP this result has been proved in [37]. If $1/2 \le s < 3/2$, then ill-posedness is due to *norm inflation*. This means that there exist DP solutions who are initially arbitrarily small and eventually arbitrarily large with respect to the H^s norm, in an arbitrarily short time. Since DP solutions conserve a quantity equivalent to the L^2-norm, there is no norm inflation in H^0 for these solutions. In this case, ill-posedness is caused by failure of uniqueness. For all other $s < 1/2$, the situation is similar to H^0. For the CH we have norm inflation and thus failure of continuity for $s \in (1, \frac{3}{2})$, and failure of uniqueness for $s < 1$ (see [13]). For DP norm inflation is described by the following result in [37].

Proposition 1.1 (Norm Inflation) *If $s \in [\frac{1}{2}, \frac{3}{2})$, then for any $\varepsilon > 0$ there is $T > 0$ such that the DP Cauchy problem has a solution $u \in C([0,T); H^s)$ satisfying the following three properties:*

(1) *Lifespan $T < \varepsilon$,*
(2) *$\|u_0\|_{H^s} < \varepsilon$,*
(3) *$\lim_{t\to T^-} \|u(t)\|_{H^s} = \infty$ (norm inflation).*

The proof of Proposition 1.1 exploits the properties of appropriately constructed two-peakon solutions, which are called peakon-antipeakon solutions. We shall describe our approach for the case of the non-periodic DP equation. In this case we have that the peakon-antipeakon traveling wave

$$u(x,t) = p(t)e^{-|x+q(t)|} - p(t)e^{-|x-q(t)|}$$

is a weak solution to the DP equation if the momentum $p = p(t)$ and the position $q = q(t)$ are solutions to the following system of ordinary differential equations

$$q' = p(e^{-2q} - 1) \quad \text{and} \quad p' = 2p^2 e^{-2q}. \tag{1.3}$$

Furthermore, if for given $\varepsilon > 0$ we choose p_0 and q_0 so that $p_0 \ge 1/\varepsilon$ and $q_0 < \ln 2/8$, then there exist $0 < T < \varepsilon$ such that the ODE system (1.3) has a unique smooth solution $(q(t), p(t))$ in the interval $[0, T)$ with $p(t)$ increasing, $q(t)$ decreasing, and

$$\lim_{t\to T^-} p(t) = \infty \quad \text{and} \quad \lim_{t\to T^-} q(t) = 0.$$

Using these properties of peakon-antipeakon solutions one can derive the following fundamental estimate on the H^s size of the solution $u(t)$ for $t \in [0, T)$

$$\|u(t)\|_{H^s} \approx \begin{cases} p(t)q(t)^{3/2-s}, & 1/2 < s < 3/2, \\ p(t)q(t)\sqrt{\ln(1/q(t))}, & s = 1/2, \\ p(t)q(t), & s < 1/2. \end{cases}$$

Then, using this estimate one proves Proposition 1.1.

Also, we describe the proof of DP ill-posedness for $1/2 \le s < 3/2$, where the continuity of the data-to-solution map fails. Let $u_n(t)$ be the peakon-antipeakon DP solution corresponding to the choice of $\varepsilon = 1/n$ and let $u_\infty(t) = 0$. Then, by property (2) in norm inflation result we have $\|u_n(0)\|_{H^s} < 1/n$. So, $u_n(0)$ converges to $u_\infty(0) = 0$ in H^s. Furthermore, by property (1) the lifespan T_n of each solution $u_n(t)$ satisfies the inequality $T_n < 1/n$, whereas the lifespan T_∞ of $u_\infty(t)$ is equal to ∞. Since, by property (3) in norm inflation result there is no $T > 0$ such that the solutions $u_n(t)$ can be extended to the interval $[0, T]$ for all sufficiently large n we see that continuity condition (iii) of well-posedness fails.

Remark 1.1 Theorem 1.1 about the local well-posedness and nonuniform dependence for CH and DP is also true for the b-family equation (which contains CH when $b = 2$ and DP when $b = 3$)

$$(1 - \partial_x^2)u_t = -(b+1)uu_x + bu_xu_{xx} + uu_{xxx}$$

for $s > 3/2$ and all $b \in \mathbb{R}$ (see [21, 27, 30]). For $s < 3/2$, in [33] it is shown that the Cauchy problem for the b-family of equations is ill-posed in Sobolev spaces H^s on both the torus and the line when $b > 1$. For $b \le 1$ it remains an **open** question.

The Euler equations The CH equation apart from being interesting because of its integrability properties and breaking traveling wave solutions, it can serve as a "toy" model for the Euler equations. This is demonstrated by the following result in [46].

Theorem 1.3 *The solution map $u_0 \mapsto u$ of the Cauchy problem for the Euler equations*

$$\partial_t u + \nabla_u u + \nabla p = 0, \quad \operatorname{div} u = 0, \quad u(0, x) = u_0(x),$$

is not uniformly continuous from the unit ball in $H^s(\mathbb{T}^n, \mathbb{R}^n)$ into the space $C([0, T]; H^s(\mathbb{T}^n, \mathbb{R}^n))$ for any $s \in \mathbb{R}$, in the periodic case. While, in the non-periodic case it is not uniformly continuous from the unit ball in $H^s(\mathbb{R}^n, \mathbb{R}^n)$ into $C([0, T]; H^s(\mathbb{R}^n, \mathbb{R}^n))$ for any $s > 0$.

The proof of Theorem 1.3 is based on well-posedness theory (as it has been developed by Ebin and Marsden [25], Kato and Ponce [54], Majda and Bertozzi [65]) and approximate solutions. Furthermore, it suffices to prove it in two dimensions, since filling the rest of the velocity components with zeros gives the proof in higher dimensions. Here, we shall provide the proof in the periodic case. Motivated by CH, one would think that the following approximate solutions

$$u^{\omega,n}(t, x) = \big(\omega n^{-1} + n^{-s}\cos(nx_2 - \omega t),\ \omega n^{-1} + n^{-s}\cos(nx_1 - \omega t)\big)$$

is a natural choice. They are divergence free and in one-dimension they collapse to CH approximate solutions. In fact, these are **actual solutions!** to the Euler equations. The corresponding to $\omega = \pm 1$ sequences $u^{+1,n}(t, x)$ and $u^{-1,n}(t, x)$ satisfy the conditions for nonuniform dependence of the periodic Euler equations in two dimensions. The proof in the non-periodic case is technical since one needs to take care of the errors introduced by the cut-offs that localize the approximate solutions. The localized approximate solutions are **not** solutions. Bourgain and Li [9] improved this result in the non-periodic case and included the case $s = 0$.

The NE and FORQ equations The well-posedness theory developed for the CH equations with quadratic nonlinearities has been extended to the ones with cubic nonlinearities. Next result summarizes this development for NE [35] and FORQ [41].

Theorem 1.4 (Well-posedness for NE and FORQ) *If $s > 3/2$ in the case of NE and $s > 5/2$ in the case of FORQ, then for $u_0 \in H^s$ there exist $T > 0$ and a unique solution $u \in C([0, T]; H^s)$ of the initial value problem for NE or FORQ which depends continuously on the initial data u_0. Furthermore,*

$$\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s} \quad \text{for } t \geq \frac{1}{4c_s\|u_0\|_{H^s}^2} \doteq T,$$

where $c_s > 0$ is a constant depending on s.

(Nonuniform dependence) Also, the data-to-solution map is not uniformly continuous from any bounded subset in H^s into $C([0, T]; H^s)$.

The presence of the cubic nonlinearities in these equations makes the proofs more technical. At the idea level, an important difference from DP and CH is that for NE the approximate solutions used

$$u^{\omega,n} = \omega n^{-1/2} + n^{-s}\cos(nx - \omega t) \quad \text{for } \omega = 0, 1,$$

(on the circle) are asymmetric. That is, one of them ($\omega = 0$) has **no low frequency**. Also, the low frequency term in the other is $n^{-1/2}$ instead of n^{-1}, which is the case for CH and DP.

Ill-posedness of the Novikov equation For the Novikov equation on both the line and the circle, in [38] the first author, Holliman and Kenig constructed a 2-peakon solution with an asymmetric antipeakon-peakon initial profile whose H^s-norm for $s < 3/2$ is arbitrarily small. Immediately after the initial time, both the antipeakon and peakon move in the positive direction, and a collision occurs in arbitrarily small time. Moreover, at the collision time the H^s-norm of the solution becomes arbitrarily large when $5/4 < s < 3/2$, thus resulting in norm inflation and ill-posedness. However, when $s < 5/4$, the solution at the collision time coincides with a second solitary antipeakon solution. This scenario thus results in nonuniqueness and ill-posedness. Finally, when $s = 5/4$ ill-posedness follows either from a failure of convergence or a failure of uniqueness. Considering that the Novikov equation is well-posed for $s > 3/2$ [35], these results put together establish $3/2$ as the critical index of well-posedness for this equation. This is summarized in the following result.

Theorem 1.5 *The Cauchy problem for the Novikov equation on the line and the circle is ill-posed in Sobolev spaces H^s for $s < 3/2$. More precisely, if $5/4 < s < 3/2$ then the data-to-solution map is not continuous while if $s < 5/4$ then solution is not unique. When $s = 5/4$ then either continuity or uniqueness fails.*

Remark 1.2 The case $s = 3/2$ has for CH and NE has been studied in [32] where it is shown that the phenomenon of norm inflation occurs which implies ill-posedness when $s = 3/2$. For FORQ and $s < 3/2$ it is shown in [36] that we have non-uniqueness of solution and thus ill-posedness. The case $3/2 \leq s \leq 5/2$ remains an **open** question.

Modified Novikov equation Next, we consider the Cauchy problem on the line for the modified by a third order dispersion Novikov equation denoted here by (mNE),

$$\partial_t u + \kappa\partial_x^3 u + \frac{1}{3}\partial_x(u^3) + (1-\partial_x^2)^{-1}\partial_x\left[u^3 + \frac{3}{2}u(\partial_x u)^2\right] + (1-\partial_x^2)^{-1}\left[\frac{1}{2}(\partial_x u)^3\right] = 0, \tag{1.4a}$$

$$u(x,0) = \varphi(x), \quad t\in\mathbb{R}, \quad x\in\mathbb{R}, \tag{1.4b}$$

and study its well-posedness in the Sobolev spaces $H^s(\mathbb{R})$. Here, $\kappa > 0$ is a parameter. Note that, like in the case of NE, solutions u to this equation conserve their H^1 norm. In fact, using the local form of mNE, which results from multiplying mNE by $(1-\partial_x^2)$,

$$\partial_t u - \partial_t\partial_x^2 u + \kappa\partial_x^3 u - \kappa\partial_x^5 u + 4u^2\partial_x u - 3u\partial_x u\partial_x^2 u - u^2\partial_x^3 u = 0, \tag{1.5}$$

and doing integration by part we get

$$\begin{aligned}\frac{d}{dt}\int[u^2+(\partial_x u)^2]dx &= 2\int u\cdot\left[\partial_t u - \partial_x^2\partial_t u\right]dx\\ &\overset{(1.5)}{=} 2\int u\cdot\left[-\kappa\partial_x^3 u + \kappa\partial_x^5 u - 4u^2\partial_x u + 3u\partial_x u\partial_x^2 u + u^2\partial_x^3 u\right]dx\\ &= 2\int\left[\partial_x\left(u^3\partial_x^2 u\right) - \partial_x\left(u^4\right)\right]dx = 0.\end{aligned}$$

Next, to state our results for mNE Cauchy problem (1.4) precisely, we recall the definition of Bourgain spaces used in the well-posedness theory of dispersive equations (see, for example, [8, 15, 56]). For any $s, b\in\mathbb{R}$, the Bourgain space $X^{s,b} = X^{s,b}(\mathbb{R}^2)$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$\|u\|_{s,b} \doteq \|u\|_{X^{s,b}} \doteq \left(\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|)^{2s}(1+|\tau-\xi^3|)^{2b}|\widehat{u}(\xi,\tau)|^2 d\xi d\tau\right)^{1/2},$$

where $\widehat{u}$ denotes the space-time Fourier transform defined by

$$\widehat{u}(\xi,\tau) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i(x\xi+t\tau)}u(x,t)dxdt.$$

Also, for $T>0$, $X_T^{s,b}$ denotes the restricted Bourgain space defined by

$$X_T^{s,b} = \{u : u(x,t) = v(x,t) \text{ on } \mathbb{R}\times(-T,T) \text{ with } v\in X^{s,b}(\mathbb{R}^2)\}. \tag{1.6}$$

Finally, we recall that the norm in $X_T^{s,b}$ is defined by

$$\|u\|_{X_T^{s,b}} = \inf_{v\in X^{s,b}}\left\{\|v\|_{s,b} : v(x,t) = u(x,t) \text{ on } \mathbb{R}\times(-T,T)\right\}. \tag{1.7}$$

With the needed definitions in place, next we state our well-posedness result for mNE proved here.

Theorem 1.6 (Local well-posedness) *If $s > \frac{2}{3}$, then for any $\varphi\in H^s(\mathbb{R})$ there exist a time $T_0 = T_0(\|\varphi\|_{H^s}) > 0$ and a unique solution u of the Cauchy problem (1.4) on the time interval $[-T_0, T_0]$, such that $u\in X_{T_0}^{s,b}\cap C([-T_0,T_0]; H^s(\mathbb{R}))$, with*

$$T_0 = \frac{c_0}{(1+\|\varphi\|_{H^s}^2)^{\frac{2}{\beta}}}, \quad \beta \doteq \min\left\{\frac{1}{6}(s-\frac{2}{3}), \frac{1}{8}\right\}, \tag{1.8}$$

for some constants $c_0 = c_{s,b} > 0$ depending only on s and b. Moreover, the local solution u satisfies the following size estimate in Bourgain spaces

$$\|u\|_{X^{s,b}_{T_0}} \leq C(s,b)\|\varphi\|_{H^s},$$

for some $b \in (1/2, 1)$. Also, we have the Hadamard space estimate

$$\sup_{|t|\leq T_0} \|u(t)\|_{H^s} \leq C_0(s,b)\|\varphi\|_{H^s}. \tag{1.9}$$

Finally, the solution depends Lip-continuously on the data φ.

Considering that mNE solutions conserve their H^1 norm, from Theorem 1.6 we get the next result.

Theorem 1.7 (Global well-posedness) *The mNE is globally well-posed in the Sobolev space H^1.*

Theorems 1.6 and 1.7 are motivated by the corresponding results obtained in [13, 42, 43] for the following dispersive modification of the CH equation, denoted by mCH,

$$\partial_t u + \kappa\partial_x^3 u + \frac{1}{2}\partial_x(u^2) + (1-\partial_x^2)^{-1}\partial_x\left[u^2 + \frac{1}{2}(\partial_x u)^2\right] = 0,$$

which is analogous to the KdV regularization $u_t - 6uu_x + \varepsilon^2 u_{xxx} = 0$ of the Burgers equation $u_t - 6uu_x = 0$, with $\kappa = \varepsilon^2$, by Lax and Levermore [58–61]. Although the question of what happens in mCH when the parameter κ goes to zero remains **open**, some numerical results in the periodic case have been obtained in [31]. The above-mentioned work of Lax and Levermore on the small dispersion limit of the Korteweg–de Vries equation provides a good motivation for investigating the corresponding problem for both mCH and mNE. Concerning the proof of well-posedness for mCH, we note that it is based on the KdV bilinear estimate $\|\partial_x(f\cdot g)\|_{X^{s,b-1}} \leq c_s\|f\|_{X^{s,b}}\|g\|_{X^{s,b}}$ in [56] that holds for $s > -3/4$ and some $b \in (\frac{1}{2}, 1)$, with f and g replaced with $\partial_x f$ and $\partial_x g$ and therefore is valid for $s > 1 - \frac{3}{4} = \frac{1}{4}$. This explains why mCH is well-posed for $s > \frac{1}{4}$. Now, considering that the mKdV is well-posed for $s \geq \frac{1}{4}$ [55] and its trilinear estimates hold for $s \geq \frac{1}{4}$ [75], this idea transferred to the mNE situation will give well-posedness for $s \geq 1+\frac{1}{4} = \frac{5}{4}$, which is worse than the well-posedness for $s > \frac{2}{3}$ obtained here. This seems to be an interesting phenomenon for mNE, which is explained by the improved trilinear estimates for the nonlocal nonlinearities stated in the next section. Concerning the mKdV, we mention that its well-posedness in H^s was proved first by Kenig, Ponce and Vega [55] for $s \geq \frac{1}{4}$ with a different method without using its trilinear estimate in Bourgain spaces (also, see [71]). Also, we mention that our proof of the mKdV trilinear estimate provided here for $s > \frac{1}{4}$ is different than the one of Tao in [75]. Concluding, we mention that the sharpness of Theorem 1.6 is an interesting **open** question. Constructing a counterexample to the nonlocal trilinear estimates (2.18) could provide useful information.

The rest of the paper is organized as follows. In Section 2 we state the three trilinear estimates needed and use them to prove Theorem 1.6. In Section 3 we prove first the trilinear estimate for the local nonlinearity $\partial_x(u^3)$, and then we use it to prove the trilinear estimates for nonlocal nonlinearities $(1-\partial_x^2)^{-1}\partial_x[u(\partial_x u)^2]$ and $(1-\partial_x^2)^{-1}[(\partial_x u)^3]$.

2 Well-posedness in Sobolev spaces—Proof of Theorem 1.6

In what follows we will assume the coefficient κ of the dispersion $\partial_x^3 u$ is equal to 1, that is $\kappa = 1$. When $\kappa > 0$ and $\kappa \neq 1$, then the constants appearing in some of the estimates depend on k.

