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ORIGINAL ARTICLE

Group Elements Whose Character Values are Roots of Unity

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Abstract
We classify all finite groups G that possess an element $x \in G$ such that every irreducible character of G takes a root of unity value at x.

Keywords Nonvanishing · Root of unity

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1 Introduction

Let G be a finite group. Following [5], an element $x \in G$ is called a nonvanishing element if $\chi(x) \neq 0$ for all irreducible complex characters χ of G. This concept has been widely studied in recent years. In this paper, we consider nonvanishing elements of finite groups which satisfy certain minimal condition as follows. Given a nonvanishing element x of a finite group G, it is not hard to show that $|\mathbf{C}_G(x)| \geq k(G)$ and that the equality holds if and only if $|\chi(x)| = 1$ for all irreducible characters χ of G (see Lemma 2.3), where $\mathbf{C}_G(x)$ is the centralizer of x in G and $k(G)$ is the number of conjugacy classes of G. Note that if $|\chi(x)| = 1$ for some character χ of G, then $\chi(x)$ is a root of unity (see, for example,

Dedicated to Pham Huu Tiep on his 60th birthday.

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Problem 3.2 in [4]). We will call an element $x \in G$ a *root of unity element* if $|\chi(x)| = 1$ for all irreducible characters of G. The condition $|\mathbf{C}_G(x)| = k(G)$ alone does not characterize root of unity elements. For example, if $G = \mathrm{A}_5$, the alternating group of degree 5, and $x \in G$ is an element of order 5, then $|\mathbf{C}_G(x)| = 5 = k(G)$ but x is not a root of unity element. We note that root of unity elements are called totally unitary or TU-elements by S. Ostrovskaya and E. M. Zhmud' and they classify all finite metabelian groups with trivial center that contain a root of unity element in [1, Chapter XXII].

Write $\mathrm{Irr}(G)$ for the set of irreducible complex characters of G and $\mathbf{F}(G)$ for the Fitting subgroup of G, that is, the largest normal nilpotent subgroup of G. In our first result, we prove the following.

Theorem A *Let G be a finite group and let $x \in G$. If $|\chi(x)| = 1$ for all irreducible characters χ of G, then $x \in \mathbf{F}(G)$ and both $\mathbf{F}(G)$ and $G/\mathbf{F}(G)$ are abelian. In particular, G is abelian or metabelian.*

Thus if a finite group G has a root of unity element, then it is abelian or metabelian. In particular, such a group is solvable. Theorem A confirms a conjecture proposed in [5] for root of unity elements. This conjecture states that every nonvanishing element of a finite solvable groups G must lie in $\mathbf{F}(G)$.

Our interest in root of unity elements stems from an observation that if $\chi \in \mathrm{Irr}(G)$ and $g \in G$ such that $|\chi(g)| = 1$, then the size of the conjugacy class g^G containing g is always divisible by $\chi(1)$ (see Lemma 2.1). Consequently, if x is a root of unity element of G, then $|x^G|$ is divisible by $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$. This is related to Conjecture C in [6] asserting that if $\chi \in \mathrm{Irr}(G)$ is a primitive character of a finite group G, then $\chi(1)$ divides $|g^G|$ for some $g \in G$. Thus the observation above gives us a way to locate the required element $g \in G$. However, not every primitive irreducible character admits a root of unity value. For example, if G is the sporadic simple group O'N, then G has a primitive irreducible character of degree 64790 which does not admit any root of unity value.

In the next result, we classify all finite groups with a root of unity element. Clearly, if G is abelian, then every element of G is a root of unity element. Let $q > 2$ be a prime power. We denote by Γ_q the unique doubly transitive Frobenius group with a cyclic complement of order $q - 1$ and degree q. So $\Gamma_q \cong \mathrm{AGL}_1(\mathbb{F}_q) = \mathbb{F}_q \rtimes \mathbb{F}_q^*$, where $\mathbb{F}_q$ is a finite field with q elements.

