



On the Equations Defining Some Hilbert Schemes

Jonathan D. Hauenstein¹ · Laurent Manivel² · Balázs Szendrői³

Received: 5 April 2021 / Accepted: 19 November 2021 / Published online: 22 January 2022
© Vietnam Academy of Science and Technology (VAST) and Springer Nature Singapore Pte Ltd. 2022

Abstract

We work out details of the extrinsic geometry for two Hilbert schemes of some contemporary interest: the Hilbert scheme $\text{Hilb}^2 \mathbb{P}^2$ of two points on \mathbb{P}^2 and the dense open set parametrizing non-planar clusters in the punctual Hilbert scheme $\text{Hilb}_0^4(\mathbb{A}^3)$ of clusters of length four on \mathbb{A}^3 with support at the origin. We find explicit equations in projective, respectively affine, embeddings for these spaces. In particular, we answer a question of Bernd Sturmfels who asked for a description of the latter space that is amenable to further computations. While the explicit equations we find are controlled in a precise way by the representation theory of SL_3 , our arguments also rely on computer algebra.

Keywords Binomial ideals · Sparse polynomials · Numerical algebraic geometry · Witness sets · Macaulay dual spaces

Mathematics Subject Classification (2010) 14Q65 · 65H14 · 68W30

1 Introduction

The Hilbert scheme $\text{Hilb}^2 \mathbb{P}^2$ of two points on the projective plane has a projective embedding

$$\text{Hilb}^2 \mathbb{P}^2 \hookrightarrow \mathbb{P}^{14} \quad (1)$$

This article is dedicated to Prof. Bernd Sturmfels on the occasion of his 60th birthday.

✉ Jonathan D. Hauenstein
hauenstein@nd.edu

Laurent Manivel
laurent.manivel@math.cnrs.fr

Balázs Szendrői
szendroi@maths.ox.ac.uk

¹ Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, Notre Dame, IN 46556, USA

² Institut de Mathématiques de Toulouse; UMR 5219, Université de Toulouse, CNRS, F-31062, Toulouse Cedex 9, France

³ Mathematical Institute, University of Oxford, OX1 2DL, Oxford, UK

defined as a composition of a natural map into $\text{Gr}(2, 6)$ followed by the Plücker embedding (for details, see Section 2.1). On the other hand, the Hilbert scheme $\text{Hilb}^4(\mathbb{A}^3)$ of 4 points on affine 3-space has a distinguished affine open subset $\text{Hilb}^4(\mathbb{A}^3)_{\text{np}}$ of non-planar clusters, which has an affine embedding

$$\text{Hilb}^4(\mathbb{A}^3)_{\text{np}} \hookrightarrow \mathbb{A}^{15},$$

whose image is the cone over the same Grassmannian (see Section 3.1). This latter Hilbert scheme has a natural closed subset, the space $\text{Hilb}_0^4(\mathbb{A}^3)_{\text{np}}$ of non-planar, punctual clusters, where punctual means that the scheme-theoretic support of the subscheme parametrized is at a single point (chosen to be the origin; see Section 3.2 for details). We thus obtain an affine embedding

$$\text{Hilb}_0^4(\mathbb{A}^3)_{\text{np}} \hookrightarrow \mathbb{A}^{15}. \tag{2}$$

The connection between these Hilbert schemes goes back to Tikhomirov’s [18, Theorem 3]: the affine cone over $\text{Hilb}^2 \mathbb{P}^2 \subset \mathbb{P}^{14}$ is the singular locus of $\text{Hilb}_0^4(\mathbb{A}^3)_{\text{np}}$.

Our aim in this paper is twofold: we give explicit equations to describe the images of the embeddings (1) and (2), and we recover this relationship between the spaces. We begin in Section 2.2 by identifying various spaces of interest as PGL_3 -orbit closures in $\text{Gr}(2, 6)$. Their defining polynomials arise from the representation theory of SL_3 (see Section 2.3). We use computer algebra to derive specific polynomials, as well as to check various properties of the resulting systems of equations, in particular that they define reduced ideals. We also provide a more synthetic way to derive the equations in Section 2.4.

Our main result is Theorem 6 in Section 3.2. Section 3.3 presents an explicit calculation to illuminate, and yet again reprove, some of the main results. The explicit polynomials defining the embeddings (1) and (2) are listed in the Appendix.

2 The Hilbert Scheme of Two Points on the Projective Plane

2.1 Basics on the Hilbert Scheme of Two Points on the Projective Plane

Let U be a 3-dimensional vector space. In this section, we recall some well-known facts about the Hilbert scheme $\text{Hilb}^2 \mathbb{P}(U)$ of two points on the projective plane $\mathbb{P}(U)$. We refer to [1, Sections 2 and 3] for more details and further references.

Let $h : \text{Hilb}^2 \mathbb{P}(U) \rightarrow \text{Sym}^2 \mathbb{P}(U)$ be the Hilbert–Chow morphism, $H = h^*(\mathcal{O}(1))$ the pullback of the natural generator of $\text{Pic}(\text{Sym}^2 \mathbb{P}(U))$. Let $\Delta \subset \text{Sym}^2 \mathbb{P}(U)$ be the diagonal and $B = h^{-1}(\Delta)$ the exceptional locus of h , a threefold ruled over $\Delta \cong \mathbb{P}(U)$. It is known that

$$\text{Pic}(\text{Hilb}^2 \mathbb{P}(U)) \cong \mathbb{Z}[H] \oplus \mathbb{Z} \left[\frac{B}{2} \right].$$

Let \mathcal{L} denote the line bundle $\mathcal{O}(2)$ on $\mathbb{P}(U)$. There is an associated rank two bundle $\mathcal{L}^{[2]}$ on $\text{Hilb}^2 \mathbb{P}(U)$ whose fiber over a length 2 subscheme $\zeta \subset \mathbb{P}(U)$ is $H^0(\zeta, \mathcal{L})$. Its space of sections is

$$H^0(\text{Hilb}^2 \mathbb{P}(U), \mathcal{L}^{[2]}) \cong H^0(\mathbb{P}(U), \mathcal{L}) = S^2 U^*$$

and it is globally generated. Its determinant

$$D = \det(\mathcal{L}^{[2]}) = 2[H] - \left[\frac{B}{2} \right] \in \text{Pic}(\text{Hilb}^2 \mathbb{P}(U))$$

is very ample and yields a $\text{PGL}(U)$ -equivariant chain of embeddings

$$\varphi_D : \text{Hilb}^2 \mathbb{P}(U) \hookrightarrow \text{Gr}(4, S^2 U^*) = \text{Gr}(2, S^2 U) \hookrightarrow \mathbb{P}(\wedge^2 S^2 U) \cong \mathbb{P}^{14}.$$