We begin the well-posedness proof by solving the initial value problem (ivp) for the linear mNE

$$\partial_t u + \partial_x^3 u = -w(x,t), \tag{2.1}$$

$$u(x,0) = \varphi(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{2.2}$$

with forcing w the mNE nonlinearity $w = w_{uuu}$, where

$$\begin{aligned} w = w_{fgh} &\doteq \frac{1}{3}\partial_x(f \cdot g \cdot h) + (1-\partial_x^2)^{-1}\partial_x \left[f \cdot g \cdot h + \frac{3}{2}(f \cdot g_x \cdot h_x)\right] \\ &+ (1-\partial_x^2)^{-1}\left[\frac{1}{2}(f_x \cdot g_x \cdot h_x)\right]. \end{aligned} \tag{2.3}$$

With this definition, we have the following formula to be used later

$$\begin{aligned} w_{uuu} - w_{vvv} = &\frac{1}{3}\partial_x[(u-v)(u^2+uv+v^2)] + (1-\partial_x^2)^{-1}\partial_x[(u-v)(u^2+uv+v^2)] \\ &+ \frac{3}{2}(1-\partial_x^2)^{-1}\partial_x\left[(u-v)u_x^2 + v(u-v)_x(u+v)_x\right] \\ &+ \frac{1}{2}(1-\partial_x^2)^{-1}\left[(u-v)_x(u_x^2 + u_x v_x + v_x^2)\right]. \end{aligned} \tag{2.4}$$

We solve initial value problem (ivp) (2.1) by taking Fourier transform with respect to x, which for a test function $\phi(x)$ is defined by the familiar formula

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}} e^{-i\xi x}\phi(x)dx,$$

and solving the resulting DE ivp in the t variable. Thus, we obtain the Duhamel's formula

$$u(x,t) = W(t)\varphi(x) - \int_0^t W(t-t')w(x,t')\,dt', \tag{2.5}$$

where $W(t)\varphi(x)$ denotes the solution to the homogeneous Cauchy problem for the linear mNE with initial data φ, that is

$$W(t)\varphi(x) \doteq \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\xi x + \xi^3 t)}\widehat{\varphi}(\xi)d\xi.$$

Now, to solve the mNE ivp locally, we introduce the usual time localizer, which is a cut-off function

$$\psi \in C_0^\infty(-1,1), \quad 0 \leq \psi \leq 1, \quad \text{and} \quad \psi(t) = 1 \ \text{ for } |t| \leq 1/2. \tag{2.6}$$

Multiplying the right-hand side (2.5) by $\psi(t)$ we obtain the following global form

$$u(x,t) = \psi(t)W(t)\varphi(x) - \psi(t)\int_0^t W(t-t')w(x,t')\,dt' \doteq \Phi u(x,t). \tag{2.7}$$

Note that for $|t| \leq 1/2$ global formulation (2.7) coincides with local formulation (2.5). Thus, our strategy for proving existence of a local solution of our ivp is to show that the map Φ defined in (2.7) has a fixed point in appropriate solution space, via a fixed point argument. For this, we assume that the forcing w is globally defined and using the relation $\widehat{w}^x(\xi, t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau t'}\widehat{w}(\xi,\tau)d\tau$ we obtain the following decomposition of the map Φ, like in [8],

$$\Phi u(x,t) = \frac{1}{2\pi}\psi(t)\int_{-\infty}^{\infty} e^{i(\xi x+\xi^3 t)}\widehat{\varphi}(\xi)d\xi \tag{2.8}$$

$$+\frac{i}{4\pi^2}\psi(t)\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\xi x+\tau t)}\frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3}\widehat{w}(\xi,\tau)d\tau d\xi \tag{2.9}$$

$$-\frac{i}{4\pi^2}\psi(t)\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\xi x+\xi^3 t)}\frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3}\widehat{w}(\xi,\tau)d\tau d\xi \tag{2.10}$$

$$+\frac{i}{4\pi^2}\psi(t)\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\xi x+\xi^3 t)}\frac{\psi(\tau-\xi^3)[e^{i(\tau-\xi^3)t}-1]}{\tau-\xi^3}\widehat{w}(\xi,\tau)d\tau d\xi. \tag{2.11}$$

Next, using this convenient for estimation in Bourgain spaces form of the iteration map Φ, we show that it is a contraction in an appropriate ball of the Bourgain $X^{s,b}$ for $s > \frac{2}{3}$ and some $b \in (\frac{1}{2}, 1)$. For proving that the map Φ is onto we begin with the inequality

$$\|\Phi u\|_{X^{s,b}} \le \|\psi(t)W(t)\varphi(x)\|_{s,b} + \left\|\psi(t)\int_0^t W(t-t')w(x,t')\,dt'\right\|_{s,b}, \quad w = w_{uuu}, \tag{2.12}$$

where w_{uuu} is given by formula (2.3), while to show that Φ is a contraction we begin with inequality

$$\|\Phi u - \Phi v\|_{X^{s,b}} \le \left\|\psi(t)\int_0^t W(t-t')w(x,t')\,dt'\right\|_{s,b}, \quad w = w_{uuu} - w_{vvv}, \tag{2.13}$$

where $w_{uuu} - w_{vvv}$ is given by formula (2.4). From inequalities (2.12) and (2.13) we see that we need the following linear estimates, whose proof follows from using the convenient writing (2.8)–(2.11) of Φu, and the definition of $X^{s,b}$-norm.

Lemma 2.1 (Linear estimates) *For any $s \in \mathbb{R}$ and $b > 0$ there is $c_1 = c_1(\psi, s, b)$ such that*

$$\|\psi(t)W(t)\varphi(x)\|_{s,b} \le c_1\|\varphi\|_{H^s}. \tag{2.14}$$

Also, for $s \in \mathbb{R}$ and $1/2 < b < 1$ we have

$$\left\|\psi(t)\int_0^t W(t-t')w(x,t')\,dt'\right\|_{s,b} \le c_1\|w\|_{s,b-1}. \tag{2.15}$$

Now, taking into consideration the form of mNE nonlinearities (2.3) we see that the following three trilinear estimates are the key ingredients for showing that our iteration map is a contraction, thus proving Theorem 1.6.

Proposition 2.1 (mKdV trilinear estimate) *If $s > \frac{1}{4}$, then there exist b and b' with $1/2 < b' \le b < 1$ such that the following trilinear estimate holds:*

$$\|\partial_x(fgh)\|_{X^{s,b-1}} \le c_2\|f\|_{X^{s,b'}}\|g\|_{X^{s,b'}}\|h\|_{X^{s,b'}}, \tag{2.16}$$

where $c_2 = c_2(s,b)$. In fact, b and b' depend on s and can be chosen as follows:

$$\frac{1}{2} < b' \le b \le \frac{1}{2} + \beta_1, \quad \text{where } \beta_1 = \beta_1(s) \doteq \min\left\{\frac{1}{6}\left(s-\frac{1}{4}\right), \frac{1}{8}\right\}.$$

Proposition 2.2 (First nonlocal trilinear estimate) *If $s > \frac{1}{2}$, then there exist b and b' with $1/2 < b' \le b < 1$ such that the following trilinear estimate holds:*

$$\|(1+\partial_x^2)^{-1}\partial_x[f(\partial_x g)(\partial_x h)]\|_{X^{s,b-1}} \le c_2\|f\|_{X^{s,b'}}\|g\|_{X^{s,b'}}\|h\|_{X^{s,b'}}, \tag{2.17}$$

where $c_2 = c_2(s, b)$. Also, b and b' depend on s and can be chosen as follows:

$$\frac{1}{2} < b' \le b \le \frac{1}{2} + \beta_2, \quad \textit{where } \beta_2 = \beta_2(s) \doteq \min\left\{\frac{1}{6}\left(s-\frac{1}{2}\right), \frac{1}{8}\right\}.$$

Proposition 2.3 (Second nonlocal trilinear estimate) *If $s > \frac{2}{3}$, then there exist b and b' with $1/2 < b' \le b < 1$ such that the following trilinear estimates holds:*

$$\|(1+\partial_x^2)^{-1}[(\partial_x f)(\partial_x g)(\partial_x h)]\|_{X^{s,b-1}} \le c_2\|f\|_{X^{s,b'}}\|g\|_{X^{s,b'}}\|h\|_{X^{s,b'}}, \tag{2.18}$$

where $c_2 = c_2(s, b)$. Finally, b and b' depend on s and can be chosen as follows:

$$\frac{1}{2} < b' \le b \le \frac{1}{2} + \beta_3, \quad \textit{where } \beta_3 = \beta_3(s) \doteq \min\left\{\frac{1}{6}\left(s-\frac{2}{3}\right), \frac{1}{8}\right\}.$$

Remark 2.1 Note that β_3 is the smallest of all parameters β_j used in the three trilinear estimates above. Therefore, all these estimates hold for

$$\frac{1}{2} < b' \le b \le \frac{1}{2} + \beta, \quad \text{where } \beta = \beta(s) \doteq \min\left\{\frac{1}{6}\left(s-\frac{2}{3}\right), \frac{1}{8}\right\},$$

and this is what we will use in the proof of our well-posedness Theorem 1.6.

As we have mentioned earlier, our proof of the mKdV trilinear estimate (2.16) is different than the one given in [75]. Also, a good part of the proof of each one of the nonlocal trilinear estimates (2.17) and (2.18) is reduced to the proof of estimate (2.16).

We shall prove the trilinear estimates in the next section. Here, we shall use them to complete the proof of Theorem 1.6. For any size initial data $\varphi \in H^s$ and lifespan T such that

$$0 < T < 1/2,$$

and to be determined later, we further localize integral equation (2.7) as follows

$$\begin{aligned} u(x,t) &= \psi(t)W(t)\varphi - \psi(t)\int_0^t W(t-t')\psi_{2T}(t')w_{uuu}(x,t')\,dt' \\ &\doteq \Phi_T(u)(x,t) = \Phi_{T,\varphi}(u)(x,t), \end{aligned} \tag{2.19}$$

where $\psi_T(t) = \psi(t/T)$, with ψ being the standard localization function defined in (2.6). If $|t| < T$, then we see that $\Phi_T(u) = \Phi(u)$ and the fixed point of the iteration map (2.19) is the solution to the modified Novikov equation ivp (1.4). Next, we show that the iteration map (2.19) is contraction on the ball

$$B(r) = \{u \in X^{s,b} : \|u\|_{X^{s,b}} \le r\},$$

if we choose the radius r and the lifespan T appropriately. For this we will need following multiplier estimate in [74] (see Lemma 2.11).

Lemma 2.2 *Let $\eta(t)$ be a function in the Schwartz space $\mathcal{S}(\mathbb{R})$. If $-\frac{1}{2} < \gamma' \le \gamma < \frac{1}{2}$, then for any $0 < T \le 1$ we have*

$$\|\eta(t/T)u\|_{X^{s,\gamma'}} \le c_3(\eta,\gamma,\gamma')T^{\gamma-\gamma'}\|u\|_{X^{s,\gamma}}. \tag{2.20}$$

Also, we will use the trilinear estimates (2.16), (2.17), and (2.18), with the following choice of b' and b

$$b = \frac{1}{2} + \frac{1}{2}\beta \ \ \text{(in place of } b') \quad \text{and} \quad b_1 = \frac{1}{2} + \beta \ \ \text{(in place of } b), \quad \text{with } \beta = \beta_3,$$

in which they read as follows

$$\|\partial_x(fgh)\|_{s,b_1-1} \le c_2\|f\|_{s,b}\|g\|_{s,b}\|h\|_{s,b}, \tag{2.21a}$$

$$\|(1+\partial_x^2)^{-1}\partial_x[f(\partial_x g)(\partial_x h)]\|_{s,b_1-1} \le c_2\|f\|_{s,b}\|g\|_{s,b}\|h\|_{s,b}, \tag{2.21b}$$

$$\|(1+\partial_x^2)^{-1}[(\partial_x f)(\partial_x g)(\partial_x h)]\|_{s,b_1-1} \le c_2\|f\|_{s,b}\|g\|_{s,b}\|h\|_{s,b}. \tag{2.21c}$$

Φ_T is onto: For $u \in B(r)$, applying the linear estimates (2.14), (2.15) and the multiplier estimate (2.20) with $\gamma = b_1 - 1$, $\gamma' = b - 1$, we get

$$\begin{aligned} \|\Phi_T(u)\|_{s,b} &\le c_1\|\varphi\|_{H^s} + c_1\|\psi_{2T}(t)w_{uuu}\|_{s,b-1} \\ &\le c_1\|\varphi\|_{H^s} + c_1c_3T^{\frac{1}{2}\beta}\|w_{uuu}\|_{s,b_1-1}, \end{aligned} \tag{2.22}$$

since $b_1 - b = \frac{\beta}{2}$. Then, applying trilinear estimates (2.21) to the nonlinearity w_{uuu} in (2.3), we get

$$\begin{aligned} \|w_{uuu}\|_{s,b_1-1} &= \left\| \frac{1}{3}\partial_x(u^3) + (1-\partial_x^2)^{-1}\partial_x\left[u^3 + \frac{3}{2}u(\partial_x u)^2\right] + (1-\partial_x^2)^{-1}\left[\frac{1}{2}(\partial_x u)^3\right]\right\|_{s,b-1} \\ &\le 4c_2\|u\|_{s,b}^3. \end{aligned} \tag{2.23}$$

Thus, combining (2.22) and (2.23) gives the onto estimate

$$\|\Phi_T(u)\|_{s,b} \le c_1\|\varphi\|_{H^s} + 4c_1c_2c_3T^{\frac{1}{2}\beta}\|u\|_{s,b}^3. \tag{2.24}$$

From (2.24) we see that for the map Φ_T to be onto, it suffices to have $c_1\|\varphi\|_{H^s} + 4c_1c_2c_3T^{\frac{1}{2}\beta}\|u\|_{s,b}^3 \le r$. And, since $u \in B(r)$ it suffices for the lifespan T and the radius r to satisfy the condition

$$c_1\|\varphi\|_{H^s} + 4c_1c_2c_3T^{\frac{1}{2}\beta}r^3 \le r. \tag{2.25}$$

Φ_T is contraction For $u, v \in B(r)$, applying the linear estimate (2.15) and the multiplier estimate (2.20) with $\gamma = b_1 - 1$, $\gamma' = b - 1$, we have

$$\begin{aligned} \|\Phi_T(u) - \Phi_T(v)\|_{s,b} &\le c_1\|\psi_{2T}(t)(w_{uuu} - w_{vvv})\|_{s,b-1} \\ &\le c_1c_3T^{\frac{1}{2}\beta}\|w_{uuu} - w_{vvv}\|_{s,b_1-1}. \end{aligned} \tag{2.26}$$

Next, applying trilinear estimates (2.21) to the nonlinearities $w_{uuu} - w_{vvv}$ defined by (2.4), we get

$$\begin{aligned} \|w_{uuu} - w_{vvv}\|_{s,b_1-1} \le\ & \frac{4}{3}c_2\|u-v\|_{s,b}\left(\|u\|_{s,b}^2 + \|u\|_{s,b}\|v\|_{s,b} + \|v\|_{s,b}^2\right) \\ & + \frac{3}{2}c_2\|u-v\|_{s,b}\left(\|u\|_{s,b}^2 + \|v\|_{s,b}\|u+v\|_{s,b}\right) \\ & + \frac{1}{2}c_2\|u-v\|_{s,b}\left(\|u\|_{s,b}^2 + \|u\|_{s,b}\|v\|_{s,b} + \|v\|_{s,b}^2\right) \\ \le\ & 4c_2\left(\|u\|_{s,b}^2 + \|u\|_{s,b}\|v\|_{s,b} + \|v\|_{s,b}^2\right)\|u-v\|_{s,b}, \end{aligned}$$

which combined with (2.26) gives the contraction estimate

$$\|\Phi_T(u) - \Phi_T(v)\|_{s,b} \leq 12c_1c_2c_3r^2T^{\frac{1}{2}\beta}\|u - v\|_{s,b}.$$

Thus, in order to make the iteration map Φ_T a contraction map, it suffices to have

$$12c_1c_2c_3r^2T^{\frac{1}{2}\beta} \leq \frac{1}{2}. \tag{2.27}$$

Combining conditions (2.25) with (2.27), we see that it suffices to have $c_1\|\varphi\|_{H^s} + \frac{1}{6}r \leq r$ or $r \geq \frac{6}{5}c_1\|\varphi\|_{H^s}$. So, we choose the radius to be

$$r \doteq 2c_1\|\varphi\|_{H^s}. \tag{2.28}$$

Then, from (2.27), it suffices to have $T^{\frac{1}{2}\beta} \leq (24c_1c_2c_3r^2)^{-1}$ or $T \leq (24c_1c_2c_3r^2)^{-2/\beta}$, which follows from choosing $T = \frac{1}{2}\frac{1}{(1+24c_1c_2c_3r^2)^{\frac{2}{\beta}}} < \frac{1}{2}$. Combining this choice of T together with choice (2.28) for r gives

$$T = \frac{1}{2}\frac{1}{(1 + 96c_1^3c_2c_3\|\varphi\|_{H^s}^2)^{\frac{2}{\beta}}} \geq \frac{c_0}{(1 + \|\varphi\|_{H^s}^2)^{\frac{2}{\beta}}} \doteq T_0, \tag{2.29}$$

for some c_0 depending on c_1, c_2, c_3 and β, that is $c_0 = c_0(s, b)$. This is the estimate for the lifespan (1.8) stated in Theorem 1.6.

Now, observe that the fixed-point $u \in X^{s,b}(\mathbb{R}^2)$ of the iteration map (2.19) restricted to $|t| \leq T_0$ satisfies the non-ψ integral equation (2.5), that is

$$u(x,t) = W(t)\varphi(x) - \int_0^t W(t - t')w(x, t')\,dt', \quad |t| \leq T_0.$$

Therefore, we have a solution u for the modified Novikov equation ivp (1.4a) in the space $X_{T_0}^{s,b}$ defined by (1.6), and having norm $\|u\|_{X_T^{s,b}}$ defined by (1.7).

Solution Bound Since the solution $u \in X_{T_0}^{s,b}$ is the restriction of the fixed-point $v(x,t) = \Phi_{T_0}(v)$ (which in our discussion above we were calling u), we have

$$\|u\|_{X_{T_0}^{s,b}} \leq \|v\|_{s,b} = \|\Phi_{T_0}(v)\|_{s,b} \leq 2c_1\|\varphi\|_{H^s},$$

where the last step follows from fact that v is a fixed-point in the ball $B(2c_1\|\varphi\|_{H^s})$. Finally, in order to prove estimate (1.9) we need the following basic result, which follows the definition of $\|\cdot\|_{H^s}$ and the Cauchy–Schwarz inequality.