Theorem B *Let G be a finite group. Then G has a root of unity element $x \in G$ if and only if one of the following holds:*

- *G is abelian;*
- *$\mathbf{F}(G) = G'\mathbf{Z}(G)$ is abelian, $G' \cap \mathbf{Z}(G) = 1$; and $G/\mathbf{Z}(G) \cong \Gamma_{q_1} \times \Gamma_{q_2} \times \cdots \times \Gamma_{q_m}$, where each $q_i > 2$ is a prime power and $m \geq 1$ is an integer.*

For each i with $1 \leq i \leq m$, write $\Gamma_{q_i} = V_i \ltimes A_i$, where V_i is the Frobenius kernel and A_i is the Frobenius complement. Let $U := \prod_{i=1}^{m}(V_i - \{1\})$ and let $\mathcal{U} = \pi^{-1}(U)$ where $\pi : G \to G/\mathbf{Z}(G)$ is the natural homomorphism. Then every element of U is a root of unity element of $G/\mathbf{Z}(G)$ by Lemma 3.17, Chapter XXII of [1] and from the proof of Theorem B, every element of $\mathcal{U}$ is a root of unity element of G.

Our notation is standard and we follow [4] for the character theory of finite groups.

2 Preliminaries

We collect some properties of root of unity elements in the next lemmas.

Lemma 2.1 *Let G be a finite group and let $g \in G$. If $\chi \in \mathrm{Irr}(G)$ and $|\chi(g)| = 1$, then $\chi(1)$ divides $|g^G|$. In particular, if $x \in G$ is a root of unity element, then $\chi(1)$ divides $|x^G|$ for all $\chi \in \mathrm{Irr}(G)$.*

Proof Assume that $\chi \in \mathrm{Irr}(G)$ and $g \in G$ such that $|\chi(g)| = 1$. Let K be the class sum of the conjugacy class g^G, that is, $K = \sum_{y \in g^G} y$. Then

$$\omega_\chi(K) = \frac{|g^G|\chi(g)}{\chi(1)}$$

is an algebraic integer by [4, Theorem 3.7]. Since $\chi(g)$ is an algebraic integer,

$$\frac{|g^G|}{\chi(1)} = \omega_\chi(K)\overline{\chi(g)}$$

is a rational algebraic integer, so it is an integer and hence $\chi(1)$ divides $|g^G|$.

If $x \in G$ is a root of unity element, then for any $\chi \in \mathrm{Irr}(G)$, we have $|\chi(x)| = 1$ and hence $\chi(1)$ divides $|x^G|$ as wanted. □

Lemma 2.2 *Let G be a finite group and let $x \in G$ be a root of unity element. Then*

(a) $k(G) = |\mathbf{C}_G(x)|$.
(b) $G' \leq \langle x^G \rangle$.
(c) *If $N \trianglelefteq G$, then xN is a root of unity in G/N.*
(d) *If $z \in \mathbf{Z}(G)$, then xz is also a root of unity element.*

Proof From the Second Orthogonality relation, we have

$$|\mathbf{C}_G(x)| = \sum_{\chi \in \mathrm{Irr}(G)} |\chi(x)|^2 = \sum_{\chi \in \mathrm{Irr}(G)} 1 = k(G).$$

Let $L = \langle x^G \rangle$. For any $\chi \in \mathrm{Irr}(G/L)$, we see that $x \in L \subseteq \mathrm{Ker}(\chi)$. Hence $1 = |\chi(x)| = \chi(1)$ and thus all characters $\chi \in \mathrm{Irr}(G/L)$ are linear which implies that G/L is abelian and so $G' \leq L$. Since $\mathrm{Irr}(G/N) \subseteq \mathrm{Irr}(G)$ whenever $N \trianglelefteq G$, if x is a root of unity of G then xN is a root of unity of G/N.

Finally, let $z \in \mathbf{Z}(G)$ and $\chi \in \mathrm{Irr}(G)$. Then $\chi_{\mathbf{Z}(G)} = \chi(1)\lambda$ for some $\lambda \in \mathrm{Irr}(\mathbf{Z}(G))$. We have $\chi(xz) = \lambda(z)\chi(x)$ and thus if x is a root of unity element, then so is xz as $|\lambda(z)| = 1$. □

The next lemma follows from the proof of Lemma 3.17 in [1, Chapter XXII] and the previous lemma. For completeness, we include the proof here.

Lemma 2.3 *Let G be a finite group and let $x \in G$ be a nonvanishing element of G. Then $|\mathbf{C}_G(x)| \geq k(G)$; and the equality holds if and only if x is a root of unity element.*

Proof We first claim that if $x \in G$ is a nonvanishing element, then $|\mathbf{C}_G(x)| \geq k(G)$. Let $\alpha = \prod_{\chi \in \mathrm{Irr}(G)} \chi(x)$. Let n be the exponent of G and let $\mathbb{Q}_n = \mathbb{Q}(\xi)$, where ξ is a primitive nth-root of unity. Let $\mathcal{G}$ be the Galois group of $\mathbb{Q}_n$ over $\mathbb{Q}$. Then $\mathcal{G}$ acts on $\mathrm{Irr}(G)$ and we see that $\chi^\sigma \in \mathrm{Irr}(G)$ if and only if $\chi \in \mathrm{Irr}(G)$ for all $\sigma \in \mathcal{G}$. Hence $\alpha^\sigma = \alpha$ for all $\sigma \in \mathcal{G}$.