The image of φ_D is of degree

$$D^4 = 16H^4 - 16H^3B + 6H^2B^2 - HB^3 + \frac{B^4}{16} = 21$$

which arises from the known intersection numbers [15] on the Hilbert scheme, namely

$$(H^4, H^3B, H^2B^2, HB^3, B^4) = (3, 0, -8, -24, -48).$$

The embedding φ_D restricts to a $\text{PGL}(U)$ -equivariant chain of embeddings

$$\varphi_D|_B: B \hookrightarrow \text{Gr}(2, S^2U) \hookrightarrow \mathbb{P}(\wedge^2 S^2U) \cong \mathbb{P}^{14}$$

of degree

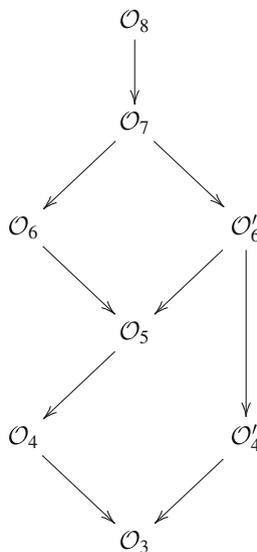
$$(D|_B)^3 = D^3B = 8H^3B - 6H^2B^2 + \frac{3HB^3}{2} - \frac{B^4}{8} = 18.$$

2.2 Pencils of Conics and Orbit Structure

Consider the space $\text{Gr}(2, S^2U)$ parametrizing pencils of conics on U^* . The decomposition of this space into $\text{PGL}(U)$ -orbits is classical, see e.g. [14]. There are eight types with normal forms as in the following table, yielding a decomposition of the Grassmannian $\text{Gr}(2, S^2U)$ into $\text{PGL}(U)$ -orbits \mathcal{O}_i (and sometimes \mathcal{O}'_i) of dimension i .

\mathcal{O}_3	$\langle x^2, xy \rangle$
\mathcal{O}_4	$\langle x^2, y^2 \rangle$
\mathcal{O}'_4	$\langle xy, xz \rangle$
\mathcal{O}_5	$\langle x^2, y^2 + xz \rangle$
\mathcal{O}_6	$\langle x^2, yz \rangle$
\mathcal{O}'_6	$\langle x^2 + yz, xz \rangle$
\mathcal{O}_7	$\langle x^2 + y^2, xz \rangle$
\mathcal{O}_8	$\langle x^2 + y^2, x^2 + z^2 \rangle$

Let Y_i, Y'_i denote the closure of $\mathcal{O}_i, \mathcal{O}'_i$ in $\text{Gr}(2, S^2U)$. Clearly, $Y_8 = \text{Gr}(2, S^2U)$. The Hasse diagram, where \mathcal{O}_i is connected to \mathcal{O}_j if the latter is open in $Y_i \setminus \mathcal{O}_i$, is the following:



- Proposition 1**
1. $Y_6 \subset \text{Gr}(2, S^2U)$ is birational to a \mathbb{P}^4 -bundle over \mathbb{P}^2 .
 2. $Y_5 \subset \text{Gr}(2, S^2U)$ is birational to a quadric bundle over \mathbb{P}^2 and is a non-normal projective variety with singular locus equal to Y_4 .
 3. $Y'_4 \simeq \mathbb{P}(U) \times \mathbb{P}(U^*)$ which is embedded in \mathbb{P}^{14} via the linear series $\mathcal{O}(2, 1)$.
 4. The chain of embeddings

$$Y_3 \subset Y_4 \subset \text{Gr}(2, S^2U) \subset \mathbb{P}(\wedge^2 S^2U) \cong \mathbb{P}^{14}$$

is isomorphic to the chain

$$B \subset \text{Hilb}^2 \mathbb{P}(U) \subset \text{Gr}(2, S^2U) \subset \mathbb{P}(\wedge^2 S^2U) \cong \mathbb{P}^{14}$$

discussed in the previous section.

Proof First, note that the variety Y_6 parametrizes pencils of conics containing a double line. We thus have a birational model $\tilde{Y}_6 \rightarrow Y_6$ parametrizing pairs (P, ℓ) of a pencil P of conics on U^* and a line $\ell \subset \mathbb{P}(U^*)$ such that ℓ^2 belongs to P . For the bundle $E = S^2U/\mathcal{O}(-2)$ over $\mathbb{P}(U)$, we see that $\tilde{Y}_6 = \mathbb{P}(E)$, a \mathbb{P}^4 -bundle over \mathbb{P}^2 .

Next, Y_5 parametrizes pencils generated by a double line and a conic tangent to this line. Consider the birational model $\tilde{Y}_5 \rightarrow Y_5$ defined as the preimage of Y_5 in \tilde{Y}_6 . Let $Q = U/\mathcal{O}(-1)$ which is a bundle over $\mathbb{P}(U)$. Note that there is a natural morphism $E \rightarrow S^2Q$ and hence a natural rank three quadratic form $S^2E \rightarrow S^2(S^2Q) \rightarrow \det(Q)^2$. Then, $\tilde{Y}_5 \subset \tilde{Y}_6 = \mathbb{P}(E)$ is the corresponding quadric bundle over $\mathbb{P}(U)$. The morphism $\tilde{Y}_5 \rightarrow Y_5$ is birational and is a double cover over Y_4 . Thus, Y_5 is not normal along Y_4 and, since the latter is the complement of the open orbit in Y_5 , it has to be the singular locus.

The variety Y'_4 parametrizes pencils of reducible conics containing a fixed line ℓ . Such a pencil is of the form ℓL where L is a hyperplane of U . Hence, we have an isomorphism $Y'_4 \cong \mathbb{P}(U) \times \mathbb{P}(U^*)$. The restriction of the tautological bundle of $\text{Gr}(2, S^2U)$ is the product of the rank one tautological bundle on $\mathbb{P}(U)$ with the rank two tautological bundle on the dual $\mathbb{P}(U^*)$. Its determinant is $\mathcal{O}(2, 1)$ showing that the degree of Y'_4 is $(2h + h')^4 = 24$.

In this description, we see that $Y_3 \subset Y'_4 = \mathbb{P}(U) \times \mathbb{P}(U^*)$ is the incidence quadric, the full flag variety of U . Its degree is $(h + h')(2h + h')^3 = 18$.