Lemma 2.3 *Let $b > \frac{1}{2}$. The inclusion $X^{s,b}(\mathbb{R}^2) \hookrightarrow C(\mathbb{R}, H^s(\mathbb{R}))$ is continuous, that is*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^s} \leq C(b)\|u\|_{X^{s,b}}.$$

Now, the desired solution bound follows from Lemma 2.3 since

$$\sup_{|t|\leq T_0} \|u(t)\|_{H^s} = \sup_{|t|\leq T_0} \|v(t)\|_{H^s} \leq \sup_{t\in\mathbb{R}} \|v(t)\|_{H^s} \leq C\|v\|_{X^{s,b}} \leq 2c_1C\|\varphi\|_{H^s}.$$

Lip-continuous dependence on initial data For any $\varphi_0 \in H^s$ and $R > 0$ let us consider the neighborhood

$$U = B(\varphi_0, R) \doteq \{\varphi \in H^s : \|\varphi - \varphi_0\|_{H^s} \leq R\}.$$

Then, by the triangle inequality we have $\|\varphi\|_{H^s} \leq \|\varphi - \varphi_0\|_{H^s} + \|\varphi_0\|_{H^s} = \|\varphi_0\|_{H^s} + R$. Using this and lifespan estimate (2.29) we see that

$$T^* \doteq \frac{1}{2}\frac{1}{[1 + 96c_1^3c_2c_3(\|\varphi_0\|_{H^s} + R)^2]^{\frac{2}{\beta}}} \tag{2.30}$$

is common lifespan of solutions to modified Novikov equation ivp (1.4) with initial data in U. Now, for $\varphi_k \in U$, $k = 1, 2$, we denote the solution of the modified Novikov equation ivp (1.4a) with the initial data φ_k by u_k, that is u_k satisfies

$$u_k(x,t) \doteq \psi(t)W(t)\varphi_k - \psi(t)\int_0^t W(t-t')\psi_{2T^*}(t')w_{u_k u_k u_k}dt', \quad |t| < T^*.$$

Moreover, we denote the fixed point of iteration map $\Phi_{T^*,\varphi_k}(v_k)$ by v_k, that is

$$v_k(x,t) \doteq \psi(t)W(t)\varphi_k - \psi(t)\int_0^t W(t-t')\psi_{2T^*}(t')w_{v_k v_k v_k}dt', \quad t \in \mathbb{R}.$$

Since v_k is extensions of u_k from $\mathbb{R} \times [-T^*, T^*]$ to $\mathbb{R} \times \mathbb{R}$, we have $\|u_1 - u_2\|_{X^{s,b}_{T^*}} \leq \|v_1 - v_2\|_{s,b}$. Thus, to prove the Lip-continuous dependence on initial data, i.e. $\|u_1 - u_2\|_{X^{s,b}_{T^*}} \leq C\|\varphi_1 - \varphi_2\|_{H^s}$, where C is a constant depending on φ_k, it suffices to show that

$$\|v_1 - v_2\|_{s,b} \leq C\|\varphi_1 - \varphi_2\|_{H^s}. \tag{2.31}$$

Now, as before, we choose the balls

$$B_k = B_k(2c_1\|\varphi_k\|_{H^s}) \doteq \left\{v \in X^{s,b} : \|v\|_{s,b} \leq 2c_1\|\varphi_k\|_{H^s}\right\}, \quad k = 1, 2. \tag{2.32}$$

From the proof of existence, we know that $v_k \in B_k$. So, applying the linear estimates (2.14), (2.15) and the multiplier estimate (2.20) with $\gamma = b_1 - 1$, $\gamma' = b - 1$, we get

$$\begin{aligned}\|v_1 - v_2\|_{s,b} &\leq c_1\|\varphi_1 - \varphi_2\|_{H^s} + c_1\|\psi_{2T^*}(t)(w_{v_1v_1v_1} - w_{v_2v_2v_2})\|_{s,b-1}\\ &\leq c_1\|\varphi_1 - \varphi_2\|_{H^s} + c_1c_3T^{*\frac{1}{2}\beta}\|w_{v_1v_1v_1} - w_{v_2v_2v_2}\|_{s,b_1-1}.\end{aligned}$$

Then, applying trilinear estimates (2.21a) to the nonlinearities $w_{v_1v_1v_1} - w_{v_2v_2v_2}$ defined by (2.4), we get

$$\begin{aligned}\|w_{v_1v_1v_1} - w_{v_2v_2v_2}\|_{s,b_1-1} &\leq \frac{4}{3}c_2\|v_1 - v_2\|_{s,b}\left(\|v_1\|_{s,b}^2 + \|v_1\|_{s,b}\|v_2\|_{s,b} + \|v_2\|_{s,b}^2\right)\\ &\quad + \frac{3}{2}c_2\|v_1 - v_2\|_{s,b}\left(\|v_1\|_{s,b}^2 + \|v_1\|_{s,b}\|v_1 + v_2\|_{s,b}\right)\\ &\quad + \frac{1}{2}c_2\|v_1 - v_2\|_{s,b}\left(\|v_1\|_{s,b}^2 + \|v_1\|_{s,b}\|v_2\|_{s,b} + \|v_2\|_{s,b}^2\right)\\ &\leq 4c_2\left(\|v_1\|_{s,b}^2 + \|v_1\|_{s,b}\|v_2\|_{s,b} + \|v_2\|_{s,b}^2\right)\|v_1 - v_2\|_{s,b},\end{aligned} \tag{2.33}$$

which combined with definition (2.32) and estimate (2.33) implies that

$$\|v_1 - v_2\|_{s,b} \leq c_1\|\varphi_1 - \varphi_2\|_{H^s} + 48c_1^3c_2c_3T^{*\frac{1}{2}\beta} \cdot (\|\varphi_0\|_{H^s} + R)^2\|v_1 - v_2\|_{s,b}.$$

Combining this with the expression (2.30) for the common lifespan T^*, we obtain

$$\begin{aligned}\|v_1 - v_2\|_{s,b} &\leq c\|\varphi_1 - \varphi_2\|_{H^s} + \frac{48c_1^3c_2c_3(\|\varphi_0\|_{H^s} + R)^2}{1 + 96c_1^3c_2c_3(\|\varphi_0\|_{H^s} + R)^2}\|v_1 - v_2\|_{s,b}\\ &\leq c\|\varphi_1 - \varphi_2\|_{H^s} + \frac{1}{2}\|v_1 - v_2\|_{s,b},\end{aligned}$$

which is the desired estimate (2.31). Now, using Lemma 2.3, we can also get the estimate in the space $C(\mathbb{R}; H^s)$, i.e.

$$\sup_{t\in\mathbb{R}} \|v_1(t) - v_2(t)\|_{H^s} \leq \|v_1 - v_2\|_{s,b} \leq 2c_1 C\|\varphi_1 - \varphi_2\|_{H^s}.$$

Finally, concerning the proof of uniqueness of solution in the space $X^{s,b}_{T_0}$, it is similar to one presented in [6]. This completes the proof of Theorem 1.6. □

3 Proof of Trilinear Estimates in Bourgain Spaces

In this section, we prove the trilinear estimates (2.16), (2.17) and (2.18). For this, we need the following calculus estimates [51, 56].

Lemma 3.1 *If $\ell > 1/2$ and $\ell' > \frac{1}{2}$ then*

$$\int_{\mathbb{R}} \frac{dx}{(1+|x-a|)^{2\ell}(1+|x-c|)^{2\ell}} \lesssim \frac{1}{(1+|a-c|)^{2\ell}}, \tag{3.1}$$

$$\int_{\mathbb{R}} \frac{dx}{(1+|x-a|)^{2(1-\ell)}(1+|x-c|)^{2\ell'}} \lesssim \frac{1}{(1+|a-c|)^{2(1-\ell)}}. \tag{3.2}$$

In addition, if $\frac{1}{4} < \ell, \ell' < \frac{1}{2}$, then

$$\int_{\mathbb{R}} \frac{dx}{(1+|x-a|)^{2\ell}(1+|x-c|)^{2\ell'}} \lesssim \frac{1}{(1+|a-c|)^{2\ell+2\ell'-1}}. \tag{3.3}$$

3.1 Proof of mKdV Trilinear Estimate (2.16)

We start with expressing trilinear estimate (2.16), i.e. $\|\partial_x(fgh)\|_{X^{s,b-1}} \leq c_{s,b}\|f\|_{X^{s,b'}}\|g\|_{X^{s,b'}}$ $\|h\|_{X^{s,b'}}$, in its L^2 form. Using the following notation

$$c_u \doteq |\widehat{u}(\xi,\tau)|(1+|\xi|)^s(1+|\tau-\xi^3|)^{b'}, \tag{3.4}$$

we get $\|u\|_{X^{s,b'}} = \|c_u\|_{L^2}$. Next, we form the $X^{s,b-1}$-norm, that is

$$\begin{aligned}
&\|\partial_x(fgh)\|^2_{X^{s,b-1}} \\
&= \int_{\mathbb{R}^2} \frac{|\xi|^2(1+|\xi|)^{2s}}{(1+|\tau-\xi^3|)^{2(1-b)}} \\
&\quad \times \left[\int_{\mathbb{R}^4} \widehat{f}(\xi_1,\tau_1)\widehat{g}(\xi_2,\tau_2)\widehat{h}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2 d\tau_2 d\xi_1 d\tau_1\right]^2 d\xi d\tau \\
&\leq \int_{\mathbb{R}^2} \frac{|\xi|^2(1+|\xi|)^{2s}}{(1+|\tau-\xi^3|)^{2(1-b)}} \\
&\quad \times \left(\int_{\mathbb{R}^4} \left|\widehat{f}(\xi_1,\tau_1)\widehat{g}(\xi_2,\tau_2)\widehat{h}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)\right| d\xi_2 d\tau_2 d\xi_1 d\tau_1\right)^2 d\xi d\tau \\
&= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} Q\cdot c_f(\xi_1,\tau_1)c_g(\xi_2,\tau_2)c_h(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2 d\tau_2 d\xi_1 d\tau_1\right)^2 d\xi d\tau,
\end{aligned}$$

where Q is defined in (3.6) below. Thus, to prove trilinear estimate (2.16), it suffices to show the L^2 inequality

$$\begin{aligned}&\left\| \int_{\mathbb{R}^4} Q \cdot c_f(\xi_1, \tau_1) c_g(\xi_2, \tau_2) c_h(\xi - \xi_1 - \xi_2, \tau - \tau_1 - \tau_2) d\xi_2 d\tau_2 d\xi_1 d\tau_1 \right\|^2_{L^2_{\xi,\tau}} \\ &\lesssim \|c_f\|^2_{L^2_{\xi,\tau}} \|c_g\|^2_{L^2_{\xi,\tau}} \|c_h\|^2_{L^2_{\xi,\tau}},\end{aligned} \tag{3.5}$$

where the multiplier $Q = Q(\xi, \tau, \xi_1, \tau_1, \xi_2, \tau_2)$ is defined by

$$Q \doteq \frac{|\xi|(1+|\xi|)^s(1+|\xi_1|)^{-s}(1+|\xi_2|)^{-s}(1+|\xi-\xi_1-\xi_2|)^{-s}}{(1+|\tau-\xi^3|)^{1-b}(1+|\tau_1-\xi_1^3|)^{b'}(1+|\tau_2-\xi_2^3|)^{b'}(1+|\tau-\tau_1-\tau_2-(\xi-\xi_1-\xi_2)^3|)^{b'}}. \tag{3.6}$$

In the multiplier Q we recognize the familiar Bourgain quantity

$$\begin{aligned}d_3(\xi, \xi_1, \xi_2) &\doteq (\tau - \xi^3) - (\tau_1 - \xi_1^3) - (\tau_2 - \xi_2^3) - [\tau - \tau_1 - \tau_2 - (\xi - \xi_1 - \xi_2)^3] \\ &= -\xi^3 + \xi_1^3 + \xi_2^3 + (\xi - \xi_1 - \xi_2)^3.\end{aligned} \tag{3.7}$$

Below, we list two useful and elementary properties for this quantity.

Lemma 3.2 *The Bourgain quantity $d_3(\xi, \xi_1, \xi_2)$ satisfies the following properties:*

$$d_3(\xi, \xi_1, \xi_2) = -3(\xi - \xi_1)(\xi - \xi_2)(\xi_1 + \xi_2), \tag{3.8}$$

$$\begin{aligned}\frac{\partial d_3}{\partial \xi_1} &= 3\xi_1^2 - 3(\xi - \xi_1 - \xi_2)^2 = 3(\xi - \xi_2)(2\xi_1 - \xi + \xi_2) \\ &= 6(\xi - \xi_2)\left[\xi_1 - \frac{1}{2}(\xi - \xi_2)\right].\end{aligned} \tag{3.9}$$

Now, we estimate the left-hand side of estimate (3.5) in a way similar to the case of trilinear estimates for the cubic nonlinear Schrödinger equation [26]. First applying the Cauchy–Schwarz inequality in $\xi_1, \xi_2, \tau_1, \tau_2$, and then using Hölder's inequality we get

$$\begin{aligned}&\left\| \int_{\mathbb{R}^4} Q \cdot c_f(\xi_1, \tau_1) c_g(\xi_2, \tau_2) c_h(\xi - \xi_1 - \xi_2, \tau - \tau_1 - \tau_2) d\xi_2 d\tau_2 d\xi_1 d\tau_1 \right\|^2_{L^2_{\xi,\tau}} \\ &\leq \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^4} Q^2 d\xi_2 d\tau_2 d\xi_1 d\tau_1 \right) \\ &\quad \times \left(\int_{\mathbb{R}^4} c_f^2(\xi_1, \tau_1) c_g^2(\xi_2, \tau_2) c_h^2(\xi - \xi_1 - \xi_2, \tau - \tau_1 - \tau_2) d\xi_2 d\tau_2 d\xi_1 d\tau_1 \right) d\xi d\tau \\ &= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^4} Q^2 d\xi_2 d\tau_2 d\xi_1 d\tau_1 \right) \left(c_f^2 * c_g^2 * c_h^2 \right)(\xi, \tau) d\xi d\tau \\ &\leq \left\| \int_{\mathbb{R}^4} Q^2 d\xi_2 d\tau_2 d\xi_1 d\tau_1 \right\|_{L^\infty_{\xi,\tau}} \left\| c_f^2 * c_g^2 * c_h^2 \right\|_{L^1_{\xi,\tau}}.\end{aligned} \tag{3.10}$$

Furthermore, applying Young's convolution inequality $\|f * g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}$ twice, we get

$$\begin{aligned}\|c_f^2 * c_g^2 * c_h^2\|_{L^1_{\xi,\tau}} &\leq \|c_f^2 * c_g^2\|_{L^1_{\xi,\tau}} \cdot \|c_h^2\|_{L^1_{\xi,\tau}} \\ &\leq \|c_f^2\|_{L^1_{\xi,\tau}} \|c_g^2\|_{L^1_{\xi,\tau}} \|c_h^2\|_{L^1_{\xi,\tau}} = \|c_f\|^2_{L^2_{\xi,\tau}} \|c_g\|^2_{L^2_{\xi,\tau}} \|c_h\|^2_{L^2_{\xi,\tau}}.\end{aligned} \tag{3.11}$$

Combining estimate (3.11) with (3.10), we get

$$\left\| \int_{\mathbb{R}^4} Q \cdot c_f(\xi_1, \tau_1) c_g(\xi_2, \tau_2) c_h(\xi - \xi_1 - \xi_2, \tau - \tau_1 - \tau_2) d\xi_2 d\tau_2 d\xi_1 d\tau_1 \right\|_{L^2}^2$$
$$\lesssim \|\Theta_1\|_{L^\infty_{\xi,\tau}} \|c_f\|_{L^2}^2 \|c_g\|_{L^2}^2 \|c_h\|_{L^2}^2.$$

Thus, to prove our trilinear estimate (3.5) it suffices to show that the quantity Θ_1 defined below is bounded, that is the following result holds.

Lemma 3.3 *If $s > \frac{1}{4}$ and $\frac{1}{2} < b' \le b < \min\{\frac{1}{3} + \frac{2}{3}s, \frac{3}{4}\}$, then for all ξ and τ we have*

$$\begin{aligned} \Theta_1(\xi, \tau) \doteq & \frac{|\xi|^2 (1+|\xi|)^{2s}}{(1+|\tau - \xi^3|)^{2(1-b)}} \\ & \times \int_{\mathbb{R}^4} \frac{(1+|\xi_1|)^{-2s}(1+|\xi_2|)^{-2s}(1+|\xi-\xi_1-\xi_2|)^{-2s} \, d\xi_2 d\tau_2 d\xi_1 d\tau_1}{(1+|\tau_1 - \xi_1^3|)^{2b'}(1+|\tau_2 - \xi_2^3|)^{2b'}(1+|\tau - \tau_1 - \tau_2 - (\xi-\xi_1-\xi_2)^3|)^{2b'}} \\ \lesssim & \ 1. \end{aligned} \tag{3.12}$$

Proof We begin the proof by taking advantage of the symmetry in convolution writing of $\widehat{f} * \widehat{g} * \widehat{h}$ to assume the following order of $|\xi_1|$, $|\xi_2|$ and $|\xi - \xi_1 - \xi_2|$

$$|\xi_1| \ge |\xi_2| \ge |\xi - \xi_1 - \xi_2|. \tag{3.13}$$

Then, in our L^2 formulation of the trilinear estimate (3.5), we can replace the multiplier Q with $\chi_A Q$, where A is defined by

$$A \doteq \{(\xi, \xi_1, \xi_2) \in \mathbb{R}^3 : |\xi_1| \ge |\xi_2| \ge |\xi - \xi_1 - \xi_2|\}.$$

This results in having χ_A as a factor of the integrand in the quantity Θ_1 defined in (3.12) above. Applying calculus estimate (3.1) in τ_1 with $a = \tau - \tau_2 - (\xi - \xi_1 - \xi_2)^3$, $c = \xi_1^3$ and $\ell = b'$, from (3.12) we get

$$\begin{aligned} \Theta_1(\xi, \tau) \lesssim & \frac{|\xi|^2 (1+|\xi|)^{2s}}{(1+|\tau - \xi^3|)^{2(1-b)}} \\ & \times \int_{\mathbb{R}^3} \frac{\chi_A(\xi, \xi_1, \xi_2)(1+|\xi_1|)^{-2s}(1+|\xi_2|)^{-2s}(1+|\xi-\xi_1-\xi_2|)^{-2s}}{(1+|\tau_2 - \xi_2^3|)^{2b'}(1+|\tau - \tau_2 - (\xi-\xi_1-\xi_2)^3 - \xi_1^3|)^{2b'}} d\tau_2 d\xi_2 d\xi_1. \end{aligned}$$

Furthermore, applying calculus estimate (3.1) in τ_2 with $a = \tau - (\xi - \xi_1 - \xi_2)^3 - \xi_1^3$, $c = \xi_2^3$, $\ell = b'$ gives

$$\begin{aligned} \Theta_1(\xi, \tau) \lesssim & \frac{|\xi|^2 (1+|\xi|)^{2s}}{(1+|\tau - \xi^3|)^{2(1-b)}} \\ & \times \int_{\mathbb{R}^2} \frac{\chi_A(\xi, \xi_1, \xi_2)(1+|\xi_1|)^{-2s}(1+|\xi_2|)^{-2s}(1+|\xi-\xi_1-\xi_2|)^{-2s}}{(1+|\tau - \xi^3 - d_3(\xi, \xi_1, \xi_2)|)^{2b'}} d\xi_2 d\xi_1, \end{aligned} \tag{3.14}$$

where $d_3(\xi, \xi_1, \xi_2)$ is the Bourgain quantity given by (3.7). Also, since for $(\xi, \xi_1, \xi_2) \in A$ we have the ordering relation $|\xi_1| \ge |\xi_2| \ge |\xi - \xi_1 - \xi_2|$, we obtain the following useful bound for $|\xi|$

$$|\xi| = |(\xi - \xi_1 - \xi_2) + \xi_2 + \xi_1| \le |\xi - \xi_1 - \xi_2| + |\xi_2| + |\xi_1| \le 3|\xi_1|. \tag{3.15}$$

Next, we consider the following two cases.