It follows that $\alpha \in \mathbb{Q}$. Since α is an algebraic integer, we must have that $\alpha \in \mathbb{Z}$. As x is nonvanishing, $\alpha \neq 0$ and so $|\alpha| \geq 1$.

By the inequality between arithmetic and geometric means, we have that

$$1 \leq |\alpha|^2 = \prod_{\chi \in \mathrm{Irr}(G)} |\chi(x)|^2 \leq \left(\frac{\sum_{\chi \in \mathrm{Irr}(G)} |\chi(x)|^2}{k(G)}\right)^{k(G)} = \left(\frac{|\mathbf{C}_G(x)|}{k(G)}\right)^{k(G)}.$$

It follows that $|\mathbf{C}_G(x)| \geq k(G)$ as wanted.

Next, assume that x is a nonvanishing element of G and $|\mathbf{C}_G(x)| = k(G)$. Then $|\alpha| = 1$ from the inequality above. Hence

$$\prod_{\chi \in \mathrm{Irr}(G)} |\chi(x)|^2 = \left(\frac{\sum_{\chi \in \mathrm{Irr}(G)} |\chi(x)|^2}{k(G)}\right)^{k(G)} = 1.$$

Therefore, $|\chi(x)| = 1$ for all $\chi \in \mathrm{Irr}(G)$. So $x \in G$ is a root of unity element.

Conversely, if x is a root of unity, then clearly x is nonvanishing and $|\mathbf{C}_G(x)| = k(G)$ by Lemma 2.2(a). □

A consequence of the previous lemma is that if $x \in G$ and $k(G) > |\mathbf{C}_G(x)|$ or equivalently $|x^G| > |G|/k(G)$, the average of the conjugacy class size of G, then x is a vanishing element of G, that is, $\chi(x) = 0$ for some $\chi \in \mathrm{Irr}(G)$. Also, if G has a root of unity element x, then the commuting probability

$$\mathrm{cp}(G) = \frac{|\{(a, b) \in G \times G : ab = ba\}|}{|G|^2} = \frac{k(G)}{|G|}$$

is equal to $1/|x^G|$.

In the next two lemmas, we quote some results in Chapter XXII of [1].

Lemma 2.4 *Let G be a finite group and suppose that $x \in G$ is a root of unity element.*

(a) *If $x \in \mathbf{F}(G)$, then $\mathbf{F}(G)$ is abelian and $G' \leq \mathbf{F}(G)$. In particular, G is abelian or metabelian.*
(b) *Conversely, if G is metabelian, then $x \in \mathbf{F}(G)$.*

Proof This is Lemma 1.5 in [1, Chapter XXII]. □

The following is the main result of Chapter XXII in [1]. Recall that the socle of a finite group G, denoted by $\mathrm{Soc}(G)$, is a product of all minimal normal subgroups of G.

Lemma 2.5 *Let G be a finite metabelian group with trivial center. Then G has a root of unity element if and only if $G \cong \Gamma_{q_1} \times \Gamma_{q_2} \times \cdots \times \Gamma_{q_m}$, where $q_1, q_2, \ldots, q_m$ are prime power > 2. Moreover, if $x \in G$ is a root of unity, then $\mathbf{F}(G) = \mathrm{Soc}(G) = G' = \langle x^G \rangle$ and $\mathbf{C}_G(x) = \mathbf{F}(G)$.*

Proof The equivalent statements follow from Theorem 1.12 and Corollary 1.11 and the last claim follows from Lemmas 3.8 and 3.14 in [1, Chapter XXII]. □

For a finite group G, recall that $\mathbf{F}(G)$, the Fitting subgroup of G, is the largest nilpotent normal subgroup of G. The Fitting series of a finite group G is defined by $\mathbf{F}_1(G) := \mathbf{F}(G)$ and for any integer $i \geq 1$, $\mathbf{F}_{i+1}(G)/\mathbf{F}_i(G) = \mathbf{F}(G/\mathbf{F}_i(G))$. Similarly, the upper central series of G is defined by $\mathbf{Z}_1(G) := \mathbf{Z}(G)$ and for $i \geq 1$, we have $\mathbf{Z}_{i+1}(G)/\mathbf{Z}_i(G) =$

$\mathbf{Z}(G/\mathbf{Z}_i(G))$. The last term of the upper central series of G is called the hypercenter (or hypercentral) of G and is denoted by $\mathbf{Z}_\infty(G)$.

