Vietnam Journal of Mathematics (2022) 50:261–274
https://doi.org/10.1007/s10013-021-00502-x

ORIGINAL ARTICLE

On Generalized Inverses of $m \times n$ Matrices Over a Pseudoring

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Received: 20 August 2020 / Accepted: 23 December 2020 / Published online: 19 May 2021

Abstract

A generalized inverse of an $m \times n$ matrix A over a pseudoring means an $n \times m$ matrix G satisfying $AGA = A$. In this paper we give a characterization of matrices having generalized inverses. Also, we introduce and study a space decomposition of a matrix, and prove that a matrix is decomposable if and only if it has a generalized inverse. Finally, we establish necessary and sufficient conditions for a matrix to possess various types of g-inverses including Moore–Penrose inverse.

Keywords Idempotent matrix · Regular matrix · Generalized inverse · Space decomposition

Mathematics Subject Classification (2010) 15B99 · 06E75 · 15A09 · 15A23

1 Introduction

Recall that [12] a pseudoring is a triple $(D, +, \cdot)$ where D is a completely ordered set with a minimal element $\mathbf{0}$; $+$ and $\cdot$ are two binary operations on D satisfying:

$$a + b = \max\{a, b\},$$

$(D^*, \cdot)$ is a group with identity element $\mathbf{1}$ (where $D^* = D \setminus \mathbf{0}$);

$$(\forall a \in D) \quad \mathbf{0} \cdot a = a \cdot \mathbf{0} = \mathbf{0};$$

“ · ” is distributive with respect to “+”.

The following algebras are all pseudorings:

(1) $(\mathbb{R} \cup \{-\infty\}, \max, +)$ (tropical algebra);
(2) $(\mathbb{Z} \cup \{-\infty\}, \max, +)$ (extended integers);

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(3) $(\mathbb{Q} \cup \{-\infty\}, \max, +)$(extended integers);
(4) $(\mathbb{R}_+ \cup \{0\}, \max, \cdot)$ (max algebra);
(5) $(\mathbb{B}, +, \cdot)$ (2-element Boolean algebra).

Matrix theory over the above pseudorings has broad applications in combinatorial optimization, control theory, automata theory and many other areas of science (see [2–4, 6, 8]). As usual, the set of all $m \times n$ matrices over a pseudoring D is denoted by $M_{m\times n}(D)$. In particular, we will use $M_n(D)$ instead of $M_{n\times n}(D)$. The operations "$+$" and "$\cdot$" on D induce corresponding operations on matrices in the obvious way. It is easy to see that $(M_n(D), \cdot)$ is a semigroup. For brevity, we will write AC in place of $A \cdot C$. Unless otherwise stated, we refer to matrix as a matrix over a pseudoring in the remainder of this paper.

For a matrix A in $M_{m\times n}(D)$, suppose that there exists a matrix G in $M_{n\times m}(D)$ such that $AGA = A$. Then A is called regular, and call G a generalized inverse of A. There are a series of papers in the literature considering the structure of regular matrices over some algebraic systems. In 1975, Rao and Rao [11] study the generalized inverses of Boolean matrices. In 1981, Hall and Katz [7] characterize the generalized inverses of nonnegative integer matrices. In 1998, Bapat [1] obtains the generalized inverses of nonnegative real matrices. And in 2015, Kang and Song [10] determine the general form of matrices over the max algebra having generalized inverses.

The main purpose of this paper is to study regular matrices and their generalized inverses over a pseudoring. The forms of regular matrices in this paper are extended the corresponding results in [10] and [11]. This paper will be divided into five sections. In Section 2 we introduce some preliminary notions and notation. A normal form of an idempotent matrix is given in Section 3. After characterizing idempotent matrices, we obtain the general form of matrices which have generalized inverses. In Section 4 we define a space decomposition of a matrix and prove that a matrix has a g-inverse if and only if it has a space decomposition. In Section 5 we establish necessary and sufficient conditions for the existence of the Moore–Penrose inverse and other types of g-inverses of a matrix.

2 Preliminaries

The following notation of linear dependence can be found in [12]. We will be interested in the space D^n consisting of n-tuples x with entries in D. We write x_i for the ith component of x. The D^n admits an addition and a scaling action of D given by $(x + y)_i = x_i + y_i$ and $(\lambda x)_i = \lambda x_i$, respectively. These operations give D^n the structure of an D-pseudomodule. Each element of this pseudomodule is called a vector. A vector α in D^n is called a *linear combination* of a subset $\{\alpha_1, \ldots, \alpha_k\}$ of D^n, if there exist $d_1, \ldots, d_k \in D$ such that