- Case 1: $|\xi_1| \le 100$.

– Case 2: $|\xi_1| > 100$.

Proof in Case 1. Since $|\xi_1| \le 100$ by estimate (3.15) and relation (3.13) we see that all of $|\xi|$, $|\xi_1|$ and $|\xi_2|$ are bounded. Furthermore, since $b' \ge 0$ and $(1-b) \ge 0$, we see that the multiplier and the integrand in (3.14) are bounded. Therefore, $\Theta_1(\xi,\tau) \lesssim 1$, since the integration is over a bounded set.
Proof in Case 2. We consider the following two subcases.

– Subcase 2.1: $|\xi_1| > 100$ and $|\xi_2| \le \frac{1}{4}|\xi_1|$.
– Subcase 2.2: $|\xi_1| > 100$ and $|\xi_2| > \frac{1}{4}|\xi_1|$.

Proof in Subcase 2.1. Since, by (3.15), $|\xi| \le 3|\xi_1|$ and also $|\xi| = |\xi_1+\xi_2+(\xi-\xi_1-\xi_2)| \ge |\xi_1|-|\xi_2|-|\xi-\xi_1-\xi_2| \ge |\xi_1|-\frac{1}{4}|\xi_1|-\frac{1}{4}|\xi_1| = \frac{1}{2}|\xi_1|$, we have $|\xi| \ge \frac{1}{2}|\xi_1|$, and therefore,

$$|\xi| \simeq |\xi_1|. \tag{3.16}$$

Using estimate (3.16) we have $(1+|\xi_1|)^{-2s} \simeq (1+|\xi|)^{-2s}$. Also, using $|\xi_2| \ge |\xi-\xi_1-\xi_2|$, for $s \ge 0$ we get $(1+|\xi_2|)^{-2s}(1+|\xi-\xi_1-\xi_2|)^{-2s} \lesssim (1+|\xi-\xi_1-\xi_2|)^{-4s}$. Combining these estimates with (3.14) gives

$$\Theta_1(\xi,\tau) \lesssim \frac{|\xi|^2}{(1+|\tau-\xi^3|)^{2(1-b)}} \int_{\mathbb{R}^2} \frac{\chi_A(\xi,\xi_1,\xi_2)(1+|\xi-\xi_1-\xi_2|)^{-4s}}{(1+|\tau-\xi^3-d_3(\xi,\xi_1,\xi_2)|)^{2b'}} d\xi_2 d\xi_1. \tag{3.17}$$

Now, making the change of variables $\widetilde{\xi}_2 = \xi-\xi_1-\xi_2$ (or $\xi_2 = \xi-\xi_1-\widetilde{\xi}_2$) and $\widetilde{\xi}_1 = \xi_1$, and using the relation $d_3(\xi,\xi_1,\xi_2) = -\xi^3+\xi_1^3+\xi_2^3+(\xi-\xi_1-\xi_2)^3 = d_3(\xi,\widetilde{\xi}_1,\widetilde{\xi}_2)$, from (3.17) we get

$$\Theta_1(\xi,\tau) \lesssim \frac{|\xi|^2}{(1+|\tau-\xi^3|)^{2(1-b)}} \int_{\mathbb{R}^2} \frac{\chi_{\widetilde{A}}(\xi,\widetilde{\xi}_1,\widetilde{\xi}_2)(1+|\widetilde{\xi}_2|)^{-4s}}{(1+|\tau-\xi^3-d_3(\xi,\widetilde{\xi}_1,\widetilde{\xi}_2)|)^{2b'}} d\widetilde{\xi}_2 d\widetilde{\xi}_1, \tag{3.18}$$

where the domain $\widetilde{A}$ is given by $\widetilde{A} \doteq \{(\xi,\widetilde{\xi}_1,\widetilde{\xi}_2) \in \mathbb{R}^3 : |\widetilde{\xi}_1| \ge |\xi-\widetilde{\xi}_1-\widetilde{\xi}_2| \ge |\widetilde{\xi}_2|\}$. Furthermore, using Fubini's theorem to switch the integration of $d\widetilde{\xi}_2$ and $d\widetilde{\xi}_1$, from (3.18) we get

$$\begin{aligned} \Theta_1(\xi,\tau) \lesssim{} & \frac{|\xi|^2}{(1+|\tau-\xi^3|)^{2(1-b)}} \\ & \times \int_{\mathbb{R}} (1+|\widetilde{\xi}_2|)^{-4s} \left[\int_{\mathbb{R}} \frac{\chi_{\widetilde{A}}(\xi,\widetilde{\xi}_1,\widetilde{\xi}_2)}{(1+|\tau-\xi^3-d_3(\xi,\widetilde{\xi}_1,\widetilde{\xi}_2)|)^{2b'}} d\widetilde{\xi}_1 \right] d\widetilde{\xi}_2. \end{aligned} \tag{3.19}$$

For the $d\widetilde{\xi}_1$-integral in (3.19) our strategy is to make the change of variables $\mu = \mu(\widetilde{\xi}_1) = d_3(\xi,\widetilde{\xi}_1,\widetilde{\xi}_2)$. For this change to be good, we need to split the $\widetilde{\xi}_1$-integral at the critical point of $\mu(\widetilde{\xi}_1)$, which is $p = (\xi-\widetilde{\xi}_2)/2$, since $\mu'(\widetilde{\xi}_1) = \partial d_3(\xi,\widetilde{\xi}_1,\widetilde{\xi}_2)/\partial\widetilde{\xi}_1 = 6(\xi-\widetilde{\xi}_2)\big[\widetilde{\xi}_1 - \frac{1}{2}(\xi-\widetilde{\xi}_2)\big]$ (see property (3.9)). Thus, using $\widetilde{\xi}_1$ intervals $I_1^{\widetilde{\xi}_1} \doteq \big(-\infty, (\xi-\widetilde{\xi}_2)/2\big)$ and $I_2^{\widetilde{\xi}_1} \doteq \big((\xi-\widetilde{\xi}_2)/2, \infty\big)$, making the change of variables $\mu = \mu(\widetilde{\xi}_1) = d_3(\xi,\widetilde{\xi}_1,\widetilde{\xi}_2)$ on each one of these two intervals and defining I_k^{μ} to be the range of μ when $\widetilde{\xi}_1 \in I_k^{\widetilde{\xi}_1}$, from (3.19) we get

$$\Theta_1 \lesssim J_1 + J_2, \tag{3.20}$$

with

$$J_k \doteq \frac{|\xi|^2}{(1+|\tau-\xi^3|)^{2(1-b)}} \int_{\mathbb{R}} (1+|\widetilde{\xi}_2|)^{-4s} \left(\int_{I_k^{\mu}} \frac{d\mu}{(1+|\tau-\xi^3-\mu|)^{2b'}|\mu'|} \right) d\widetilde{\xi}_2.$$

Now, we need the following bound of $|\mu'|$ from below

$$|\mu'| = 6|\xi - \widetilde{\xi}_2|\,\big|\widetilde{\xi}_1 - (\xi - \widetilde{\xi}_2)/2\big| \gtrsim |\xi|^2, \tag{3.21}$$

which follows from $|\xi - \widetilde{\xi}_2| \geq |\widetilde{\xi}_1| - |\xi - \widetilde{\xi}_1 - \widetilde{\xi}_2| \geq |\xi_1| - \frac{1}{4}|\xi_1| = \frac{3}{4}|\xi_1| \gtrsim |\xi|$ and $|\widetilde{\xi}_1 - \frac{1}{2}(\xi - \widetilde{\xi}_2)| = |\frac{1}{2}\widetilde{\xi}_1 - \frac{1}{2}(\xi - \widetilde{\xi}_1 - \widetilde{\xi}_2)| \geq \frac{1}{2}|\widetilde{\xi}_1| - \frac{1}{2}|\xi - \widetilde{\xi}_1 - \widetilde{\xi}_2| \geq \frac{1}{2}|\xi_1| - \frac{1}{8}|\xi_1| = \frac{3}{8}|\xi_1| \gtrsim |\xi|$, since $\widetilde{\xi}_1 = \xi_1$ and $|\xi - \widetilde{\xi}_1 - \widetilde{\xi}_2| = |\xi_2| \leq \frac{1}{4}|\xi_1|$. Next, combining estimate (3.21) with (3.20) and integrating μ over $\mathbb{R}$, we get

$$J_k \lesssim \frac{1}{(1 + |\tau - \xi^3|)^{2(1-b)}} \int_{\mathbb{R}} \frac{d\widetilde{\xi}_2}{(1 + |\widetilde{\xi}_2|)^{4s}} \cdot \int_{\mathbb{R}} \frac{d\mu}{(1 + |\tau - \xi^3 - \mu|)^{2b'}}, \quad k = 1, 2.$$

The multiplier $\frac{1}{(1+|\tau-\xi^3|)^{2(1-b)}}$ is bounded if $2(1-b) \geq 0$ or $b \leq 1$. Also, the first integral is bounded if $4s > 1$ or $s > \frac{1}{4}$, and the second integral is bounded if $2b' > 1$ or $b' > \frac{1}{2}$. Therefore, for J_k to be bounded it suffices to have $\frac{1}{2} < b' \leq b \leq 1$ and $s > \frac{1}{4}$. This completes the proof in Subcase 2.1.

Proof in Subcase 2.2. Since $\frac{1}{2} < b' \leq b < 1$ we have $0 < 2(1-b) < 1 < 2b'$, so we can move $(1 + |\tau - \xi^3|)^{2(1-b)}$ inside the integral and replace $2b'$ with $2(1-b)$. Since $(1 + |\tau - \xi^3|)(1 + |\tau - \xi^3 - d_3(\xi, \xi_1, \xi_2)|) \geq |\tau - \xi^3| + |\tau - \xi^3 - d_3(\xi, \xi_1, \xi_2)|$ and also $|\tau - \xi^3| + |\tau - \xi^3 - d_3(\xi, \xi_1, \xi_2)| \geq |\tau - \xi^3 - d_3(\xi, \xi_1, \xi_2) - (\tau - \xi^3)| = |d_3(\xi, \xi_1, \xi_2)|$, we have

$$(1 + |\tau - \xi^3|)^{2(1-b)}(1 + |\tau - \xi^3 - d_3(\xi, \xi_1, \xi_2)|)^{2b'} \gtrsim |d_3(\xi, \xi_1, \xi_2)|^{2(1-b)}. \tag{3.22}$$

Using (3.22), from (3.14) we obtain

$$\Theta_1(\xi, \tau) \lesssim |\xi|^2(1 + |\xi|)^{2s} \times \int_{\mathbb{R}^2} \frac{\chi_A(\xi, \xi_1, \xi_2)(1 + |\xi_1|)^{-2s}(1 + |\xi_2|)^{-2s}(1 + |\xi - \xi_1 - \xi_2|)^{-2s}}{|d_3(\xi, \xi_1, \xi_2)|^{2(1-b)}} d\xi_2 d\xi_1. \tag{3.23}$$

Recalling the factorization $d_3 = -3(\xi - \xi_1)(\xi - \xi_2)(\xi_1 + \xi_2)$, from (3.23) we obtain

$$\Theta_1(\xi, \tau) \lesssim |\xi|^2(1 + |\xi|)^{2s} \times \int_{\mathbb{R}^2} \frac{\chi_A(\xi, \xi_1, \xi_2)(1 + |\xi_1|)^{-2s}(1 + |\xi_2|)^{-2s}(1 + |\xi - \xi_1 - \xi_2|)^{-2s}}{|(\xi - \xi_1)(\xi - \xi_2)(\xi_1 + \xi_2)|^{2(1-b)}} d\xi_2 d\xi_1. \tag{3.24}$$

Next, we consider the following two possibilities about the size of $|\xi - \xi_1 - \xi_2|$.

- Subcase 2.2.1: $|\xi_1| > 100$, $|\xi_2| > \frac{1}{4}|\xi_1|$ and $|\xi - \xi_1 - \xi_2| \leq \frac{1}{16}|\xi_1|$.
- Subcase 2.2.2: $|\xi_1| > 100$, $|\xi_2| > \frac{1}{4}|\xi_1|$ and $|\xi - \xi_1 - \xi_2| > \frac{1}{16}|\xi_1|$.

Proof in Subcase 2.2.1. In this situation we have $|\xi - \xi_1| = |\xi_2 + (\xi - \xi_1 - \xi_2)| \geq |\xi_2| - |\xi - \xi_1 - \xi_2| \geq (\frac{1}{4} - \frac{1}{16})|\xi_1| \gtrsim |\xi_1|$ and $|\xi - \xi_2| = |\xi_1 + (\xi - \xi_1 - \xi_2)| \geq |\xi_1| - |\xi - \xi_1 - \xi_2| \geq (1 - \frac{1}{16})|\xi_1| \gtrsim |\xi_1|$. Thus,

$$|\xi - \xi_1| \gtrsim |\xi_1| \quad \text{and} \quad |\xi - \xi_2| \gtrsim |\xi_1|. \tag{3.25}$$

Now, using estimates (3.25), $(1 + |\xi_1|)^{-2s} \simeq |\xi_1|^{-2s}$ and $(1 + |\xi_2|)^{-2s} \lesssim |\xi_1|^{-2s}$, putting $|\xi|^2(1 + |\xi|)^{2s}$ inside the integral, from (3.24) we obtain

$$\Theta_1(\xi, \tau) \lesssim \int_{\mathbb{R}} \frac{|\xi|^2(1 + |\xi|)^{2s}}{|\xi_1|^{4-4b+4s}} \left[\int_{\mathbb{R}} \frac{\chi_A(\xi, \xi_1, \xi_2)(1 + |\xi - \xi_1 - \xi_2|)^{-2s}}{|\xi_1 + \xi_2|^{2(1-b)}} d\xi_2\right] d\xi_1. \tag{3.26}$$

Furthermore, we split the ξ_2-integral in (3.26) for $|\xi_1+\xi_2| \le 1$ and for $|\xi_1+\xi_2| > 1$ by using the ξ_2 intervals $I_1^{\xi_2} = (-1-\xi_1, 1-\xi_1)$ and $I_2^{\xi_2} = (-\infty, -1-\xi_1) \cup (1-\xi_1, \infty)$. Thus, from (3.26) we have

$$\Theta_1 \lesssim J_1 + J_2 \tag{3.27}$$

with

$$J_k \doteq \int_{\mathbb{R}} \frac{|\xi|^2(1+|\xi|)^{2s}}{|\xi_1|^{4-4b+4s}} \left[\int_{I_k^{\xi_2}} \frac{\chi_A(\xi,\xi_1,\xi_2)(1+|\xi-\xi_1-\xi_2|)^{-2s}}{|\xi_1+\xi_2|^{2(1-b)}} d\xi_2\right] d\xi_1.$$

Estimate for J_1. Since $\xi_2 \in I_1^{\xi_2}$ we have $|\xi_1+\xi_2| \lesssim 1$, which implies that $1+|\xi| \simeq 1+|\xi-(\xi_1+\xi_2)|$. This gives $(1+|\xi-\xi_1-\xi_2|)^{-2s} \simeq (1+|\xi|)^{-2s}$. Thus, from (3.27) we obtain

$$J_1 \lesssim \int_{\mathbb{R}} \frac{|\xi|^2}{|\xi_1|^{4-4b+4s}} \left[\int_{I_1^{\xi_2}} \frac{\chi_A(\xi,\xi_1,\xi_2)}{|\xi_1+\xi_2|^{2(1-b)}} d\xi_2\right] d\xi_1. \tag{3.28}$$

Now, making the change of variables $\mu = \mu(\xi_2) = \xi_1+\xi_2$ and using $|\mu| \le 1$, from (3.28) we obtain

$$J_1 \lesssim \int_{|\mu|\le 1} \frac{1}{|\mu|^{2(1-b)}} d\mu \cdot \int_{\mathbb{R}} \frac{|\xi|^2}{|\xi_1|^{4-4b+4s}} d\xi_1. \tag{3.29}$$

For the first integral in the above estimate, since $2(1-b) < 1$ or $b > \frac{1}{2}$, we have $\int_{|\mu|\le 1} \frac{1}{|\mu|^{2(1-b)}} d\mu \simeq (\mu^{2b-1})\big|_0^1 = 1$. For the second integral in (3.29), using estimate (3.15), i.e. $|\xi| \le 3|\xi_1|$, we get $\int_{\mathbb{R}} \frac{|\xi|^2}{|\xi_1|^{4-4b+4s}} d\xi_1 \lesssim \int_{\mathbb{R}} \frac{1}{|\xi_1|^{2-4b+4s}} d\xi_1$, which is bounded if $2-4b+4s > 1$ or $b < \frac{1}{4}+s$, since $|\xi_1| > 100$. For $b > \frac{1}{2}$, it suffices to have $\frac{1}{4}+s > \frac{1}{2}$ or $s > \frac{1}{4}$. Combining the above computations with (3.29), for J_1 to be bounded, it suffices to have $\frac{1}{2} < b < \frac{1}{4}+s$ and $s > \frac{1}{4}$. This completes the proof of boundedness for J_1.

Estimate for J_2. Since $\xi_2 \in I_2^{\xi_2}$, we have $|\xi_1+\xi_2| > 1$, which implies that $|\xi_1+\xi_2| \simeq (1+|\xi_1+\xi_2|)$ or $|\xi_1+\xi_2|^{2(1-b)} \simeq (1+|\xi_1+\xi_2|)^{2(1-b)}$. Using this, from (3.27) we get

$$J_2 \lesssim \int_{\mathbb{R}} \frac{|\xi|^2(1+|\xi|)^{2s}}{|\xi_1|^{4-4b+4s}} \left[\int_{I_2^{\xi_2}} \frac{\chi_A(\xi,\xi_1,\xi_2)}{(1+|\xi-\xi_1-\xi_2|)^{2s}(1+|\xi_1+\xi_2|)^{2(1-b)}} d\xi_2\right] d\xi_1. \tag{3.30}$$

Now, for the ξ_2-integral in (3.30), making the change of variables $\mu = \mu(\xi_2) = \xi_1+\xi_2$ and using $|\mu| \le |\xi_1|+|\xi_2| \le 2|\xi_1|$, we get

$$J_2 \lesssim \int_{\mathbb{R}} \frac{|\xi|^2(1+|\xi|)^{2s}}{|\xi_1|^{4-4b+4s}} \left[\int_{|\mu|\le 2|\xi_1|} \frac{1}{(1+|\xi-\mu|)^{2s}(1+|\mu|)^{2(1-b)}} d\mu\right] d\xi_1. \tag{3.31}$$

Next, we consider the following two cases concerning s.

