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ORIGINAL ARTICLE

On a Diophantine Equation

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Abstract
This paper proves that for all positive integers n, the equation

$$\frac{x}{y} + p\frac{y}{z} + \frac{z}{w} + p\frac{w}{x} = 8np,$$

where $p = 1$ or p is a prime congruent to $1 \pmod 8$, does not have solutions in positive integers.

Keywords Diophantine equations · Hilbert symbol · Sum of fractions

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1 Introduction

The problem concerning the sum of rationals whose product is 1 has been studied by many authors. Cassels [5] showed that the equation

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 1$$

does not have solutions in integers. Bremner and Guy [3] found integer solutions to the equation

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = n$$

for many values of n in the range $|n| \leq 1000$. Sierpinski [6] asked if the equation

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 4$$

has solutions in positive integers? Bondarenko [1] showed that the equation

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 4k^2$$

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does not have solutions in positive integers if $3 \nmid k$. Using the technique developed by Bremner and Tho [4], which is based on Stoll's idea [8], we will prove the following results:

Theorem 1 *Let n be a positive integer. Then the equation*

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{w} + \frac{w}{x} = 8n$$

does not have solutions in positive integers.

Theorem 2 *Let n be a positive integer, p - a prime congruent to* 1 (mod 8). *Then the equation*

$$\frac{x}{y} + p\frac{y}{z} + \frac{z}{w} + p\frac{w}{x} = 8pn$$

does not have solutions in positive integers.

An equivalent form of Theorem 1 is that there are no four positive rationals whose product is 1 and sum is an integer divisible by 8. In the next section, we give a proof for Theorem 2. Theorem 1 can be proven in a similar (and simpler) way. All computations in the paper are done in Magma [2].

2 Proof of Theorem 2

2.1 Notation

For a prime q and a nonzero q-adic number a, denote $v_q(a)$ the highest power of q dividing a. By definition, $\mathbb{Q}_\infty = \mathbb{R}$. Let $k = \mathbb{Q}_q$ or $k = \mathbb{R}$. For a, b in k^*, the Hilbert symbol $(a, b)_q$ is defined by

$$(a, b)_q = \begin{cases} 1 & \text{if } ax^2 + by^2 = z^2 \text{ has a solution } (x, y, z) \neq (0, 0, 0) \text{ in } k^3, \\ -1 & \text{otherwise.} \end{cases}$$

When $k = \mathbb{Q}_\infty$, the symbol $(a, b)_\infty$ is defined similarly. The following properties of Hilbert symbol are true, see Serre [7, Chap. III]:

(i) For $a, b, c \in \mathbb{Q}_q^*$,

$$\begin{aligned} (a, bc)_q &= (a, b)_q (a, c)_q, \\ (a, b^2)_q &= 1. \end{aligned}$$

(ii) For $a, b \in \mathbb{Q}^*$,

$$(a, b)_\infty \prod_{q \text{ prime}} (a, b)_q = 1.$$

(iii) For $a, b \in \mathbb{Q}_q^*$, let $a = q^\alpha u$, $b = q^\beta v$, where $\alpha = v_q(a)$ and $\beta = v_q(b)$. Then

$$\begin{aligned} (a, b)_q &= (-1)^{\alpha\beta(q-1)/2} \left(\frac{u}{q}\right)^\beta \left(\frac{v}{q}\right)^\alpha, \quad &\text{if } q \neq 2, \\ (a, b)_q &= (-1)^{\frac{(u-1)(v-1)}{4} + \frac{\alpha(v^2-1)}{8} + \frac{\beta(u^2-1)}{8}}, \quad &\text{if } q = 2, \end{aligned}$$

where $\left(\frac{u}{q}\right)$ denotes the Legendre symbol.

2.2 Proof

Assume that (x, y, z, w) is a positive integer solution to

$$\frac{x}{y} + p\frac{y}{z} + \frac{z}{w} + p\frac{w}{x} = 8pn \tag{1}$$

with $\gcd(x, y, z, w) = 1$.