$$\alpha = d_1\alpha_1 + \cdots + d_k\alpha_k.$$

For a subset S of D^n, let span(S) denote

$$\left\{\sum_{i=1}^{k} d_i\alpha_i \mid k \in \mathbb{N}, \alpha_i \in S, d_i \in D, i = 1, 2, \ldots, k\right\},$$

where $\mathbb{N}$ denotes the set of all natural numbers. The set S is called *linearly dependent* if there exists a vector $\alpha \in S$ such that α is a linear combination of elements in $S \setminus \{\alpha\}$. Otherwise, S is called *linearly independent*. A subset $\{\alpha_i \mid i \in I\}$ of a subpseudomodule $\mathcal{V}$ of D^n is called a *basis* of $\mathcal{V}$ if span$(\{\alpha_i \mid i \in I\}) = \mathcal{V}$ and $\{\alpha_i \mid i \in I\}$ is linearly independent.

In the sequel, the following notions and notation are needed for us.

- I_n denotes the *identity matrix*, i.e., the diagonal entries are all $\mathbf{1}$ and the other entries are all $\mathbf{0}$.
- An $n \times n$ matrix A is called to be *invertible* if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. In this case, B is called an inverse of A and is denoted by A^{-1}.
- An $n \times n$ matrix is called a *monomial matrix* if it has exactly one entry in each row and column which is not equal to $\mathbf{0}$.
- An $n \times n$ matrix is called a *permutation matrix* if it is formed from the identity matrix by reordering its columns and/or rows.
- O denotes the *zero matrix*, i.e., the matrix whose entries are all 0.

Proposition 2.1 *A matrix A is invertible if and only if A is a monomial matrix.*

Proof Suppose that $A = (a_{ij})_{n\times n}$ is an invertible matrix. Then there exists a matrix $B = (b_{ij})_{n\times n}$ such that

$$AB = BA = (c_{ij})_{n\times n} = I_n.$$

Thus

$$a_{i1}b_{1i} + a_{i2}b_{2i} + \cdots + a_{in}b_{ni} = c_{ii} = \mathbf{1}.$$

It follows that there exists $1 \leq k \leq n$, such that $a_{ik}b_{ki} = \mathbf{1}$, and so

$$a_{ik} = (b_{ki})^{-1} \neq \mathbf{0}. \tag{2.1}$$

Since for any $j \neq i$,

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} + \cdots + a_{in}b_{nj} = \mathbf{0}, \\ c_{ji} &= a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jk}b_{ki} + \cdots + a_{jn}b_{ni} = \mathbf{0}, \end{aligned}$$

we have

$$b_{kj} = \mathbf{0} \quad \text{and} \quad a_{jk} = \mathbf{0}. \tag{2.2}$$

Since for any $k \neq j$,

$$\begin{aligned} c_{kj} &= b_{k1}a_{1j} + b_{k2}a_{2j} + \cdots + b_{ki}a_{ij} + \cdots + b_{kn}a_{nj} = \mathbf{0}, \\ c_{jk} &= b_{j1}a_{1k} + b_{j2}a_{2k} + \cdots + b_{ji}a_{ik} + \cdots + b_{jn}a_{nk} = \mathbf{0}, \end{aligned}$$

then

$$a_{ij} = \mathbf{0} \quad \text{and} \quad b_{ji} = \mathbf{0}. \tag{2.3}$$

By (2.1), (2.2) and (2.3), we obtain that A is a monomial matrix.

It is easy to see that the converse is true. □

A permutation matrix is also a monomial matrix. And the inverse of a permutation matrix is its transpose.

Let A be an $m \times n$ matrix. The *column space* (*row space*, resp.) of an $m \times n$ matrix A is the subpseudomodule of D^m (D^n, resp.) spanned by all its columns (rows, resp.) and is denoted by $\mathrm{Col}(A)$ ($\mathrm{Row}(A)$, resp.). By Corollary 4.7 in [12], we know that every finitely generated pseudomodule has a basis. For an $m \times n$ matrix A, let $\mathbf{a}_{i*}$ and $\mathbf{a}_{*j}$ denote the ith row and the jth column of A, respectively. As a consequence, by Theorem 5 in [12] we immediately have

Lemma 2.2 *Let $\{\mathbf{a}_{*i_1}, \ldots, \mathbf{a}_{*i_r}\}$ and $\{\mathbf{a}_{*j_1}, \ldots, \mathbf{a}_{*j_r}\}$ be any two bases of* $\mathrm{Col}(A)$. *Then there exists an $r \times r$ monomial matrix M such that*

$$\left[\, \mathbf{a}_{*i_1} \ \cdots \ \mathbf{a}_{*i_r} \,\right] = \left[\, \mathbf{a}_{*j_1} \ \cdots \ \mathbf{a}_{*j_r} \,\right] M.$$

Lemma 2.2 tells us that the cardinalities of any two bases for $\mathrm{Col}(A)$ are same. The cardinality is called the *column rank* of A, denoted by $c(A)$. Dually, we can define the *row rank* of A, denoted by $r(A)$. The column rank and the row rank of a matrix need not be equal in general. Let A be an $m \times n$ matrix. If $c(A) = r(A) = r$, then r is called the *rank* of A. If $c(A) = n$ and $r(A) = m$, then A is called *nonsingular*, and *singular* otherwise.