- Subcase 2.2.1.1: $\frac{1}{4} < s < \frac{1}{2}$.
- Subcase 2.2.1.2: $s \ge \frac{1}{2}$.

Proof in Subcase 2.2.1.1. For the $d\mu$-integral in (3.31), integrating it over $\mathbb{R}$ and using calculus estimate (3.3) with $\frac{1}{4} < \ell = s < \frac{1}{2}$ and $\frac{1}{4} < \ell' = 1-b < \frac{1}{2}$ (or $\frac{1}{2} < b < \frac{3}{4}$), $a = \xi$ and $c = 0$, we get

$$\int_{|\mu|\le 2|\xi_1|} \frac{1}{(1+|\xi-\mu|)^{2s}(1+|\mu|)^{2(1-b)}} d\mu \le \int_{\mathbb{R}} \frac{1}{(1+|\xi-\mu|)^{2s}(1+|\mu|)^{2(1-b)}} d\mu$$
$$\lesssim \frac{1}{(1+|\xi|)^{2s+1-2b}}.$$

Combining the above estimate with (3.31), we obtain

$$J_2 \lesssim \int_{\mathbb{R}} \frac{|\xi|^2(1+|\xi|)^{2s}}{|\xi_1|^{4-4b+4s}} \frac{1}{(1+|\xi|)^{2s+1-2b}} d\xi_1 = \int_{\mathbb{R}} \frac{|\xi|^2(1+|\xi|)^{2b-1}}{|\xi_1|^{4-4b+4s}} d\xi_1. \tag{3.32}$$

Again, using estimate (3.15), i.e. $|\xi| \leq 3|\xi_1|$, and $2b-1>0$ from (3.32) we get $J_2 \lesssim \int_{\mathbb{R}} \frac{1}{|\xi_1|^{3-6b+4s}} d\xi_1$, which is bounded if $3-6b+4s>1$ or $b<\frac{2}{3}s+\frac{1}{3}$. For $b>\frac{1}{2}$, it suffices to have $\frac{2}{3}s+\frac{1}{3}>\frac{1}{2}$ or $\frac{2}{3}s>\frac{1}{6}$, which implies that $s>\frac{1}{4}$. This completes the estimate for J_2 when $\frac{1}{4}<s<\frac{1}{2}$.

Proof in Subcase 2.2.1.2 ($s \geq \frac{1}{2}$). Using μ-intervals $I_1^\mu = \{\mu \in \mathbb{R} : |\mu| \leq 2|\xi_1| \text{ and } |\xi-\mu| \leq \frac{1}{2}|\xi|\}$, $I_2^\mu = \{\mu \in \mathbb{R} : |\mu| \leq 2|\xi_1| \text{ and } |\xi-\mu| > \frac{1}{2}|\xi|\}$, we split J_2 as $J_2 \lesssim J_{21}+J_{22}$, where

$$J_{2k} \doteq \int_{\mathbb{R}} \frac{|\xi|^2(1+|\xi|)^{2s}}{|\xi_1|^{4-4b+4s}} \left[\int_{I_k^\mu} \frac{d\mu}{(1+|\xi-\mu|)^{2s}(1+|\mu|)^{2(1-b)}}\right] d\xi_1, \quad k=1,2. \tag{3.33}$$

Estimate for J_{21}. For $\mu \in I_1^\mu$, we have

$$|\mu| \gtrsim |\xi|, \tag{3.34}$$

which follows from $|\mu| = |\xi-(\xi-\mu)| \geq |\xi|-|\xi-\mu| \geq |\xi|-\frac{1}{2}|\xi| = \frac{1}{2}|\xi|$, since $|\xi-\mu| \leq \frac{1}{2}|\xi|$. Also, since $2(1-b) \geq 0$ or $b \leq 1$, using estimate (3.34) we get $\frac{1}{(1+|\mu|)^{2(1-b)}} \lesssim \frac{1}{(1+|\xi|)^{2(1-b)}}$, which combined with estimate (3.33) gives

$$J_{21} \lesssim \int_{\mathbb{R}} \frac{|\xi|^2(1+|\xi|)^{2s-2(1-b)}}{|\xi_1|^{4-4b+4s}} \left[\int_{I_1^\mu} \frac{d\mu}{(1+|\xi-\mu|)^{2s}}\right] d\xi_1. \tag{3.35}$$

For the $d\mu$-integral in (3.35), we make the change of variables $\mu_1 = \xi-\mu$. Using estimate (3.15) and $|\mu| \leq 2|\xi_1|$, we get $|\mu_1| \leq |\xi|+|\mu| \leq 5|\xi_1|$. Also, since $2s \geq 1$ we obtain $\int_{I_1^\mu} \frac{d\mu}{(1+|\xi-\mu|)^{2s}} \lesssim \int_{|\mu_1| \lesssim |\xi_1|} \frac{d\mu_1}{(1+|\mu_1|)^{2s}} \lesssim \ln|\xi_1|$. Again, using estimate (3.15) we get $|\xi|^2(1+|\xi|)^{2s-2(1-b)} \lesssim (1+|\xi|)^{2s+2b} \lesssim |\xi_1|^{2s+2b}$. Combining the above computations with (3.35), we obtain $J_{21} \lesssim \int_{\mathbb{R}} \frac{|\xi_1|^{2s+2b}}{|\xi_1|^{4-4b+4s}} \cdot \ln|\xi_1| d\xi_1 = \int_{\mathbb{R}} \frac{1}{|\xi_1|^{4-6b+2s}} \cdot \ln|\xi_1| d\xi_1$, which is bounded if $4-6b+2s>1$ or $b<\frac{1}{2}+\frac{s}{3}$. For $b>\frac{1}{2}$, it suffices to have $s>0$. This completes the estimate for J_{21}.

Estimate for J_{22}. Since $|\xi-\mu| \geq \frac{1}{2}|\xi|$, for $s \geq 0$ we have $\frac{1}{(1+|\xi-\mu|)^{2s}} \lesssim \frac{1}{(1+|\xi|)^{2s}}$ or $\frac{(1+|\xi|)^{2s}}{(1+|\xi-\mu|)^{2s}} \lesssim 1$. Putting $(1+|\xi|)^{2s}$ inside the $d\mu$ integral, from (3.33) we get

$$J_{22} \lesssim \int_{\mathbb{R}} \frac{|\xi|^2}{|\xi_1|^{4-4b+4s}} \left[\int_{I_2^\mu} \frac{d\mu}{(1+|\mu|)^{2(1-b)}}\right] d\xi_1. \tag{3.36}$$

For the $d\mu$-integral in the above estimate, since $2(1-b)<1$ or $b>\frac{1}{2}$ and $|\mu| \leq 2|\xi_1|$ we have $\int_{I_2^\mu} \frac{d\mu}{(1+|\mu|)^{2(1-b)}} \simeq [(1+\mu)^{2b-1}]\big|_0^{2|\xi_1|} \simeq |\xi_1|^{2b-1}$. Again, using estimate (3.15) and the above computation, from (3.36) we obtain

$$J_{22} \lesssim \int_{\mathbb{R}} \frac{|\xi_1|^2}{(1+|\xi_1|)^{4-4b+4s}} |\xi_1|^{2b-1} d\xi_1 = \int_{\mathbb{R}} \frac{1}{(1+|\xi_1|)^{3-6b+4s}} d\xi_1,$$

which is bounded if $3-6b+4s>1$ or $b<\frac{1}{3}+\frac{2}{3}s$. For $b>\frac{1}{2}$, it suffices to have $\frac{1}{3}+\frac{2}{3}s>\frac{1}{2}$ or $\frac{2}{3}s>\frac{1}{6}$, which implies that $s>\frac{1}{4}$. This completes the proof of Subcase 2.2.1.

Proof in Subcase 2.2.2. Since $|\xi_1| > 100$, $|\xi_2| > \frac{1}{4}|\xi_1|$ and $|\xi - \xi_1 - \xi_2| > \frac{1}{16}|\xi_1|$, we have the relations

$$|\xi - \xi_1 - \xi_2| \simeq |\xi_2| \simeq |\xi_1| > 100. \tag{3.37}$$

Now, moving $|\xi|^2(1+|\xi|)^{2s}$ inside the integral in (3.24), we get

$$\begin{aligned}&\Theta_1(\xi,\tau)\\ &\lesssim \int_{\mathbb{R}^2} \frac{\chi_A(\xi,\xi_1,\xi_2)|\xi|^2(1+|\xi|)^{2s}(1+|\xi_1|)^{-2s}(1+|\xi_2|)^{-2s}(1+|\xi-\xi_1-\xi_2|)^{-2s}}{|(\xi-\xi_1)(\xi-\xi_2)(\xi_1+\xi_2)|^{2(1-b)}} d\xi_2 d\xi_1\end{aligned}$$

and simplifying the numerator as follows, by using estimate (3.15), i.e. $|\xi| \le 3|\xi_1|$, and (3.37),

$$|\xi|^2(1+|\xi|)^{2s}(1+|\xi_1|)^{-2s}(1+|\xi_2|)^{-2s}(1+|\xi-\xi_1-\xi_2|)^{-2s} \lesssim |\xi_1|^{1-2s}|\xi_2|^{1-2s},$$

we are reduced to the inequality

$$\Theta_1(\xi,\tau) \lesssim \int_{\mathbb{R}^2} \frac{\chi_A(\xi,\xi_1,\xi_2)|\xi_1|^{1-2s}|\xi_2|^{1-2s}}{|(\xi-\xi_1)(\xi-\xi_2)(\xi_1+\xi_2)|^{2(1-b)}} d\xi_2 d\xi_1. \tag{3.38}$$

Next, we consider the following two possibilities.

- Subcase 2.2.2.1: $|\xi_1| > 100$, $|\xi_2| > \frac{1}{4}|\xi_1|$, $|\xi-\xi_1-\xi_2| > \frac{1}{16}|\xi_1|$ and $\big(|\xi| \le \frac{1}{64}|\xi-\xi_1-\xi_2|$ or $|\xi| \ge \frac{65}{64}|\xi_1|\big)$.
- Subcase 2.2.2.2: $|\xi_1| > 100$, $|\xi_2| > \frac{1}{4}|\xi_1|$, $|\xi-\xi_1-\xi_2| > \frac{1}{16}|\xi_1|$ and $\frac{1}{64}|\xi-\xi_1-\xi_2| < |\xi| < \frac{65}{64}|\xi_1|$.

Proof in Subcase 2.2.2.1. For each factor of the denominator in (3.38) we have the following estimate

$$|\xi-\xi_1| \ge \big||\xi|-|\xi_1|\big| \ge \frac{1}{64}|\xi_1| \gtrsim |\xi_1| \simeq |\xi_2|, \tag{3.39a}$$

$$|\xi-\xi_2| \ge \big||\xi|-|\xi_2|\big| \ge \frac{1}{64}|\xi_2| \gtrsim |\xi_2| \simeq |\xi_1|, \tag{3.39b}$$

$$\begin{aligned}|\xi_1+\xi_2| &= |\xi-(\xi-\xi_1-\xi_2)| \ge \big||\xi|-|\xi-\xi_1-\xi_2|\big|\\ &\ge \frac{1}{64}|\xi-\xi_1-\xi_2| \simeq |\xi_1| \simeq |\xi_2|.\end{aligned} \tag{3.39c}$$

This follows from estimate (3.37) and $|\xi| \le \frac{1}{64}|\xi-\xi_1-\xi_2| \le \frac{1}{64}|\xi_2| \le \frac{1}{64}|\xi_1|$ or $|\xi| \ge \frac{65}{64}|\xi_1| \ge \frac{65}{64}|\xi_2| \ge \frac{65}{64}|\xi-\xi_1-\xi_2|$. Using estimates (3.39a)–(3.39c), we get $|(\xi-\xi_1)(\xi-\xi_2)(\xi_1+\xi_2)| \gtrsim |\xi_1|^{\frac{3}{2}}|\xi_2|^{\frac{3}{2}}$. Using this and estimate (3.37), from (3.38) we obtain

$$\begin{aligned}\Theta_1(\xi,\tau) &\lesssim \int_{|\xi_2|\simeq|\xi_1|>100} \frac{|\xi_1|^{1-2s}|\xi_2|^{1-2s}}{|\xi_1|^{3(1-b)}|\xi_2|^{3(1-b)}} d\xi_2 d\xi_1\\ &= \int_{|\xi_1|\gtrsim 1} \frac{1}{|\xi_1|^{3(1-b)-1+2s}} d\xi_1 \int_{|\xi_2|\gtrsim 1} \frac{1}{|\xi_2|^{3(1-b)-1+2s}} d\xi_2,\end{aligned}$$

which is bounded if $3(1-b)-1+2s > 1$ or $3(1-b) > 2-2s$. It suffices to have $1-b > \frac{2}{3}-\frac{2}{3}s$ or $b < \frac{1}{3}+\frac{2}{3}s$. For $b > \frac{1}{2}$, it suffices to have $\frac{1}{3}+\frac{2}{3}s > \frac{1}{2}$ or $\frac{2}{3}s > \frac{1}{6}$, which implies $s > \frac{1}{4}$. This completes the proof in Subcase 2.2.2.1.

Proof in Subcase 2.2.2.2. Using estimate (3.37) and $\frac{1}{64}|\xi-\xi_1-\xi_2| < |\xi| < \frac{65}{64}|\xi_1|$, we have

$$|\xi| \simeq |\xi-\xi_1-\xi_2| \simeq |\xi_2| \simeq |\xi_1| > 100. \tag{3.40}$$

Using (3.40) we bound the numerator in (3.24) as follows $|\xi_1|^{1-2s}|\xi_2|^{1-2s} \simeq |\xi|^{2-4s}$. Taking out $|\xi|^{2-4s}$, from (3.38) we get

$$\Theta_1(\xi,\tau) \lesssim |\xi|^{2-4s}\int_{\mathbb{R}^2}\frac{1}{|(\xi-\xi_1)(\xi-\xi_2)(\xi_1+\xi_2)|^{2(1-b)}}d\xi_1 d\xi_2. \tag{3.41}$$

Now, it suffices to estimate (3.41) in the case we integrate over E_1 (the other two cases are similar)

$$E_1 = \{(\xi_1,\xi_2)\in\mathbb{R}^2 : |\xi-\xi_1| \geq \max\{|\xi-\xi_2|, |\xi_1+\xi_2|\}\}. \tag{3.42}$$

Using the assumption (3.42) and the triangle inequality we get

$$\begin{aligned} |\xi-\xi_1| &\geq \frac{1}{3}\left[|\xi-\xi_1|+|\xi-\xi_2|+|\xi_1+\xi_2|\right] \\ &\geq \frac{1}{3}|(\xi-\xi_1)+(\xi-\xi_2)+(\xi_1+\xi_2)| = \frac{2}{3}|\xi| \gtrsim |\xi|. \end{aligned} \tag{3.43}$$

Now, using estimate (3.43), from (3.41) we get

$$\begin{aligned} \Theta_1 &\lesssim |\xi|^{2-4s}\int_{E_1}\frac{d\xi_1 d\xi_2}{|\xi|^{2(1-b)}|(\xi-\xi_2)(\xi_1+\xi_2)|^{2(1-b)}} \\ &= |\xi|^{2b-4s}\int_{E_1}\frac{d\xi_1 d\xi_2}{|(\xi-\xi_2)(\xi_1+\xi_2)|^{2(1-b)}}. \end{aligned} \tag{3.44}$$

Furthermore, making the change of variables $\mu_1 = \xi-\xi_2$, $\mu_2 = \xi_1+\xi_2$ and using estimate (3.40), we get $|\mu_1| = |\xi-\xi_2| \leq |\xi|+|\xi_2| \lesssim |\xi|$, $|\mu_2| = |\xi_1+\xi_2| \leq |\xi_1|+|\xi_2| \lesssim |\xi|$. Thus, from (3.44) we obtain

$$\Theta_1(\xi,\tau) \lesssim |\xi|^{2b-4s}\int_{|\mu_1|\lesssim|\xi|}\frac{1}{|\mu_1|^{2(1-b)}}d\mu_1\int_{|\mu_2|\lesssim|\xi|}\frac{1}{|\mu_2|^{2(1-b)}}d\mu_2. \tag{3.45}$$

For the first integral in the above estimate, since $2(1-b) < 1$, we have $\int_{|\mu_1|\lesssim|\xi|}\frac{1}{|\mu_1|^{2(1-b)}} d\mu_1 \simeq (\mu_1^{2b-1})|_0^{|\xi|} = |\xi|^{2b-1}$. Similarly, we have $\int_{|\mu_2|\lesssim|\xi|}\frac{1}{|\mu_2|^{2(1-b)}}d\mu_1 \simeq |\xi|^{2b-1}$. Using these computations, from (3.45) we get

$$\Theta_1(\xi,\tau) \lesssim |\xi|^{2b-4s}\cdot|\xi|^{2b-1}\cdot|\xi|^{2b-1} = |\xi|^{6b-4s-2}.$$

Using (3.40), we get $|\xi| \gtrsim 1$. Thus, the above quantity is bounded if $6b-4s-2 \leq 0$ or $b \leq \frac{1}{3}+\frac{2}{3}s$. For $b > \frac{1}{2}$ it suffices to have $\frac{1}{3}+\frac{2}{3}s > \frac{1}{2}$ or $\frac{2}{3}s > \frac{1}{6}$, which gives $s > 1/4$. This completes the proof of Lemma 3.3, and also the proof of the mKdV trilinear estimate (2.16). □