To conclude, consider the variety Y_4 parametrizing pencils generated by two double lines and their degeneracies. Its dense open subset \mathcal{O}_4 is isomorphic to $\text{Sym}^2\mathbb{P}(U) - \Delta$ where Δ denotes the diagonal. For dimension reasons, its closure Y_4 must be $\text{Hilb}^2 \mathbb{P}(U) \subset \text{Gr}(2, S^2U)$. Finally, note that the diagonal in $\text{Hilb}^2 \mathbb{P}(U) = Y_4$ is $\mathbb{P}(T_{\mathbb{P}(U)}) \cong F \cong Y_3$ as claimed which completes the proof. □

2.3 Explicit Equations

Consider the group $\text{SL}(U) \cong \text{SL}_3(\mathbb{C})$. Denote by $S_{a,b}$ the representation of $\text{SL}(U)$ with highest weight $(a - b)\omega_1 + b\omega_2$, where ω_1 and ω_2 are the fundamental weights and $a \geq b \geq 0$. Clearly, $S_{0,0} = \mathbb{C}$ is the trivial module, $S_{1,0} = U$ is the three-dimensional “vector” representation, and $S_{1,1} = U^*$ is its dual. Consider the $\text{SL}(U)$ -module $W = \wedge^2 S^2U^* = S_{3,2}$ of dimension 15. Its second symmetric square splits into irreducibles as

$$S^2W \cong S_{6,4} \oplus S_{4,3} \oplus S_{4,0} \oplus S_{3,1} \oplus S_{2,2} \tag{3}$$

of dimension $120 = 60 + 24 + 15 + 15 + 6$. This decomposition can be easily checked computationally, e.g., via the SchurRings [16] package in Macaulay2 [9].

Consider the 14-dimensional projective space $\mathbb{P}W^* = \mathbb{P}(\wedge^2 S^2 U) \cong \mathbb{P}^{14}$ with coordinate ring $S^*(W)$. Regard elements of $S^2 W$ as quadrics on $\mathbb{P}W^*$. Using the decomposition (3) above, define ideals of the ring $S^*(W)$ generated by spaces of quadrics as follows:

$$\begin{aligned}
 I_8 &= \langle S_{3,1} \rangle \triangleleft S^*(W), \\
 I_5 &= \langle S_{3,1}, S_{2,2} \rangle \triangleleft S^*(W), \\
 I_4 &= \langle S_{3,1}, S_{2,2}, S_{4,3} \rangle \triangleleft S^*(W), \\
 I_3 &= \langle S_{3,1}, S_{2,2}, S_{4,3}, S_{4,0} \rangle \triangleleft S^*(W).
 \end{aligned}
 \tag{4}$$

These ideals are generated by 15, 21, 45 and 60 quadrics, respectively. The first ideal I_8 consists of quadrics parametrized by $S_{3,1} = \wedge^2 S^2 U$ itself and it is well known that its vanishing locus is the Grassmannian:

$$\mathbb{V}(I_8) = \text{Gr}(2, S^2 U) \subset \mathbb{P}(\wedge^2 S^2 U).$$

Proposition 2 *For $i = 3, 4, 5$ we have, using the notations of the previous section,*

$$\mathbb{V}(I_i) \cong Y_i \subset \text{Gr}(2, S^2 U) \subset \mathbb{P}(\wedge^2 S^2 U) \cong \mathbb{P}^{14}.$$

In particular, the ideals of the orbit closures $Y_3, Y_4, Y_5 \subset \text{Gr}(2, S^2 U)$ discussed before are generated by quadrics.

Proof Our proof uses computational methods, with some details omitted.

Fix a basis $\{v_i\}$ of U^* . The key to finding explicit equations is to derive an explicit form of the decomposition (3) yielding basis elements for each of the modules on the right hand side in terms of the obvious basis of the left hand side $S^2 W \cong S^2 (\wedge^2 S^2 U^*)$ consisting of symmetric pairs of elements of the form $[(v_i \otimes v_j) \wedge (v_k \otimes v_l)]$. This can be done using SLA [6] in GAP [17]. We obtain explicit generators of these four ideals, which are listed in the Appendix. As expected, the 15-dimensional space $S_{3,1}$ has a basis whose elements can readily be identified with the quadratic Plücker relations defining $\mathbb{V}(I_8) = \text{Gr}(2, 6) \subset \mathbb{P}^{14}$.

Given the explicit polynomial generators, the dimension and degree of the remaining ideals can be computed using Macaulay2 [9], namely:

ideal	dimension	degree	Hilbert series
I_5	5	56	$(1 + 9t + 24t^2 + 19t^3 + 3t^4)/(1 - t)^6$
I_4	4	21	$(1 + 10t + 10t^2)/(1 - t)^5$
I_3	3	18	$(1 + 11t + 6t^2)/(1 - t)^4$

We next claim that each of these ideals defines an irreducible and reduced subscheme in \mathbb{P}^{14} .

We first verify that each top-dimensional component in each scheme is irreducible of multiplicity 1. To that end, certifiable witness point sets can be computed in numerical algebraic geometry using Bertini [2] and alphaCertified [12] via well-constrained subsystems [8]. Each witness point set consists of degree-many points along a complementary dimensional linear space, which are nonsingular with respect to the resulting system. Certifiable monodromy loops following [10] yield that all of the witness points for each system are smoothly connected, showing that each of the top-dimensional components is irreducible of multiplicity 1 with respect to the corresponding ideal.

In each case, showing that the scheme is irreducible and reduced is now equivalent to showing that the top-dimensional irreducible component has the same Hilbert series as

the entire scheme; this precludes the existence of embedded components. This was verified using [5] with certified numerical Hilbert function computations [11]. In fact, this also showed that each scheme is arithmetically Cohen–Macaulay via [5].

The ideals $I_5 \subset I_4 \subset I_3$ thus define $SL(3)$ -invariant irreducible reduced subschemes of $\mathbb{V}(I_8) = \text{Gr}(2, 6) \subset \mathbb{P}^{14}$ of dimensions 5, 4, 3 respectively. As there is only one five-dimensional orbit closure in $\text{Gr}(2, 6)$ by our results in the previous section, we must have $\mathbb{V}(I_5) = Y_5 \subset \mathbb{P}^{14}$, and then necessarily $\mathbb{V}(I_4) = Y_4 \supset \mathbb{V}(I_3) = Y_3$ as claimed. \square

Remark 1 The 45 quadrics defining $Y_4 \cong \text{Hilb}^2 \mathbb{P}^2 \subset \text{Gr}(2, 6)$ were also determined, using a different method, in [3, Sect. 2]. That paper also studied the corresponding tropicalization.