Let $A = (a_{ij})$ be an $n \times n$ matrix, the submatrix

$$\begin{bmatrix} \mathbf{a}_{j_1 i_1} & \mathbf{a}_{j_1 i_2} & \cdots & \mathbf{a}_{j_1 i_s} \\ \mathbf{a}_{j_2 i_1} & \mathbf{a}_{j_2 i_2} & \cdots & \mathbf{a}_{j_2 i_s} \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_{j_r i_1} & \mathbf{a}_{j_r i_2} & \cdots & \mathbf{a}_{j_r i_s} \end{bmatrix}$$

is called a *basis submatrix* of A, if $\{\mathbf{a}_{*i_1}, \ldots, \mathbf{a}_{*i_r}\}$ is a basis of $\mathrm{Col}(A)$ and $\{\mathbf{a}_{j_1 *}, \ldots, \mathbf{a}_{j_s *}\}$ is a basis of $\mathrm{Row}(A)$. If A_1 and A_2 are both bases submatrices of A, then there exist monomial matrices M_1 and M_2 such that $A_1 = M_1 A_2 M_2$. We denote the basis submatrix by $\bar{A}$. It is easy to see that $c(A) = c(\bar{A}) =$ the number of columns of $\bar{A}$, and that $r(A) = r(\bar{A}) =$ the number of rows of $\bar{A}$. The Rao "normal form" of a matrix is to be needed for us.

Lemma 2.3 ([5] Lemma 101 Rao "normal form") *Let A be an $m \times n$ matrix. Then there exists an $m \times m$ permutation matrix Q and an $n \times n$ permutation matrix P such that*

$$A = Q \begin{bmatrix} \bar{A} & \bar{A}U \\ V\bar{A} & V\bar{A}U \end{bmatrix} P,$$

where U is an $r \times (n-r)$ matrix and V is an $(m-s) \times s$ matrix.

In the following we will introduce and study the equivalences $\mathscr{R}^*$ and $\mathscr{L}^*$ on the set $M(D)$ $(= \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} M_{m\times n}(D))$ of all matrices. The equivalences $\mathscr{R}^*$ and $\mathscr{L}^*$ are, respectively, defined by

$$(\forall A,\ B \in M(D))\ \ A\mathscr{R}^*B \iff (\exists X,\ Y \in M(D))\ \ A = BX \text{ and } B = AY;$$
$$(\forall A,\ B \in M(D))\ \ A\mathscr{L}^*B \iff (\exists X,\ Y \in M(D))\ \ A = XB \text{ and } B = YA.$$

It is easy to see that if $A\mathscr{R}^*B$ ($A\mathscr{L}^*B$, resp.), then the number of rows (columns, resp.) of A is equal to that of B. In particular, the restriction of $\mathscr{R}^*$ and $\mathscr{L}^*$ to $M_n(D)$ coincide with Green's relations $\mathscr{R}$ and $\mathscr{L}$ on the semigroup $(M_n(D), \cdot)$, respectively, which play a key role in the algebraic theory of semigroups.

Proposition 2.4 *Let A and B be matrices in $M(D)$. Then*

$$A\mathscr{R}^*B \iff \mathrm{Col}(A) = \mathrm{Col}(B) \quad \textit{and} \quad A\mathscr{L}^*B \iff \mathrm{Row}(A) = \mathrm{Row}(B).$$

Proof Suppose that $A\mathscr{R}^*B$. If A is an $m \times n$ matrix and B is an $m \times q$ matrix, then by the definition of $\mathscr{R}^*$, there exists a $q \times n$ matrix X such that $BX = A$. Now, it follows that $\mathrm{Col}(BX) = \mathrm{Col}(A) \subseteq \mathrm{Col}(B)$, since the columns of BX are contained in $\mathrm{Col}(B)$. Dually, $\mathrm{Col}(B) \subseteq \mathrm{Col}(A)$. Thus $\mathrm{Col}(A) = \mathrm{Col}(B)$.

Conversely, suppose that $\mathrm{Col}(A) = \mathrm{Col}(B)$. If A is an $m \times n$ matrix and B is an $m \times q$ matrix, then each column of A is in $\mathrm{Col}(A)$ and so is in $\mathrm{Col}(B)$, since the pseudomodule D has a multiplicative identity $\mathbf{1}$. This implies that each column of A can be written as a linear combination of the columns of B. Thus there exists a $q \times n$ matrix X such that $BX = A$. Similarly, we can prove that there exists an $n \times q$ matrix Y such that $AY = B$. Hence, we have shown that $A\mathscr{R}^*B$.

