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ORIGINAL ARTICLE

Singularities of Normal Quartic Surfaces I (char=2)

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Abstract
We show, in this first part, that the maximal number of singular points of a normal quartic surface $X \subset \mathbb{P}^3_K$ defined over an algebraically closed field K of characteristic 2 is at most 16. We produce examples with 14, respectively 12, singular points and show that, under several geometric assumptions ($\mathfrak{S}_4$-symmetry, or behaviour of the Gauss map, or structure of tangent cone at one of the singular points P, separability/inseparability of the projection with centre P), we can obtain smaller upper bounds for the number of singular points of X.

Keywords Quartic surfaces · Singularities · Positive characteristic (= 2) · Gauss map · Inseparability · Symmetry · K3 surfaces

Mathematics Subject Classification (2010) 14J17 · 14J25 · 14J28 · 14N05

1 Introduction

Given an irreducible surface X of degree d in $\mathbb{P}^3$, defined over an algebraically closed field K, which is normal, that is, with only finitely many singular points, one important question is to determine the maximal number $\mu(d)$ of singular points that X can have (observe however, see for instance [3, 17, 29, 35, 41], that the research has been more focused on the seemingly simpler question of finding the maximal number of nodes, that is, ordinary quadratic singularities). The case of $d = 1, 2$ being trivial ($\mu(1) = 0$, $\mu(2) = 1$), the first interesting cases are for $d = 3, 4$.

We have that $\mu(3) = 4$, while $\mu(4) = 16$ if $\mathrm{char}(K) \neq 2$.

For $d = 3$ (see Proposition 1 and for instance [5] for more of classical references), a normal cubic surface X can have at most 4 singular points, no three of them can be collinear, and if it does have 4 singular points, these are linearly independent, hence X is projectively

Dedicated to Bernd Sturmfels on the occasion of his 60th birthday.

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equivalent to the so-called Cayley cubic, first apparently found by Schläfli, see [8, 10, 34] (and then the singular points are nodes).

The Cayley cubic has the simple equation

$$X := \left\{ x := (x_0, x_1, x_2, x_3) \mid \sigma_3(x) := \sum_i \frac{1}{x_i} x_0 x_1 x_2 x_3 = 0 \right\}.$$

Here σ_3 is the third elementary symmetric function (the four singular points are the 4 coordinate points).

Even the classical case of cubic surfaces still offers plenty of open questions [30]: in this article we go up to the case of quartic surfaces.

The main purpose of this paper is indeed to show with elementary methods (Theorem 14) that, if $\mathrm{char}(K) = 2$, then $\mu(4) \leq 16$, and to provide easy examples which lead to the conjecture that $\mu(4) \leq 14$, which will be proven in Part II of this article, in cooperation with Matthias Schütt (using the special features of elliptic fibrations in characteristic 2).

Whereas symmetric functions produce surfaces with the maximal number of singularities for degree $d = 3$, or for $d = 4$ in characteristic $\neq 2$ (see for instance [5, 26]) we show in the last section that for $d = 4$ and char $= 2$ symmetric functions produce quartics with at most 12 singular points, and explicit examples with 0,1,4,5,6,10,12 singular points.

In this paper we produce the following explicit example, of quartic surfaces with 14 singular points[1], and producing what we call 'the inseparable case':

$$X := \{(z, x_1, x_2, x_3) \mid z^2(x_1x_2 + x_3^2) + (y_3 + x_1)(y_3 + x_2)y_3(y_3 + x_1 + x_2) = 0,$$
$$y_3 := a_3x_3 + a_1x_1 + a_2x_2,\ a_3 \neq 0, a_1, a_2, a_3 \text{ general}\}.$$

A normal quartic surface can have, if $\mathrm{char}(K) \neq 2$, at most 16 singular points. Indeed, if $\mathrm{char}(K) \neq 2$, and X is a normal quartic surface, by Proposition 1 it has at most 7 singular points if it has a triple point, else it suffices to project from a double point of the quartic to the plane, and to use the bound for the number of singular points for a plane curve of degree 6, which equals 15, to establish that X has at most 16 singular points.

Quartics with 16 singular points ($\mathrm{char}(K) \neq 2$) have necessarily nodes as singularities, and they are the so called Kummer surfaces [22] (the first examples were found by Fresnel, 1822).

There is a long history of research on Kummer quartic surfaces in $\mathrm{char}(K) \neq 2$, for instance it is well known that if $d = 4, \mu = 16$, then X is the quotient of a principally polarized Abelian surface A by the group $\{\pm 1\}$.

But in char $= 2$ [38] Kummer surfaces behave differently, and have at most 4 singular points.[2]

In this paper we show among other results that, if X is a normal quartic surface defined over an algebraically closed field K of characteristic 2, then:

i) If X has a point of multiplicity 3, then $|\mathrm{Sing}(X)| \leq 7$ (Proposition 1).
ii) If X has a point of multiplicity 2 such that the projection with centre P is inseparable, then $|\mathrm{Sing}(X)| \leq 16$ (Proposition 5, see steps I) and II) for smaller upper bounds under special assumptions).

[1] and one more in Section 2.3, suggested by Matthias Schütt.

[2] A similar construction, introduced in [21], and based on other group schemes, leads to 'Kummer' surfaces X with 17 singular points: these, by the results of this paper, cannot be isomorphic to quartic surfaces in $\mathbb{P}^3$.

iii) $|\mathrm{Sing}(X)| \leq 16$ (Theorem 14), and equality holds only if the singularities are all nodes (A_1-singularities, double points with smooth projective tangent cone) or 15 nodes and an A_2-singularity.

This paper has a big overlapping with the previous preprint [6], improves the main result and most results in loc. cit., and supersedes it.

In Part II, in cooperation with Matthias Schütt [7] we show:

i) the better upper bound $|\mathrm{Sing}(X)| \leq 14$ holds, with equality only if the singular points are nodes and the minimal resolution of X is a supersingular K3 surface.
ii) if the Gauss image is a plane, equivalently if the equation contains only even powers of one of the variables, then, counting multiplicities, $|\mathrm{Sing}(X)| \geq 14$ and generically the surface X has 14 nodes as singularities; one may ask whether conversely all quartics with 14 nodes arise in this way.

1.1 Notation and Preliminaries

For a point in projective space, we shall freely use the vector notation $(a_1, \ldots, a_{n+1})$, instead of the more precise notation $[a_1, \ldots, a_{n+1}]$, which denotes the equivalence class of the above vector.

Let $Q(x_1, x_2, x_3) = 0$ be a conic over a field K of characteristic 2.

Then we can write

$$Q(x_1, x_2, x_3) = \sum_i b_i^2 x_i^2 + \sum_{i<j} a_{ij} x_i x_j = \left(\sum_i b_i x_i\right)^2 + \sum_{i<j} a_{ij} x_i x_j.$$

One finds that, unless Q is the square of a linear form, $[a] := (a_{23}, a_{13}, a_{12})$ is the only point where the gradient of Q vanishes.

Taking coordinates such that $[a] = (0, 0, 1)$ we have that $Q(x) = x_1 x_2 + b(x)^2$, where $b(x)$ is a (new) linear form: for instance, if $Q(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3$, then $Q = (x_1 + x_2)(x_1 + x_3) + x_1^2$.

We have two cases $Q[a] = 0$, hence $b_3 = 0$, hence $Q(x) = x_1 x_2 + b(x_1, x_2)^2$, hence changing again coordinates we reach the normal form $Q = x_1 x_2$; while if $b_3 \neq 0$ we reach the normal form $Q = x_1 x_2 + x_3^2$.

Hence we have just the three normal forms (as in the classical case)

$$x_1^2, \quad x_1 x_2, \quad x_1 x_2 + x_3^2.$$

We also use the standard notation for partial derivatives, given a polynomial $G(x_1, \ldots, x_n)$, we denote

$$G_i := \frac{\partial G}{\partial x_i}.$$

2 Singular Points of Quartic Surfaces in Characteristic 2

We consider a quartic surface $X = \{F = 0\} \subset \mathbb{P}^3_K$, where K is an algebraically closed field of characteristic equal to 2, and such that X is normal, that is, $\mathrm{Sing}(X)$ is a finite set.

If X has a point of multiplicity 4, then this is the only singular point, while if X contains a triple point P, we can write the equation, assuming that the point P is the point $x_1 = x_2 = x_3 = 0$:

$$F(x_1, x_2, x_3, z) = zG(x) + B(x),$$

where of course $G(x)$ is homogeneous of degree 3 and $B(x)$ is homogeneous of degree 4.

Recalling that $G_i := \frac{\partial G}{\partial x_i}$, $B_i := \frac{\partial B}{\partial x_i}$, we have

$$\mathrm{Sing}(X) = \{G(x) = B(x) = G_i z + B_i = 0, \ i = 1, 2, 3\}.$$

If $(x, z) \in \mathrm{Sing}(X)$ and $x \in \{G(x) = B(x) = 0\}$, then $x \notin \mathrm{Sing}(\{G = 0\})$, since $x \in \mathrm{Sing}(\{G = 0\}) \Rightarrow x \in \mathrm{Sing}(\{B = 0\})$ and then the whole line $(\lambda_0 z, \lambda_1 x) \subset \mathrm{Sing}(X)$. Hence $\nabla(G)(x) \neq 0$ and there exists a unique singular point of X in the above line. Since the two curves $\{G(x) = 0\}$, $\{B(x) = 0\}$ have the same tangent at x their intersection multiplicity at x is at least 2, and we conclude:

Proposition 1 *Let X be quartic surface $X = \{F = 0\} \subset \mathbb{P}^3_K$, where K is an algebraically closed field, and suppose that* $\mathrm{Sing}(X)$ *is a finite set. If X has a triple point then* $|\mathrm{Sing}(X)| \leq 7$.

More generally, if X is a degree d surface $X = \{F = 0\} \subset \mathbb{P}^3_K$, where K is an algebraically closed field, and we suppose that

- $\mathrm{Sing}(X)$ *is a finite set, and*
- *X has a point of multiplicity $d - 1$*

then

$$|\mathrm{Sing}(X)| \leq 1 + \frac{d(d-1)}{2}.$$

Proof The second assertion follows by observing that in proving the first we never used the degree d, except for concluding that the total intersection number (with multiplicity) of G, B equals $d(d - 1)$. □

Assume now that we have a double point P of X and we take coordinates such that $P = \{x := (x_1, x_2, x_3) = 0, z = 1\}$, thus we can write the equation

$$\textbf{(Taylor development)}: \quad F(x_1, x_2, x_3, z) = z^2 Q(x) + zG(x) + B(x),$$

where of course Q, G, B are homogeneous of respective degrees 2, 3, 4.