Remark 2 In the language of the previous section, the quadrics in $S_{3,1}$ and $S_{4,0}$ define a four-dimensional variety of degree 24 inside of $\text{Gr}(2, S^2U)$ which is $Y'_4 \cong \mathbb{P}^2 \times \mathbb{P}^2$ embedded by $\mathcal{O}(2, 1)$ as above.

2.4 A Different Way to Derive Equations for Orbit Closures

We explain here an alternative, synthetic way to re-derive the quadratic equations of the orbit closures obtained above by a computer-based calculation. In this section, we will use the language of $GL(U)$ -modules to respect the full symmetry of the problem. By a slight abuse of notation, we will use $\psi(-)$ as shorthand for the quadratic form associated to a symmetric bilinear form $\psi(-, -)$.

The Equations of Y_5 The equations of $Y_5 \subset \mathbb{P}^{14}$ are the Plücker equations together with another irreducible module of quadratic equations that is $GL(U)$ -isomorphic to $S^2U^* \otimes \det(U^*)^2$. Those equations have a simple description in terms of the discriminant of ternary quadratic forms (or symmetric tensors), which is a $GL(U)$ -equivariant map

$$\delta : S^3(S^2U) \rightarrow \det(U)^2.$$

Polarizing yields a morphism

$$\Delta : S^2(S^2U) \rightarrow S^2U^* \otimes \det(U)^2.$$

Using the same trick for U^* and twisting by $\det(U)$ appropriately yields another $GL(U)$ -equivariant map

$$\Delta^* : S^2(S^2U^* \otimes \det(U)^2) \rightarrow S^2U \otimes \det(U)^2.$$

In order to describe the equations of $Y_5 \subset \text{Gr}(2, S^2U)$, since it is the unique irreducible $GL(U)$ -component of $S^2(\wedge^2 S^2U^*)$ isomorphic to $S^2U^* \otimes \det(U^*)^2$, it suffices to exhibit a nonzero morphism

$$\Psi : S^2(\wedge^2 S^2U) \rightarrow S^2U \otimes \det(U)^2$$

from the dual of the former to the dual of the latter. For q_1, q_2, q_3, q_4 belonging to S^2U , we claim that the following formula defines such a morphism:

$$\Psi(q_1 \wedge q_2, q_3 \wedge q_4) = \Delta^*(\Delta(q_1, q_3), \Delta(q_2, q_4)) - \Delta^*(\Delta(q_1, q_4), \Delta(q_2, q_3)).$$

Since this is nonzero (see below) and has all the required properties, it expresses in a compact form all the equations of Y_5 apart from the Plücker relations. In the following statement,

we use the same notations Ψ and Δ for the quadratic forms associated to the symmetric bilinear forms defined by these symbols.

Proposition 3 *The subvariety $Y_5 \subset \text{Gr}(2, S^2U)$ of the Grassmannian $\text{Gr}(2, S^2U)$ is defined by*

$$\Psi(q_1 \wedge q_2) = \Delta^*(\Delta(q_1), \Delta(q_2)) - \Delta^*(\Delta(q_1, q_2)) = 0$$

for $q_1, q_2 \in S^2U$.

As a sanity check, let us evaluate Ψ on the representatives of the PGL_3 -orbits of $G(2, S^2U)$. For this we normalize Δ by letting $\Delta(u^2, v^2) = (u \wedge v)^2$.

- For \mathcal{O}_5 , we let $q_1 = x^2$ and $q_2 = y^2 + xz$. Hence, $\Delta(q_1) = 0$ and $\Delta(q_1, q_2) = (x \wedge y)^2$ has rank one, so that $\Psi(q_1 \wedge q_2) = -\Delta^*(\Delta(q_1, q_2)) = 0$.
- For \mathcal{O}_6 , we let $q_1 = x^2$ and $q_2 = yz$. Hence, $\Delta(q_1) = 0$ and writing $4q_2 = (y + z)^2 - (y - z)^2$, we get $\Delta(q_1, q_2) = (x \wedge y)(x \wedge z)$. Since this has rank two, $\Psi(q_1 \wedge q_2) = -\Delta^*(\Delta(q_1, q_2)) \neq 0$.
- For \mathcal{O}'_4 , we let $q_1 = xy$, $q_2 = xz$, $u = x \wedge y$, and $v = x \wedge z$. Then, $2\Delta(q_1) = -u^2$, $2\Delta(q_2) = -v^2$, and $2\Delta(q_1, q_2) = -uv$. Therefore, $\Delta^*(\Delta(q_1), \Delta(q_2)) = \Delta^*(u^2, v^2)/4 = (u \wedge v)^2/4$ together with $\Delta^*(\Delta(q_1, q_2)) = \Delta^*(uv)/4 = -(u \wedge v)^2/8$ yields $\Psi(q_1 \wedge q_2) = 3(v \wedge w)^2/8 \neq 0$.

The Other Quadratic Equations Consider the remaining modules of equations defining $Y_4, Y_3 \subset \mathbb{P}^{14}$. These irreducible modules are generated by highest weight vectors. We can write down explicit quadratic polynomials on the Grassmannian in terms of those highest weight vectors as follows.

The 15-dimensional module is the irreducible $\text{GL}(U)$ -module $S^4(\wedge^2 U^*)$, whose highest weights vectors are the tensors of the form $(e \wedge f)^4$ for $e, f \in U^*$. We want to associate to such a vector a quadratic polynomial P on the cone of tensors of the form $q_1 \wedge q_2$ for $q_1, q_2 \in S^2U$. In other words, we need to find a polynomial in e, f, q_1, q_2 , which has degree four in e and f , degree two in q_1 and q_2 , invariant under the action of $\text{GL}(U)$, and compatible with the skew-symmetry conditions. By the Fundamental Theorems of Invariant Theory, such an invariant polynomial has to be expressed in terms of contractions of q_1 and q_2 by e and f . A straightforward computation shows that up to scalar, there is only one possibility, namely

$$P(q_1 \wedge q_2) = (q_1(e)q_2(f) - q_2(e)q_1(f))^2 + 4(q_1(e, f)q_2(f) - q_2(e, f)q_1(f))(q_1(e, f)q_2(e) - q_2(e, f)q_1(e)).$$

One can easily test this polynomial on representatives of the orbits in $\text{Gr}(2, S^2U)$ and check that it vanishes identically only on the orbit $\mathcal{O}_3 = Y_3$.