Dually, we can show that $A\mathscr{L}^*B$ if and only if $\text{Row}(A) = \text{Row}(B)$. □

We now immediately deduce the following result.

Corollary 2.5 *Let A and B be matrices in* $M(\mathbb{B})$. *Then*

$$A\mathscr{R}^*B \implies c(A) = c(B) \quad \text{and} \quad A\mathscr{L}^*B \implies r(A) = r(B).$$

For an $m \times n$ matrix A, we will introduce various types of inverses of A. Consider an $n \times m$ matrix G in the following equations:

(G-1) $AGA = A$;
(G-2) $GAG = G$;
(G-3) $(AG)^T = AG$;
(G-4) $(GA)^T = GA$.

A matrix G satisfying (G-1) is called a *generalized inverse* (g-inverse for short) of A. If G satisfies (G-1) and (G-3) ((G-1) and (G-4), resp.), then it is called a $\{1,3\}$-g-*inverse* ($\{1,4\}$-g-*inverse*, resp.) of A. Finally, if G satisfies all from (G-1) to (G-4), then it is called a *Moore–Penrose inverse* of A.

We note that if G_1 and G_2 are any two g-inverses of A, then $G_1 + G_2$ is also a g-inverse of A, since

$$A(G_1 + G_2)A = AG_1A + AG_2A = A + A = A.$$

Also, it is well known that G is a $\{1,3\}$-g-inverse of A if and only if G^T is a $\{1,4\}$-g-inverse of A^T.

Proposition 2.6 *Let A and G be an* $m \times n$ *matrix and an* $n \times m$ *matrix, respectively. Then the following statements are equivalent:*

(i) $AGA = A$;
(ii) $(AG)^2 = AG$ *and* $A\mathscr{R}^*AG$;
(iii) $(GA)^2 = GA$ *and* $A\mathscr{L}^*GA$.

Proof (i) ⇒ (ii). Assume that (i) holds. Then $(AG)^2 = (AGA)G = AG$. Also, it is clear that $A\mathscr{R}^*AG$.

(ii) ⇒ (i). Assume that (ii) holds. Then $A = AGX$ for some $m \times n$ matrix X. This implies that $AGA = AGAGX = AGX = A$ since $(AG)^2 = AG$. Thus (i) holds.

Similarly, we can show that (i) ⇔ (iii). □

3 Normal Form of an Idempotent Matrix

In this section, we will give the Rao normal form of an idempotent matrix which plays a very important role in the studying the maximal subgroup of the semigroup $M_n(D)$. The following results are inspired by Kang and Song [10], in which they studied the idempotent matrices over the max-algebra.

Define the partial order $\leq$ on $M_{m\times n}(D)$ by

$$A \leq B \iff A + B = B.$$

Lemma 3.1 *Let A, B, C be* $m \times n$ *matrices, let* X_1, Y_1 *be* $n \times p$ *matrices and let* X_2, Y_2 *be* $p \times m$ *matrices. Then the following statements hold.*

(i) *If* $A + B = C$, *then* $A \leq C$ *and* $B \leq C$.
(ii) *If* $X_1 \leq Y_1$, *then* $AX_1 \leq AY_1$. *If* $X_2 \leq Y_2$, *then* $X_2A \leq Y_2A$.

Proof (i) Suppose that $A + B = C$. Then we have

$$A + C = A + (A + B) = A + B = C.$$

Hence $A \leq C$. Similarly, $B \leq C$.

(ii) Suppose that $X_1 \leq Y_1$. Then $X_1 + Y_1 = Y_1$. Thus it follows that

$$AX_1 + AY_1 = A(X_1 + Y_1) = AY_1,$$

and so $AX_1 \leq AY_1$. Similarly, if $X_2 \leq Y_2$, then $X_2A \leq Y_2A$. □

Lemma 3.2 *If* $E = (e_{ij})$ *be an* $n \times n$ *idempotent matrix, then* $e_{ii} \leq \mathbf{1}$ *for all* $1 \leq i \leq n$.