Then

$$\mathrm{Sing}(X) = \{(x, z) \mid G(x) = z^2 Q(x) + B(x) = z^2 Q_i(x) + zG_i(x) + B_i(x) = 0, \ i = 1, 2, 3\}.$$

We consider then the projection

$$\pi_P : X \setminus \{P\} \to \mathbb{P}^2 := \{(x_1, x_2, x_3)\}.$$

Lemma 2 1) *If P is a singular point of the quartic X, consider the projection* $\mathrm{Sing}(X) \setminus \{P\} \to \mathbb{P}^2$.

Two singular points (different from P) can have the same image only if they map to the same point x of the finite subscheme $\Sigma \subset \mathbb{P}^2$ defined by $Q = G = B = 0$, the three gradients $\nabla Q(x)$, $\nabla G(x)$, $\nabla B(x)$ are all proportional and moreover $\nabla Q(x) \neq 0$, $\nabla G(x) \neq 0$.

2) *If $x \in \Sigma$, then there is a z such that $(x, z) \in \mathrm{Sing}(X)$ if*

$$z^2 \nabla Q(x) + z \nabla G(x) + \nabla B(x) = 0.$$

Proof In fact, if a line L through P intersects X in 2 other singular points, then $L \subset X$, hence

$$L \subset \{(x, z) \mid Q(x) = G(x) = B(x) = 0\} =: P * \Sigma,$$

where $\Sigma \subset \mathbb{P}^2$ is the subscheme defined by $Q = G = B = 0$, and Σ is a 0-dimensional subscheme since X is irreducible.

If $x \in \Sigma$ and (x, z), $(x, z + w) \in \mathrm{Sing}(X)$ are different points, from the equations

$$z^2 \nabla Q(x) + z \nabla G(x) + \nabla B(x) = (z^2 + w^2) \nabla Q(x) + (z + w) \nabla G(x) + \nabla B(x) = 0$$

follows $\nabla G(x) = w \nabla Q(x)$, $\nabla B(x) = z(z+w) \nabla Q(x)$. In particular, it cannot be $\nabla Q(x) = 0$, and the three curves are all tangent at x.

Finally, if the three gradients are proportional, then we can find (z, w) solving the equations $\nabla G(x) = w \nabla Q(x)$, $\nabla B(x) = z(z + w) \nabla Q(x)$; and $w \neq 0$ if $\nabla G(x) \neq 0$. □

2.1 The 7 Strange Points of a Plane Quartic in Characteristic = 2

In general, if $\{g(x) = 0\}$ is a plane curve of even degree $d = 2k$, from the Euler formula $\sum_i x_i g_i \equiv 0$ we infer that if the critical scheme $\mathcal{C}_g := \{\nabla g(x) = 0\}$ is finite, then the first trivial estimate is that its cardinality is at most $(d - 1)^2$, by the theorem of Bézout.

Take in fact a line L not intersecting this scheme $\mathcal{C}_g$, and assume that $L = \{x_3 = 0\}$: then $\{\nabla g(x) = 0\} = \{g_1 = g_2 = 0,\ x_3 \neq 0\}$, and this set has, with multiplicity, cardinality $(d - 1)^2$ or less if $g_1 = g_2 = x_3 = 0$ is non empty.

We consider first the case of a general plane quartic curve $\{B(x) = 0\}$.

The crucial observation is that the space of (homogeneous) quartic polynomials $K[x_1, x_2, x_3]_4$ splits as a direct sum

$$K[x_1, x_2, x_3]_4 = \mathcal{Q} \oplus V,$$

where $\mathcal{Q}$ consists of squares of (homogeneous) quadratic polynomials, and V is spanned as follows:

$$V := \langle x_i^3 x_j, x_i x_1 x_2 x_3 \rangle.$$

V has dimension 9 and the group $\mathrm{GL}(3, K)$ acts with finite stabilizer on the Klein curve

$$B^0 := x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1$$

hence the orbit of B^0 is dense in V.

The gradient ∇B^0 vanishes at precisely 7 points, the seven points $\{(1, \epsilon, \epsilon^5) | \epsilon^7 = 1\}$.

In fact,

$$B_1^0 = x_1^2 x_2 + x_3^3, \quad B_2^0 = x_2^2 x_3 + x_1^3, \quad B_3^0 = x_3^2 x_1 + x_2^3.$$

Using the first equation and taking the cubes of $x_2^2 x_3 = x_1^3$, we find (since $x_i \neq 0$ for the points of $\mathcal{C}$) that $y = \epsilon x$, with $\epsilon^7 = 1$, and $z = \epsilon^5 x$. Hence we get the seven points $\{(1, \epsilon, \epsilon^5) | \epsilon^7 = 1\}$.

We derive the following property.

Proposition 3 *For a homogeneous quartic polynomial $B \in K[x_1, x_2, x_3]_4$ let $\mathcal{C}_B$ be the critical locus of B (where the gradient ∇B vanishes). If $\mathcal{C}_B$ is a finite set, then it consists of at most 7 points.*

For B general, $\mathcal{C}_B$ consists of exactly 7 reduced points.

Proof Since $x_3 B_3 = x_1 B_1 + x_2 B_2$, we get that if $B_1 = B_2 = 0$, then $x_3 B_3 = 0$.

If $\{B_1 = B_2 = 0\}$ is infinite, then on its divisorial part C we have $x_3 = 0$, since $\mathcal{C}_B$ is finite.

This means that $B = x_3 B' + \beta(x_1, x_2)^2$, hence $\nabla B = 0 \Leftrightarrow \nabla(x_3 B') = 0$.

If x_3 does not divide B', then $|\mathcal{C}_B| \leq 6$, since the gradient of B' vanishes at most in 3 points. If x_3 divides B', we get a contradiction to the finiteness of $\mathcal{C}_B$.

We can therefore assume that we are in the case where $\mathcal{B}' := \{B_1 = B_2 = 0\}$ is finite. Take a line containing $\delta \geq 2$ points of $\mathcal{C}_B \subset \mathcal{B}'$, say $x_3 = 0$, and let x_1, x_2 be coordinates both not vanishing at these points.
Then by Bezout's theorem $\mathcal{B}'$ consists of at most 9 points, counted with multiplicity.

At any of the $\delta \geq 2$ points of $\mathcal{C}_B \cap \{x_3 = 0\}$, the equation $x_3 B_3 = x_1 B_1 + x_2 B_2$ shows that, since $x_3 B_3$ vanishes of multiplicity at least 2, a linear combination of ∇B_1, ∇B_2 vanishes, implying that the local intersection multiplicity of B_1, B_2 is ≥ 2.
Hence $|\mathcal{C}_B| \leq 9 - \delta \leq 7$.

For the second assertion, use the decomposition $K[x_1, x_2, x_3]_4 = \mathcal{Q} \oplus V$, which allows us to write a general polynomial, after a change of variables, in the form

$$B = q^2 + B_0, \quad q \in K[x_1, x_2, x_3]_2.$$

We conclude since then $\nabla B = \nabla B_0$. □

Remark 4 (a) In Part II of this article we show more generally in a more elementary way that if B is a polynomial of even degree $= 2m$, and $\mathcal{C}_B$ is finite, then its cardinality is at most $(d-1)(d-2)+1$, and that this estimate is sharp, that is, equality holds in a Zariski open set of the spaces of such polynomials.

(b) A referee points out that a more general result (also in other characteristics) is contained in Theorem 2.4 of [23], whose statement however does neither mention derivatives nor critical sets, so that our clear statement about $\mathcal{C}_B$ is not easy to extract from the statement of Theorem 2.4 of [23].

2.2 The Inseparable Case

We consider first the inseparable case where $G \equiv 0$, hence

$$X = \{z^2 Q(x) + B(x) = 0\}, \quad \mathrm{Sing}(X) = X \cap \{z^2 \nabla Q(x) = \nabla B(x)\}.$$

Proposition 5 *Let X be a normal quartic surface $X = \{F = 0\} \subset \mathbb{P}^3_K$, where K is an algebraically closed field of characteristic* 2, *hence* $\mathrm{Sing}(X)$ *is a finite set. If X has a double point P as in (Taylor) such that the projection with centre P is an inseparable double cover of $\mathbb{P}^2$, i.e., $G \equiv 0$, then* $|\mathrm{Sing}(X)| \leq 16$, *and there exists a case with* $|\mathrm{Sing}(X)| = 14$.

Proof As in the proof of Proposition 1, if a singular point (x, z) satisfies $Q(x) = 0$, then $B(x) = 0$, and since $\mathrm{Sing}(X)$ is finite it must be $\nabla(Q)(x) \neq 0$: under this assumption (x, z) is the only singular point lying above the point $x \in \Sigma = \{Q(x) = B(x) = 0\}$.
Hence, in view of Lemma 2 the projection $\mathrm{Sing}(X) \setminus \{P\} \to \mathbb{P}^2$ is injective.

Conversely, if $x \in \Sigma = \{Q(x) = B(x) = 0\}$, it is not possible that $\nabla Q(x) = \nabla B(x) = 0$, while for $\nabla Q(x) = 0$, $\nabla B(x) \neq 0$ there is no singular point lying over x, and for $\nabla Q(x) \neq 0$ there is at most one singular point lying over x, and one if and only if $\nabla Q(x)$, $\nabla B(x)$ are proportional vectors.

Hence in the last case the intersection multiplicity of Q, B at x is at least 2, and in particular over Σ lie at most 4 singular points.

We proceed now in the proof considering several different cases, according to the normal form of Q.

Step I) We first consider the case where Q is a double line, and show that in this case X has at most 8 singular points.

We can in fact choose coordinates such that $F = z^2 x_1^2 + B(x)$, hence the singular points (x, z) are determined by the equations $F = \nabla B(x) = 0$. And we have a bijection between $\mathrm{Sing}(X)$ and the points of the plane with coordinates x where $\nabla B(x) = 0$ and $x_1 \neq 0$. In fact the points where $\nabla B(x) = x_1 = 0$, $B \neq 0$, do not come from singular points, while the points with $\nabla B(x) = x_1 = B = 0$ would provide infinitely many singular points.

We are done by Proposition 3 if $\mathcal{C}_B$ is finite. In the contrary case, since $\mathrm{Sing}(X)$ is finite, the only common divisor of the B_i's is x_1^a.

Since x_1 divides all the partial derivatives B_i, we can write $B = q(x_2, x_3)^2 + x_1^2 B'$, and $\nabla B = x_1^2 \nabla B'$, and since B' is not a square, we get at most 2 singular points.
Hence Step I) is proven.

Step II) We consider next the case where Q consists of two lines, and show that in this case X has at most 13 singular points.