Consider two quadratic forms:

$$D(X, Z) = X^2 + Z^2 - 2XZ(8pn^2 - 1),$$
$$H(X, Z) = X^2 + Z^2 - 2XZ(8pn^2 + 1).$$

Lemma 1

$$D(x, z) < 0, \quad H(y, w) < 0.$$

Proof From (1) and the AM-GM inequality, we have

$$8pn = \left(\frac{x}{y} + \frac{pw}{x}\right) + \left(\frac{py}{z} + \frac{z}{w}\right) \geq 2\sqrt{\frac{x}{y}\frac{pw}{x}} + 2\sqrt{\frac{py}{z}\frac{z}{w}} = 2\sqrt{p}\frac{y + w}{\sqrt{yw}}.$$

Thus,

$$4n\sqrt{pyw} \geq y + w.$$

Hence,

$$y^2 - 2(8pn^2 - 1)yw + w^2 \leq 0.$$

Similarly,

$$8np = \left(\frac{x}{y} + \frac{py}{z}\right) + \left(\frac{z}{w} + \frac{pw}{x}\right) \geq 2\left(\sqrt{\frac{x}{y}\frac{py}{z}} + \sqrt{\frac{z}{w}\frac{pw}{x}}\right) = 2\sqrt{p}\frac{x + z}{\sqrt{xz}}.$$

Thus,

$$4n\sqrt{pxz} \geq x + z.$$

Hence,

$$x^2 - 2(8pn^2 - 1)xz + z^2 \leq 0.$$

Since $(8pn^2 - 1)^2 - 1$ is not a perfect square, we have $y^2 - 2(8pn^2 - 1)yw + w^2 < 0$ and $x^2 - 2(8pn^2 - 1)xz + z^2 < 0$. Hence $D(x, z) < 0$ and $H(y, w) < D(y, w) < 0$. □

From (1):

$$x^2zw + py^2wx + z^2xy + pw^2yz - 8npxyzw = 0. \tag{2}$$

Fix x, z and consider the projective curve $F_{x,z}(Y, W, d) = 0$, where

$$F_{x,z}(Y, W, d) = pxWY^2 + pW^2Yz + (xz^2Y + x^2zW)d^2 - 8npxzYWd.$$

Then, $F_{x,z}(y, w, 1) = F_{x,z}(0, 1, 0) = 0$. So, $F_{x,z}(Y, W, d) = 0$ is isomorphic to the elliptic curve

$$C_{x,z}\colon \omega^2 = u(u^2 + pxz(16n^2pxz - x^2 - z^2)u + p^2x^4z^4) \tag{3}$$

via the rational maps $\phi\colon F_{x,z} \to C_{x,z}$,

$$\phi(Y:W:d) = \left(\frac{-x^2z^2Wp}{Y}, \frac{x^2z^2W(4nxzd - xY - zW)}{Yd}\right),$$

and $\psi\colon C_{x,z} \to F_{x,z}$,

$$\psi(u,\omega) = \big(px^2z^2(4nxzu + p\omega) : -u(4nxznu + p\omega) : zu(u - px^3z)\big).$$

Let $D = D(x,z)$, $H = H(x,z)$. Let

$$\begin{aligned} A &= pxz(16n^2pxz - x^2 - z^2), \\ B &= p^2x^4z^4. \end{aligned}$$

Lemma 2 *If q is an odd prime, then*

$$(u, D)_q = 1.$$

Proof Let $d = \gcd(x,z)$, $x = dx_1$, $y = dy_1$, where $x_1, z_1 \in \mathbb{Z}^+$ with $\gcd(x_1, z_1) = 1$, $u_1 = \frac{u}{d^4}$ and $\omega_1 = \frac{\omega}{d^6}$. Then

$$\begin{aligned} (D(x,z), u)_q &= (d^2(x_1^2 + z_1^2 - 2(8pn^2 - 1)x_1z_1), d^4u_1)_q \\ &= (D(x_1, z_1), u_1)_q. \end{aligned}$$

From (3), we also have

$$\omega_1^2 = u_1\left(u_1^2 + px_1z_1(16pn^2x_1z_1 - x_1^2 - z_1^2)u_1 + p^2x_1^4z_1^4\right).$$

Therefore, it is enough to prove Lemma 2 in the case $\gcd(x,z) = 1$.