The second, 24-dimensional module is the $\text{GL}(U)$ -submodule of $S^3U^* \otimes \wedge^2 U^* \otimes \wedge^3 U^*$ whose highest weights vectors are the tensors of the form $e^3(e \wedge f)(e \wedge f \wedge g)$ for $e, f, g \in U^*$. Again, we need to associate to such a vector a quadratic polynomial Q on the cone of tensors of the form $q_1 \wedge q_2$ for $q_1, q_2 \in S^2U$. Thus, we need to find a polynomial in e, f, g, q_1 , and q_2 which has degree five in e , degree two in f , degree one in g , degree two in q_1 and q_2 , invariant under the action of $\text{GL}(U)$, and compatible with the skew-symmetry

conditions. Another straightforward computation shows that up to scalar, there is only one possibility, namely

$$\begin{aligned}
 Q(q_1 \wedge q_2) = & \left(q_1(e, g)q_2(e) - q_2(e, g)q_1(e) \right) \left(q_1(e)q_2(f) - q_2(e)q_1(f) \right) \\
 & + \left(q_1(e, f)q_2(e) - q_2(e, f)q_1(e) \right) \\
 & \times \left(q_1(f, g)q_2(e) - q_2(f, g)q_1(e) + q_1(e, g)q_2(e, f) - q_2(e, g)q_1(e, f) \right).
 \end{aligned}$$

Again, one can easily test this polynomial on representatives of the orbits in $\text{Gr}(2, S^2U)$ and check that it vanishes identically only on \mathcal{O}_3 and \mathcal{O}_4 . We deduce

Proposition 4 *The subvarieties $Y_3 \subset Y_4 \subset \text{Gr}(2, S^2U)$ of the Grassmannian $\text{Gr}(2, S^2U)$ are defined by the sets of equations*

$$\Psi(q_1 \wedge q_2) = Q(q_1 \wedge q_2) = 0$$

and

$$\Psi(q_1 \wedge q_2) = P(q_1 \wedge q_2) = Q(q_1 \wedge q_2) = 0$$

for $q_1, q_2 \in S^2U$, respectively.

3 The Punctual Hilbert Scheme of Four Points on \mathbb{A}^3

3.1 Non-planar Clusters of Length Four on Affine Three-space

Recall the fixed three-dimensional vector space U from the previous section. In this section, we will think of its dual $T = U^*$ as a copy of affine 3-space \mathbb{A}^3 with ring of functions $\mathbb{C}[T] = \text{Sym}^\bullet U$.

Let $\text{Hilb}^m(T) \cong \text{Hilb}^m(\mathbb{A}^3)$ denote the Hilbert scheme of m points on affine three-space. The additive structure on T yields a center-of-mass morphism $c: \text{Hilb}^m(T) \rightarrow T$. As is true in all dimensions, $\text{Hilb}^m(T)$ is a nonsingular variety for $m \leq 3$. It is also known [13] that $\text{Hilb}^4(T)$ is an irreducible and reduced variety of dimension 12, singular along the locus of length-four subschemes of T given by the squares $[\mathfrak{m}_p^2] \in \text{Hilb}^4(T)$ of maximal ideals of points $p \in T$. It has a dense affine open subset $\text{Hilb}^4(T)_{\text{np}}$ containing all its singularities defined by the condition that the clusters parametrized by its points are non-planar, i.e., not scheme-theoretically contained in a plane.

Although the following was already proved in [7], we present a variant of the proof suited to the present narrative.

Theorem 5 *There is an $\text{SL}(U)$ -equivariant isomorphism*

$$\text{Hilb}^4(T)_{\text{np}} \cong T \times \mathcal{C}\text{Gr}(2, S^2U),$$

where $\mathcal{C}\text{Gr}(2, S^2U) \subset \wedge^2 S^2U$ is the affine cone over the Grassmannian $\text{Gr}(2, S^2U) \subset \mathbb{P}(\wedge^2 S^2U) \cong \mathbb{P}^{14}$.

Proof If $I_\xi \triangleleft \mathbb{C}[T] = \text{Spec Sym}^\bullet U$ is the ideal corresponding to a point $\xi \in \text{Hilb}^4(T)_{\text{np}}$, then the map $\mathbb{C} \oplus U \rightarrow \mathbb{C}[T]/I_\xi$ is an isomorphism of vector spaces. The resulting algebra structure on the vector space $\mathbb{C} \oplus U$ is encoded by two symmetric bilinear maps:

$$a : U \otimes U \rightarrow \mathbb{C} \quad \text{and} \quad m : U \otimes U \rightarrow U.$$

Requiring the product on $\mathbb{C} \oplus U$ to be associative leads to the following equations for any $x, y, z \in U$:

$$a(x, m(y, z)) = a(y, m(x, z)), \tag{5}$$

$$m(x, m(y, z)) + a(y, z)x = m(y, m(x, z)) + a(x, z)y. \tag{6}$$

Using the highest weight notation for SL_3 -modules introduced in the previous section, first observe that m is a tensor in $S^2U^* \otimes U \cong U^* \oplus S_{3,2}$ and the projection to $T = U^*$ is the center of mass map c . Using a translation by the evident action of U^* on the whole setup, it suffices to restrict to the case where the center of mass of the ideal I_ξ is at the origin in T so that $m \in S_{3,2}$.

Equation (6) shows that $a(y, z)x - a(x, z)y$ is determined by $m, x, y,$ and z which implies that a is uniquely determined by m and depends quadratically on it. As confirmed by [16], there exists a unique $SL(U)$ -equivariant map up to scale, namely

$$\Theta : S^2(S_{3,2}) \rightarrow S^2U^*.$$

Fixing the correct normalization, (6) implies that $a = \Theta(m)$. Now, for any $x, y, z \in U$, we can rewrite (5)–(6) in the following form:

$$\Theta(m)(x, m(y, z)) = \Theta(m)(y, m(x, z)), \tag{7}$$

$$m(x, m(y, z)) + \Theta(m)(y, z)x = m(y, m(x, z)) + \Theta(m)(x, z)y. \tag{8}$$

These equations are families of cubic and quadratic equations on $m \in S_{3,2}$. We claim that

- (a) the quadratic equation (8) on m are equivalent to the Plücker equations on $\wedge^2 S^2U$;
- (b) the cubic equation (7) are implied by the quadratic ones.

In order to prove (a), recall the SL_3 -decomposition

$$S^2S_{3,2} \cong S_{6,4} \oplus S_{4,3} \oplus S_{4,0} \oplus S_{3,1} \oplus S_{2,2} \tag{9}$$

already used above in (3). Equation (8) asks for the vanishing of a cubic tensor in $x, y,$ and z , skew-symmetric in x and y , and takes values in U , i.e., an element of the SL_3 -decomposition

$$\text{Hom}(\wedge^2U \otimes U, U) \cong 2S_{1,1} \oplus S_{3,1} \oplus S_{2,2}. \tag{10}$$

Comparing (9) with (10), the common terms are the last two irreducible components; they are the only ones imposing non-trivial conditions. The component $S_{2,2}$ has already been taken into account by letting $a = \Theta(m)$. The remaining conditions are the quadratic equations parametrized by $S_{3,1}$. This means that we get the quadratic equations spanning the ideal $I_8 \triangleleft S^*(\wedge^2 S^2U^*)$ from (4), parametrized by $\wedge^2 S^2U$, which are precisely the Plücker equations.