We can in fact choose coordinates such that $F = z^2 x_1 x_2 + B(x)$, hence

$$\mathrm{Sing}(X) = \{z^2 x_1 x_2 + B(x) = 0, z^2 x_2 + B_1(x) = z^2 x_1 + B_2(x) = B_3 = 0\}.$$

Hence the singular points satisfy

$$(**) \qquad B_3 = B + x_1 B_1 = B + x_2 B_2 = 0,$$

where the last equation follows form the first two, in view of the Euler relation.

Let (x) be one of the solutions of $(**)$: we find at most one singular point lying above it if $x_1 \neq 0$, or $x_2 \neq 0$. If instead $x_1 = x_2 = 0$ for this point, then $B = B_3 = 0$, and either there is no singular point of X lying over it, or (x) is a singular point of B, and we get infinitely many singular points for X, a contradiction. Hence it suffices to bound the cardinality of this set.

If the solutions of $(**)$ are a finite set, then their cardinality is ≤ 12, and also $|\mathrm{Sing}(X)| \leq 13$.

If instead $(**)$ contains an irreducible curve C, this factor C cannot be $x_1 = 0$ or $x_2 = 0$, since for instance in the first case then $x_1 | B \Rightarrow x_1 | F$, a contradiction.

We find then infinitely many points satisfying $(**)$ and with $x_1 x_2 \neq 0$. Over these lies a singular point if $z^2 = B_1/x_2 = B_2/x_1 = B/(x_1 x_2)$ is satisfied.
But if the first equation is verified, then also the others follow from $x_1 B_1 + x_2 B_2 = x_3 B_3 = 0$, respectively $B + x_1 B_1 = 0$.
Hence the existence of such a curve C leads to the existence of infinitely many singular points of X and Step II) is proven by contradiction.

Step III) We consider next the case where Q is smooth, and show that in this case X has at most 16 singular points.

We can in fact choose coordinates such that $Q(x) = x_1 x_2 + x_3^2$, hence $F = z^2 Q(x) + B(x) = z^2 x_1 x_2 + z^2 x_3^2 + B(x)$.
Here

$$\mathrm{Sing}(X) = \{z^2 x_1 x_2 + z^2 x_3^2 + B(x) = 0, z^2 x_2 + B_1(x) = z^2 x_1 + B_2(x) = B_3 = 0\}.$$

Hence the singular points satisfy

$$(*) \quad B_3 = B + x_1 B_1 + z^2 x_3^2 = B + x_2 B_2 + z^2 x_3^2 = 0,$$

and again here the last equation follows form the first two, in view of the Euler relation.

If $x_1 \neq 0$, or $x_2 \neq 0$, there is at most one singular point lying over (we mean always: a point different from P) the point x. The same for $x_1 = x_2 = 0$ since then $x_3^2 \neq 0$.

Multiplying the second equation of (*) by x_2, and the last by x_1, we get, for the singular points,

$$Bx_2 + QB_1 = Bx_2 + x_1x_2B_1 + B_1x_3^2 = 0,$$
$$Bx_1 + QB_2 = Bx_1 + x_1x_2B_2 + B_2x_3^2 = 0.$$

Hence the singular points different from P project injectively into the set

$$\mathcal{B} := \{x | B_3 = Bx_2 + QB_1 = Bx_1 + QB_2 = 0\} \supset \{x \mid B_3 = B = Q = 0\}.$$

By Bézout $\mathcal{B}$ consists of at most 15 points, unless B_3, $Bx_2 + QB_1$, $Bx_1 + QB_2$ have a common component.

If $\mathcal{B}$ contains an irreducible curve C, since either $x_1 \neq 0$ or $x_2 \neq 0$ for the general point of C, there exists $i \in \{1, 2\}$ such that the cone Γ over C has as open nonempty subset the cone Γ' over $C' := C \setminus \{x_i = 0\}$ which is contained in the set of solutions of $(*)$. Hence Γ is contained in the set of solutions of $(*)$.

Arguing similarly, if $C \neq \{x_1 = 0\}$, $C \neq \{x_2 = 0\}$, the cone Γ over C has as nonempty open subset the cone Γ'' over $C'' := C \setminus \{x_1x_2 = 0\}$ which is contained in the solution set of $z^2x_2 + B_1(x) = z^2x_1 + B_2(x) = B_3 = 0$, hence Γ is contained in this solution set. We conclude that $\mathrm{Sing}(X)$ contains $X \cap \Gamma$, hence it is an infinite set, a contradiction.

By symmetry, it suffices to exclude the possibility that $\mathcal{B}$ contains $C = \{x_1 = 0\}$. In this case we would have that $x_1 | B_3, B_2, Bx_2 + QB_1$. Since x_1 does not divide Q, it follows that the plane $\Gamma = \{x_1 = 0\}$ has as nonempty open subset the cone over $C \setminus \{Q = 0\}$ which is contained in the set $\{z^2x_2 + B_1(x) = z^2x_1 + B_2(x) = B_3 = 0\}$, hence Γ is contained in this set and $\mathrm{Sing}(X)$ contains $X \cap \Gamma$, again a contradiction.
Hence Step III) is proven.

Remark 6 In part II, using the Hilbert-Burch theorem, we show that, in the case of Step III), the number of singular points is at most 14.

Step IV) We construct now a case where there are other 13 singular points beyond P, hence X has 14 singular points.

By the previous steps, we may assume that $Q(x) = x_1x_2 + x_3^2$, and we take $B = y_1y_2y_3y_4$, where y_1, y_2, y_3 are independent linear forms and $y_4 = y_1 + y_2 + y_3$.

Recall that

$$\mathrm{Sing}(X) = \{z^2Q(x) + B(x) = 0,\ z^2\nabla Q(x) = \nabla B(x)\}.$$

Multiplying the second (vector) equation by Q we get the equation

$$(*) \quad B(x)\nabla Q(x) = Q(x)\nabla B(x) \quad \Leftrightarrow \quad \nabla(QB)(x) = 0.$$

The solutions of (*) consist of

i) the points where $Q(x) = 0$, hence $Q = B = 0$: these are precisely 8 points, for general choice of the linear forms y_i; and they are not projections of singular points of $X \setminus P$, as the gradients ∇B, ∇Q are linearly independent at them;
ii) the points where $Q(x) \neq 0$, $B(x) = 0$, hence $B = \nabla B = 0$; that is, the 6 singular points $y_i = y_j = 0$ of $\{B(x) = 0\}$, giving rise to the 6 singular points of X with $z = 0$;
iii) (possibly) a point where $\nabla Q(x) = \nabla B(x) = 0$ but $Q \neq 0$, $B \neq 0$, this comes from exactly one singular point of X;
iv) points satisfying

$$(**) \quad Q(x) \neq 0 \neq B(x), \quad \nabla(Q) \neq 0 \neq \nabla(B), \quad B(x)\nabla Q(x) = Q(x)\nabla B(x).$$

Observe that $\nabla Q(x) = (x_2, x_1, 0)$ vanishes exactly at the point $x_1 = x_2 = 0$, while in the coordinates (y_1, y_2, y_3) we have

$$^t\, \nabla(B)(y) = \big(y_2y_3(y_2+y_3),\ y_1y_3(y_1+y_3),\ y_1y_2(y_1+y_2)\big),$$

hence the gradient $\nabla(B)(y)$ vanishes exactly at the 6 singular points of B, and at the point $y_1 + y_2 = y_1 + y_3 = 0$.

We have that this point is the point $x_1 = x_2 = 0$ as soon as $y_1 = y_3 + x_1$, $y_2 = y_3 + x_2$ (then $y_4 = y_3 + x_1 + x_2$).

Going back to the notation of Step III), we consider then the set

$$\mathcal{B} := \{x \mid B_3 = Bx_2 + QB_1 = Bx_1 + QB_2 = 0\} \supset \{x \mid B_3 = B = Q = 0\}.$$

Since $B = (y_3 + x_1)(y_3 + x_2)y_3(y_3 + x_1 + x_2)$, if we set $y_3 = a_1x_1 + a_2x_2 + a_3x_3$, with $a_3 \neq 0$, then using

$$B_i = \partial(y_1y_2y_3y_4)/\partial x_i = \sum_1^4 \frac{B}{y_j}(y_j)_i,$$

$$B_3 = 0 \ \Leftrightarrow\ \sum_1^4 \frac{B}{y_j} = 0,\ B_1 = a_1B_3 + y_2y_3y_4 + y_1y_2y_3,\ B_2 = a_2B_3 + y_1y_3y_4 + y_1y_2y_3,$$

the equations of $\mathcal{B}$ simplify to

$$B_3 = 0, \quad y_2y_3(y_1y_4x_2 + Q(y_1 + y_4)) = 0, \quad y_1y_3(y_2y_4x_1 + Q(y_2 + y_4)) = 0.$$

Since we are left with finding solutions where $B \neq 0$, the equations reduce to

$$B_3 = 0,\ [(y_3 + x_1)(y_3 + x_1 + x_2) + Q]x_2 = 0,\ [(y_3 + x_2)(y_3 + x_1 + x_2) + Q]x_1 = 0.$$

We already counted the point $x_1 = x_2 = 0$. If $x_1 = 0$ and $x_2 \neq 0$, we find

$$B_3 = y_3^2 + y_3x_2 + x_3^2 = 0,$$

but since we observe that $B_3 \equiv 0$ on the line $x_1 = 0$, we get the two points

$$x_1 = y_3^2 + y_3x_2 + x_3^2 = 0 \quad \Leftrightarrow \quad x_1 = 0,\ (a_3^2 + 1)x_3^2 + (a_2^2 + a_2)x_2^2 + a_3x_2x_3 = 0.$$

Similarly if $x_2 = 0$ and $x_1 \neq 0$ we get the two points

$$x_2 = y_3^2 + y_3x_1 + x_3^2 = 0 \quad \Leftrightarrow \quad x_2 = 0,\ (a_3^2 + 1)x_3^2 + (a_1^2 + a_1)x_1^2 + a_3x_1x_3 = 0.$$

If both $x_1 \neq 0$, $x_2 \neq 0$, we find

$$B_3 = x_1^2 + x_2^2 + y_3(x_1 + x_2) = (y_3 + x_1)^2 + y_3x_2 + x_3^2 = 0.$$

The second equation (of the three above) is reducible, it equals

$$(x_1 + x_2 + y_3)(x_1 + x_2) = 0.$$

Again $B_3 \equiv 0$ on the line $x_1 = x_2$, while the points with $(x_1 + x_2 + y_3) = 0$ yield the line $y_4 = 0$ which is contained in B, hence we do not need to consider these points.