3.2 Proof of the First Nonlocal Trilinear Estimates (2.17)

We recall that this estimate reads as follows: $\|(1+\partial_x^2)^{-1}\partial_x[(\partial_x f)(\partial_x g)h]\|_{X^{s,b-1}} \leq c_{s,b}\|f\|_{X^{s,b'}}\|g\|_{X^{s,b'}}\|h\|_{X^{s,b'}}$. Since $(\partial_x f)\cdot(\partial_x g)\cdot h$ is not symmetric, which can be seen clearly if we write it in its L^2 form using convolution, we will symmetrize it by introducing the following symmetric quantity

$$\begin{aligned}\mathcal{T}(f,g,h) &\doteq \|(1+\partial_x^2)^{-1}\partial_x[(\partial_x f)(\partial_x g)h]\|_{X^{s,b-1}} + \|(1+\partial_x^2)^{-1}\partial_x[(\partial_x g)(\partial_x h)f]\|_{X^{s,b-1}} \\ &\quad + \|(1+\partial_x^2)^{-1}\partial_x[(\partial_x h)(\partial_x f)g]\|_{X^{s,b-1}} \\ &\le c_{s,b}\|f\|_{X^{s,b'}}\|g\|_{X^{s,b'}}\|h\|_{X^{s,b'}}.\end{aligned} \tag{3.46}$$

Then proving the trilinear estimate for $\mathcal{T}(f,g,h)$ gives the desired nonlocal trilinear estimates (2.17). Using $a^2+b^2+c^2 \le (|a|+|b|+|c|)^2 \le 3(a^2+b^2+c^2)$ we bound $\mathcal{T}(f,g,h)$ as follows

$$\begin{aligned}|\mathcal{T}(f,g,h)|^2 \lesssim \int_{\mathbb{R}^2} &\frac{|\xi|^2(1+|\xi|)^{2s}}{(1+\xi^2)^2(1+|\tau-\xi^3|)^{2(1-b)}} \\ &\times\Bigg[\int_{\mathbb{R}^4}(|\xi_1||\xi_2|+|\xi_2||\xi-\xi_1-\xi_2|+|\xi-\xi_1-\xi_2||\xi_1|) \\ &|\widehat{f}(\xi_1,\tau_1)\widehat{g}(\xi_2,\tau_2)\widehat{h}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)|d\xi_2 d\tau_2 d\xi_1 d\tau_1\Bigg]^2 d\xi d\tau.\end{aligned}$$

Furthermore, using notation (3.4) we see that estimate (3.46) follows from its L^2 formulation

$$\begin{aligned}&\left\|\int_{\mathbb{R}^4} Q\cdot c_f(\xi_1,\tau_1)c_g(\xi_2,\tau_2)c_h(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2 d\tau_2 d\xi_1 d\tau_1\right\|_{L^2_{\xi,\tau}} \\ &\lesssim \|c_f\|_{L^2_{\xi,\tau}}\|c_g\|_{L^2_{\xi,\tau}}\|c_h\|_{L^2_{\xi,\tau}},\end{aligned} \tag{3.47}$$

where the multiplier $Q = Q(\xi,\tau,\xi_1,\tau_1,\xi_2,\tau_2)$ is defined as follows

$$Q \doteq \frac{|\xi|(1+|\xi|)^s(|\xi_1||\xi_2|+|\xi_2||\xi-\xi_1-\xi_2|+|\xi-\xi_1-\xi_2||\xi_1|)(1+|\xi_1|)^{-s}(1+|\xi_2|)^{-s}(1+|\xi-\xi_1-\xi_2|)^{-s}}{(1+\xi^2)(1+|\tau-\xi^3|)^{1-b}(1+|\tau_1-\xi_1^3|)^{b'}(1+|\tau_2-\xi_2^3|)^{b'}(1+|\tau-\tau_1-\tau_2-(\xi-\xi_1-\xi_2)^3|)^{b'}}.$$

Thanks to our symmetric writing (3.46), we can assume the following order of $|\xi_1|$, $|\xi_2|$ and $|\xi-\xi_1-\xi_2|$ (similar to assumption (3.13))

$$|\xi_1| \ge |\xi_2| \ge |\xi-\xi_1-\xi_2|. \tag{3.48}$$

Thus by ordering relation (3.48), we get $|\xi_1||\xi_2|+|\xi_2||\xi-\xi_1-\xi_2|+|\xi-\xi_1-\xi_2||\xi_1| \le 3|\xi_1||\xi_2|$, which combined with $(1+\xi^2)^{-1}(1+|\xi|)^s \simeq (1+|\xi|)^{s-2}$, reduces Q as follows

$$\begin{aligned}Q &\lesssim \widetilde{Q} \\ &\doteq \frac{\chi_{|\xi_1|\ge|\xi_2|\ge|\xi-\xi_1-\xi_2|}|\xi|(1+|\xi|)^{s-2}|\xi_1\xi_2|(1+|\xi_1|)^{-s}(1+|\xi_2|)^{-s}(1+|\xi-\xi_1-\xi_2|)^{-s}}{(1+|\tau-\xi^3|)^{1-b}(1+|\tau_1-\xi_1^3|)^{b'}(1+|\tau_2-\xi_2^3|)^{b'}(1+|\tau-\tau_1-\tau_2-(\xi-\xi_1-\xi_2)^3|)^{b'}}.\end{aligned} \tag{3.49}$$

Furthermore, for $|\xi| \ge 10^{-3}|\xi_1|$, we have $|\xi_1\xi_2| \lesssim (1+|\xi|)^2$ or $(1+|\xi|)^{-2}|\xi_1\xi_2| \lesssim 1$, which gives

$$\widetilde{Q} \lesssim \frac{\chi_{|\xi_1|\ge|\xi_2|\ge|\xi-\xi_1-\xi_2|}|\xi|(1+|\xi|)^{s}(1+|\xi_1|)^{-s}(1+|\xi_2|)^{-s}(1+|\xi-\xi_1-\xi_2|)^{-s}}{(1+|\tau-\xi^3|)^{1-b}(1+|\tau_1-\xi_1^3|)^{b'}(1+|\tau_2-\xi_2^3|)^{b'}(1+|\tau-\tau_1-\tau_2-(\xi-\xi_1-\xi_2)^3|)^{b'}},$$

where the right-hand side of the above inequality is bounded by the multiplier defined in (3.6). So, the L^2 inequality (3.47) is reduced to the L^2 inequality (3.5) for the mKdV trilinear estimate. Hence, we assume that $|\xi| < 10^{-3}|\xi_1|$ and the multiplier $\widetilde{Q}$ defined in (3.49) becomes

$$Q_1 \doteq \frac{\chi_B(\xi,\xi_1,\xi_2)\cdot|\xi|(1+|\xi|)^{s-2}|\xi_1\xi_2|(1+|\xi_1|)^{-s}(1+|\xi_2|)^{-s}(1+|\xi-\xi_1-\xi_2|)^{-s}}{(1+|\tau-\xi^3|)^{1-b}(1+|\tau_1-\xi_1^3|)^{b'}(1+|\tau_2-\xi_2^3|)^{b'}(1+|\tau-\tau_1-\tau_2-(\xi-\xi_1-\xi_2)^3|)^{b'}},$$

with the domain B given by

$$B \doteq \{(\xi,\xi_1,\xi_2)\in\mathbb{R}^3 : |\xi_1|\geq|\xi_2|\geq|\xi-\xi_1-\xi_2| \text{ and } |\xi|<10^{-3}|\xi_1|\}. \tag{3.50}$$

Now, using duality, the left-hand side of L^2 form (3.47) (with $Q = Q_1$) is bounded as follows

$$\left\|\int_{\mathbb{R}^4} Q_1\cdot c_f(\xi_1,\tau_1)c_g(\xi_2,\tau_2)c_h(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2d\tau_2d\xi_1d\tau_1\right\|_{L^2_{\xi,\tau}} \tag{3.51}$$

$$\lesssim \sup_{\|d\|_{L^2}=1}\int_{\mathbb{R}^2} d(\xi,\tau)\left[\int_{\mathbb{R}^4} Q_1c_f(\xi_1,\tau_1)c_g(\xi_2,\tau_2)c_h(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2d\tau_2d\xi_1d\tau_1\right]d\xi d\tau.$$

Furthermore, using Fubini's Theorem in (3.51) to switch integrations $d\xi_1d\tau_1$ and $d\xi d\tau$, we get

$$\left\|\int_{\mathbb{R}^4} Q_1\cdot c_f(\xi_1,\tau_1)c_g(\xi_2,\tau_2)c_h(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2d\tau_2d\xi_1d\tau_1\right\|_{L^2_{\xi,\tau}} \tag{3.52}$$

$$\lesssim \sup_{\|d\|_{L^2}=1}\int_{\mathbb{R}^2} c_f(\xi_1,\tau_1)\left[\int_{\mathbb{R}^4} Q_1d(\xi,\tau)c_g(\xi_2,\tau_2)c_h(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2d\tau_2d\xi d\tau\right]d\xi_1d\tau_1.$$

In addition, using Cauchy–Schwarz inequality for the integral $d\xi_1d\tau_1$ in (3.52) we get

$$\left\|\int_{\mathbb{R}^4} Q_1\cdot c_f(\xi_1,\tau_1)c_g(\xi_2,\tau_2)c_h(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2d\tau_2d\xi_1d\tau_1\right\|_{L^2_{\xi,\tau}} \tag{3.53}$$

$$\lesssim \sup_{\|d\|_{L^2}=1}\|c_f\|_{L^2}\left(\int_{\mathbb{R}^2}\left|\int_{\mathbb{R}^4} Q_1d(\xi,\tau)c_g(\xi_2,\tau_2)c_h(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2d\tau_2d\xi d\tau\right|^2 d\xi_1d\tau_1\right)^{\frac{1}{2}}.$$

Moreover, for the $d\xi_2d\tau_2d\xi d\tau$ integral in (3.53), applying the Cauchy–Schwarz inequality we get

$$\left\|\int_{\mathbb{R}^4} Q_1\cdot c_f(\xi_1,\tau_1)c_g(\xi_2,\tau_2)c_h(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2d\tau_2d\xi_1d\tau_1\right\|_{L^2_{\xi,\tau}}$$

$$\lesssim \sup_{\|d\|_{L^2}=1}\|c_f\|_{L^2}\left(\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^4} Q_1^2(\xi,\xi_1,\xi_2,\tau,\tau_1,\tau_2)d\xi_2d\tau_2d\xi d\tau\right]\right. \tag{3.54}$$

$$\left.\times\left[\int_{\mathbb{R}^4} d^2(\xi,\tau)c_g^2(\xi_2,\tau_2)c_h^2(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2d\tau_2d\xi d\tau\right]d\xi_1d\tau_1\right)^{1/2}.$$

In (3.54), taking the supremum of $\int_{\mathbb{R}^4} Q_1^2(\xi,\xi_1,\xi_2,\tau,\tau_1,\tau_2)d\xi_2 d\tau_2 d\xi d\tau$ over ξ_1, τ_1, we get

$$\left\| \int_{\mathbb{R}^4} Q_1 \cdot c_f(\xi_1,\tau_1)c_g(\xi_2,\tau_2)c_h(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2 d\tau_2 d\xi_1 d\tau_1 \right\|_{L^2_{\xi,\tau}}$$
$$\lesssim \sup_{\|d\|_{L^2}=1} \|c_f\|_{L^2} \left\| \int_{\mathbb{R}^4} Q_1^2(\xi,\xi_1,\xi_2,\tau,\tau_1,\tau_2)d\xi_2 d\tau_2 d\xi d\tau \right\|^{\frac{1}{2}}_{L^\infty_{\xi_1,\tau_1}} \tag{3.55}$$
$$\times \left(\int_{\mathbb{R}^6} d^2(\xi,\tau)c_g^2(\xi_2,\tau_2)c_h^2(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2 d\tau_2 d\xi d\tau d\xi_1 d\tau_1 \right)^{1/2}.$$

Next, using the following estimate (which similar to (3.11))

$$\int_{\mathbb{R}^6} d^2(\xi,\tau)c_g^2(\xi_2,\tau_2)c_h^2(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2 d\tau_2 d\xi d\tau d\xi_1 d\tau_1$$
$$\lesssim \|d\|^2_{L^2}\|c_g\|^2_{L^2}\|c_h\|^2_{L^2},$$

from estimate (3.55) we arrive at

$$\left\| \int_{\mathbb{R}^4} Q_1 \cdot c_f(\xi_1,\tau_1)c_g(\xi_2,\tau_2)c_h(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2 d\tau_2 d\xi_1 d\tau_1 \right\|_{L^2_{\xi,\tau}}$$
$$\lesssim \sup_{\|d\|_{L^2}=1} \|c_f\|_{L^2} \left\| \int_{\mathbb{R}^4} Q_1^2(\xi,\xi_1,\xi_2,\tau,\tau_1,\tau_2)d\xi_2 d\tau_2 d\xi d\tau \right\|^{\frac{1}{2}}_{L^\infty_{\xi_1,\tau_1}} \|d\|_{L^2}\|c_g\|_{L^2}\|c_h\|_{L^2}$$
$$\lesssim \|\Theta_2\|^{\frac{1}{2}}_{L^\infty_{\xi_1,\tau_1}} \|c_f\|_{L^2_{\xi,\tau}} \|c_g\|_{L^2_{\xi,\tau}} \|c_h\|_{L^2_{\xi,\tau}}, \tag{3.56}$$

where Θ_2 is as in the following result, which provides its estimate.

Lemma 3.4 *If $s > \frac{1}{2}$ and $\frac{1}{2} < b' \le b \le \min\{\frac{s}{3}+\frac{1}{3}, 1\}$, then for all ξ_1 and τ_1 we have*

$$\Theta_2(\xi_1,\tau_1) \doteq \frac{(1+|\xi_1|)^{-2s}}{(1+|\tau_1-\xi_1^3|)^{2b'}}$$
$$\times \int_{\mathbb{R}^4} \frac{\chi_B(\xi,\xi_1,\xi_2)\cdot|\xi|^2(1+|\xi|)^{2s-4}|\xi_1\xi_2|^2(1+|\xi_2|)^{-2s}(1+|\xi-\xi_1-\xi_2|)^{-2s}d\tau_2 d\xi_2 d\tau d\xi}{(1+|\tau-\xi^3|)^{2-2b}(1+|\tau_2-\xi_2^3|)^{2b'}(1+|\tau-\tau_1-\tau_2-(\xi-\xi_1-\xi_2)^3|)^{2b'}}$$
$$\lesssim 1. \tag{3.57}$$

Proof For the $d\tau_2$-integral in (3.57), applying estimate (3.1) with $a = \tau-\tau_1-(\xi-\xi_1-\xi_2)^3$, $c = \xi_2^3$, $\ell = b'$, we get

$$\Theta_2 \lesssim \frac{(1+|\xi_1|)^{-2s}}{(1+|\tau_1-\xi_1^3|)^{2b'}} \tag{3.58}$$
$$\times \int_{\mathbb{R}^3} \frac{\chi_B \cdot |\xi|^2(1+|\xi|)^{2s-4}|\xi_1\xi_2|^2(1+|\xi_2|)^{-2s}(1+|\xi-\xi_1-\xi_2|)^{-2s}d\tau d\xi_2 d\xi}{(1+|\tau-\xi^3|)^{2(1-b)}(1+|\tau-\tau_1-(\xi-\xi_1-\xi_2)^3-\xi_2^3|)^{2b'}}.$$

Furthermore, for the $d\tau$-integral in (3.58), applying estimate (3.2) with $c = \tau_1 + (\xi - \xi_1 - \xi_2)^3 + \xi_2^3$, $a = \xi^3$, $\ell = b$, $\ell' = b'$ we get

$$\Theta_2 \lesssim \frac{(1+|\xi_1|)^{-2s}}{(1+|\tau_1-\xi_1^3|)^{2b'}} \times \int_{\mathbb{R}^2} \frac{\chi_B |\xi|^2 (1+|\xi|)^{2s-4} |\xi_1\xi_2|^2 (1+|\xi_2|)^{-2s} (1+|\xi-\xi_1-\xi_2|)^{-2s} d\xi_2 d\xi}{(1+|\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2)|)^{2(1-b)}}, \tag{3.59}$$

where $d_3(\xi, \xi_1, \xi_2)$ is the Bourgain quantity defined by (3.8). Next, we consider the following cases.

- Case 1: $|\xi_1| \le 100$.
- Case 2: $|\xi_1| > 100$.

Proof in Case 1. Since $|\xi| < 10^{-3}|\xi_1|$ and $|\xi_1| \le 100$, by the ordering relation (3.48), all of $|\xi|$, $|\xi_1|$ and $|\xi_2|$ are bounded. Furthermore, since $b' \ge 0$ and $(1-b) \ge 0$, we see that the multiplier and the integrand in Θ_2 are bounded. Therefore, $\Theta_2(\xi_1, \tau_1) \lesssim 1$, since the integration is over a bounded set.