The following results are well-known.

Lemma 2.6 *Let G be a finite group and let N be a normal subgroup of G such that $N \leq \mathbf{Z}(G)$.*

(1) *If G/N is nilpotent, then G is nilpotent.*
(2) $\mathbf{F}(G/N) = \mathbf{F}(G)/N$.
(3) $\mathbf{F}(G/\mathbf{Z}_i(G)) = \mathbf{F}(G)/\mathbf{Z}_i(G)$ *for all $i \geq 1$.*

Proof The first two claims are well-known. The last claim follows from the second claim and induction. □

The next result is Corollary 2.3 in [8].

Lemma 2.7 *Let G be a finite solvable group and assume that the Sylow 2-subgroups of $\mathbf{F}_{i+1}(G)/\mathbf{F}_i(G)$ are abelian for $1 \leq i \leq 9$. Then every nonvanishing element of G lies in $\mathbf{F}(G)$.*

3 Solvability of Finite Groups with a Root of Unity Element

We first prove Theorem A for finite solvable groups.

Proposition 3.1 *Let G be a finite solvable group and suppose that $x \in G$ is a root of unity element. Then G is abelian or G is metabelian, $x \in \mathbf{F}(G)$ and both $\mathbf{F}(G)$ and $G/\mathbf{F}(G)$ are abelian.*

Proof If G is abelian, then we are done. So assume that G is nonabelian. If G is metabelian, then the conclusion follows from Lemma 2.4. Thus we only need to show that G is metabelian. We will prove this by induction on $|G|$.

Let N be a minimal normal subgroup of G. Let $\overline{G} = G/N$ and use the 'bar' notation. By Lemma 2.2 (c), $\bar{x}$ is a root of unity element in $\overline{G}$ and thus by induction, $\overline{G}$ is abelian or metabelian. If $\overline{G}$ is abelian, then G is metabelian. So assume that $\overline{G}$ is metabelian (but G is neither metabelian nor abelian). Then $\bar{x} \in \mathbf{F}(\overline{G})$ and both $\mathbf{F}(\overline{G})$ and $\overline{G}/\mathbf{F}(\overline{G})$ are abelian by Lemma 2.4. Furthermore, N is abelian since it is a minimal normal subgroup and G (and so also N) is solvable. Thus $N \leq \mathbf{F}(G)$.

Therefore, $G/\mathbf{F}(G)$ is also metabelian. Again by Lemma 2.4 we have that $\mathbf{F}(G/\mathbf{F}(G)) = \mathbf{F}_2(G)/\mathbf{F}_1(G)$ and $G/\mathbf{F}_2(G)$ are abelian. It follows that $\mathbf{F}_3(G) = G$ and $\mathbf{F}_{i+1}(G)/\mathbf{F}_i(G)$ is abelian for all $i \geq 1$. Now by Lemma 2.7, we have $x \in \mathbf{F}(G)$ as x is nonvanishing and so G is metabelian. This contradiction completes the proof. □

The following result follows from the proof of Theorem A in [7].

Lemma 3.2 *Let G be a finite group. Assume that G has a unique minimal normal subgroup N. If N is non-abelian and G/N is solvable, then every element in $G - N$ is a vanishing element.*

Let $n \geq 2$ be an integer and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of n. For $1 \leq i \leq k$ and $1 \leq j \leq \lambda_i$, we denote by $h^{\lambda}_{i,j}$ the hook length of the node (i, j) of the Young diagram of λ. Let λ and μ be partitions of n. We use the notation χ^{λ}_{μ} to denote the value of the irreducible character of S_n labeled by λ evaluated at the conjugacy class with cycle type μ.