Let $u = q^ru_0$, where $r = v_q(u)$. We consider the following cases:

Case 1: $r < 0$. From (3), we have

$$\omega^2 = q^{3r}u_0(u_0^2 + q^{-r}Au_0 + q^{-2r}B).$$

Thus

$$3r = v_q(\omega^2) = 2v_q(\omega).$$

Therefore, $2 \mid r$. Now

$$(q^{-3r/2}\omega)^2 = u_0(u_0^2 + q^{-r}Au_0 + q^{-2r}B).$$

Taking reduction (mod p) shows u_0 is a square (mod q). Thus $u_0 \in \mathbb{Z}_q^2$. Because r is even, we have $u = q^ru_0 \in \mathbb{Q}_q^2$. Hence $(D,u)_q = 1$.

Case 2: $r = 0$. Let $D = q^kD_1$, where $k = v_q(D)$. Because $D \in \mathbb{Z}$, we have $k \geq 0$. Suppose $2 \mid k$. Because u and D_1 are units in $\mathbb{Z}_q$, we have $(u, D_1)_q = 1$. Thus

$$(u, D)_q = (u, q^kD_1)_q = (u, D_1)_q = 1.$$

We consider the case $2 \nmid k$. Then,

$$x^2 + z^2 \equiv 2(8pn^2 - 1)xz \pmod{q}.$$

Hence,

$$(x+z)^2 \equiv 16pn^2xz \pmod{q}.$$

We have

$$u^2 + Au + B = \left(u + \frac{A}{2}\right)^2 - \frac{p^2x^2z^2HD}{4}$$
$$\equiv \left(u + \frac{A}{2}\right)^2 \pmod q.$$

(i) Suppose $q \nmid u + \frac{A}{2}$. Then $u^2 + Au + B \in \mathbb{Z}_q^2$. Because $\omega^2 = u(u^2 + Au + B)$, we have $u \in \mathbb{Z}_q^2$. Thus $(u, D)_q = 1$.

(ii) Suppose $q \mid u + \frac{A}{2}$. Thus $u \equiv -\frac{A}{2} \pmod q$. Therefore, $q \nmid A$. Because $q \mid D$, we have

$$A = pxz(2xz - D) \equiv 2px^2z^2 \pmod q.$$

Because $q \nmid u$ and $q \nmid A$, we have $q \nmid 2px^2z^2$. Now $q \nmid 2pxz$, so $\gcd(D, H) = 1$. Let $S = u + \frac{A}{2}$ and $T = \frac{HD}{4}$. Because $\gcd(H, D) = 1$ and $q \mid D$, we have $v_q(T) = v_q(D) = k$. Let $S = q^l S_1$, $T = q^k T_1$, where $l = v_q(S)$. From $\omega^2 = u(u^2 + Au + B)$ and $q \nmid u$, we have

$$v_q(u^2 + Au + B) = 2v_q(\omega).$$

Thus $2 \mid v_q(u^2 + Au + B)$. On the other hand,

$$v_q(u^2 + Au + B) = v_q(S^2 + T) = v_q(q^{2l}S_1^2 + q^kT_1).$$

Since $2 \nmid k$, we must have $2l < k$. Thus,

$$u^2 + Au + B = q^{2l}(S_1^2 + q^{k-2l}T_1) \in \mathbb{Q}_q^2.$$

Hence, $u = \frac{\omega^2}{u^2+Au+B} \in \mathbb{Q}_q^2$. So $(u, D)_q = 1$.

Case 3: $r > 0$.

(a) Suppose $q \nmid pxz$. Since $(u^2 + Au + B)\alpha^2 + D\beta^2 = \gamma^2$ has a solution $(1, 0, px^2z^2) \pmod q$, it has a nontrivial solution in $\mathbb{Q}_q$. Therefore,

$$(u^2 + Au + B, D)_q = 1.$$

Because $u(u^2 + Au + B) = \omega^2 \neq 0$, we have $(u, D)_q = 1$.

(b) Suppose $q \mid xz$. Then $q \mid x$ or $q \mid z$. If $q \mid x$, then

$$D = x^2 + z^2 - 2p(8n^2 - 1)xz \equiv z^2 \pmod q.$$

Note that $\gcd(x, z) = 1$, thus $D \in \mathbb{Z}_q^2$. Therefore, $(u, D)_q = 1$. Similarly, we also have $(u, D)_q = 1$ if $q \mid z$.