We conclude that $\text{Hilb}^4(T)_{\text{np}}$ is contained in $T \times \mathcal{CGr}(2, S^2U)$ which implies, by a dimension count, that these two reduced schemes must be equal. Since the ideal of $\mathcal{CGr}(2, S^2U)$ is radical, we also deduce claim (b): the cubic relations (7) do not impose any further conditions on m . □

3.2 The Space of Non-planar Punctual Clusters

Let $\text{Hilb}_0^m(T) \subset \text{Hilb}^m(T)$ denote the punctual Hilbert scheme of m points, the subscheme of the Hilbert scheme $\text{Hilb}^m(T)$ given by the condition that the support of the zero-dimensional subscheme being parametrized is at the origin $0 \in T$. It carries a natural projective scheme structure as the scheme-theoretic fiber of the Hilbert–Chow morphism

$\text{Hilb}^m(T) \rightarrow S^m(T)$ over $m \cdot 0 \in S^m(T)$, but here we consider it in its reduced scheme structure.

Clearly, $\text{Hilb}_0^2(T) \cong \mathbb{P}(T)$, and $\text{Hilb}_0^3(T)$ and $\text{Hilb}_0^4(T)$ are known to be irreducible but singular projective varieties of dimensions 4 and 6 respectively. Descriptions of these spaces as well as natural desingularizations are given in [18] from a sheaf-theoretic perspective. We will describe the affine open set $\text{Hilb}_0^m(T)_{\text{np}} \subset \text{Hilb}_0^m(T)$ obtained by intersecting $\text{Hilb}_0^m(T)$ with the set of non-planar clusters. Note that $\text{Hilb}_0^m(\mathbb{A}^3)_{\text{np}}$ is dense in $\text{Hilb}_0^m(\mathbb{A}^3)$ and forms an affine neighbourhood of its most interesting point $[\mathbf{m}_0^m] \in \text{Hilb}_0^m(\mathbb{A}^3)$, the length-four subscheme of \mathbb{A}^3 given by the square of the maximal ideal of the origin $0 \in \mathbb{A}^3$. The following is our main result.

Theorem 6 *The reduced space $\text{Hilb}_0^4(T)_{\text{np}}$ of non-planar, punctual clusters of length 4 on T is $\text{SL}(U)$ -equivariantly isomorphic to the cone $\mathcal{C}Y_5 \subset \wedge^2 S^2 U \cong \mathbb{A}^{15}$ over the projective variety $Y_5 \subset \mathbb{P}(\wedge^2 S^2 U)$ described in Propositions 1–2. In particular, $\text{Hilb}_0^4(T)_{\text{np}}$ is a non-normal subvariety of \mathbb{A}^{15} cut out by 21 explicitly computable quadrics and has a codimension one singular locus isomorphic to the affine cone $\mathcal{C}\text{Hilb}^2(\mathbb{P}^2) \subset \wedge^2 S^2 U$ with the apex of the cone corresponding to the distinguished ideal $[\mathbf{m}_0^2] \in \text{Hilb}_0^4(T)$.*

Proof The classification of $\text{PGL}(U)$ -orbits in $\text{Gr}(2, U)$ explained in Section 2.2 above shows that $\mathcal{C}Y_5$ is the only six-dimensional $\text{SL}(U)$ -stable subvariety of $\mathcal{C}\text{Gr}(2, U)$. □

Remark 3 The description of the singular locus of (a neighbourhood of $[\mathbf{m}_0^2]$ in) $\text{Hilb}_0^4(T)$ is not a new result as it was also obtained in [18] using sheaf-theoretic methods. The main advantage of our approach is that we can describe all these spaces by explicit affine equations which may be useful in applications [4].

Remark 4 It is also possible to prove this result using the language used in [7] followed by some computer calculations. In coordinates, we can represent a non-planar cluster of length four using three four-by-four matrices φ_1, φ_2 , and φ_3 which describe the action of the coordinate functions of T on the 4-dimensional vector space $\mathbb{C} \oplus U$. The conditions for a triple (φ_i) to describe a cluster become explicit equations which reduce to the Plücker relations. To describe a punctual cluster based at 0, the matrices φ_i should additionally be nilpotent. It can be checked computationally that the reduced ideal of conditions arising from $\text{tr } \wedge^k \varphi_i = 0$ for $1 \leq k \leq 4$ are generated by the linear relations $\text{tr } \varphi_i = 0$ and the quadratic conditions $\text{tr } \wedge^2 \varphi_i = \text{tr } \wedge^2 (\varphi_i + \varphi_j) = 0$, the latter being the 6 extra quadrics of I_5 defining Y_5 inside $\text{Gr}(2, S^2 U)$ in Proposition 2 above.

Remark 5 The orbit decomposition of $\text{Gr}(2, S^2 U)$ from Section 2.2 yields a decomposition of $\mathcal{C}\text{Gr}(2, S^2 U)$. It is easy to check that the various orbits correspond to different length four subschemes in T as follows.

1. $\mathcal{C}O_8$ parametrizes four general points;
2. $\mathcal{C}O_7$ parametrizes two reduced general points $p, q \in T$ and a degree two scheme at $-(p + q)/2$;
3. $\mathcal{C}O_6$ parametrizes two degree two schemes supported on opposite general points;
4. $\mathcal{C}O'_6$ parametrizes one reduced point at $p \in T$ and a degree three scheme supported on $-p/3$;
5. $\mathcal{C}O_5$ parametrizes an open subset of $\text{Hilb}_0^4(T)$;

6. \mathcal{CO}_4 parametrizes clusters with normal form $xy = z$, all other products being equal to zero;
7. \mathcal{CO}'_4 parametrizes one reduced point p and a fat point in a plane at $-p/3$;
8. \mathcal{CO}_3 parametrizes clusters with normal form $x^2 = z$, all other products being equal to zero.