Hence we get two more solutions:

$$\begin{aligned} & x_1 = x_2, \quad y_3^2 + x_1^2 + y_3x_1 + x_3^2 = 0, \\ \Leftrightarrow \quad & x = (x_1, x_1, x_3), \quad (a_3^2 + 1)x_3^2 + (a_2^2 + a_1^2 + a_2 + a_1 + 1)x_1^2 + a_3x_1x_3 = 0 \end{aligned}$$

and X has exactly $1+6+1+2+2+2 = 14$ singular points,

$$\begin{aligned}
(1):&\quad x = 0, z = 1,\\
(6):&\quad z = 0, x = (0,1,a_2), (0,1,1+a_2), (1,0,a_1), (1,0,1+a_1),\\
&\quad (1,1,a_1+a_2), (1,1,1+a_1+a_2),\\
(1):&\quad (1,0,0,1),\\
(2):&\quad x = (0,1,b),\ (a_3^2+1)b^2 + (a_2^2+a_2) + a_3 b = 0,\\
(2):&\quad x = (1,0,c),\ (a_3^2+1)c^2 + (a_1^2+a_1) + a_3 c = 0,\\
(2):&\quad x = (1,1,d),\ (a_3^2+1)d^2 + (a_2^2+a_1^2+a_2+a_1+1) + a_3 d = 0.
\end{aligned}$$

One can now verify that, for general choice of the a_i's (which can be made explicit requiring $a_3 \neq 1, b \neq a_2, 1+a_2, c \neq a_1, 1+a_1, d \neq a_1+a_2, 1+a_1+a_2$), we obtain 14 distinct points. □

An interesting question posed by Matthias Schütt is: how many of the 14 singular points may be defined over $\mathbb{F}_2, \mathbb{F}_4$?

Over $\mathbb{F}_2$, there are altogether 15 points in $\mathbb{P}^3_{\mathbb{F}_2}$, and each coordinate plane contains 7 points: but a plane section of X cannot have 7 singular points, hence there cannot be ≥ 12 singular points defined over $\mathbb{F}_2$.

For $\mathbb{F}_4 = \mathbb{F}_2[u]/(u^2+u+1)$, we try with $a_3 = u, a_1 = u^2, a_2 = u^2$.

The above conditions mean that $c, d \neq u, u^2, d \neq 0, 1$ and we see that u, u^2 are not roots of $b^2+u^2+b = 0$, while 0,1 are not roots of $ud^2+1+ud = 0$, which when multiplied by u^2 becomes $d^2+u^2+d = 0$.

We conclude that

Proposition 7 *There exists a normal quartic surface X with* 14 *singular points defined over $\mathbb{F}_{16}$,* 8 *of them defined over $\mathbb{F}_4$,* 4 *of them defined over $\mathbb{F}_2$.*

Proof Choosing $a_3 = u, a_1 = u^2, a_2 = u^2$ we get, on top of the two points $x = 0$, $z = 1$, $(1,0,0,1)$, the 12 points with $z = 0$ and with

$$x = (0,1,u^2), (0,1,u), (1,0,u^2), (1,0,u), (1,1,0), (1,1,1), (0,1,b), (1,0,c), (1,1,d),$$

where b, c, d are roots of the quadratic equation $z^2+z+u^2 = 0$.

This equation has no root in $\mathbb{F}_4$, hence it defines a quadratic extension of $\mathbb{F}_4$. □

2.3 More on the Inseparable Case

Here is another construction of quartics with 14 singular points, due to Matthias Schütt.

Consider the quartic X of equation

$$X := \{F(w, x_1, x_2, x_3) := w^4 + w^2 x_1^2 + B(x) = 0\}.$$

Here $B(x) := B(x_1, x_2, x_3)$ is homogeneous of degree 4 and the singular points are the solutions of

$$\nabla B(x) = 0, w^4 + w^2 x_1^2 + B(x) = 0.$$

For each x, the polynomial $w^4 + w^2 x_1^2 + B(x)$ is the square of a quadratic polynomial in w, which is separable if $x_1 \neq 0$.

Let

$$\mathcal{C} := \{x \in \mathbb{P}^2 \mid \nabla B(x) = 0\}.$$

Hence $|\mathrm{Sing}(X)| = 2 \mid \mathcal{C}|$ provided $\mathcal{C} \cap \{x_1 = 0\} = \emptyset$, a condition which can be realized for the choice of a general linear form once we find a quartic B with $\mathcal{C}$ finite.

By Proposition 3, follows that, for a general quartic polynomial B, the locus $\mathcal{C}$ consists of 7 reduced points, hence we get X with 14 singular points, which are nodes.

If we want to be more explicit, the first choice is to take B the product of 4 general linear forms, as in Step IV) of Proposition 5.

The second explicit choice, as already mentioned, is to take the Klein quartic

$$B := x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1.$$

Recall that the set $\mathcal{C}$ has always cardinality at most 7 in view of Proposition 3, so this construction leads to no more than 14 singular points, and in general to 14 singular points.

2.4 The Case Where a Variable Appears only with even Multiplicity

In Part II of this article, using elaborate arguments, we shall show that if X is defined by

$$X = \{(x_1, x_2, x_3, z) \mid \lambda z^4 + z^2 Q(x) + B(x) = 0\},$$

and X is normal, then X has at most 14 singular points, and in general it has 14 singular points which are nodes.

It is still an open question whether all the quartics with 14 nodes belong to this family.

We prove here a weaker result with an elementary proof.

Theorem 8 *Assume that the normal quartic surface X is defined by an equation of the form*

$$X = \{(x_1, x_2, x_3, z) \mid F(x, z) := \lambda z^4 + z^2 Q(x) + B(x) = 0\}.$$

Then $|\mathrm{Sing}(X)| \leq 16$.

Proof 1) The case $\lambda = 0$ was dealt with in Proposition 5.

2) The case where $Q \equiv 0$, $\lambda = 1$ gives rise, again by Proposition 3, to at most 7 singular points, which are in general 7 A_3-singularities.

3) The case where $Q(x)$ is the square of a linear form was dealt with in the previous subsection.

4) There remains to treat the case where $\lambda = 1$ and Q is a smooth conic, $Q(x) = x_1 x_2 + \mu x_3^2$ in suitable coordinates.

Then

$$\nabla F = 0 \;\Leftrightarrow\; z^2 x_2 + B_1 = z^2 x_1 + B_2 = B_3 = 0.$$

If a singular point has $x_2 \neq 0$, or $x_1 \neq 0$, then z is uniquely determined by its projection in the plane with coordinates (x), which must lie in the set

$$\mathcal{F}_2 := \{B_3 = B_1^2 + Q B_1 x_2 + x_2^2 B = 0\},$$

and also in the set

$$\mathcal{F}_1 := \{B_3 = B_2^2 + Q B_2 x_1 + x_1^2 B = 0\}.$$

Actually, $\mathcal{F}_2 \cap \{x_2 \neq 0\}$ is in bijection with $\mathrm{Sing}(X) \cap \{x_2 \neq 0\}$, and similarly $\mathcal{F}_2 \cap \{x_1 \neq 0\}$ is in bijection with $\mathrm{Sing}(X) \cap \{x_1 \neq 0\}$, hence $\mathcal{F}_1$, $\mathcal{F}_2$ are finite unless x_1 divides B_2, B_3, respectively x_2 divides B_1, B_3.

Assume that $\mathcal{F}_2$ is finite: then by the theorem of Bézout it consists of 18 points counted with multiplicity. Since x_2 does not divide B_1, B_3, and we notice that

$$B_1 \equiv c x_1^2 x_3 + d x_3^3 \pmod{x_2}, \quad B_3 \equiv c x_1^3 + d x_1 x_3^2 \pmod{x_2},$$

the length two subscheme $\mathcal{A} := \{x_2 = cx_1^2 + dx_3^2 = 0\}$ is contained in $\mathcal{F}_2$.

Since $B_1^2 + QB_1x_2 + x_2^2B$ lies in the square of the ideal generated by x_2, B_1, at the point of $\mathcal{A}$ the intersection multiplicity is at least 4, hence $\mathcal{F}_2$ consists of at most 15 points.

Similarly, if $\mathcal{F}_1$ is finite, it consists of at most 15 points.

Finally, if x_2 divides B_1, B_3, and x_1 divides B_2, B_3, then we can write

$$B(x) = q(x)^2 + cx_1x_2^3 + dx_1^3x_2.$$

Hence B_3 is identically zero, and, setting $\phi(x) := cx_2^2 + dx_1^2$, we get

$$B_1 = x_2\phi, \quad B_2 = x_1\phi.$$

Then

$$\mathrm{Sing}(X) = \{F = 0,\ x_2(z^2 + \phi(x) = 0,\ x_1(z^2 + \phi(x) = 0\} \supset \{F = z^2 + \phi(x) = 0\},$$

X is not normal.

We conclude since over the point $\{x_1 = x_2 = 0\}$ lie at most 2 singular points, hence $|\mathrm{Sing}(X)| \leq 16$. □

Remark 9 Case 2) above (equation $z^4 + B(x) = 0$), which in general gives rise to 7 A_3-singularities is quite interesting.

Because the minimal resolution S of X has then Picard number 22, hence one sees immediately here that S is a supersingular K3 surface.

3 Inequalities Provided by the Gauss Map

The Gauss map $\gamma : X \dashrightarrow \mathcal{P} := (\mathbb{P}^3)^\vee$ is the rational map given by

$$\gamma(x) := \nabla F(x), \quad x \in X^* := X \setminus \mathrm{Sing}(X).$$

We let $Y := \gamma(X)$ be the image of the Gauss map, which is a morphism on X^*, and becomes a morphism $\tilde{\gamma}$ on a suitable blow up $\tilde{S}$ of the minimal resolution S of X. The image $Y = \gamma(X)$ is called the dual variety of X.

In order to compute the degree of Y (this is defined to be equal to zero if Y is a curve), we consider a line $\Lambda \subset \mathcal{P}$ such that Λ is transversal to Y, this means:

1) $\Lambda \cap Y = \emptyset$ if Y is a curve,
2) Λ is not tangent to Y at any smooth point, and neither contains any singular point of Y, nor any point y where the dimension of the fibre $\tilde{\gamma}^{-1}(y)$ is $= 1$, so that
3) $\Lambda \cap Y$ is in particular a subscheme consisting of $\deg(Y)$ distinct points, and its inverse image in $\tilde{S}$ is a finite set.

By a suitable choice of the coordinates, we may assume that

$$\gamma^{-1}(\Lambda) \subset X \cap \{F_1 = F_2 = 0\}.$$

The latter is a finite set, hence by Bezout's theorem it consists of $4 \cdot 3^2 = 36$ points counted with multiplicity, including the singular points of X.