(c) Suppose $q \nmid xz$ and $q = p$. Then $u = p^r u_0$. So

$$\omega^2 = p^r u_0(p^{2r}u_0^2 + Ap^r u_0 + p^2x^4z^4).$$

We have two subcases:

(i) $r \geq 2$. Then $2v_p(\omega) = r + 2$. Thus $2 \mid r$. We now have

$$(\omega p^{-r/2})^2 = u_0(p^{2r-2}u_0^2 + Ap^{r-2}u_0 + x^4z^4).$$

Because $p \mid A$, a reduction $\pmod p$ gives $u_0x^4z^4$ is a square $\pmod p$. Therefore, $u_0 \in \mathbb{Z}_q^2$. Thus,

$$(u, D)_p = (2^s u_0, D)_p = 1.$$

(ii) $r = 1$. Then

$$\begin{aligned}\omega^2 &= pu_0(p^2u_0^2 + pAu_0 + p^2x^4z^4)\\ &= p^3u_0(u_0^2 + xz(16pn^2xz - x^2 - z^2)u_0 + x^4z^4).\end{aligned}$$

Thus p divides

$$u_0^2 + xz(-x^2 - z^2)u_0 + x^4z^4 = (u_0 - x^3z)(u_0 - xz^3).$$

Therefore,

$$u_0 \equiv x^3z \pmod p \qquad \text{or} \qquad u_0 \equiv xz^3 \pmod p.$$

If $u_0 \equiv x^3z \pmod p$, then

$$\begin{aligned}(pu_0, p)_p &= (-1)^{(p-1)/2}\left(\frac{u_0}{p}\right) = \left(\frac{x^3z}{p}\right)\\ &= \left(\frac{-x^4}{p}\right) = \left(\frac{-1}{p}\right)\\ &= (-1)^{\frac{p-1}{2}}\\ &= 1.\end{aligned}$$

Similarly, if $xz^3 \equiv 0 \pmod p$, then $(pu_0, p)_p = 1$. So it is always true that

$$(pu_0, p)_p = 1. \tag{4}$$

Now

$$D = (x+z)^2 - 16pn^2xz \equiv (x+z)^2 \pmod p.$$

Suppose $p \nmid x + z$. Then $D \in \mathbb{Z}_p^2$. Hence $(u, D)_p = 1$. We consider the case $p \mid x + z$. Let $x + z = p^s f$, where $s = v_p(x + y) > 0$. Then

$$D = p(p^{2s-1}f^2 - 16n^2xz).$$

– Suppose that $p \nmid n$, then

$$p^{2s-1}f^2 - 16n^2xz \equiv (4nx)^2 \pmod p.$$

Hence, $p^{2s-1}f^2 - 16n^2xz \in \mathbb{Z}_p^2$. Let $D = pD_1^2$, where $D_1 \in \mathbb{Z}_p$. Then from (4),

$$(u, D)_p = (pu_0, pD_1^2)_p = (pu_0, p)_p = 1.$$

– Suppose that $p \mid n$. Let $n = p^t n_1$, where $t = v_p(n) > 0$. Then

$$D = p^{2s}f^2 - 16p^{2t+1}xz.$$

If $s \le t$, then

$$D = p^{2s}(f^2 - 16p^{2t+1-2s}n_1^2xz).$$

Thus $D \in \mathbb{Z}_p^2$. Hence $(u, D)_p = 1$. We consider the case $s > t$. Then

$$D = p^{2t+1}(p^{2s-2t-1}f^2 - 16n_1^2xz).$$

Because

$$p^{2s-2t-1}f^2 - 16n_1^2xz \equiv 16n_1^2x^2 \pmod p,$$

we have $D = p^{2s+1}D_2^2$, where $D_2 \in \mathbb{Z}_p$. From (4), we have

$$(u, D)_p = (pu_0, p^{2r+1}D_2^2)_p = (pu_0, p)_p = 1.$$

□

Lemma 3 *If* $4 \nmid x+z$, *then*

$$(D,u)_2 = 1.$$

Proof Let $d = \gcd(x,z)$, let $x = dx_1$, $y = dy_1$, where $x_1, z_1 \in \mathbb{Z}^+$ with $\gcd(x_1,z_1) = 1$, let $u_1 = \frac{u}{d^4}$ and $\omega_1 = \frac{\omega}{d^6}$. Then

$$\begin{aligned}(D(x,z),u)_q &= \left(d^2(x_1^2+z_1^2-2(8pn^2-1)x_1z_1), d^4u_1\right)_q \\ &= \left(D(x_1,z_1),u_1\right)_q.\end{aligned}$$