Proof in Case 2. Here we have

$$|\xi_2| \simeq |\xi_1|. \tag{3.60}$$

By the ordering relation $|\xi_1| \ge |\xi_2| \ge |\xi - \xi_1 - \xi_2|$, it suffices to show $|\xi_2| > \frac{1}{4}|\xi_1|$. In fact, if $|\xi_2| \le \frac{1}{4}|\xi_1|$, then we would get $|\xi| = |\xi_1 + \xi_2 + (\xi - \xi_1 - \xi_2)| \ge |\xi_1| - |\xi - \xi_1 - \xi_2| - |\xi_2| \ge |\xi_1| - \frac{1}{4}|\xi_1| - \frac{1}{4}|\xi_1| \ge \frac{1}{2}|\xi_1|$, which is a contradiction to $|\xi| < 10^{-3}|\xi_1|$, a condition of $(\xi, \xi_1, \xi_2) \in B$. Using (3.60) we get $|\xi_1\xi_2|^2 \simeq |\xi_1|^4$, and $(1+|\xi_2|)^{-2s} \simeq |\xi_1|^{-2s}$ since $|\xi_1| > 100$. Also, we have $(1+|\xi_1|)^{-2s} \simeq |\xi_1|^{-2s}$. Combining this with (3.59) we get

$$\Theta_2(\xi_1, \tau_1) \lesssim \frac{|\xi_1|^{4-4s}}{(1+|\tau_1-\xi_1^3|)^{2b'}} \int_{\mathbb{R}^2} \frac{\chi_B(\xi,\xi_1,\xi_2) \cdot (1+|\xi|)^{2s-2} (1+|\xi-\xi_1-\xi_2|)^{-2s}}{(1+|\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2)|)^{2(1-b)}} d\xi_2 d\xi. \tag{3.61}$$

Since $\frac{1}{2} < b' \le b < 1$ we have $0 < 2(1-b) < 1 < 2b'$, so we can move $(1+|\tau_1 - \xi_1^3|)^{2b'}$ inside the integral and replace $2b'$ with $2(1-b)$. Since $(1+|\tau_1-\xi_1^3|)(1+|\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2)|) \ge |\tau_1-\xi_1^3| + |\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2)|$ and also $|\tau_1-\xi_1^3| + |\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2)| \ge |\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2) - (\tau_1-\xi_1^3)| = |d_3(\xi,\xi_1,\xi_2)|$, we have

$$(1+|\tau_1-\xi_1^3|)^{2b'}(1+|\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2)|)^{2(1-b)} \gtrsim |d_3(\xi,\xi_1,\xi_2)|^{2(1-b)},$$

which combined with (3.61) gives

$$\Theta_2(\xi_1, \tau_1) \lesssim |\xi_1|^{4-4s} \int_{\mathbb{R}^2} \frac{\chi_B(\xi,\xi_1,\xi_2) \cdot (1+|\xi|)^{2s-2} (1+|\xi-\xi_1-\xi_2|)^{-2s}}{|d_3(\xi,\xi_1,\xi_2)|^{2(1-b)}} d\xi_2 d\xi.$$

Combining this with property (3.8), i.e. $d_3 = -3(\xi-\xi_1)(\xi-\xi_2)(\xi_1+\xi_2)$, we get

$$\Theta_2(\xi_1, \tau_1) \lesssim |\xi_1|^{4-4s} \int_{\mathbb{R}^2} \frac{\chi_B(\xi,\xi_1,\xi_2) \cdot (1+|\xi|)^{2s-2} (1+|\xi-\xi_1-\xi_2|)^{-2s}}{|(\xi-\xi_1)(\xi-\xi_2)(\xi_1+\xi_2)|^{2(1-b)}} d\xi_2 d\xi. \tag{3.62}$$

Next, using $(1+|\xi-\xi_1-\xi_2|)^{-2s} \lesssim 1$ and the inequalities $|\xi-\xi_1| \gtrsim |\xi_1|$ and $|\xi-\xi_2| \gtrsim |\xi_1|$, which follows from $|\xi-\xi_1| \geq |\xi_1| - |\xi| \geq |\xi_1| - 10^{-3}|\xi_1| \gtrsim |\xi_1|$ and $|\xi-\xi_2| \geq |\xi_2| - |\xi| \geq \frac{1}{4}|\xi_1| - 10^{-3}|\xi_1| \gtrsim |\xi_1|$, from (3.62) we obtain

$$\Theta_2(\xi_1,\tau_1) \lesssim |\xi_1|^{4b-4s} \int_{\mathbb{R}^2} \frac{\chi_B(\xi,\xi_1,\xi_2)\cdot(1+|\xi|)^{2s-2}}{|\xi_1+\xi_2|^{2(1-b)}} d\xi_2 d\xi$$
$$= |\xi_1|^{4b-4s} \int_{|\xi|\leq 10^{-3}|\xi_1|} \frac{d\xi}{(1+|\xi|)^{2-2s}} \int_{|\xi_2|\leq|\xi_1|} \frac{d\xi_2}{|\xi_1+\xi_2|^{2(1-b)}}. \tag{3.63}$$

Since $s > \frac{1}{2}$, we have $2-2s < 1$, which implies the following bound for the first integral

$$\int_{|\xi|\leq 10^{-3}|\xi_1|} \frac{d\xi}{(1+|\xi|)^{2-2s}} \simeq \left[(1+\xi)^{2s-1}\right]\Big|_0^{10^{-3}|\xi_1|} \lesssim |\xi_1|^{2s-1}. \tag{3.64}$$

Concerning the second integral in (3.63), we make the change of variables $\mu = \mu(\xi_2) = \xi_1+\xi_2$, and using the inequalities $|\mu| \leq |\xi_1| + |\xi_2| \leq 2|\xi_1|$, for $2(1-b) < 1$ or $b > \frac{1}{2}$, we have

$$\int_{|\xi_2|\leq|\xi_1|} \frac{d\xi_2}{|\xi_1+\xi_2|^{2(1-b)}} = \int_{|\mu|\leq 2|\xi_1|} \frac{d\mu}{|\mu|^{2(1-b)}} \simeq (\mu^{2b-1})\Big|_0^{2|\xi_1|} \simeq |\xi_1|^{2b-1}. \tag{3.65}$$

Combining estimates (3.64), (3.65) with (3.63), we obtain

$$\Theta_2(\xi_1,\tau_1) \lesssim |\xi_1|^{4b-4s}|\xi_1|^{2s-1}|\xi_1|^{2b-1} = |\xi_1|^{6b-2s-2}.$$

Since $|\xi_1| > 100$, the above quantity is bounded if $6b-2s-2 \leq 0$ or $b \leq \frac{1}{3}s + \frac{1}{3}$. For $b > \frac{1}{2}$, it suffices to have $\frac{1}{3}s + \frac{1}{3} > \frac{1}{2}$ or $\frac{1}{3}s > \frac{1}{6}$, which implies that $s > 1/2$. This completes the proof of Lemma 3.4. □

3.3 Proof of the Second Nonlocal Trilinear Estimates (2.18)

Using notation (3.4), we see that to prove the estimate (2.18), i.e.

$$\|(1+\partial_x^2)^{-1}[(\partial_x f)(\partial_x g)(\partial_x h)]\|_{X^{s,b-1}} \leq c_{s,b}\|f\|_{X^{s,b'}}\|g\|_{X^{s,b'}}\|h\|_{X^{s,b'}},$$

it suffices to prove the following L^2 estimate

$$\left\| \int_{\mathbb{R}^4} Q\cdot c_f(\xi_1,\tau_1)c_g(\xi_2,\tau_2)c_h(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2 d\tau_2 d\xi_1 d\tau_1 \right\|_{L^2_{\xi,\tau}}$$
$$\lesssim \|c_f\|_{L^2_{\xi,\tau}}\|c_g\|_{L^2_{\xi,\tau}}\|c_h\|_{L^2_{\xi,\tau}}, \tag{3.66}$$

where the multiplier $Q = Q(\xi,\tau,\xi_1,\tau_1,\xi_2,\tau_2)$ is defined by

$$Q \doteq \frac{(1+\xi^2)^{-1}(1+|\xi|)^s|\xi_1||\xi_2||\xi-\xi_1-\xi_2|(1+|\xi_1|)^{-s}(1+|\xi_2|)^{-s}(1+|\xi-\xi_1-\xi_2|)^{-s}}{(1+|\tau-\xi^3|)^{1-b}(1+|\tau_1-\xi_1^3|)^{b'}(1+|\tau_2-\xi_2^3|)^{b'}(1+|\tau-\tau_1-\tau_2-(\xi-\xi_1-\xi_2)^3|)^{b'}}.$$

By symmetry in convolution writing of $\widehat{\partial_x f} * \widehat{\partial_x g} * \widehat{\partial_x h}$, we can assume the following order of $|\xi_1|$, $|\xi_2|$ and $|\xi-\xi_1-\xi_2|$

$$|\xi_1| \geq |\xi_2| \geq |\xi-\xi_1-\xi_2|. \tag{3.67}$$

Therefore, the multiplier Q is reduced to $\chi_{|\xi_1|\geq|\xi_2|\geq|\xi-\xi_1-\xi_2|}Q$. Also, using $(1+\xi^2)^{-1}(1+|\xi|)^s \lesssim (1+|\xi|)^{s-2}$, Q becomes

$$Q \lesssim \widetilde{Q} \doteq \frac{\chi_{|\xi_1|\geq|\xi_2|\geq|\xi-\xi_1-\xi_2|}(1+|\xi|)^{s-2}|\xi_1\xi_2(\xi-\xi_1-\xi_2)|(1+|\xi_1|)^{-s}(1+|\xi_2|)^{-s}(1+|\xi-\xi_1-\xi_2|)^{-s}}{(1+|\tau-\xi^3|)^{1-b}(1+|\tau_1-\xi_1^3|)^{b'}(1+|\tau_2-\xi_2^3|)^{b'}(1+|\tau-\tau_1-\tau_2-(\xi-\xi_1-\xi_2)^3|)^{b'}}.$$

Furthermore, for $|\xi| \geq 10^{-3}|\xi_1|$, we have $|\xi_1\xi_2(\xi-\xi_1-\xi_2)| \lesssim |\xi|(1+|\xi|)^2$ or $(1+|\xi|)^{-2}|\xi_1\xi_2(\xi-\xi_1-\xi_2)| \lesssim |\xi|$, which implies that

$$\widetilde{Q} \lesssim \frac{\chi_{|\xi_1|\geq|\xi_2|\geq|\xi-\xi_1-\xi_2|}|\xi|(1+|\xi|)^{s}(1+|\xi_1|)^{-s}(1+|\xi_2|)^{-s}(1+|\xi-\xi_1-\xi_2|)^{-s}}{(1+|\tau-\xi^3|)^{1-b}(1+|\tau_1-\xi_1^3|)^{b'}(1+|\tau_2-\xi_2^3|)^{b'}(1+|\tau-\tau_1-\tau_2-(\xi-\xi_1-\xi_2)^3|)^{b'}},$$

where the right-hand side of the above inequality is bounded by the multiplier defined in (3.6). So, the L^2 inequality (3.66) is reduced to the L^2 inequality (3.5) for the local trilinear estimate. Hence, we assume that $|\xi| < 10^{-3}|\xi_1|$ and the multiplier $\widetilde{Q}$ defined in (3.49) becomes

$$Q_2 \doteq \frac{\chi_B(\xi,\xi_1,\xi_2)\cdot(1+|\xi|)^{s-2}|\xi_1\xi_2(\xi-\xi_1-\xi_2)|(1+|\xi_1|)^{-s}(1+|\xi_2|)^{-s}(1+|\xi-\xi_1-\xi_2|)^{-s}}{(1+|\tau-\xi^3|)^{1-b}(1+|\tau_1-\xi_1^3|)^{b'}(1+|\tau_2-\xi_2^3|)^{b'}(1+|\tau-\tau_1-\tau_2-(\xi-\xi_1-\xi_2)^3|)^{b'}},$$

where B is the domain defined by (3.50) incorporating the order relation and the condition $|\xi| < 10^{-3}|\xi_1|$. Like in the proof for the first nonlocal trilinear estimate (see (3.56)), using duality we bound the left-hand side in the L^2 formulation (3.66) (with $Q = Q_2$) as follows

$$\begin{aligned}&\left\| \int_{\mathbb{R}^4} Q_2 \cdot c_f(\xi_1,\tau_1)c_g(\xi_2,\tau_2)c_h(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_2 d\tau_2 d\xi_1 d\tau_1 \right\|_{L^2}\\ &\lesssim \|\Theta_3\|_{L^\infty_{\xi_1,\tau_1}}^{\frac{1}{2}} \|c_f\|_{L^2}\|c_g\|_{L^2}\|c_h\|_{L^2},\end{aligned}$$

where Θ_3 is as in the following result, which provides its L^∞ estimate.

Lemma 3.5 *If $s > \frac{2}{3}$ and $\frac{1}{2} < b' \leq b < \min\{s-\frac{1}{6}, \frac{s}{3}+\frac{1}{3}, \frac{5}{6}\}$, then for all ξ_1 and τ_1 we have*

$$\begin{aligned}\Theta_3(\xi_1,\tau_1) &\doteq \frac{(1+|\xi_1|)^{-2s}}{(1+|\tau_1-\xi_1^3|)^{2b'}}\\ &\times\int_{\mathbb{R}^4} \frac{\chi_B(\xi,\xi_1,\xi_2)(1+|\xi|)^{2s-4}|\xi_1\xi_2(\xi-\xi_1-\xi_2)|^2(1+|\xi_2|)^{-2s}(1+|\xi-\xi_1-\xi_2|)^{-2s}d\tau_2 d\xi_2 d\tau d\xi}{(1+|\tau-\xi^3|)^{2-2b}(1+|\tau_2-\xi_2^3|)^{2b'}(1+|\tau-\tau_1-\tau_2-(\xi-\xi_1-\xi_2)^3|)^{2b'}}\\ &\lesssim 1. \end{aligned} \tag{3.68}$$

Proof For the $d\tau_2$-integral in (3.68), applying estimate (3.1) with $a = \tau-\tau_1-(\xi-\xi_1-\xi_2)^3$, $c = \xi_2^3$, $\ell = b'$, we get

$$\begin{aligned}\Theta_3 &\lesssim \frac{(1+|\xi_1|)^{-2s}}{(1+|\tau_1-\xi_1^3|)^{2b'}}\\ &\times\int_{\mathbb{R}^3} \frac{\chi_B(\xi,\xi_1,\xi_2)(1+|\xi|)^{2s-4}|\xi_1\xi_2(\xi-\xi_1-\xi_2)|^2(1+|\xi_2|)^{-2s}(1+|\xi-\xi_1-\xi_2|)^{-2s}\,d\tau d\xi_2 d\xi}{(1+|\tau-\xi^3|)^{2(1-b)}(1+|\tau-\tau_1-(\xi-\xi_1-\xi_2)^3-\xi_2^3|)^{2b'}}.\end{aligned}$$

Furthermore, for the $d\tau$-integral in the above estimate, applying estimate (3.2) with $c = \tau_1 + (\xi - \xi_1 - \xi_2)^3 + \xi_2^3$, $a = \xi^3$, $\ell = b$, $\ell' = b'$, we get

$$\begin{aligned}
\Theta_3 &\lesssim \frac{(1+|\xi_1|)^{-2s}}{(1+|\tau_1-\xi_1^3|)^{2b'}} \\
&\quad\times \int_{\mathbb{R}^2} \frac{\chi_B (1+|\xi|)^{2s-4}|\xi_1\xi_2(\xi-\xi_1-\xi_2)|^2(1+|\xi_2|)^{-2s}(1+|\xi-\xi_1-\xi_2|)^{-2s}d\xi_2 d\xi}{(1+|\tau_1+(\xi-\xi_1-\xi_2)^3+\xi_2^3-\xi^3|)^{2(1-b)}} \\
&= \frac{(1+|\xi_1|)^{-2s}}{(1+|\tau_1-\xi_1^3|)^{2b'}} \\
&\quad\times \int_{\mathbb{R}^2} \frac{\chi_B (1+|\xi|)^{2s-4}|\xi_1\xi_2(\xi-\xi_1-\xi_2)|^2(1+|\xi_2|)^{-2s}(1+|\xi-\xi_1-\xi_2|)^{-2s}}{(1+|\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2)|)^{2(1-b)}} d\xi_2 d\xi,
\end{aligned} \tag{3.69}$$

where $d_3(\xi, \xi_1, \xi_2)$ is the Bourgain quantity defined by (3.7). Next, we consider the following cases.

- Case 1: $|\xi_1| \le 100$.
- Case 2: $|\xi_1| > 100$.

Proof in Case 1. Since $|\xi| < 10^{-3}|\xi_1|$ and $|\xi_1| \le 100$, by the ordering relation (3.67), all of $|\xi|$, $|\xi_1|$ and $|\xi_2|$ are bounded. Furthermore, since $b' \ge 0$ and $(1-b) \ge 0$, we see that $\Theta_3(\xi_1, \tau_1) \lesssim 1$.

Proof in Case 2. Here we have

$$|\xi_2| \simeq |\xi_1|. \tag{3.70}$$

By the ordering relation $|\xi_1| \ge |\xi_2| \ge |\xi-\xi_1-\xi_2|$, it suffices to show $|\xi_2| > \frac{1}{4}|\xi_1|$. In fact, if $|\xi_2| \le \frac{1}{4}|\xi_1|$, then we would get $|\xi| = |\xi_1+\xi_2+(\xi-\xi_1-\xi_2)| \ge |\xi_1| - |\xi-\xi_1-\xi_2| - |\xi_2| \ge |\xi_1| - \frac{1}{4}|\xi_1| - \frac{1}{4}|\xi_1| \ge \frac{1}{2}|\xi_1|$, which is a contradiction to $|\xi| < 10^{-3}|\xi_1|$, a condition of $(\xi, \xi_1, \xi_2) \in B$. Using (3.70) we get $|\xi_1\xi_2|^2 \simeq |\xi_1|^4$, and $(1+|\xi_2|)^{-2s} \simeq |\xi_1|^{-2s}$ since $|\xi_1| > 100$. Also, we have $(1+|\xi_1|)^{-2s} \simeq |\xi_1|^{-2s}$. Combining this with $|\xi-\xi_1-\xi_2|^2 \lesssim (1+|\xi-\xi_1-\xi_2|)^2$, from (3.69) we get

$$\begin{aligned}
\Theta_3(\xi_1,\tau_1) &\lesssim \frac{|\xi_1|^{4-4s}}{(1+|\tau_1-\xi_1^3|)^{2b'}} \\
&\quad\times \int_{\mathbb{R}^2} \frac{\chi_B(\xi,\xi_1,\xi_2)\cdot(1+|\xi|)^{2s-4}(1+|\xi-\xi_1-\xi_2|)^{2-2s}}{(1+|\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2)|)^{2(1-b)}} d\xi_2 d\xi.
\end{aligned} \tag{3.71}$$

Since $\frac{1}{2} < b' \le b < 1$ we have $0 < 2(1-b) < 1 < 2b'$, so we can move $(1+|\tau_1-\xi_1^3|)^{2b'}$ inside the integral and replace $2b'$ with $2(1-b)$. Since $(1+|\tau_1-\xi_1^3|)(1+|\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2)|) \ge |\tau_1-\xi_1^3| + |\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2)|$ and also $|\tau_1-\xi_1^3| + |\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2)| \ge |\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2)-(\tau_1-\xi_1^3)| = |d_3(\xi,\xi_1,\xi_2)|$, we have

$$(1+|\tau_1-\xi_1^3|)^{2b'}(1+|\tau_1-\xi_1^3+d_3(\xi,\xi_1,\xi_2)|)^{2(1-b)} \gtrsim |d_3(\xi,\xi_1,\xi_2)|^{2(1-b)},$$

which combined with (3.71) gives

$$\Theta_3(\xi_1,\tau_1) \lesssim |\xi_1|^{4-4s} \int_{\mathbb{R}^2} \frac{\chi_B(\xi,\xi_1,\xi_2)\cdot(1+|\xi|)^{2s-4}(1+|\xi-\xi_1-\xi_2|)^{2-2s}}{|d_3(\xi,\xi_1,\xi_2)|^{2(1-b)}} d\xi_2 d\xi.$$

Then, using the factorization $d_3 = -3(\xi-\xi_1)(\xi-\xi_2)(\xi_1+\xi_2)$, we get

$$\Theta_3(\xi_1,\tau_1) \lesssim |\xi_1|^{4-4s} \int_{\mathbb{R}^2} \frac{\chi_B(\xi,\xi_1,\xi_2)\cdot(1+|\xi|)^{2s-4}(1+|\xi-\xi_1-\xi_2|)^{2-2s}}{|(\xi-\xi_1)(\xi-\xi_2)(\xi_1+\xi_2)|^{2(1-b)}} d\xi_2 d\xi. \tag{3.72}$$

Next, using the inequalities

$$|\xi-\xi_1|\gtrsim|\xi_1| \quad \text{and} \quad |\xi-\xi_2|\gtrsim|\xi_1|,$$

which follows from $|\xi-\xi_1|\geq|\xi_1|-|\xi|\geq|\xi_1|-10^{-3}|\xi_1|\gtrsim|\xi_1|$ and $|\xi-\xi_2|\geq|\xi_2|-|\xi|\geq\frac{1}{4}|\xi_1|-10^{-3}|\xi_1|\gtrsim|\xi_1|$, from (3.72), we obtain

$$\Theta_3(\xi_1,\tau_1)\lesssim|\xi_1|^{4b-4s}\int_{\mathbb{R}^2}\frac{\chi_B(\xi,\xi_1,\xi_2)\cdot(1+|\xi|)^{2s-4}(1+|\xi-\xi_1-\xi_2|)^{2-2s}}{|\xi_1+\xi_2|^{2(1-b)}}d\xi_2 d\xi. \tag{3.73}$$

Now, we consider the following two subcases arising from the sign of $(2-2s)$.