We get then the well known formula

$$(DEGREE - FORMULA) \quad \deg(\gamma)\deg(\gamma(X)) = 36 - \sum_{P \in \mathrm{Sing}(X)} (F, F_1, F_2)_P,$$

where $(F, F_1, F_2)_P$ is the local intersection multiplicity at P, equal to

$$\dim_K(\mathcal{O}_{X,P}/(F_1, F_2)) = \dim_K(\mathcal{O}_{\mathbb{P}^3,P}/(F, F_1, F_2)).$$

Since P is a singular point (actually we are interested in the case where F vanishes of order exactly 2), we have

$$(F, F_1, F_2)_P \geq 2 \quad \forall P \in \mathrm{Sing}(X).$$

Remark 10 If P is a uniplanar double point, then the Taylor development of F has the form $F = \ell(x)^2 + g(x)$, where $\ell(x)$ is linear and g vanishes of order at least 3. Hence F_1, F_2 vanish of order at least 2, so that $(F, F_1, F_2)_P \geq 8$.

If we have a biplanar double point, $F = \ell_1(x)\ell_2(x) + g(x)$, where g vanishes of order at least 3, then for general choice of coordinates x_1, x_2, F_1, F_2 vanish of order 1 and $\ell_1(x), \ell_2(x)$ are in the ideal generated by F_1, F_2, hence by semicontinuity we have always $(F, F_1, F_2)_P \geq 3$.

Indeed, a biplanar double point is ([6]) an A_n singularity, and for this singularity the contribution is at least $n + 1$.

We observe moreover:

Lemma 11 (i) *Assume that $P \in \mathrm{Sing}(X)$ is a node and E the exceptional curve in the minimal resolution S of X. Then E maps, via the Gauss map, to a line via an inseparable map of degree two. In particular the Gauss map cannot be birational if $X^\vee$ is a normal surface.*

(ii) *The Gauss image of X cannot be a line.*

Proof (i): given a node P, an A_1-singularity, then the affine Taylor development at P is given by

$$F = xy + z^2 + \psi(x, y, z) = 0$$

and the Gauss map on the exceptional conic $E \subset \mathbb{P}^2$, $E = \{xy + z^2 = 0\}$ is given by $(x, y, 0, 0)$.

If $X^\vee$ is a normal surface, then

$$\tilde{\gamma} : \tilde{S} \to X^\vee$$

is an isomorphism over the complement of a finite number of points of $X^\vee$, a contradiction since E maps 2 to 1 to a line.

(ii): if $X^\vee$ is a line, then there are projective coordinates in $\mathbb{P}^3$ such that

$$X = \{az^4 + bw^4 + cz^2w^2 + z^2D(x, y) + w^2E(x, y) + f(x, y) = 0\}.$$

Writing

$$D(x, y) = d_1x^2 + d_2y^2 + dxy, \quad E(x, y) = e_1x^2 + e_2y^2 + exy,$$
$$f(x, y) = q(x, y)^2 + f_1x^3y + f_2xy^3,$$

we see that

$$\mathrm{Sing}(X) = X \cap \{yM = xM = 0\}, \quad M = dz^2 + ew^2 + f_1x^2 + f_2y^2,$$

hence $\mathrm{Sing}(X) \supset X \cap \{M = 0\}$ and X is not normal. □

From the above considerations follows:

Proposition 12 *If X is a normal quartic surface in $\mathbb{P}^3$, with singular points of multiplicity 2, then $\nu := |\mathrm{Sing}(X)| \leq 16$.*

Equality holds only if all the singularities are nodes, except possibly a singularity of type A_2 in the case where the Gauss image Y of X is a plane.

If X has a uniplanar double point, then $\nu := |\mathrm{Sing}(X)| \leq 15$.

Proof First of all, the inequality $\nu \leq 16$ is proven in Theorem 8 if some variable, say z, occurs only with even exponent. This condition is equivalent to $F_z \equiv 0$, and to the fact that the image is contained in a plane.

If the number of singular points is ≥ 13, necessarily by the degree formula follows that we must have some node, since $13 \times 3 = 39 > 36$.

We can then apply Lemma 11 which says that the image of the Gauss map contains a line and cannot consist only of one line: hence the image will be an irreducible surface $Y := \gamma(X)$ of degree ≥ 2.
From this follows then that either $\deg(\gamma) \geq 2$, or $\deg(\gamma) = 1$ and $\deg(Y) \geq 3$, since in this case X is the dual of Y by the biduality theorem, hence Y cannot be a quadric.

The inequality $\nu \leq 16$ follows then from the degree formula, and in case of equality (since the only singular points which give a contribution $(F, F_1, F_2)_P = 2$ are the nodes) we have then at least 15 nodes and a singularity of type A_1 (a node) or A_2.
Hence the minimal resolution of X is a K3 surface, therefore it cannot be birational to a cubic surface: hence we conclude that $\deg(\gamma)\deg(Y) \geq 4$ and we must have 16 nodes. □

In this Part I we show with elementary arguments that $|\mathrm{Sing}(X)| \leq 16$, but in Part II, in collaboration with Matthias Schütt, we shall use the fine theory of elliptic fibrations in characteristic 2 and their wild ramification in order to prove the optimal bound $|\mathrm{Sing}(X)| \leq 14$.

We briefly give now the flavour of the geometric arguments which shall be used in Part II.

Consider two singular points of X, say P_1, P_2, and the pencil of planes containing the line $L := \overline{P_1 P_2}$, corresponding to the linear system $|H - P_1 - P_2|$.

Projection with centre L provides a rational map

$$\pi_L : X \dashrightarrow \mathbb{P}^1,$$

which is a fibration with fibres of arithmetic genus at most 1.

The basic observation is that, if the general plane sections with planes in $|H - P_1 - P_2|$ have a singular point which is not a point in $\mathrm{Sing}(X)$ (this means that we have a so-called quasi-elliptic fibration) then the dual line $L^\vee$ is contained in the dual variety $Y = \gamma(X)$.

Example 13 In the case where we have a double point P of inseparable type, we have the equation

$$\{F(z, x) = z^2 Q(x) + B(x) = 0\}.$$

Then $\frac{\partial F}{\partial z} = 0$ at all points of X, and the image $Y = \gamma(X)$ is contained in a plane. Moreover the Gauss map γ factors through the inseparable double cover $\pi_P : X \to \mathbb{P}^2$ such that $\pi_P(z, x) = x$, since

$$\nabla F(z, x) = z^2 \nabla Q(x) + \nabla B(x) = (B \nabla Q + Q \nabla B)(x).$$

If furthermore Q is the square of a linear form, then

$$\gamma(z, x) = (0, B_1(x), B_2(x), B_3(x)).$$

The base scheme $\{x \mid B_i(x) = 0, i = 1, 2, 3\}$ is finite (since X is normal) and has length at most 7 by Lemma 3, hence we get a map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of degree ≥ 2.

The conclusion is that we have at most 8 singular points and that $\deg(\gamma) \geq 4$.

The estimate that we get, $|\mathrm{Sing}(X)| \leq 8$, works perfectly since the point P is uniplanar and gives a contribution ≥ 18.

We reach then our final result:

Theorem 14 *Let X be quartic surface $X = \{F = 0\} \subset \mathbb{P}^3_K$, where K is an algebraically closed field of characteristic* 2*, and suppose that* $\mathrm{Sing}(X)$ *is a finite set.*
Then $|\mathrm{Sing}(X)| \leq 16$.
In fact:

(1) *if* $|\mathrm{Sing}(X)| = 16$*, then the only singularities of X are nodes, except possibly a singularity of type A_2 in the case where the Gauss image Y of X is a plane. The minimal resolution S of X is a minimal K3 surface with Picard number at least* $|\mathrm{Sing}(X)| + 1 = 17$.
(2) *If X contains a uniplanar double point, then* $|\mathrm{Sing}(X)| \leq 14$*, and if equality holds, all other singularities are nodes, and Y is a plane.*

Proof If X has a point of multiplicity 4, then $|\mathrm{Sing}(X)| = 1$.

If X has a point of multiplicity 3, then by Proposition 1 $|\mathrm{Sing}(X)| \leq 7$.

If X has a point of multiplicity 2 such that the projection with centre P is inseparable, then $|\mathrm{Sing}(X)| \leq 16$ by Proposition 5.

(ii) of Lemma 11 shows that the image $X^\vee$ of the Gauss map is a surface, and Theorem 8 shows that if $X^\vee$ is a plane, then $|\mathrm{Sing}(X)| \leq 16$.

(i) of Lemma 11 shows that the image $X^\vee$ is not a normal surface if $\deg(\gamma) = 1$, hence $\deg(X^\vee)\deg(\gamma) \geq 3$ and indeed equality holds only if $\deg(\gamma) = 1$.

The inequality $|\mathrm{Sing}(X)| \leq 16$ and items (1) and (2) follow from Proposition 12. □

Remark 15 Let $f : S \to X$ be the minimal resolution of a quartic surface with only double points $p_1, \dots, p_k$ as singularities.
Then the inverse image of each p_i is a union of irreducible curves $E_{i,1}, \dots, E_{i,r_i}$ and if $r := \sum_1^k r_j \geq k$, we have irreducible curves $E_1, \dots, E_r$ such that the intersection matrix $\langle E_i, E_j \rangle$ is negative definite [27].

Hence it follows that the Picard number $\rho(S) \geq r + 1 \geq k + 1$, keeping in consideration that the hyperplane section H is orthogonal to the E_i's. More generally, each singular point P contributes $r(P)$ to the Picard number $\rho(S)$ if the local resolution has $r(P)$ exceptional curves. Then

$$\sum_P r(P) + 1 \leq \rho(S) \leq b_2(S),$$

where the lower bound for the second Betti number is obtained using ℓ-adic cohomology, see for instance [25, Corollary 3.28, p. 216].

If S is a minimal K3 surface, we have $\chi(S) = 2$, $b_2(S) = 22$.

Now, it follows that S is a minimal K3 surface if the singular points are rational singularities (this means that $\mathcal{R}^1 f_*(\mathcal{O}_S) = 0$): because these are then rational double points (see [1]) and K_S is a trivial divisor. However this does not need to hold in general.

If instead the singular points are not rational, it follows that K_S is the opposite of an effective exceptional divisor and $h^1(\mathcal{O}_S) > 0$. K_S^2 is negative, and $\chi(S)$ nonpositive; if both are negative ($h^1(\mathcal{O}_S) \geq 2$) by Castelnuovo's theorem S is ruled and possibly non minimal. Hence in this case we do not have an explicit upper bound for $b_2(S) = -K_S^2 + 12\chi(S)$.

Remark 16 The K3 surfaces with $\rho(S) = b_2(S) = 22$ are the so-called Shioda-supersingular K3 surfaces.

Shioda observes [38] that Kummer surfaces in characteristic 2 have at most 4 singular points, and proves that the Kummer surface associated to a product of elliptic curves has $\rho(S) \leq 20$.