From (3), we also have

$$\omega_1^2 = u_1\left(u_1^2 + px_1z_1(16pn^2x_1z_1 - x_1^2 - z_1^2) + p^2x_1^4z_1^4\right).$$

Of course, $4 \nmid x_1+z_1$ if $4 \nmid x+z$. Therefore, it is enough to prove Lemma 3 in the case $\gcd(x,z) = 1$.

If $2 \nmid x+z$, then

$$D = (x+z)^2 - 16pn^2xz \equiv 1 \pmod 8.$$

So $D \in \mathbb{Z}_2^2$, hence $(D,u)_2 = 1$. Let us consider the case $2 \mid x+z$. Because $4 \nmid x+z$, we can write $x+z = 2h$ with $2 \nmid h$. Then

$$D = 4(h^2 - 4pn^2xz).$$

We consider two cases:

(a) $2 \mid n$. Then $h^2 - 4pn^2xz \equiv 1 \pmod 8$. Thus $D \in \mathbb{Z}_2^2$. Hence $(D,u)_2 = 1$.

(b) $2 \nmid n$. Then $pn^2xz \equiv 1 \pmod 4$, and so $h^2 - 4pn^2xz \equiv 5 \pmod 8$. Thus $D = 4D_1$, where $D_1 \equiv 5 \pmod 8$.

Let $u = 2^r u_1$, where $r = v_2(u)$. Then

$$\omega^2 = 2^r u_1(2^{2r}u_1^2 + 2^r Au_1 + B).$$

(i) Suppose $r \geq 3$. Then $r = 2v_2(\omega)$. We have

$$(2^{-r/2}\omega)^2 = u_1(2^{2r}u_1^2 + 2^r Au_1 + B).$$

Because

$$\begin{aligned}2^{2r}u_1^2 + 2^r Au_1 + B &\equiv B \pmod 8 \\ &\equiv p^2x^4z^4 \pmod 8 \\ &\equiv 1 \pmod 8,\end{aligned}$$

we have $u_1 \equiv 1 \pmod 8$. Thus $u_1 \in \mathbb{Z}_2^2$, so $u = 2^r u_1 \in \mathbb{Z}_2^2$. Hence $(u,D)_2 = 1$.

(ii) Suppose $r = 2$. Then

$$\left(\frac{\omega}{2}\right)^2 = u_1(2^4u_1^2 + 2^2Au_1 + B).$$

Taking a reduction (mod 8) gives $u_1 \equiv 1 \pmod 8$. Therefore, $u = 2^2u_1 \in \mathbb{Z}_2^2$. Hence $(u,D)_2 = 1$.

(iii) Suppose $r = 1$. Then

$$\omega^2 = 2u_1(4u_1^2 + 2Au_1 + B),$$

what is impossible (mod 2).

(iv) Suppose $r = 0$. Then $u = u_1$ and $D = 2^2 D_1$, where $D_1 \equiv 5 \pmod 8$. Therefore,

$$\begin{aligned}(u, D)_2 &= (u_1, 2^2 D_1)_2\\ &= (u_1, D_1)_2\\ &= (-1)^{(u_1-1)(D_1-1)/4}\\ &= 1.\end{aligned}$$

(v) Suppose $r < 0$. Then

$$\omega^2 = 2^{3r} u_1 \left(u_1^2 + 2^{-r} A u_1 + 2^{-2r} B\right).$$

Therefore, $3r = 2v_2(\omega)$. Thus $r \leq -2$. Then

$$\left(2^{-3r/2}\omega\right)^2 = u_1 \left(u_1^2 + 2^{-r} A u_1 + 2^{-2r} B\right).$$

Note that $2 \mid A$, so taking a reduction (mod 8) gives $u_1 \equiv 1 \pmod 8$. Thus $u_1 \in \mathbb{Z}_2^2$, so $u \in \mathbb{Z}_2^2$. Hence $(u, D)_2 = 1$.