Lemma 3.3 *Let $n \geq 6$ be an integer and let $x \in \mathrm{A}_n$. Then there exists a partition λ of n which is not self-conjugate such that $\chi^{\lambda}(x) \neq \pm 1$.*

Proof We will use the following fact which can follow easily from Murnaghan-Nakayama formula. If $m \geq 1$ is an integer and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_r)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_s)$ are partition of m with $h^{\gamma}_{2,1} < \beta_1$ and $\gamma_1 - \gamma_2 \geq \beta_1$, then

$$\chi^{\gamma}_{\beta} = \chi^{(\gamma_1-\beta_1,\gamma_2,\ldots,\gamma_r)}_{(\beta_2,\beta_3,\ldots,\beta_2)}.$$

Let $\alpha \vdash n$ be the cycle partition of $x \in \mathrm{A}_n$. Since $n \geq 6$, from the proof of Lemma 1.6 in [9] we may assume that all parts of α are distinct, except possibly for the part 1, which may have multiplicity 2.

Write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \vdash n$. We consider the following cases.

Case 1: $l \geq 2$ and $\alpha_{l-1} = \alpha_l$. In this case, we have $\alpha_{l-1} = \alpha_l = 1$, $l \geq 3$ (as $n \geq 6$) and $\alpha_{l-2} > 1$.

Assume first that $\alpha_{l-2} > 2$. Since $n \geq 6$, the partition $(n-2, 1, 1)$ of n is not self-conjugate and we have that

$$\chi^{(n-2,1,1)}_{\alpha} = \chi^{(\alpha_{l-2},1,1)}_{(\alpha_{l-2},1,1)} = 0.$$

The first equality holds by the observation above and the latter equality holds since $\alpha_{l-2} > 2$ so

$$\begin{aligned} h^{(\alpha_{l-2},1,1)}_{1,1} &= \alpha_{l-2} + 2 > \alpha_{l-2}, \\ h^{(\alpha_{l-2},1,1)}_{1,2} &= \alpha_{l-2} - 1 < \alpha_{l-2}, \\ h^{(\alpha_{l-2},1,1)}_{2,1} &= 2 < \alpha_{l-2}. \end{aligned}$$

Assume next that $\alpha_{l-2} = 2$. Then $l \geq 4$ and $\alpha_{l-3} \geq 3$. Then the partition $(n-2, 2)$ of n is not self-conjugate and by the observation above, we have

$$\chi^{(n-2,2)}_{\alpha} = \chi^{(2,2)}_{(2,1,1)} = 0.$$

Case 2: $l \geq 2$ and $\alpha_{l-1} \neq \alpha_l$.

Assume first that $l \geq 3$ or $\alpha_{l-1} \neq \alpha_l + 1$. Then

$$\chi^{(n-\alpha_l,1^{\alpha_l})}_{\alpha} = \chi^{(\alpha_{l-1},1^{\alpha_l})}_{(\alpha_{l-1},\alpha_l)} = 0$$

as

$$\begin{aligned} h^{(\alpha_{l-1},1^{\alpha_l})}_{1,1} &= \alpha_{l-1} + \alpha_l > \alpha_{l-1}, \\ h^{(\alpha_{l-1},1^{\alpha_l})}_{1,2} &= \alpha_{l-1} - 1 < \alpha_{l-1}, \\ h^{(\alpha_{l-1},1^{\alpha_l})}_{2,1} &= \alpha_l < \alpha_{l-1}, \end{aligned}$$

and $(n-\alpha_l, 1^{\alpha_l})' = (\alpha_l + 1, 1^{n-\alpha_l-1}) \neq (n-\alpha_l, 1^{\alpha_l})$ as

$$n - \alpha_l \geq \alpha_{l-2} + \alpha_{l-1} \geq 2\alpha_l + 3 > \alpha_l + 1$$

if $l \geq 3$; and $n - \alpha_l \geq \alpha_{l-1} > \alpha_l + 1$ if $\alpha_{l-1} \neq \alpha_l$.

Assume next that $\ell = 2$ and $\alpha_1 = \alpha_2 + 1$. Then $n = 2\alpha_2 + 1$. As $n \geq 6$, we have $\alpha_2 \geq 3$. Again $(n-2, 2)' \neq (n-2, 2)$.

If $\alpha_2 = 3$, then $n = 7$ and $\chi^{(5,2)}_{(4,3)} = 0$. Assume that $\alpha_2 \geq 4$. Then

$$\chi^{(n-2,2)}_{\alpha} = \chi^{(n-2,2)}_{(\alpha_2+1,\alpha_2)} = \chi^{(\alpha_2-2,2)}_{(\alpha_2)} = 0$$

as $h^{(\alpha_2-2,2)}_{1,1} = \alpha_2 - 1 < \alpha_2$.