Rudakov and Shafarevich [33] described the supersingular K3 surfaces in characteristic 2 according to their Artin invariant σ, which determines the intersection form on $\mathrm{Pic}(S)$.

Shimada [37] and then Dolgachev and Kondo [13] constructed a supersingular K3 surface in characteristic 2 with Artin invariant 1 and with 21 disjoint (-2) curves, but this surface does not have an embedding as a quartic surface with 21 ordinary double points (see also [19]), as we saw in the proof of Theorem 14.

In their case the orthogonal to the 21 disjoint (-2) curves is a divisor H with $H^2 = 2$, yielding a realization as a double plane.

A supersingular K3 surface can be birational to a nodal quartic surface: indeed, the examples given here with 14 nodes are supersingular, being unirational, see [39].

Artin proved [2] that K3 surfaces with height of the formal Brauer group h ($h \in \mathbb{N} \setminus \{0\} \cup \{\infty\}$) satisfy $\rho(S) \leq 22 - 2h$ if the formal Brauer group is p-divisible.

Artin observes that K3 surfaces S with $\rho(S) = 21$ do not exist, as he proves that if $h = \infty$ and S is elliptic, then the formal Brauer group is p-divisible, and moreover S is elliptic once $\rho(S) \geq 5$ by Meyer's theorem (see [36, Corollary 2, p. 77]).

Artin predicted (modulo the conjecture that $h = \infty$ implies that S is elliptic) that $h = \infty \Leftrightarrow \rho(S) = 22$.

This equivalence follows from the Tate conjecture for K3 surfaces over finite fields, as explained in [24], discussion in Section 4, especially Theorem 4.8 and Remark 4.9; the Tate conjecture was proven in char $= 2$ by Charles, Theorem 1.4 of [9], and by Kim-Madapusi Pera [20].

4 Symmetric Quartics

One can try to see whether, as it happens for cubic surfaces or for quartics in characteristic $\neq 2$, one can construct quartics with the maximum number of singular points as quartic surfaces admitting $\mathfrak{S}_4$-symmetry.

By the theorem of symmetric functions, every such quartic X has an equation of the form

$$F(x) := F(a, \beta, x) := a_1\sigma_1^4 + a_2\sigma_1^2\sigma_2 + a_3\sigma_1\sigma_3 + a_4\sigma_4 + \beta\sigma_2^2 = 0,$$

where σ_i is as usual the ith elementary symmetric function and $a := (a_1, \ldots, a_4)$.

But the main result of this section is negative in this direction:

Theorem 17 *Quartic surfaces admitting $\mathfrak{S}_4$-symmetry form a 4-dimensional projective space $\mathcal{P}$ and the general such quartic X is smooth.*

The singular quartics in $\mathcal{P}$ are contained in four irreducible subvarieties, labeled by the number of singular points of the general element in it:

- $\mathcal{P}(6)$, *defined by* $\beta = 0$*:*

$X \in \mathcal{P}(6)$ has at least 6 singular points (the $\mathfrak{S}_4$-orbit of $(0, 0, 1, 1)$ is contained in $\mathrm{Sing}(X)$ if and only if $\beta = 0$) and either

6 singular points, or

10 singular points, the 4 extra singular points being either the $\mathfrak{S}_4$-orbit of the point $(0, 0, 0, 1)$, or the $\mathfrak{S}_4$-orbit of a point $(1, 1, 1, b)$, or

infinitely many singular points.

- *$\mathcal{P}(1)$, defined by $a_4 = 0$:*
  $X \in \mathcal{P}(1)$ if and only if the point $(1, 1, 1, 1)$ is a singular point of X.
- *$\mathcal{P}(12)$, defined by $a_2a_4 + a_3^2 = 0$:*
  the general $X \in \mathcal{P}(12)$ has 12 singular points (the $\mathfrak{S}_4$-orbit of a point $(1, 1, b, c)$ with $b \neq c$ and $b, c \neq 0, 1$).
- *$\mathcal{P}(4)$, defined by $a_2(\beta + a_2 + a_3) + a_1a_4 = 0$:*
  the general $X \in \mathcal{P}(4)$ has 4 singular points (the $\mathfrak{S}_4$-orbit of a point $(1, 1, 1, b)$ with $b \neq 1$).

If we restrict to the subset $\mathcal{F}$ of normal surfaces inside $\mathcal{P}$, it turns out that

i) $\mathcal{P}(12) \cap \mathcal{P}(m) \cap \mathcal{F} = \emptyset$, $m = 1, 6$,
ii) *$X \in \mathcal{P}(12) \cap \mathcal{P}(4) \cap \mathcal{F}$ implies that X has 4 singular points,*
iii) *$X \in \mathcal{F}$ implies that $|\mathrm{Sing}(X)| \leq 12$,*
iv) $\mathcal{P}(1) \cap \mathcal{P}(6) \cap \mathcal{F} = \emptyset$,
v) $\mathcal{P}(1) \cap \mathcal{P}(4) \cap \mathcal{F} \neq \emptyset$,
vi) $\mathcal{P}(6) \cap \mathcal{P}(4) \cap \mathcal{F} \neq \emptyset$.

Moreover, a normal symmetric quartic has exactly one of the following cardinalities

$$1, 4, 5, 6, 10, 12$$

for its number of singular points.

We shall prove the theorem through a sequence of auxiliary and more precise results.

We observe preliminarily that the singular set of X is the set

$$\mathrm{Sing}(X) := \{x \mid F(x) = F_i(x) = 0, \ \forall 1 \leq i \leq 4\},$$

which is clearly a union of $\mathfrak{S}_4$ orbits.

We have the following main result:

Proposition 18 *If we have a singular point of a normal symmetric quartic surface X, then the four coordinates cannot be all different from each other.*

For the other points, they are singular for a symmetric quartic X according to the following rules:

- *$\mathcal{P}(6)$: $(0, 0, 1, 1) \in \mathrm{Sing}(X)$ if and only if $\beta = 0$;*
- *$\mathcal{P}(4, 0, 0, 0)$: $(0, 0, 0, 1) \in \mathrm{Sing}(X)$ if and only if $a_1 = a_2 = 0$;*
- *$\mathcal{P}(4, b)$: $(1, 1, 1, b) \in \mathrm{Sing}(X)$, $b \neq 1$ if and only if*

$$a_4 = a_2(1 + b)^2, \quad \beta = a_1(1 + b)^2 + a_2 + a_3;$$

- *$\mathcal{P}(1)$: $(1, 1, 1, 1) \in \mathrm{Sing}(X)$ if and only if $a_4 = 0$;*
- *$\mathcal{P}(12, 0, 0)$: $(0, 0, 1, b) \in \mathrm{Sing}(X)$, $b \neq 0, 1$ if and only if*

$$a_2 = a_3 = 0, \quad a_1(1 + b)^4 + \beta b^2 = 0;$$

- $\mathcal{P}(12, 0, 1)$*:* $(0, 1, 1, z) \in \mathrm{Sing}(X)$*,* $z \neq 0, 1$ *if and only if*

$$a_3 = z a_2, \quad a_4 = z^2 a_2, \quad \beta = a_1 z^4 + a_2 z^2 (1+z);$$

- $\mathcal{P}(12, 1, 1)$*:* $(1, 1, b, c) \in \mathrm{Sing}(X)$*,* $b, c \neq 0, 1$*,* $b \neq c$*, if and only if, setting* $z := b + c \neq 0$*,*

$$a_3 = z a_2, \quad a_4 = z^2 a_2, \quad \beta(1+bc)^2 + a_1 z^4 + a_2 z^2 (1+z) = 0.$$

The above Proposition 18 shall be proven through a sequence of lemmas which take care of the several cases, and then we shall end the proof giving the final argument.

Let us begin with a calculation of the partial derivatives, which yields:

$$F_i := \frac{\partial F}{\partial x_i} = a_2 \sigma_1^2 (\sigma_1 + x_i) + a_3 (\sigma_3 + \sigma_1 (\sigma_2 + \sigma_1 x_i + x_i^2)) + a_4 \frac{\sigma_4}{x_i}.$$

In particular, the coordinates of the singular points, since the following equations are satisfied:

$$0 = F_i x_i = a_2 x_i \sigma_1^2 (\sigma_1 + x_i) + a_3 x_i (\sigma_3 + \sigma_1 (\sigma_2 + \sigma_1 x_i + x_i^2)) + a_4 \sigma_4,$$

are roots of the equation

$$f(z) := z^3 (a_3 \sigma_1) + z^2 (a_2 + a_3) \sigma_1^2 + z (a_2 \sigma_1^3 + a_3 (\sigma_3 + \sigma_1 \sigma_2)) + a_4 \sigma_4 = 0.$$

This gives an idea for the first assertion of Proposition 18: because if this equation is not identically zero, then the four coordinates of a singular point cannot be all different, and if the equation is identically zero, we shall see in (III) of the following lemma that the three exceptional cases have at least two equal coordinates.

Lemma 19 (I) *If* $a_3 = a_4 = 0$ *then* $\mathrm{Sing}(X)$ *is infinite, since it contains* $\{x \mid \sigma_1 = \sigma_2 = 0\}$.

(II) *If a singular point of X has one coordinate equal to zero, and* $\sigma_1 = 0$*, then it has two coordinates equal to zero.*

(III) *If X is normal, the equation* $f(z) = 0$ *is not identically zero for the points of* $\mathrm{Sing}(X)$*, unless* $a_4 = 0$ *and we have the singular point* $(1, 1, 1, 1)$*, or unless we have* $\beta = 0$ *and we have the singular point* $(0, 0, 1, 1)$*, or unless we have* $a_2 = a_3 = 0$ *and a singular point with two coordinates equal to zero.*

In particular a singular point never has four different coordinates.

(IV) *If a singular point of X has two coordinates equal to zero, say* $x_1 = x_2 = 0$*, then the equation* $a_2 \sigma_1^3 + a_3 (\sigma_3 + \sigma_1 \sigma_2) = 0$ *must be satisfied by the singular point.*

Once this equation is satisfied, then for $i = 3, 4$

$$F_i = 0 \Leftrightarrow \sigma_1 x_i ((a_2 + a_3) + a_3 x_i) = 0.$$

These two equations are satisfied for

i) $\sigma_1 = 0$*, equivalently* $x_3 + x_4 = 0$*, and we get the point* $(0, 0, 1, 1)$ *(and its* $\mathfrak{S}_4$*-orbit).*

ii) $a_2 = 0$ *and we get then the point* $(0, 0, 0, 1)$ *(and its* $\mathfrak{S}_4$*-orbit).*

iii) $a_2 = a_3 = 0$*,* $x_3, x_4 \neq 0$*, and then, if we assume* $\sigma_1 \neq 0$*, it must be* $a_1 (x_3 + x_4)^4 + \beta (x_3 x_4)^2 = 0$.