Proof Let $E = (e_{ij})_{n\times n}$ be an idempotent matrix. Then for any $1 \leq i \leq n$,

$$e_{ii} \cdot e_{ii} \leq (e_{i1} \cdot e_{1i}) + \cdots + (e_{ii} \cdot e_{ii}) + \cdots + (e_{in} \cdot e_{ni}) = e_{ii}.$$

This implies that $e_{ii} \leq e_{ii}e_{ii}^{-1} = \mathbf{1}$. □

Lemma 3.3 *Let* $E = (e_{ij})$ *be an* $n \times n$ *idempotent matrix. If* $e_{ii} < \mathbf{1}$ *for some* $i \in \{1, 2, \ldots, n\}$, *then the* i*-th column* (*row, resp.*) *of* E *is a linear combination of the remaining columns* (*rows, resp.*). *Furthermore, the matrix obtained from* E *by deleting the* i*-th column and the* i*-th row is an* $(n-1) \times (n-1)$ *idempotent matrix.*

Proof Let $E = (e_{ij})_{n\times n}$ be an idempotent matrix. Suppose that $e_{ii} < \mathbf{1}$ for some $1 \leq i \leq n$. Without loss of generality, we assume that $e_{11} < \mathbf{1}$. Partition E as $\begin{bmatrix} e_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$. Then we have

$$E^2 = \begin{bmatrix} e_{11} \cdot e_{11} + E_{12}E_{21} & e_{11}E_{12} + E_{12}E_{22} \\ E_{21}e_{11} + E_{22}E_{21} & E_{21}E_{12} + E_{22}^2 \end{bmatrix} = \begin{bmatrix} e_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}.$$

This implies that

$$\begin{bmatrix} E_{12}E_{21} & E_{12}E_{22} \\ E_{22}E_{21} & E_{21}E_{12} + E_{22}^2 \end{bmatrix} = \begin{bmatrix} e_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},$$

since $e_{11} < \mathbf{1}$. Thus it follows that

$$\begin{bmatrix} e_{11} \\ E_{21} \end{bmatrix} = \begin{bmatrix} E_{12}E_{21} \\ E_{22}E_{21} \end{bmatrix} = \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix} E_{21}, \tag{3.1}$$

$$\begin{bmatrix} e_{11} & E_{12} \end{bmatrix} = \begin{bmatrix} E_{12}E_{21} & E_{12}E_{22} \end{bmatrix} = E_{12}\begin{bmatrix} E_{21} & E_{22} \end{bmatrix}, \tag{3.2}$$

$$E_{21}E_{12} + E_{22}^2 = E_{22}. \tag{3.3}$$

The equation (3.1) ((3.2), resp.) tells us that the 1-th column (the 1-th row, resp.) of E is a linear combination of the remaining columns (rows, resp.). By Lemma 3.1 and (3.3), we have

$$E_{22}^2 \leq E_{22} \quad \text{and} \quad E_{21}E_{12} \leq E_{22}. \tag{3.4}$$

Thus it follows by (3.4) and Lemma 3.1 that $E_{21}E_{12} = E_{21}E_{12}E_{22} \leq E_{22}^2$, since $E_{12}E_{22} = E_{12}$. We therefore have

$$E_{22} = E_{21}E_{12} + E_{22}^2 \leq E_{22}^2 + E_{22}^2 = E_{22}^2, \tag{3.5}$$

by Lemma 3.1. Thus (3.4) and (3.5) tell us that $E_{22}^2 = E_{22}$. □

The above lemma tells us that if $E = (e_{ij})$ is an $n \times n$ idempotent matrix and $e_{ii} < 0$ for some $1 \le i \le n$, then $c(E) < n$ and $r(E) < n$. Thus by Lemmas 3.2 and 3.3, we immediately have the following result.

Corollary 3.4 *All main diagonal entries of a nonsingular idempotent matrix are* **1**.

Lemma 3.5 *Let E be an $n \times n$ idempotent matrix whose main diagonal entries are all* **1**. *Then the i-th row of E is a linear combination of the remaining rows if and only if the i-th column of E is a linear combination of the remaining columns. Furthermore, the matrix obtained from E by deleting the i-th column and the i-th row is an $(n-1) \times (n-1)$ idempotent matrix.*

Proof Let $E = (e_{ij})$ be an $n \times n$ idempotent matrix with $e_{ii} = 0$ for all $1 \le i \le n$.

Suppose that the i-th row of A is a linear combination of the remaining rows. Without loss of generality, we assume that the 1-th row of E is a linear combination of the remaining rows. Partition E as $\begin{bmatrix} \mathbf{1} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$. Then we have

$$E^2 = \begin{bmatrix} \mathbf{1} + E_{12}E_{21} & E_{12} + E_{12}E_{22} \\ E_{21} + E_{22}E_{21} & E_{21}E_{12} + E_{22}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}.$$

Thus by Lemma 3.1 we have

$$E_{22}E_{21} \le E_{21} \quad \text{and} \quad E_{22}^2 \le E_{22}. \tag{3.6}$$

On the other hand, it is easy to see that

$$I_{n-1} \le E_{22},$$

since $e_{ii} = \mathbf{1}$ for all $2 \le i \le n$. Thus by Lemma 3.1, we can show that

$$E_{21} \le E_{22}E_{21} \quad \text{and} \quad E_{22} \le E_{22}^2. \tag{3.7}$$