Case 3: $l = 1$. Then $\alpha = (n)$. Since $n \geq 6$, $(n-2, 2)$ is not self-conjugate and $\chi^{(n-2,2)}_{(n)} = 0$ as $h^{(n-2,2)}_{1,1} = n - 1 < n$. □

We are now ready to prove Theorem A.

Proof of Theorem A Let $x \in G$ be a root of unity element. If G is solvable, then the theorem follows from Proposition 3.1. Thus it suffices to show that G is solvable. Suppose not and let G be a counterexample to the theorem with $|G|$ minimal. Then $x \in G$ is a root of unity but G is non-solvable. Let $L = \langle x^G \rangle$. Then $G' \leq L \trianglelefteq G$ by Lemma 2.2 (b).

Let N be a minimal normal subgroup of G. By Lemma 2.2 (c), G/N has a root of unity xN. Since $|G/N| < |G|$, G/N is solvable. As G is non-solvable, N is non-solvable. If G has two distinct minimal normal subgroups, say $N_1 \neq N_2$, then $N_1 \cap N_2 = 1$ and thus G embeds into $G/N_1 \times G/N_2$, where the latter group is solvable by the argument above. Therefore, G is solvable, which is a contradiction. It follows that G has a unique minimal normal subgroup N, which is non-solvable and G/N is solvable. Hence $N = S_1 \times S_2 \cdots \times S_k$, where each $S_i \cong S$ for some non-abelian simple group S. By Lemma 3.2, $x \in N$ as every element in $G - N$ is a vanishing element. It follows from Lemma 2.2 (b) that $G' = \langle x^G \rangle = N$, and so G/N is abelian.

Write $x = (x_1, x_2, \ldots, x_k) \in N$, where $x_i \in S_i$ for $1 \leq i \leq k$. As G is nonabelian, x is nontrivial and so $o(x)$, the order of x, is divisible by some prime $p \geq 2$. Clearly, $o(x_i)$ is divisible by p for some $i \geq 1$. Assume that S has an irreducible character θ of p-defect zero. Then $\lambda = \theta \times \theta \times \cdots \times \theta \in \mathrm{Irr}(N)$ has p-defect zero. Clearly, every G-conjugate of λ also has p-defect zero and hence if $\chi \in \mathrm{Irr}(G)$ lying over λ, then χ_N is a sum of G-conjugates of λ so that $\chi(x) = 0$ since every conjugate of λ vanishes at x as $o(x)$ is divisible by p. Therefore, we can assume that S has no p-defect zero character. By [3, Corollary 2], one of the following cases holds.

(i) $p = 2$ and S is isomorphic to M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_3 or A_n for some integer $n \geq 7$; or
(ii) $p = 3$ and S is isomorphic to Suz, Co_3 or A_n for some integer $n \geq 7$.

We make the following observation. Assume that S has a rational-valued irreducible character $\theta \in \mathrm{Irr}(S)$ which is extendible to $\mathrm{Aut}(S)$. Then $\varphi = \theta \times \theta \times \cdots \times \theta \in \mathrm{Irr}(N)$ extends to $\chi \in \mathrm{Irr}(G)$ and

$$1 = |\chi(x)| = |\varphi(x)| = \prod_{i=1}^{k} |\theta(x_i)|.$$

Since θ is rational, $\theta(x_i)$ is a non-zero integer and thus $|\theta(x_i)| \geq 1$ for all i. The previous equation now implies that $|\theta(x_i)| = 1$ for all i.

(a) Assume first that S is one of the sporadic simple groups in (i) but not in (ii). Then x and hence $x_i's$ must be a 2-element. Using [2], we can find an irreducible rational-valued character θ which is extendible to $\mathrm{Aut}(S)$ and does not take root of unity values on any 2-elements. So this case cannot occur.

Similarly, if S is one of the sporadic simple groups in (ii), then x and hence $x_i's$ are $\{2,3\}$-elements. Again, by using [2], we can find an irreducible rational-valued character θ which is extendible to $\mathrm{Aut}(S)$ and does not take root of unity values on any $\{2,3\}$-elements.

(b) Assume that $S \cong \mathrm{A}_n$, where $n \geq 7$ is an integer and that A_n has no block of p-defect zero for $p = 2, 3$ or both.

By the observation above, if λ is a partition of n which is not self-conjugate, then χ^λ, the irreducible character of S_n labeled by λ, remains irreducible upon reduction to A_n and thus $|\chi^\lambda(x_i)| = 1$. Note that χ^λ is rational-valued. Now Lemma 3.3 provides a contradiction.