□

Lemma 4 *If $4 \nmid x + z$, then*

$$(u, D)_\infty = 1.$$

Proof Since $(D, u)_q = 1$ for all prime q, and

$$(D, u)_\infty \prod_{q \text{ prime}, q<\infty} = 1,$$

we have

$$(u, D)_\infty = 1.$$

□

Lemma 5 *If q is an odd prime, then*

$$(H, u)_q = 1.$$

Proof Because $A^2 - 4B = p^2 x^2 z^2 DH$, we have

$$\omega^2 = u\left(\left(u + \frac{A}{2}\right)^2 - DH\left(\frac{pxz}{2}\right)^2\right).$$

So

$$u\alpha^2 - uDH\beta^2 = \omega^2,$$

where $\alpha = u + \frac{A}{2}$ and $\beta = \frac{pxz}{2}$. Thus

$$(u, -uHD)_q = 1.$$

On the other hand, $(u, -u)_q = 1$ and $(u, D)_q = 1$; therefore,

$$(u, H)_q = 1.$$

□

Lemma 6 *If* $4 \nmid x - z$*, then*

$$(u, H)_2 = 1.$$

Proof Similar to the proof of Lemma 3, it is enough to consider the case $\gcd(x, z) = 1$. Suppose $2 \nmid x - z$, then

$$H = (x - z)^2 - 16pn^2xz \equiv 1 \pmod 8.$$

Thus $H \in \mathbb{Z}_2^2$, and hence $(u, H)_2 = 1$. We consider the case $2 \mid x - z$. Because $4 \nmid x - z$, let $x - z = 2k$, where $2 \nmid k$. Then

$$H = 4(k^2 - 4pn^2xz).$$

We consider two cases:

(a) $2 \mid n$. Then $k^2 - 4pn^2xz \equiv 1 \pmod 8$. Thus $H \in \mathbb{Z}_2^2$ and $(u, H)_2 = 1$.

(b) $2 \nmid n$. Then $4pn^2xz \equiv 4 \pmod 8$. Thus $k^2 - 4pn^2xz \equiv 5 \pmod 8$, so $H = 4H_1$, where $H_1 \equiv 5 \pmod 8$. Let $u = 2^r u_1$, where $r = v_2(u)$. Then

$$\omega^2 = 2^r u_1 (2^{2r} u_1^2 + 2^r A u_1 + B).$$

(i) Suppose $r \geq 3$. Then $r = 2v_2(\omega)$. Thus $2 \mid r$. We have

$$(2^{-r/2}\omega)^2 = u_1(2^{2r}u_1^2 + 2^r A u_1 + B).$$

Taking reduction (mod 8) gives $2^{2r}u_1^2 + 2^r A u_1 + B \equiv 1 \pmod 8$. Hence $u_1 \in \mathbb{Z}_2^2$, thus $u = 2^r u_1 \in \mathbb{Z}_2^2$. So $(u, H)_2 = 1$.

(ii) Suppose $r = 2$. Then

$$\left(\frac{\omega}{2}\right)^2 = u_1(2^4 u_1^2 + 2^2 A u_1 + B).$$

Taking reduction (mod 8) gives $u_1 \equiv 1 \pmod 8$. Therefore, $u \in \mathbb{Z}_2^2$. Hence $(u, H)_2 = 1$.

(iii) Suppose $r = 1$. Then

$$\omega^2 = 2u_1(4u_1^2 + 2Au_1 + B),$$

what is impossible (mod 2).

(iv) Suppose $r = 0$. Then $u = u_1$ and $H = 2^2 H_1$, where $H_1 \equiv 5 \pmod 8$. Therefore,

$$\begin{aligned}(u, H)_2 &= (u_1, 2^2 H_1)_2 \\ &= (u_1, H_1)_2 \\ &= (-1)^{(u_1-1)(H_1-1)/4} \\ &= 1.\end{aligned}$$

(v) Suppose $r < 0$. Then

$$\omega^2 = \frac{u_1(u_1^2 + 2^{-r}Au_1 + 2^{-2r}B)}{2^{-3r}}.$$

Therefore, $2 \mid r$. Thus $r \leq -2$. Now

$$(2^{3r/2}\omega)^2 = u_1(u_1^2 + 2^{-r}Au_1 + 2^{-2r}B).$$

Note that $2 \mid A$, so taking (mod 8) gives $u_1 \equiv 1 \pmod 8$. Therefore, $u_1 \in \mathbb{Z}_2^2$, so $u \in \mathbb{Z}_2^2$. Thus $(u, H)_2 = 1$.