3.3 A Concrete Computation

Starting from a non-planar scheme with center of mass 0 corresponding to an ideal $I \triangleleft \mathbb{C}[T]$, recall that we get an algebra structure on the vector space $\mathbb{C} \oplus U$ which is partly encoded by a symmetric bilinear multiplication map

$$m \in \text{Hom}(S^2U, U) \cong S^2U^* \otimes \wedge^2U^* \otimes \det(U).$$

Apply to m the morphism

$$\begin{aligned} \Gamma : S^2U^* \otimes \wedge^2U^* &\rightarrow \wedge^2(S^2U^*) \\ a^2 \otimes b \wedge c &\mapsto ab \wedge ac. \end{aligned}$$

Then, the discussion above shows that $\Gamma(m) \in \wedge^2(S^2U^*) \otimes \det(U)$ must be a decomposable tensor.

To see this in a concrete example, fix a basis $\{x, y, z\}$ of U and consider the reduced scheme

$$\zeta = \{(-1, 0, 0), (0, -1, 0), (0, 0, -1), (1, 1, 1)\} \subset T.$$

In the ring $\mathbb{C}[x, y, z]/I_\zeta$, we have

$$\begin{aligned} x^2 &= \frac{1}{2}(1 - x + y + z), & y^2 &= \frac{1}{2}(1 + x - y + z), & z^2 &= \frac{1}{2}(1 - x + y - z), \\ xy &= yz = zx & &= \frac{1}{4}(1 + x + y + z). \end{aligned}$$

The tensor m is obtained by keeping the degree one part of the right hand side in these equations. In terms of the dual basis $\{e, f, g\}$ of U^* , note that the basis of S^2U^* dual to the basis $\{x^2, y^2, z^2, yz, xz, xy\}$ of S^2U is $\{e^2, f^2, g^2, 2fg, 2eg, 2ef\}$. So, as a tensor,

$$\begin{aligned} 4m &= 2e^2 \otimes (-x + y + z) + 2f^2 \otimes (x - y + z) + 2g^2 \otimes (x + y - z) \\ &\quad + 2(fg + ge + ef) \otimes (x + y + z) \\ &= (e^2 + f^2 + g^2 + (e + f + g)^2) \otimes (x + y + z) - 4(e^2 \otimes x + f^2 \otimes y + g^2 \otimes z). \end{aligned}$$

Note that up, to a common factor, x identifies with $f \wedge g$, y with $g \wedge e$ and z with $e \wedge f$. Substituting these expressions and applying Γ , we get, after letting $h = e + f + g$,

$$\begin{aligned} 4\Gamma(m) &= ef \wedge eg + eg \wedge e^2 + e^2 \wedge ef + f^2 \wedge fg + fg \wedge fe + fe \wedge f^2 \\ &\quad + gf \wedge g^2 + g^2 \wedge ge + ge \wedge gf \\ &\quad + hf \wedge hg + hg \wedge he + he \wedge hf - 4(ef \wedge eg + fg \wedge fe + ge \wedge gf) \\ &= hf \wedge hg + hg \wedge he + he \wedge hf - (ef \wedge eg + fg \wedge fe + ge \wedge gf) \\ &\quad + he \wedge (ef - eg) + hf \wedge (fg - ef) + hg \wedge (eg - fg), \end{aligned}$$

and the final result of our computation is

$$4\Gamma(m) = (he - hf + eg - fg) \wedge (hf - hg + ef - eg).$$

This is, as expected, a decomposable tensor. Note that since the GL_3 -orbit of our scheme is open in the subvariety of $\text{Hilb}^4(T)$ parametrizing schemes with center of mass at the origin, this yields another proof of Theorem 5.

Moreover, observe that we can rewrite $he - hf + eg - fg = e^2 - f^2 + 2eg - 2fg = (e + g)^2 - (f + g)^2$, and similarly $hf - hg + ef - eg = (e + f)^2 - (f + g)^2$, so that

$$4\Gamma(m) = (e + f)^2 \wedge (f + g)^2 + (f + g)^2 \wedge (g + e)^2 + (g + e)^2 \wedge (e + f)^2.$$

This leads to a down-to-earth interpretation of the map

$$\pi : \text{Hilb}^4(T)_{\text{np}} \mapsto \mathcal{CGr}(2, S^2U)$$

from Theorem 5 on the open set of reduced subschemes.

Proposition 7 *Consider a finite subscheme $\zeta \subset T$ consisting of four non-coplanar reduced points $p_i \in T$ for $i = 1, \dots, 4$ with center of mass p_0 . The squares of the six vectors $p_{ij} = p_i + p_j - 2p_0 \in T$ are three tensors $p_{12}^2 = p_{34}^2$, $p_{13}^2 = p_{24}^2$ and $p_{14}^2 = p_{23}^2$ in $S^2T = S^2U^*$ and*

$$\pi(\zeta) = \left(p_{12}^2 \wedge p_{13}^2 + p_{13}^2 \wedge p_{14}^2 + p_{14}^2 \wedge p_{12}^2 \right) \otimes \omega^{-1},$$

where the twist $\omega \in \det(U^*)$ is given by

$$\omega = p_1 \wedge p_2 \wedge p_3 - p_2 \wedge p_3 \wedge p_4 + p_3 \wedge p_4 \wedge p_1 - p_4 \wedge p_1 \wedge p_2.$$

Note that ω is invariant under a common translation of the four points and is non-zero exactly when $\zeta \subset T$ is non-planar. Moreover, permuting the four points multiplies ω by the sign of the permutation but $\pi(\zeta)$ itself remains invariant. Thus, it only depends on ζ and not on the order of the four points.

Proof of Proposition 7 The expression for $\pi(\zeta)$ depends equivariantly on ζ and yields the correct expression when

$$\zeta = (-e, -f, -g, e + f + g).$$

Since the orbit of ζ in $\text{Hilb}^4(T)_{\text{np}}$ is dense, this expression must be correct everywhere. \square

Appendix: Explicit Polynomials

The following set of Macaulay2 commands generates the ideals I_8, I_5, I_4 and I_3 in (4). As proved in the main body of the paper, the ideal I_4 defines the projective image of the embedding (1), whereas I_5 defines the affine image of the embedding (2).

```
S=QQ[a,b,c,d,e,f,g,h,i,j,k,l,m,n,o];
```

```
I8=ideal(a*j-b*g+c*f, a*k-b*h+d*f, a*l-b*i+e*f, a*m-c*h+d*g,a*n-c*i+e*g,
a*o-d*i+e*h, b*m-c*k+d*j, b*n-c*l+e*j, b*o-d*l+e*k, c*o-d*n+e*m,f*m-g*k+h*j,
f*n-g*l+i*j, f*o-h*l+i*k, g*o-h*n+i*m, j*o-k*n+l*m);
```

```
I5=I8+ideal(2*d*o-e*n-2*f*o-2*h*l+i*i-2*j*l+3*k*k, 2*a*i-2*a*k-2*b*h+2*b*j-c*e+d*d
+3*f*f, c*n-2*d*m-2*f*m-2*g*i-2*g*k+3*h*h+j*j, a*n-2*b*m-c*k+d*h+d*j-e*g+f*h-f*j,
2*a*o-b*n-c*l+d*i+d*k-e*h+f*i-f*k, c*o-e*m-f*n-2*g*l+h*i+h*k-i*j+j*k);
```