Therefore, G must be solvable as wanted. The proof is now complete. □

4 Finite Metabelian Groups with a Root of Unity Element

In this section, we will characterize finite metabelian groups with a root of unity element. Such a group with trivial center was classified by S. Ostrovskaya and E. M. Zhmud'. Recall that if $q > 2$ is a prime power, then Γ_q is a doubly transitive Frobenius group with a cyclic complement of order $q-1$ and degree q. Note that Γ_q has a root of unity element and every root of unity element of Γ_q lies in $\mathbf{F}(\Gamma_q)$, which is an elementary abelian p-group, where q is a power of a prime p. Moreover, if $\Gamma = \Gamma_{q_1} \times \Gamma_{q_2} \times \cdots \times \Gamma_{q_m}$, where each $q_i > 2$ are prime powers, then Γ has a root of unity element and furthermore, all Sylow subgroups of Γ are abelian.

Lemma 4.1 *Let G be a finite group and let $x \in G$ be a root of unity element. Let $K = \mathbf{Z}_\infty(G)$ be the hypercenter of G. Assume that G is nonabelian. Then*

(1) *$G/K \cong \Gamma_{q_1} \times \Gamma_{q_2} \times \cdots \times \Gamma_{q_m}$, where each $q_i > 2$ is a prime power and $m \geq 1$ is an integer. Moreover, $\mathbf{F}(G) = G'K$ is abelian, and $\mathbf{C}_{G/K}(xK) = \mathbf{F}(G/K) = \mathbf{F}(G)/K$.*

(2) *$\mathbf{F}(G) = \mathbf{C}_G(x)$ and $k(G) = |\mathbf{F}(G)| = |\mathbf{C}_G(x)|$.*

(3) *If $N = \mathbf{Z}_i(G)$ for some $i \geq 1$ or $N \leq \mathbf{Z}(G)$, then $\mathbf{C}_{G/N}(xN) = \mathbf{F}(G/N) = \mathbf{F}(G)/N$ and $k(G/N) = |\mathbf{F}(G) : N|$.*

Proof By Theorem A, $x \in F := \mathbf{F}(G)$, F is abelian and $G' \leq F$. Since $K \trianglelefteq G$ is nilpotent, we have $\mathbf{Z}(G) \leq K \leq F$. Now the center of G/K is trivial by the definition of K. Moreover $\mathbf{F}(G/K) = F/K$ by Lemma 2.6 (3). Since G/K has a root of unity element xK, part (1) follows from Lemma 2.5.

Since $x \in F$ and F is abelian, we have $K \leq F \leq \mathbf{C}_G(x)$. Let $\overline{G} = G/K$. From part (1), we have $\mathbf{C}_{\overline{G}}(\overline{x}) = \mathbf{F}(\overline{G}) = \overline{F}$. Hence

$$\overline{F} \leq \overline{\mathbf{C}_G(x)} \leq \mathbf{C}_{\overline{G}}(\overline{x}) = \overline{F}.$$

Thus $\overline{F} = \overline{\mathbf{C}_G(x)}$ and hence $F = \mathbf{C}_G(x)$. As $k(G) = |\mathbf{C}_G(x)|$ by Lemma 2.2 (a), part (2) follows.

Finally, let $N = \mathbf{Z}_i(G)$ for some $i \geq 1$ or $N \leq \mathbf{Z}(G)$. Then G/N has a root of unity and it is not nilpotent. By Lemma 2.6, $\mathbf{F}(G/N) = F/N$. Now part (3) follows by applying part (2) to G/N. □

Following P. Hall, a finite solvable group G is called an A-group if every Sylow subgroup of G is abelian. The next lemma shows that any finite group with a root of unity element is an A-group.

Proposition 4.2 *If a finite group G has a root of unity element, then G is an A-group.*

Proof Let G be a finite group with a root of unity element $x \in G$. Clearly, if G is abelian, then G is an A-group. So, we can assume that G is nonabelian and hence by Theorem A, $x \in F := \mathbf{F}(G)$ and both F and G/F are abelian so G is solvable. We prove by induction on $|G|$ that all Sylow subgroups of G are abelian.