- Subcase 2.1: $s\leq 1$.
- Subcase 2.2: $s>1$.

Proof in Subcase 2.1. Then $2-2s\geq 0$, which combined with $|\xi-\xi_1-\xi_2|\leq|\xi_1|$ and $|\xi_1|>100$ gives $(1+|\xi-\xi_1-\xi_2|)^{2-2s}\lesssim|\xi_1|^{2-2s}$. From the last inequality and estimate (3.73) we get

$$\begin{aligned}\Theta_3(\xi_1,\tau_1)&\lesssim|\xi_1|^{4b-6s+2}\int_{\mathbb{R}^2}\frac{\chi_B(\xi,\xi_1,\xi_2)\cdot(1+|\xi|)^{2s-4}}{|\xi_1+\xi_2|^{2(1-b)}}d\xi_2 d\xi\\&=|\xi_1|^{4b-6s+2}\int_{|\xi|\leq 10^{-3}|\xi_1|}\frac{d\xi}{(1+|\xi|)^{4-2s}}\cdot\int_{|\xi_2|\leq|\xi_1|}\frac{d\xi_2}{|\xi_1+\xi_2|^{2(1-b)}}.\end{aligned} \tag{3.74}$$

Since $4-2s\geq 2$ the first integral is bounded. For the second integral, making the change of variables $\mu=\mu(\xi_2)=\xi_1+\xi_2$ and using the inequalities $|\mu|\leq|\xi_1|+|\xi_2|\leq 2|\xi_1|$, for $2(1-b)<1$ or $b>\frac{1}{2}$, we have $\int_{|\xi_2|\leq|\xi_1|}\frac{d\xi_2}{|\xi_1+\xi_2|^{2(1-b)}}=\int_{|\mu|\leq 2|\xi_1|}\frac{d\mu}{|\mu|^{2(1-b)}}\simeq(\mu^{2b-1})\big|_0^{2|\xi_1|}\simeq|\xi_1|^{2b-1}$. Using the above computations, from (3.74) we get

$$\Theta_3(\xi_1,\tau_1)\lesssim|\xi_1|^{4b-6s+2}|\xi_1|^{2b-1}=|\xi_1|^{6b-6s+1}.$$

Since $|\xi_1|>100$, the above quantity is bounded if $6b-6s+1\leq 0$ or $b\leq s-\frac{1}{6}$. For $b>\frac{1}{2}$, it suffices to have $s-\frac{1}{6}>\frac{1}{2}$ or $s>2/3$. This completes the proof in Subcase 2.1.
Proof in Subcase 2.2. Then $2-2s<0$, which implies that $(1+|\xi-\xi_1-\xi_2|)^{2-2s}\lesssim 1$. Therefore,

$$\Theta_3(\xi_1,\tau_1)\lesssim|\xi_1|^{4b-4s}\int_{|\xi|\leq 10^{-3}|\xi_1|}\frac{d\xi}{(1+|\xi|)^{4-2s}}\cdot\int_{|\xi_2|\leq|\xi_1|}\frac{d\xi_2}{|\xi_1+\xi_2|^{2(1-b)}}. \tag{3.75}$$

For the first integral, we have

$$\int_{|\xi|\leq 10^{-3}|\xi_1|}\frac{d\xi}{(1+|\xi|)^{4-2s}}\lesssim\begin{cases}1, & s<\frac{3}{2},\\ \ln|\xi_1|, & s=\frac{3}{2},\\ |\xi_1|^{2s-3}, & s>\frac{3}{2}.\end{cases}$$

For the second integral in (3.75), making the change of variables $\mu=\mu(\xi_2)=\xi_1+\xi_2$ and using the inequalities $|\mu|\leq|\xi_1|+|\xi_2|\leq 2|\xi_1|$, for $2(1-b)<1$ or $b>\frac{1}{2}$, we get $\int_{|\xi_2|\leq|\xi_1|}\frac{d\xi_2}{|\xi_1+\xi_2|^{2(1-b)}}=\int_{|\mu|\leq 2|\xi_1|}\frac{d\mu}{|\mu|^{2(1-b)}}\simeq(\mu^{2b-1})\big|_0^{2|\xi_1|}\simeq|\xi_1|^{2b-1}$. Finally, using the above computations, from (3.75) we get

$$\Theta_3(\xi_1,\tau_1)\lesssim\begin{cases}|\xi_1|^{6b-4s-1}, & s<\frac{3}{2},\\ |\xi_1|^{6b-7}\ln|\xi_1|, & s=\frac{3}{2},\\ |\xi_1|^{6b-2s-4}, & s>\frac{3}{2},\end{cases}$$

which is bounded if $b \leq \frac{5}{6}$, since $s > 1$. This completes the proof for Lemma 3.5, and also the proof of our last trilinear estimate. □

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References

1. Arnold, V.: Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. (in French) Ann. Inst. Fourier (Grenoble) **16**, 319–361 (1966)
2. Alber, M., Camassa, R., Holm, D., Marsden, J.: The geometry of peaked solitons and billiard solutions of a class of integrable PDE's. Lett. Math. Phys. **32**, 137–151 (1994)
3. Barostichi, R., Himonas, A., Petronilho, G.: Autonomous Ovsyannikov theorem and applications to nonlocal evolution equations and systems. J. Funct. Anal. **270**, 330–358 (2016)
4. Beals, R., Sattinger, D., Szmigielski, J.: Multi-peakons and a theorem of Stieltjes. Inverse Probl. **15**, L1–L4 (1999)
5. Beals, R., Sattinger, D., Szmigielski, J.: Multipeakons and the classical moment problem. Adv. Math. **154**, 229–257 (2000)
6. Bekiranov, D., Ogawa, T., Ponce, G.: Interaction equations for short and long dispersive waves. J. Funct. Anal. **158**, 357–388 (1998)
7. Bona, J.L., Smith, R.: The initial-value problem for the Korteweg-de Vries equation. Philos. Trans. R. Soc. Lond. Ser. A **278**, 555–601 (1975)
8. Bourgain, J.: Fourier transform restriction phenomena for certain lattice subsets and applications to non-linear evolution equations. Part 2: KdV-equation. Geom. Funct. Anal. GAFA **3**, 209–262 (1993)
9. Bourgain, J., Li, D.: Galilean boost and non-uniform continuity for incompressible Euler. Commun. Math. Phys. **372**, 261–280 (2019)
10. Boussinesq, J.: Essai sur la Théorie des Eaux Courantes. Mémoires Présentés par Divers Savants à l'Académie des Sciences, vol. 23, no. 1. Paris Imprimerie Nationale (1877)
11. Bressan, A., Constantin, A.: Global conservative solutions of the Camassa-Holm equation. Arch. Rational Mech. Anal. **183**, 215–239 (2007)
12. Bressan, A., Constantin, A.: Global dissipative solutions of the Camassa-Holm equation. Anal. Appl. **5**, 1–27 (2007)
13. Byers, P.: Existence time for the Camassa-Holm equation and the critical Sobolev index. Indiana Univ. Math. J. **55**, 941–954 (2006)
14. Camassa, R., Holm, D.: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71**, 1661–1664 (1993)
15. Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T.: Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$. J. Amer. Math. Soc. **16**, 705–749 (2003)
16. Constantin, A.: Finite propagation speed for the Camassa-Holm equation. J. Math. Phys. **46**, 023506 (2005)
17. Constantin, A., Escher, J.: Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation. Commun. Pure Appl. Math. **51**, 475–504 (1998)
18. Constantin, A., Lannes, D.: The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations. Arch. Rational Mech. Anal. **192**, 165–186 (2009)
19. Constantin, A., McKean, H.: A shallow water equation on the circle. Commun. Pure Appl. Math. **52**, 949–982 (1999)
20. Constantin, A., Strauss, W.: Stability of the Camassa-Holm solitons. J. Nonlinear Sci. **12**, 415–422 (2002)
21. Christov, O., Hakkaev, S.: On the Cauchy problem for the periodic b-family of equations and of the non-uniform continuity of the Degasperis-Procesi equation. J. Math. Anal. Appl. **360**, 47–56 (2009)
22. Danchin, R.: A few remarks on the Camassa-Holm equation. Differ. Integral Equ. **14**, 953–988 (2001)
23. de Lellis, C., Kappeler, T., Topalov, P.: Low-regularity solutions of the periodic Camassa-Holm equation. Commun. Partial Differ. Equ. **32**, 87–126 (2007)
24. Degasperis, A., Procesi, M.: Asymptotic integrability. (English summary) Symmetry and perturbation theory (Rome, 1998), pp. 23–37. World Sci. Publ., River Edge, NJ (1999)
25. Ebin, D., Marsden, J.: Groups of diffeomorphisms and the motion of an incompressible fluid. Ann. Math. (2) **92**, 102–163 (1970)
26. Erdoğan, M.B., Tzirakis, N.: Regularity properties of the cubic nonlinear Schrödinger equation on the half line. J. Funct. Anal. **271**, 2539–2568 (2016)

27. Escher, J., Yin, Z.: Well-posedness, blow-up phenomena, and global solutions for the b-equation. J. Reine Angew. Math. **2008**, 51–80 (2008)
28. Fokas, A.: On a class of physically important integrable equations. Phys. D **87**, 145–150 (1995)
29. Fuchssteiner, B., Fokas, A.: Symplectic structures, their Bäklund transformations and hereditary symmetries. Phys. D **4**, 47–66 (1981)
30. Grayshan, K.: Continuity properties of the data-to-solution map for the periodic b-family equation. Differ. Integral Equ. **25**, 1–20 (2012)
31. Gorsky, J., Nicholls, D.P.: A small dispersion limit to the Camassa-Holm equation: A numerical study. Math. Comput. Simul. **80**, 120–130 (2009)
32. Guo, Z., Liu, X., Molinet, L., Yin, Z.: Ill-posedness of the Camassa-Holm and related equations in the critical space. J. Differ. Equ. **266**, 1698–1707 (2019)
33. Himonas, A., Grayshan, K., Holliman, C.: Ill-posedness for the b-family of equations. J. Nonlinear Sci. **26**, 1175–1190 (2016)
34. Himonas, A., Holliman, C.: On well-posedness of the Degasperis-Procesi equation. Discrete Contin. Dyn. Syst. **31**, 469–488 (2011)
35. Himonas, A., Holliman, C.: The Cauchy problem for the Novikov equation. Nonlinearity **25**, 449–479 (2012)
36. Himonas, A., Holliman, C.: Non-uniqueness for the Fokas-Olver-Rosenau-Qiao equation. J. Math. Anal. Appl. **470**, 647–658 (2019)
37. Himonas, A., Holliman, C., Grayshan, K.: Norm inflation and ill-posedness for the Degasperis-Procesi equation. Commun. Partial Differ. Equ. **39**, 2198–2215 (2014)
38. Himonas, A., Holliman, C., Kenig, C.: Construction of 2-peakon solutions and ill-posedness for the Novikov equation. SIAM J. Math. Anal. **50**, 2968–3006 (2018)
39. Himonas, A., Kenig, C.: Non-uniform dependence on initial data for the CH equation on the line. Differ. Integral Equ. **22**, 201–224 (2009)
40. Himonas, A., Kenig, C., Misiołek, G.: Non-uniform dependence for the periodic CH equation. Commun. Partial Differ. Equ. **35**, 1145–1162 (2010)
41. Himonas, A., Mantzavinos, D.: The Cauchy problem for the Fokas-Olver-Rosenau-Qiao equation. Nonlinear Anal. **95**, 499–529 (2014)
42. Himonas, A., Misiołek, G.: The Cauchy problem for shallow water type equations. Commun. Partial Differ. Equ. **23**, 123–139 (1998)
43. Himonas, A., Misiołek, G.: Global well-posedness of the Cauchy problem for a shallow water equation on the circle. J. Differ. Equ. **161**, 479–495 (2000)
44. Himonas, A., Misiołek, G.: The Cauchy problem for an integrable shallow water equation. Differ. Integral Equ. **14**, 821–831 (2001)
45. Himonas, A., Misiołek, G.: High-frequency smooth solutions and well-posedness of the Camassa-Holm equation. Int. Math. Res. Not. **2005**, 3135–3151 (2005)
46. Himonas, A., Misiołek, G.: Non-uniform dependence on initial data of solutions to the Euler equations of hydrodynamics. Commun. Math. Phys. **296**, 285–301 (2010)
47. Himonas, A., Misiołek, G., Ponce, G.: Non-uniform continuity in H^1 of the solution map of the CH equation. Asian J. Math. **11**, 141–150 (2007)
48. Himonas, A., Misiołek, G., Ponce, G., Zhou, Y.: Persistence properties and unique continuation of solutions of the Camassa-Holm equation. Commun. Math. Phys. **271**, 511–522 (2007)
49. Holden, H., Raynaud, X.: Global conservative solutions of the Camassa-Holm equation–A Lagrangian point of view. Commun. Partial Differ. Equ. **32**, 1511–1549 (2007)
50. Holden, H., Raynaud, X.: Periodic conservative solutions of the Camassa-Holm equation. Ann. Inst. Fourier (Grenoble) **58**, 945–988 (2008)
51. Holmer, J.: The initial-boundary value problem for the Korteweg-de Vries equation. Commun. Partial Differ. Equ. **31**, 1151–1190 (2006)
52. Hone, A., Lundmark, H., Szmigielski, J.: Explicit multipeakon solutions of Novivov's cubically nonlinear integrable Cammasa-Holm type equation. Dyn. Partial Differ. Equ. **6**, 253–289 (2009)
53. Kappeler, T., Topalov, P.: Global wellposedness of KdV in $H^{-1}(\mathbb{T}, \mathbb{R})$. Duke Math. J. **135**, 327–360 (2006)
54. Kato, T., Ponce, G.: Commutator estimates and the Euler and Navier-Stokes equations. Commun. Pure Appl. Math. **41**, 891–907 (1988)
55. Kenig, C., Ponce, G., Vega, L.: Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. Commun. Pure Appl. Math. **46**, 527–620 (1993)
56. Kenig, C., Ponce, G., Vega, L.: A bilinear estimate with applications to the KdV equation. J. Amer. Math. Soc. **9**, 573–603 (1996)

57. Korteweg, D.J., de Vries, G.: On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philos. Mag. **39**, 422–443 (1895)
58. Lax, P., Levermore, D.: The zero dispersion limit for the Korteweg-de Vries (KdV) equation. Proc. Natl. Acad. Sci. U.S.A. **76**, 3602–3606 (1979)
59. Lax, P., Levermore, D.: The small dispersion limit of the Korteweg-de Vries equation. I. Commun. Pure Appl. Math. **36**, 253–290 (1983)
60. Lax, P., Levermore, D.: The small dispersion limit of the Korteweg-de Vries equation. II. Commun. Pure Appl. Math. **36**, 571–593 (1983)
61. Lax, P., Levermore, D.: The small dispersion limit of the Korteweg-de Vries equation. III. Commun. Pure Appl. Math. **36**, 809–829 (1983)
62. Lenells, J.: Traveling wave solutions of the Camassa-Holm equation. J. Differ. Equ. **217**, 393–430 (2005)
63. Li, Y., Olver, P.: Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation. J. Differ. Equ. **162**, 27–63 (2000)
64. Linares, F., Ponce, G., Sideris, T.: Properties of solutions to the Camassa-Holm equation on the line in a class containing the peakons. In: Kato, K., Ogawa, T., Ozawa, T. (eds.) Asymptotic Analysis for Nonlinear Dispersive and Wave Equations. Adv. Stud. Pure Math., vol. 81, pp. 197–246. Math. Soc. Japan, Tokyo (2019)
65. Majda, A., Bertozzi, A.: Vorticity and Incompressible Flow. Cambridge Texts in Applied Mathematics, vol. 27. Cambridge University Press, Cambridge (2002)
66. McKean, H.: Breakdown of a shallow water equation. Asian J. Math. **2**, 867–874 (1998)
67. McKean, H.: Breakdown of the Camassa-Holm equation. Commun. Pure Appl. Math. **57**, 416–418 (2004)
68. Misiołek, G.: Classical solutions of the periodic Camassa-Holm equation. Geom. Funct. Anal. **12**, 1080–1104 (2002)
69. Novikov, V.: Generalizations of the Camassa-Holm equation. J. Phys. A **42**, 342002 (2009)
70. Olver, P., Rosenau, P.: Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support. Phys. Rev. E (3) **53**, 1900–1906 (1996)
71. Ponce, G., Linares, F.: Introduction to Nonlinear Dispersive Equations. Universitext. Springer, New York (2009)
72. Qiao, Z.: A new integrable equation with cuspons and W/M-shape-peaks solitons. J. Math. Phys. **47**, 112701 (2006)
73. Rodríguez-Blanco, G.: On the Cauchy problem for the Camassa-Holm equation. Nonlinear Anal. **46**, 309–327 (2001)
74. Tao, T.: Nonlinear Dispersive Equations: Local and Global Analysis. Conference Board of the Mathematical Sciences, no. 106. American Mathematical Society, Providence, RI (2006)
75. Tao, T.: Multilinear weighted convolution of L^2-functions, and applications to nonlinear dispersive equations. Amer. J. Math. **123**, 839–908 (2001)
76. Whitham, G.B.: Linear and Nonlinear Waves. J. Wiley & Sons, New York (1980)
77. Yin, Z.: Global existence for a new periodic integrable equation. J. Math. Anal. Appl. **283**, 129–139 (2003)