□

Lemma 7 *If* $4 \nmid x - z$, *then*

$$(u, H)_\infty = 1.$$

Proof Since $(u, H)_q = 1$ for all primes q and

$$(u, H)_\infty \prod_{q \text{ prime}} (u, H)_q = 1,$$

we have

$$(u, H)_\infty = 1.$$

□

We summarize Lemmas 2, 3, 4, 5, 6, and 7 into the following proposition:

Proposition 1

- $(D(x, z), u)_q = (H(x, z), u)_q = 1$ *if* q *is an odd prime or* $q = \infty$.
- $(D(x, z), u)_2 = (D(x, z), u)_\infty = 1$ *if* $4 \nmid x + z$.
- $(H(x, z), u)_2 = (H(x, z), u)_\infty = 1$ *if* $4 \nmid x - z$.

In order for (2) to have a positive integer solutions, we seek for points (u, ω) on $C_{x,z}$ such that $\psi(u, \omega) = (Y : W : d)$ satisfies $d \neq 0$, $\frac{Y}{d} > 0$ and $\frac{W}{d} > 0$. If $u = 0$, then $\omega = 0$. Because $\psi(0, 0) = (1 : 0 : 0)$, we have $u \neq 0$. Therefore,

$$\begin{cases} \dfrac{px^2z(4nxzu + p\omega)}{u(u - px^3z)} > 0, \\ -\dfrac{4xznu + p\omega}{u - px^3z} > 0. \end{cases}$$

Multiplying two inequalities gives $u < 0$. Let

$$(u, \omega) = \phi(y : w : 1) = \left(\frac{-x^2z^2wp}{y}, \frac{x^2z^2w(4nxz - xy - zw)}{y} \right). \tag{5}$$

If $\omega \neq 0$, then we consider the following cases:

- $4 \nmid x + z$. From Proposition 1, we have $(D(x, z), u)_\infty = 1$. Because $D(x, z) < 0$, by Lemma 1 we have $u > 0$, contradicting $u < 0$.
- $4 \nmid x - z$. From Proposition 1, we have $(H(x, z), u)_\infty = 1$. Because $H(x, z) < 0$, by Lemma 1 we have $u > 0$, contradicting $u_0 < 0$.
- $4 \mid x + z$ and $4 \mid x - z$. Let $x = 2x_1$ and $z = 2z_1$ with $x_1, z_1 \in \mathbb{Z}$ and $2 \nmid x_1, z_1$. Then $4 \nmid x_1 + z_1$ or $4 \nmid x_1 - z_1$. From $\omega^2 = u(u^2 + Au + B)$, we have

$$\left(\frac{\omega}{2^6}\right)^2 = \frac{u}{2^4}\left(\left(\frac{u}{2^4}\right)^2 + A_1\left(\frac{u}{2^4}\right) + B_1\right),$$

where $A_1 = px_1z_1(16pn^2x_1z_1 - x_1^2 - z_1^2)$ and $B_1 = p^2x_1^4z_1^4$. Now $4 \nmid x_1 - z_1$ or $4 \nmid x_1 + z_1$, so we have $\left(D(x_1, z_1), \frac{u}{2^4}\right)_2 = 1$ or $\left(H(x_1, z_1), \frac{u}{2^4}\right)_2 = 1$. In addition,

$$(D(x, z), u)_2 = \left(2^2 D(x_1, z_1), 2^4 \frac{u}{2^4}\right)_2 = \left(D(x_1, z_1), \frac{u}{2^4}\right)_2.$$

Similarly

$$(H(x, z), u)_2 = \left(H(x_1, z_1), \frac{u}{2^4}\right)_2.$$

So $(D(x, z), u)_2 = 1$ or $(H(x, z), u)_2 = 1$. Hence $u > 0$, contradicting $u < 0$.