Proof (I) The equation is then of the form $a_1 \sigma_1^4 + a_2 \sigma_1^2 \sigma_2 + \beta \sigma_2^2 = 0$.

(II) If $x_1 = \sigma_1 = 0$, then $a_3 \sigma_3 + a_4 x_2 x_3 x_4 = 0$, equivalently $(a_3 + a_4) \sigma_3 = 0$, hence either $\sigma_3 = 0$, that is, two coordinates are equal to zero or, since $f(x_i) = 0$, $a_3 \sigma_3 = 0$ and $a_3 = a_4 = 0$, as in (I).

(III) If $f(z)$ is identically zero, then $a_4\sigma_4 = 0$ and either $\sigma_1 = a_3\sigma_3 = 0$, or $a_2 = a_3 = 0$. Since $a_3 = a_4 = 0$ is excluded by (I), we get the three cases:
(i) $\sigma_4 = \sigma_1 = a_3\sigma_3 = 0$,
(ii) $a_4 = \sigma_1 = \sigma_3 = 0$,
(iii) $\sigma_4 = a_2 = a_3 = 0$.
(i): $\sigma_4 = 0$ implies by (II) that two coordinates are zero, and since $\sigma_1 = 0$ we have the point $(0, 0, 1, 1)$. In this case all the symmetric functions vanish except $\sigma_2 = 1$, hence it must be $\beta = 0$.
(ii): then $\beta\sigma_2^2 = 0$, hence ($\beta = a_4 = 0$ implies that σ_1 divides $F(x)$, a contradiction), we have $\sigma_1 = \sigma_2 = \sigma_3 = 0$, and the singular point must be the point $(1, 1, 1, 1)$.
(iii): since $\sigma_4 = 0$ we may assume $x_1 = 0$, and then $F_1 = a_4x_2x_3x_4 = 0$ implies that there are two coordinates equal to zero.
(IV) If a singular point of X has two coordinates equal to zero, say $x_1 = x_2 = 0$, then $a_4\frac{\sigma_4}{x_i}$ vanishes for all i, and for $i = 1, 2$ we get that the equation $a_2\sigma_1^3 + a_3(\sigma_3 + \sigma_1\sigma_2) = 0$ must be satisfied by the singular point.
Once this equation is satisfied, then for $i = 3, 4$

$$F_i = 0 \Leftrightarrow \sigma_1 x_i((a_2 + a_3)\sigma_1 + a_3x_i) = 0.$$

These two equations are satisfied for

i) $\sigma_1 = 0$, or for
ii) $x_3 = 0, x_4 \neq 0, a_2 = 0$, or for
iii) $x_3, x_4 \neq 0$,

$$a_3x_i + (a_3 + a_2)\sigma_1 = 0 \Rightarrow a_3(x_3 + x_4) = 0.$$

For $\sigma_1 = 0$, equivalently $x_3 + x_4 = 0$, we get the point $(0, 0, 1, 1)$ (and its $\mathfrak{S}_4$-orbit).

For $x_3 = 0$, $x_4 \neq 0$, the only possibility, because of $a_2x_4 = 0$, is that we get the point $(0, 0, 0, 1)$ (and its $\mathfrak{S}_4$-orbit) and $a_2 = 0$.

If $a_3 = 0$ and $\sigma_1 \neq 0$, then $a_2 = 0$ and $a_1(x_3 + x_4)^4 + \beta(x_3x_4)^2 = 0$. □

Lemma 20 *The quartic* $X_a := \{x \mid F(a, x) = 0\}$ *has the property that* $\mathrm{Sing}(X)$ *contains the* $\mathfrak{S}_4$*-orbit of the point* $(0, 0, 1, 1)$ *if and only if* $\beta = 0$*, and it contains the* $\mathfrak{S}_4$*-orbit of the point* $(0, 0, 0, 1)$ *if and only if* $a_1 = a_2 = 0$.

Proof We calculate more generally, for later use:

$$(Sym) \quad \sigma_1(b, c, 1, 1) = \sigma_3(b, c, 1, 1) = b{+}c,\ \sigma_2(b, c, 1, 1) = 1{+}bc,\ \sigma_4(b, c, 1, 1) = bc.$$

For $b = c = 0$ we get that all σ_i vanish except $\sigma_2 = 1$, hence $(0, 0, 1, 1) \in X$ if and only if $\beta = 0$; and then, we have a singular point by (III) of Lemma 19.

For the point $(0, 0, 0, 1)$ all σ_i vanish except $\sigma_1 = 1$, hence this point is in X_a if and only if $a_1 = 0$, and we apply (IV) of Lemma 19 to infer that we have a singular point if and only if $a_1 = a_2 = 0$. □

Lemma 21 *The quartic* $X := \{x \mid F(a_1, a_2, a_3, a_4, \beta, x) = 0\}$ *has the property that* $\mathrm{Sing}(X)$ *contains the* $\mathfrak{S}_4$*-orbit of a point* $(0, 1, x_3, x_4)$*, with* $x_3, x_4 \neq 0$*, if and only if this orbit is either*

i) *the* $\mathfrak{S}_4$*-orbit of a point of the form* $(0, 1, 1, z)$, $z \neq 0, 1$ *(consisting of* 12 *points) and this holds if and only if*

$$a_3 = a_2 z, \quad a_4 = a_2 z^2, \quad \beta = a_1 z^4 + a_2 z^2 (1 + z),$$

or

ii) $z = 1$ *(in this case the orbit has* 4 *points), and this holds if and only if*

$$a_2 = a_4, \quad \beta = a_1 + a_2 + a_3.$$

In particular, we must have $a_2 \neq 0$ *if* $z \neq 1$.

Proof We know from (III) of Lemma 19 that the four coordinates cannot be all distinct, hence our singular point must be of the form $(0, 1, 1, z)$, and then $\sigma_1 = z$, $\sigma_2 = 1$, $\sigma_3 = z$, $\sigma_4 = 0$.

We look at the equations derived from the vanishing of the partial derivatives, $F_j = 0$.

For $j = 1$ we know that

$$a_2 \sigma_1^3 + a_3(\sigma_3 + \sigma_1 \sigma_2)) + a_4 \sigma_3 = 0,$$

hence for $j = 2, 3, 4$

$$(a_2 + a_3)\sigma_1^2 x_i + a_3(\sigma_1 x_i^2) + a_4 \sigma_3 = 0,$$

which can be rewritten (since $z \neq 0$) as

$$(a_2 + a_3) z x_i + a_3(x_i^2) + a_4 = 0.$$

This is an equation of degree 2, and since it is not identically zero, by (I) of Lemma 19, it has at most two roots.

Since we want $z \neq 0$, the conditions that the point is in X, plus that we have a singular point (hence $1, z$ are roots of the quadratic equation) are:

$$\beta + a_1 z^4 + (a_2 + a_3) z^2 = 0, \quad (a_2 + a_3) z = a_3 + a_4, \quad a_2 z^2 = a_4,$$

for $z = 1$ the second equation is a consequence of the third one.

If $z \neq 1$, then plugging the third equation in the second and dividing by $(1 + z)$ yields $a_3 = a_2 z^2$. □

Consider next a point of the form $(b, c, 1, 1)$, with $b \neq c$, and with $b, c \neq 0, 1$. Its orbit consists of 12 points.

Proposition 22 *The quartic*

$$X_{a,\beta} = \{x \mid a_1 \sigma_1^4 + a_2 \sigma_1^2 \sigma_2 + a_3 \sigma_1 \sigma_3 + a_4 \sigma_4 + \beta \sigma_2^2 = 0\}$$

contains the $\mathfrak{S}_4$*-orbit of the point* $(b, c, 1, 1)$ *with* $b \neq c$*, and with* $b, c \neq 0, 1$ *if and only if, setting* $z := (b + c) \neq 0$*, the coefficients satisfy*

$$a_3 = z a_2, \quad a_4 = z^2 a_2, \quad a_1 z^4 + a_2 z^2 (1 + z) + \beta (1 + bc)^2 = 0.$$

In particular, if these conditions are satisfied, and moreover $\beta = 0$*, then* X *has infinitely many singular points.*

Proof By the previous Lemma 20, $\mathrm{Sing}(X)$ contains the $\mathfrak{S}_4$-orbit of the point $(0, 0, 1, 1)$ if and only if $\beta = 0$.

We are going to see first when the point $(b, c, 1, 1)$ is a point of $X_{a,\beta}$, and then when it is a singular point.

First of all we get a point of $X_{a,\beta}$, by formula (Sym), if and only if

$$a_1z^4 + a_2z^2(1+bc) + a_3z^2 + a_4bc + \beta(1+bc)^2 = 0.$$

For the partial derivatives, at the point $(b, c, 1, 1)$, since the condition for the singular points with all coordinates different from zero boils down, for the given point, to $1, b, c$ being roots of the equation

$$f(w) := w^3(a_3\sigma_1) + w^2(a_2+a_3)\sigma_1^2 + w(a_2\sigma_1^3 + a_3(\sigma_3 + \sigma_1\sigma_2)) + a_4\sigma_4 = 0,$$

equivalently of the equation

$$f(w) := w^3(a_3z) + w^2(a_2+a_3)z^2 + w(a_2z^3 + a_3(zbc)) + a_4bc = 0.$$

Since $f(w) = (a_3z)(w+1)(w+b)(w+c)$, it must be $a_4, a_3 \neq 0$, and then $a_4 = a_3z$, and dividing by a_4,

$$1 + z = z(a_2/a_3 + 1) \;\Leftrightarrow\; a_3 = a_2z,$$

and then the third claimed equality holds automatically, since we are then left with the requirement that $bc = bc$.

We can then rewrite the condition that the point lies in $X_{a,\beta}$ as

$$a_1z^4 + a_2z^2 + a_2z^3 + \beta(1+bc)^2 = 0.$$

To finish the proof, we observe that if $\beta = 0$, since the equations depend only on z, we get infinitely many singular points varying b, c with $b + c = z$. □

End of the proof of Proposition 18. The singular points with some coordinate equal to zero have been considered in the previous Lemmas 19, 20, 21, hence we are only left with the case where all coordinates are non zero, but only two values are achieved (if the coordinates take three distinct values, we obtain the situation of the previous Proposition 22).

Therefore only two possibilities remain.

If the singular point is of the form $(1, 1, b, b)$ then $\sigma_1 = \sigma_3 = 0$, $\sigma_2 = 1 + b^2$, $\sigma_4 = b^2$.

From the equation $f(w) = 0$, since $\sigma_1 = \sigma_3 = 0$, we infer $a_4\sigma_4 = 0$, and since we assume $b \neq 0$, we obtain $a_4 = 0$. Then $0 = F(1, 1, b, b) = \beta(1+b^2)^2$ implies $\beta = 0$, and then X is reducible, (F is divisible by σ_1), or $b = 1$, and this is a singular point of X if and only if $a_4 = 0$.