Hence by summing (3.6) and (3.7), we have

$$E_{21} = E_{22}E_{21} \quad \text{and} \quad E_{22} = E_{22}^2. \tag{3.8}$$

Since $\begin{bmatrix} \mathbf{1} & E_{12} \end{bmatrix}$ is a linear combination of the rows of $\begin{bmatrix} E_{21} & E_{22} \end{bmatrix}$, there exists a row vector X such that $\begin{bmatrix} \mathbf{1} & E_{12} \end{bmatrix} = X \begin{bmatrix} E_{21} & E_{22} \end{bmatrix}$. That is to say,

$$\mathbf{1} = XE_{21} \quad \text{and} \quad E_{12} = XE_{22}. \tag{3.9}$$

Thus it follows from (3.8) and (3.9) that $\mathbf{1} = XE_{21} = XE_{22}E_{21} = E_{12}E_{21}$. Therefore,

$$\begin{bmatrix} \mathbf{1} \\ E_{21} \end{bmatrix} = \begin{bmatrix} E_{12}E_{21} \\ E_{22}E_{21} \end{bmatrix} = \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix} E_{21}.$$

This shows that the 1-th column of E is a linear combination of the remaining columns.

Dually, we can show that the converse is true. This completes our proof. □

By Lemmas 3.3 and 3.5 we immediately have

Corollary 3.6 *If E is an idempotent matrix, then $c(E) = r(E)$.*

Lemma 3.7 *Let A and B be $n \times n$ matrices. If all main diagonal entries of A are* **1** *and $ABA \le A$, then $B \le A$.*

Proof Suppose that $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices. Assume that $ABA = (c_{ij})$. If all main diagonal entries of A are $\mathbf{1}$ and $ABA \leq A$, then for any $1 \leq i,\ j \leq n$,

$$b_{ij} = \mathbf{1} \cdot b_{ij} \cdot \mathbf{1} = a_{ii} \cdot b_{ij} \cdot a_{jj} \leq \sum_{k=1}^{n}\sum_{l=1}^{n}(a_{ik} \cdot b_{kl} \cdot a_{lj}) = c_{ij} \leq a_{ij}.$$

That is to say, $B \leq A$. □

The following gives the normal form of an idempotent matrix.

Theorem 3.8 *Let E be an $n \times n$ matrix. Then E is an idempotent matrix of rank r if and only if there exists an $n \times n$ permutation matrix P such that*

$$E = P\begin{bmatrix} \bar{E} & \bar{E}U \\ V\bar{E} & V\bar{E}U \end{bmatrix}P^T,$$

where $\bar{E}$ is a basis submatrix of E and is an $r \times r$ nonsingular idempotent matrix, U is an $r \times (n-r)$ matrix and V is an $(n-r) \times r$ matrix such that $UV \leq \bar{E}$.

Proof Suppose that E is an $n \times n$ matrix. If E is an idempotent matrix of rank r ($r < n$), then by Lemmas 3.3 and 3.5 we have that the i-th row of E is a linear combination of the remaining rows if and only if the i-th column of E is a linear combination of the remaining columns. Thus by carrying out the same row permutations and column permutations of E, we can find a matrix E' such that the first r columns and the first r rows of E' are linearly independent. That is to say, there exists an $n \times n$ permutation matrix P such that

$$P^T EP = E' = \begin{bmatrix} \bar{E} & X \\ Y & Z \end{bmatrix},$$

where $\bar{E}$ is an basis submatrix of E, X, Y and Z are matrices of appropriate sizes. Also, by Lemmas 3.3 and 3.5 we can show that $\bar{E}$ is an $r \times r$ nonsingular idempotent matrix. $\begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} \bar{E} \\ Y \end{bmatrix}U$ for some $r \times (n-r)$ matrix U, since each column of $\begin{bmatrix} X \\ Z \end{bmatrix}$ is a linear combination of the columns of $\begin{bmatrix} \bar{E} \\ Y \end{bmatrix}$. Dually, we can show that $\begin{bmatrix} Y & YU \end{bmatrix} = V\begin{bmatrix} \bar{E} & \bar{E}U \end{bmatrix}$ for some $(n-r) \times r$ matrix V. Hence, we have

$$E = P\begin{bmatrix} \bar{E} & \bar{E}U \\ V\bar{E} & V\bar{E}U \end{bmatrix}P^T.$$

This is a Rao "normal form" of E. Since $E^2 = E$, we have that $E' = E'^2$. That is to say,

$$\begin{bmatrix} \bar{E} & \bar{E}U \\ V\bar{E} & V\bar{E}U \end{bmatrix} = \begin{bmatrix} \bar{E}^2 + \bar{E}UV\bar{E} & \bar{E}^2U + \bar{E}UV\bar{E}U \\ V\bar{E}^2 + V\bar{E}UV\bar{E} & V\bar{E}U + V\bar{E}UV\bar{E}U \end{bmatrix}.$$

Hence, $\bar{E} = \bar{E}^2 + \bar{E}UV\bar{E}$. Thus by Lemma 3.1, we can show that $\bar{E}UV\bar{E} \leq \bar{E}$. By Corollary 3.4 and Lemma 3.7, we have that $UV \leq \bar{E}$.