Therefore, $\omega = 0$. From (5), we have

$$4nxz - xy - zw = 0. \tag{6}$$

Thus

$$\frac{y}{z} + \frac{w}{x} = 4n.$$

Hence

$$\frac{x}{y} + \frac{z}{w} = 4np. \tag{7}$$

Now fix y, w and consider the equation $F_{y,w}(X, Z, d) = 0$, where

$$F_{y,w}(X, Z, d) = X^2yZ + py^2Zwd^2 + Z^2wX + pw^2Xyd^2 - 8npXZdyw.$$

Then $F_{y,w}(0, 1, 0) = F_{y,w}(x, z, 1) = 0$. So $F_{y,w}(X, Z, d) = 0$ is isomorphic to the elliptic curve

$$C_{y,w}\colon \omega'^2 = u'\left(u'^2 + pyw\left(16n^2pyw - y^2 - w^2\right)u' + p^2y^4w^4\right),$$

via the rational maps $\alpha\colon F_{y,w} \to C_{y,w}$,

$$\alpha(X : Z : d) = \left(\frac{-w^2pZy^2}{X}, \frac{-py^2w^2Z(Xy + Zw - 4npywd)}{Xd}\right),$$

and $\beta\colon E_{y,w} \to F_{y,w}$,

$$\beta(u', \omega') = \left(pw^2(4npywu' + \omega') : -u'(4npywu' + \omega') : wu'(u' - py^3w)\right).$$

We have the following result:

Proposition 2

- $(D(y, w), u')_q = (H(y, w), u')_q = 1$ *if q is an odd prime.*
- $(D(y, w), u')_2 = (D(y, w), u')_\infty = 1$ *if $4 \nmid y + w$.*
- $(H(y, w), u')_2 = (H(y, w), u')_\infty = 1$ *if $4 \nmid y - w$.*

Proof The same as for Proposition 1. □

In order for (2) to have positive integer solutions, we seek for points (u', ω') on $C_{y,w}$ such that $\beta(u', \omega') = (X : Z : d)$ satisfies $d \neq 0$, $\frac{X}{d} > 0$ and $\frac{Z}{d} > 0$. If $u = 0$, then $\omega = 0$. Because $\beta(0, 0) = (1 : 0 : 0)$. We must have $u \neq 0$. So

$$\begin{cases} \dfrac{pw(4npywu' + \omega')}{u'(u' - py^3w)} > 0, \\ -\dfrac{4npywu' + \omega'}{w(u' - py^3w)} > 0. \end{cases}$$

Multiplying together the two inequalities gives $u' < 0$. Assume

$$(u', \omega') = \alpha(x : z : 1) = \left(\frac{-py^2w^2z}{x}, \frac{-py^2w^2z(yx + wz - 4npyw)}{x} \right). \tag{8}$$

If $\omega' \neq 0$, we then consider the following cases:

- $4 \nmid y + w$. From Proposition 2, we have $(D(y, w), u')_\infty = 1$, thus $u' > 0$ because $D(y, w) < 0$ by Lemma 1. This contradicts $u' < 0$.
- $4 \nmid y - w$. From Proposition 2, we have $(H(y, w), u')_\infty = 1$. Because $H(y, w) < 0$, we have $u' > 0$, contradicting $u' < 0$.
- $4 \mid y + w$ and $4 \mid y - w$. Then $y = 2y_1$ and $w = 2w_1$, where $2 \nmid y_1, w_1$. Then $4 \nmid y_1 + w_1$ or $4 \nmid y_1 - w_1$. Then similar to the case $4 \nmid x + z$ and $4 \nmid x - z$, we have $(D(y, w), u')_\infty = 1$ or $(H(y, w), u')_\infty = 1$; the either case implies $u' > 0$, which contradicts $u' < 0$.

Therefore, $\omega' = 0$. From (8), we have

$$xy + zw - 4npyw = 0. \tag{9}$$

From (6) and (9), we have

$$4nxz = 4npyw.$$

Thus,

$$\frac{x}{y}\frac{z}{w} = p. \tag{10}$$

From (7) and (10), we have

$$(4np)^2 - 4p = \left(\frac{x}{y} - \frac{z}{w} \right)^2.$$

Thus $4n^2p^2 - p \in \mathbb{Q}^2$, hence $4n^2p^2 - p \in \mathbb{Z}^2$, impossible because $p^2 \nmid 4n^2p^2 - p$. Therefore, there are no positive integer solutions to (1).

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