Notice first that if $1 < N \trianglelefteq G$, then G/N has a root of unity element xN and so by induction, every Sylow subgroup of G/N is abelian. Now let N be a minimal normal subgroup of G. Since G is solvable, N is an elementary abelian p-group for some prime p. If Q is a Sylow r-subgroup of G for some prime $r \neq p$, then $QN/N \cong Q/Q \cap N \cong Q$ is abelian. Thus it remains to show that every Sylow p-subgroup of G is abelian. Let $P \in \mathrm{Syl}_p(G)$. Then $N \trianglelefteq P$ and P/N is abelian as $P/N \in \mathrm{Syl}_p(G/N)$. Hence $P' \leq N$. Now if G has another minimal normal subgroup, say $M \neq N$, then $P' \leq N \cap M = 1$ and hence P is abelian as wanted if M is also a p-group. If instead M is not a p-group then we can conclude as in the $r \neq p$ case that P is abelian. Therefore, we may assume that N is the unique minimal normal subgroup of G.

Since F is abelian and $N \leq F$ is the unique minimal normal subgroup of G, F must be a p-group and so $F = \mathbf{O}_p(G)$. Since P/F is a Sylow p-subgroup of an abelian group G/F, we deduce that $P/F \trianglelefteq G/F$ which implies that $P \trianglelefteq G$. Hence $P \leq \mathbf{O}_p(G) = F$ and thus $P = F$ is abelian. Therefore, G is an A-group as wanted. □

Corollary 4.3 *Let G be a finite group and let $x \in G$ be a root of unity. Then $\mathbf{Z}(G) = \mathbf{Z}_\infty(G)$, that is, $G/\mathbf{Z}(G)$ has a trivial center, and $G' \cap \mathbf{Z}(G) = 1$.*

Proof By Proposition 4.2, G is a finite solvable A-group. The result now follows from (3.8) and Theorem 4.1 in [10]. □

Proof of Theorem B Let G be a finite group, let $Z := \mathbf{Z}(G)$ and $F := \mathbf{F}(G)$. Suppose first that $x \in G$ is a root of unity element. If G is abelian, then we are done. Assume that G is non-abelian. By Theorem A, $x \in F$, F and G/F are abelian and G is metabelian. By Corollary 4.3, $\mathbf{Z}_\infty(G) = \mathbf{Z}(G)$, G/Z has trivial center and $G' \cap \mathbf{Z}(G) = 1$. Now, the conclusion follows from Lemma 4.1 (1).

For the converse, assume that G is nonabelian. So $F = G'Z$ is abelian, $G' \cap Z = 1$ and $G/Z \cong \prod_{i=1}^m \Gamma_{q_i}$ for some integer $m \geq 1$ and prime powers $q_i > 2$. In particular, G is a metabelian group. Write $\overline{G} = G/Z$ and use the 'bar' notation. By Lemma 2.5, $\overline{G}$ has a root of unity element $\overline{x}$ for some $x \in G$. Note that the hypothesis above implies that $F = G' \times Z$.

We claim that x is also a root of unity element of G, that is, $|\chi(x)| = 1$ for all $\chi \in \mathrm{Irr}(G)$. As $|\chi(x)| = |\chi(\overline{x})| = 1$ for every $\chi \in \mathrm{Irr}(G/Z)$, it suffices to show that if $1 \neq \lambda \in \mathrm{Irr}(Z)$ and $\chi \in \mathrm{Irr}(G)$ lying over λ, then $|\chi(x)| = 1$.

Let $1 \neq \lambda \in \mathrm{Irr}(Z)$. Since $F = Z \times G'$, $\theta = \lambda \times 1_{G'} \in \mathrm{Irr}(F)$ is an extension of λ and $G' \leq \mathrm{Ker}(\theta)$. So θ can be considered an irreducible character of F/G' and thus θ extends to $\phi \in \mathrm{Irr}(G/G')$. Thus λ extends to $\phi \in \mathrm{Irr}(G)$. By Gallagher's theorem, every $\chi \in \mathrm{Irr}(G)$ lying above λ has the form $\phi\mu$ for some irreducible character $\mu \in \mathrm{Irr}(\overline{G})$. Since ϕ is linear, we have $|\phi(x)| = 1$. We also have $|\mu(x)| = |\mu(\overline{x})| = 1$ as $\mu \in \mathrm{Irr}(\overline{G})$ and $\overline{x}$ is a root of unity in $\overline{G}$. Therefore

$$|\chi(x)| = |\phi(x)\mu(x)| = |\phi(x)| \cdot |\mu(x)| = 1,$$

hence x is a root of unity element of G. □

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