The converse is easily to be verified. □

Corollary 3.9 *Let E be an $n \times n$ matrix. Then E is a symmetric idempotent matrix of rank r if and only if there exists an $n \times n$ permutation matrix P such that*

$$E = P\begin{bmatrix} \bar{E} & \bar{E}U \\ V\bar{E} & V\bar{E}U \end{bmatrix}P^T,$$

where $\bar{E}$ is a basis submatrix of E and is an $r \times r$ symmetric nonsingular idempotent matrix, U is an $r \times (n-r)$ matrix and V is an $(n-r) \times r$ matrix such that $UV \leq \bar{E}$.

4 Generalized Inverses of a Regular Matrices

In this section we will give a characterization of regular matrices. Also, we will define a space decomposition of a matrix and prove that a matrix A is regular if and only if A is space decomposable.

Theorem 4.1 *Let A be an $m \times n$ matrix. Then A is regular if and only if there exists an $m \times m$ permutation matrix P and an $n \times n$ permutation matrix Q such that*

$$A = P \begin{bmatrix} FM & FC \\ VFM & VFC \end{bmatrix} Q,$$

where F is a nonsingular idempotent matrix, M is a diagonal monomial matrix and C, V are matrices of appropriate sizes.

Proof Suppose that A is an $m \times n$ matrix. If A is regular, then there exists an $n \times m$ matrix G such that G is a g-inverse of A. Thus AG is idempotent and $A\mathscr{R}^*AG$ by Proposition 2.6. Let the rank of AG be r. Now, by Theorem 3.8, there exists an $m \times m$ permutation matrix P such that $AG = P \begin{bmatrix} F & FU \\ VF & VFU \end{bmatrix} P^T$, where F is an $r \times r$ nonsingular idempotent matrix, U and V are matrices of appropriate sizes, and so

$$AGP = P \begin{bmatrix} F & FU \\ VF & VFU \end{bmatrix}. \tag{4.1}$$

Notice that the set of the first r columns of AGP is a basis of $\mathrm{Col}(AGP)$. Since $A\mathscr{R}^*AG\mathscr{R}^*(AGP)$, it follows from (4.1), Lemma 2.2 and Proposition 2.4 that there exists a permutation matrix Q such that

$$A = AGP \begin{bmatrix} M & X \\ -\infty & Y \end{bmatrix} Q = P \begin{bmatrix} F & FU \\ VF & VFU \end{bmatrix} \begin{bmatrix} M & X \\ O & Y \end{bmatrix} Q = P \begin{bmatrix} FM & FC \\ VFM & VFC \end{bmatrix} Q,$$

where M is an $r \times r$ monomial matrix and $C = X + UY$.

Conversely, it is easy to verify $Q^T \begin{bmatrix} M^{-1} & O \\ O & O \end{bmatrix} P^T$ is a g-inverse of A. □

As a consequence, we have

Corollary 4.2 *If A is a regular matrix, then $c(A) = r(A)$.*

Notice that for a matrix A, A is regular if and only if A^T is regular, since G is a g-inverse of A if and only if G^T is a g-inverse of A^T.

A nonzero matrix A is said to be *space decomposable* if there exist matrices L and R such that

$$A = LR, \quad A\mathscr{R}^*L \quad \text{and} \quad A\mathscr{L}^*R. \tag{4.2}$$

The decomposition LR will be called a *space decomposition* of A. Thus $\mathrm{Col}(A) = \mathrm{Col}(L)$ and $\mathrm{Row}(A) = \mathrm{Row}(R)$. This means the matrix A is decomposed two matrices, which satisfy a matrix preserves column space and another matrix preserves row space.

Proposition 4.3 *A nonzero matrix is regular if and only if it is space decomposable.*

Proof Suppose that A is a nonzero $m \times n$ matrix.

If A is a regular matrix of rank r, then by Theorem 4.1, A is of the form

$$P \begin{bmatrix} FM & FC \\ VFM & VFC \end{bmatrix} Q,$$

where P and Q are permutation matrices and M is a monomial matrix. Let

$$L_A = P \begin{bmatrix} F \\ VF \end{bmatrix} \quad \text{and} \quad R_A = \begin{bmatrix} FM & FC \end{bmatrix} Q. \tag{4.3}$$

Then

$$A = L_A R_A, \quad L_A = AQ^T \begin{bmatrix} M^{-1} \\ O \end{bmatrix} \quad \text{and} \quad R_A = \begin{bmatrix} I_r & O \end{bmatrix} P^T A.$$

This implies that $A \mathscr{R}^* L_A$ and $A \mathscr{L}^* R_A$. Thus L_A and R_A satisfy the condition (4.2) and so A is space decomposable.