If instead the point is $(1, 1, 1, b)$ it follows that $\sigma_1 = 1 + b$, $\sigma_4 = b$, $\sigma_2 = 1 + b$, $\sigma_3 = 1 + b$.

Since the case $b = 1$ was already treated, we assume that $b \neq 1$.

The condition that $1, b$ are roots of the cubic equation $f(w) = 0$ is easily seen to be equivalent to the single condition

$$a_2(1+b)^2 = a_4.$$

The condition that $(1, 1, 1, b) \in X$ boils then down to

$$a_1(1+b)^4 + a_2(1+b)^3 + a_3(1+b)^2 + a_4b + \beta(1+b)^2 = 0,$$

which, after using $a_2(1+b)^2 = a_4$ and after dividing by $(1+b)^2$ boils down to

$$\beta + a_3 + a_2 + a_1(1+b)^2 = 0.$$

With the customary notation $z := (1 + b)$, we get

$$a_4 = a_2z^2, \quad \beta + a_3 + a_2 + a_1z^2 = 0.$$

Remark 23 In the above equation $a_4 = a_2(1+b)^2$, since $(1+b) \neq 0$, $a_4 \neq 0$, it cannot be $a_2 = 0$.

It follows then, since $a_2 \neq 0$, that b is uniquely determined by this equation.

Proposition 24 *In the pencil of quartics*

$$X_c := \{c\sigma_1\sigma_3 + \sigma_4 = 0\}$$

the quartic has always 10 *singular points except for* $c = 0$.
The singularities are nodes (A_1*-singularities).*

Proof By Lemma 19 the only singular points with at least two coordinates equal to zero are just the orbits of $(0, 0, 1, 1)$ and $(0, 0, 0, 1)$ if $a_3 = c \neq 0$.

Points with just one coordinate equal to zero are excluded by Lemma 21, while singular points with nonzero coordinates taking three values are excluded by Proposition 22.

For points of type $(1, 1, b, b)$, $\sigma_1 = 0$ hence they cannot lie in X_c, since $a_4 = 1$.

For points of type $(1, 1, 1, b)$, $b \neq 1$, they lie in X_c if and only if $c(1+b)^2 = b$, but as we saw the condition that we have a singular point boils down to

$$a_2(1+b)^2 = a_4 = 1,$$

impossible since $a_2 = 0$.

For the last assertion, at the point $(0, 0, 1, 1)$ we have $x_3 = 1$ and local coordinates x_1, x_2, σ_1: then the quadratic part of the equation is

$$x_1x_2 + \sigma_1(x_1 + x_2)$$

and we have a node.

Likewise, at the point $(0, 0, 0, 1)$ we have $x_4 = 1$ and local coordinates x_1, x_2, x_3: then the quadratic part of the equation is

$$\sigma_3 = x_1x_2 + x_1x_3 + x_2x_3$$

and again we have a node. □

Proof of Theorem 17 We have already seen in Proposition 18 that the only singular orbit with 6 elements is the orbit of the point $(0, 0, 1, 1)$, and this point is a singular point if and only if $\beta = 0$, that is, $X \in \mathcal{P}(6)$. Similarly the only orbit with one element is the point $(1, 1, 1, 1)$, which is singular if and only if $a_4 = 0$, that is, $X \in \mathcal{P}(1)$.

We prove directly now that $\mathcal{P}(1) \cap \mathcal{P}(6)$ contains no normal surface: since $\beta = a_4 = 0$ imply that σ_1 divides the equation F of X.

We pass now to consider $\mathcal{P}(12)$, the closure of the locus of quartics with an orbit of singular points having 12 elements. This locus consists of three sets, $\mathcal{P}(12, 0, 0)$, $\mathcal{P}(12, 0, 1)$, $\mathcal{P}(12, 1, 1)$, and for the second and third set there must exist $z \neq 0$ such that $a_3 = za_2$, $a_4 = z^2a_2$. In particular we must have $a_2a_4 + a_3^2 = 0$.

The above equation holds in particular if $a_2 = a_3 = 0$, and then we can find a 'unique' $b \neq 0, 1$ such $a_1(1 + b^4) + \beta b^2 = 0$ provided $a_1 \neq 0$, $\beta \neq 0$ (in fact, the two roots b, $\frac{1}{b}$ yield the same $\mathfrak{S}_4$-orbit in projective space).

If instead $a_2a_4 + a_3^2 = 0$ and $a_2, a_3 \neq 0$, we find $z \neq 0$ such $a_3 = za_2$, $a_4 = z^2a_2$, and then the pair b, c is determined by the conditions that $b + c = z$, and that bc is the solution of the equation $\beta(1 + bc)^2 + a_1z^4 + a_2z^2(1 + z) = 0$. Of course $bc = 0$ if and only if we are in case $\mathcal{P}(12, 0, 1)$.

The locus $\mathcal{P}(4)$, defined by $a_2(\beta + a_2 + a_3) + a_1a_4 = 0$, clearly contains the loci $\mathcal{P}(4, 0, 0, 0)$, $\mathcal{P}(4, b)$: moreover if the above equation is satisfied, we can find a unique b

(equivalently, a unique $(1+b)^2$) such that $a_4 = a_2(1+b)^2$, $\beta = a_1(1+b)^2 + a_2 + a_3$, if $a_1 \neq 0$ or $a_2 \neq 0$ (observe that the equation of $\mathcal{P}(4)$ is the condition for the simultaneous solvability of both equations for $(1+b)^2$).

We pass now to further intersection properties of these loci.

We have seen that $\mathcal{P}(6) \cap \mathcal{P}(1)$ contains no normal surface, while $\mathcal{P}(6) \cap \mathcal{P}(4)$ contains normal surfaces by Proposition 24.

$\mathcal{P}(4) \cap \mathcal{P}(1)$ is the union of $a_2 = a_4 = 0$ and of $a_4 = \beta + a_2 + a_3 = 0$. These are two components whose general element has 5 singular points.

That $\mathcal{P}(6) \cap \mathcal{P}(12)$ contains no normal surface follows since if $\beta = 0$ the locus $\mathcal{P}(6) \cap \mathcal{P}(12, 0, 0)$ consists of the surface $\{\sigma_4 = 0\}$, while the locus $\mathcal{P}(6) \cap (\mathcal{P}(12, 0, 1) \cup \mathcal{P}(12, 1, 1))$ consists of surfaces with infinitely many singular points (for each choice of b, c with $b + c = z$ we get a singular point).

That $\mathcal{P}(1) \cap \mathcal{P}(12)$ contains no normal surface follows by (I) of Lemma 19, since then $a_3 = a_4 = 0$.

We consider now $\mathcal{P}(4) \cap \mathcal{P}(12)$, defined by

$$a_2(\beta + a_2 + a_3) + a_1 a_4 = 0, \quad a_2 a_4 + a_3^2 = 0.$$

First of all, if $a_2 = 0$, then $a_3 = a_1 a_4 = 0$, and $a_2 = a_3 = a_4 = 0$ yields surfaces of the form $a_1 \sigma_1^4 + \beta \sigma_2^2$, hence with singular curve $\sigma_1 = \sigma_2 = 0$.

Instead, the case $a_1 = a_2 = a_3 = 0$ yields a surface of the form (if irreducible) $X = \{\sigma_4 + \beta \sigma_2^2 = 0\}$.

At the four coordinate points both σ_2, σ_4 vanish, and we see easily that these are uniplanar double points (if $x_1 = 1, x_j = 0,\ j \geq 2$, then the local equation is

$$(x_2 + x_3 + x_4)^2 + x_2 x_3 x_4 + (x_2 x_3 + x_2 x_4 + x_3 x_4)^2 = 0).$$

An easy inspection of the cases of Proposition 18 shows that there are no other singular points for $\beta \neq 0$.

We may now assume that $a_2 = 1$, hence $a_3 = z \neq 0$, $a_4 = z^2$, and

$$(*) \quad a_1 z^2 + (1 + z + \beta) = 0.$$

The point $(1, 1, 1, 1 + z)$ and its orbit are then singular points of X. We claim that these are all.

In fact, there is no orbit consisting of 12 singular points by the following arguments.

Case $\mathcal{P}(12, 0, 0)$ is excluded since we have $a_2 \neq 0$.

In case $\mathcal{P}(12, 0, 1)$ $(*)$ and $\beta = a_1 z^4 + z^2(1 + z)$ imply $\beta = \beta z^2$, hence $\beta = 0$, since $z \neq 1$. We are then done since we have shown $\mathcal{P}(6) \cap \mathcal{P}(12) \cap \mathcal{F} = \emptyset$.

In case $\mathcal{P}(12, 1, 1)$ $(*)$ and $\beta(1 + bc)^2 = a_1 z^4 + z^2(1 + z)$ imply $\beta(1 + z^2 + (bc)^2) = 0$. Since $\beta = 0$ leads to a contradiction as above, we have

$$0 = 1 + b + c + bc = (1 + b)(1 + c),$$

a contradiction since $b, c \neq 0, 1$.

Likewise, there is no other orbit of singular points with cardinality 4,6,1 by Proposition 18.

To finish the proof that there are no more than 12 singular points, it suffices to observe that cases $\mathcal{P}(12, 0, 0)$, $\mathcal{P}(12, 0, 1)$, $\mathcal{P}(12, 1, 1)$ are mutually exclusive for $\beta \neq 0$, since $a_2 = 0$ in the first case, and $a_2 \neq 0$ for the other two, while the second and third case are exclusive because $bc \neq 0$.

Finally, one sees also that the two subcases of $\mathcal{P}(4)$ are mutually exclusive, hence the possible cardinalities of $\mathrm{Sing}(X)$, for X a normal symmetric surface, are only 1, 4, 5, 6, 10, 12. □

We summarize in the next corollary some result obtained so far:

Corollary 25 *The maximal number of singular points that a normal symmetric quartic X can have is exactly* 12.

The symmetric quartics of the form $X = \{\sigma_4 + \beta\sigma_2^2 = 0\}$, $\beta \neq 0$ have 4 *uniplanar double points, which are singularities of type D_4.*

Proof We have the local equation

$$(x_2 + x_3 + x_4)^2 + x_2x_3x_4 + (x_2x_3 + x_2x_4 + x_3x_4)^2 = 0.$$

Blowing up the singular point, we obtain a line in the exceptional $\mathbb{P}^2$, with three singular points which are nodes, since we get the equation

$$x^2 + uyz(x + y + z) + u^2(\dots) = 0,$$

where $u = 0$ is the equation of the exceptional divisor. □

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