$$I4=I5+ideal(a*d+a*f-b*c, a*e-b*d+b*f, g*n-h*m-j*m, c*m-g*h+g*j, e*o-i*l+k*l, i*o+k*o-l*n, 3*a*i+a*k-b*h-3*b*j-2*d*f, 2*a*k+b*h-3*b*j-3*c*e+3*d*d-d*f-6*f*f, 2*a*l+b*i-3*b*k-e*f, 2*a*m+c*h-d*g-3*f*g, a*n+2*b*m-c*i+c*k+3*d*h-d*j-2*e*g-6*f*h, 2*b*o+d*l-e*k-3*f*l, 2*a*o+3*b*n+d*i+3*d*k-e*h-3*e*j-6*f*i-6*f*k, 2*b*m-3*c*i+c*k+6*d*h-d*j-3*e*g-6*f*h+3*f*j, b*n-c*l+3*d*k-2*e*j-3*f*i, 3*c*n-9*d*m+5*f*m+12*g*i-2*g*k-6*h*h-7*h*j+3*j*j, c*o+2*d*n+e*m-3*h*i+3*j*k, 3*d*m-f*m-3*g*i+g*k+2*h*j, 2*d*o+e*n-h*l-i*i-i*k+j*l+2*k*k, 3*d*n+3*e*m-f*n-2*g*l-6*h*i+6*h*k-i*j+3*j*k, 3*e*n-2*f*o-h*l-3*i*i+i*k-3*j*l+6*k*k, 2*g*o+h*n-i*m-3*k*m, 3*h*o+j*o-k*n-2*l*m, 3*a*h-a*j-2*b*g-c*f);$$

$$I3=I4+ideal(4*a*g-c*c, 4*b*l-e*e, 4*m*o-n*n, 2*a*h+a*j+b*g-c*d, a*l+b*i+2*b*k-d*e, a+m+c*j-d*g+2*f*g, b*o+d*l-e*i+2*f*l, g*o-i*m-j*n+2*k*m, 2*h*o-i*n-j*o+l*m, 2*a*i+4*a*k+4*b*h+2*b*j-c*e-2*d*d, c*n-2*d*m+4*f*m-2*g*i+4*g*k-2*j*j, 2*d*o-e*n+4*f*o+4*h*l-2*i*i-2*j*l, a*n+b*m+2*c*k-2*d*h+d*j-e*g+4*f*h+2*f*j, a*o+b*n+c*l-d*i+2*d*k-2*e*h+2*f*i+4*f*k, c*o-e*m+2*f*n+g*l-2*h*i+4*h*k-i*j-2*j*k);$$

Acknowledgements The authors would like to thank Bernd Sturmfels for asking the question that lead to this work, Heather Harrington for introducing some of the collaborators to each other, and Willem de Graaf, Jack Huizinga, Miles Reid, Tim Ryan, and Anna Seigal for helpful advice and correspondence. J.D.H. and B.Sz. acknowledge support from NSF grant CCF-1812746 and EPSRC grant EP/R045038/1 respectively.

Declarations

Conflict of Interests The authors have no relevant financial or non-financial interests to disclose.

References

1. Arcara, D., Bertram, A., Coskun, I., Huizinga, J.: The minimal model program for the Hilbert scheme of points on \mathbb{P}^2 and Bridgeland stability. *Adv. Math.* **235**, 580–626 (2013)
2. Bates, D.J., Hauenstein, J.D., Sommese, A.J., Wampler, C.W.: Bertini: software for numerical algebraic geometry. <https://bertini.nd.edu> (2013)
3. Brodsky, S., Sturmfels, B.: Tropical quadrics through three points. *Linear Algebra Appl.* **435**, 1778–1785 (2011)
4. Cid-Ruiz, Y., Homs, R., Sturmfels, B.: Primary ideals and their differential equations. *Found. Comput. Math.* **21**, 1363–1399 (2021)
5. Daleo, N.S., Hauenstein, J.D.: Numerically deciding the arithmetically Cohen–Macaulayness of a projective scheme. *J. Symb. Comput.* **72**, 128–146 (2016)
6. de Graaf, W.: SLA: computing with simple Lie algebras. a GAP package
7. Dimca, A., Szendrői, B.: The Milnor fibre of the Pfaffian and the Hilbert scheme of four points on \mathbb{C}^3 . *Math. Res. Lett.* **16**, 1037–1055 (2009)
8. Duff, T., Hein, N., Sottile, F.: Certification for polynomial systems via square subsystems. *J. Symb. Comput.* **109**, 367–385 (2022)
9. Grayson, D.R., Stillman, M.E.: Macaulay2, a software system for research in algebraic geometry. <https://www.math.uiuc.edu/Macaulay2/>
10. Hauenstein, J.D., Haywood, I., Liddell, A.C.: An a posteriori certification algorithm for Newton homotopies. In: *Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation*, pp. 248–255. ACM (2014)
11. Hauenstein, J.D., Helmer, M.: Probabilistic saturations and Alt’s problem. *Exp Math.* <https://doi.org/10.1080/10586458.2020.1740835> (2020)
12. Hauenstein, J.D., Sottile, F.: Algorithm 921: alphaCertified: certifying solutions to polynomial systems. *ACM Trans. Math. Softw.* **38**, 28 (2012)
13. Katz, S.: The desingularization of $Hilb^4(\mathbb{P}^3)$ and its Betti numbers. In: Orecchia, F., Chiantini, L. (eds.) *Zero-Dimensional Schemes*, pp. 231–242. de Gruyter, Berlin, Boston (1994)
14. Kraśkiewicz, W., Weyman, J.: Algebra of coinvariants and the action of a Coxeter element. *Bayreuth. Math. Schr.* **63**, 265–284 (2001)
15. Ryan, T.: The geometry of Hilbert schemes of two points on projective space. [arXiv:2103.12674](https://arxiv.org/abs/2103.12674) (2021)

16. Stillman, M., Schenck, H., Raicu, C.: SchurRings, a Macaulay2 package
17. The GAP Group, GAP – Groups, Algorithms, and Programming. <https://www.gap-system.org>
18. Tikhomirov, S.A.: Punctual Hilbert schemes of small length in dimensions 2 and 3. *Math. Notes* **67**, 348–364 (2000)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.