Conversely, assume that A is space decomposable. Then it follows from (4.2) that there exist matrices L and R such that $A = LR$, $A \mathscr{R}^* L$ and $A \mathscr{L}^* R$. This implies that $L = AX$ and $R = YA$ for some matrices X and Y. Hence

$$A = LR = A(XY)A,$$

and so A is regular. □

Notice that a matrix of rank less than 3 is space decomposable.

Corollary 4.4 *If the rank of a matrix A is less than* 3, *then A is regular.*

In [9], Johnson and Kambites proved that the all 2×2 tropical matrix are regular. Corollary 4.4 extend this result.

Lemma 4.5 *Let A be a nonsingular idempotent matrix. If G is a* g*-inverse of A, then $AG = GA = A$.*

Proof Suppose that $A = (a_{ij})$ is an $n \times n$ nonsingular idempotent matrix and that G is a g-inverse of A. Then $AGA = A$. It follows from Proposition 2.6 that AG is idempotent and $A \mathscr{R}^* AG$. Since A is nonsingular we can show by Corollary 2.5 that AG is also nonsingular. Thus by Corollary 3.4 it follows that main diagonal entries of A and AG are all **1**. This implies by Lemma 3.7 that $G \leq A$, since $AGA = A$. Hence

$$AG \leq A^2 = A, \tag{4.4}$$

by Lemma 3.1(ii). Since $(AG)A(AG) = AG$, by Lemma 3.7 we have

$$A \leq AG. \tag{4.5}$$

(4.4) and (4.5) tell us that $AG = A$. Notice that A^T is idempotent and $A^T G^T A^T = A^T$. Similarly, we have that $A^T G^T = A^T$, and hence $GA = A$. □

Proposition 4.6 *Let A be of the form in Theorem 4.1 and let $L_A R_A$ be the space decomposition of A in* (4.3)*. Then for any $m \times s$ matrix L and any $s \times n$ matrix R, LR is a space decomposition of A if and only if $L = L_A M_1$, $R = M_2 R_A$ and $FM_1M_2 = M_1M_2F = F$ for some $r \times s$ matrix M_1 and some $s \times r$ matrix M_2.*

Proof If A is an $m \times n$ regular matrix of rank r, then by Theorem 4.1, A is of the form

$$P\begin{bmatrix} FM & FC \\ VFM & VFC \end{bmatrix}Q,$$

where P and Q are permutation matrices, M is a monomial matrix and F is a nonsingular idempotent matrix. Let $L_A R_A$ be the space decomposition of A in (4.3). Then it follows from (4.2) that $A = L_A R_A$, $A\mathscr{R}^* L_A$ and $A\mathscr{L}^* R_A$.

For any $m \times s$ matrix L and any $s \times n$ matrix R, if LR is a space decomposition of A, then $A = LR$, $L\mathscr{R}^* A$ and $R\mathscr{L}^* A$. This implies that $L\mathscr{R}^* L_A$ and $R\mathscr{L}^* R_A$ and so $L = L_A M_1$ and $R = M_2 R_A$ for some $r \times s$ matrix M_1 and some $s \times r$ matrix M_2. Now we have

$$A = LR = P\begin{bmatrix} F \\ VF \end{bmatrix} M_1 M_2 \begin{bmatrix} FM & FC \end{bmatrix} Q = P\begin{bmatrix} FM_1M_2FM & FM_1M_2FC \\ VFM_1M_2FM & VFM_1M_2FC \end{bmatrix} Q.$$

Since P and Q are permutation matrices, $FM = FM_1M_2FM$. It follows that $F = FM_1M_2F$, since M is a monomial matrix. Notice that F is a nonsingular idempotent matrix. By Lemma 4.5 we have that $FM_1M_2 = M_1M_2F = F$.

Conversely, assume that there exists an $r \times s$ matrix M_1 and an $s \times r$ matrix M_2 such that $L = L_A M_1$, $R = M_2 R_A$ and $FM_1M_2 = M_1M_2F = F$. Then we have

$$\begin{aligned} LM_2 &= L_A M_1 M_2 = P\begin{bmatrix} F \\ VF \end{bmatrix} M_1 M_2 = P\begin{bmatrix} F \\ VF \end{bmatrix} = L_A, \\ M_1 R &= M_1 M_2 R_A = M_1 M_2 \begin{bmatrix} FM & FC \end{bmatrix} Q = \begin{bmatrix} FM & FC \end{bmatrix} Q = R_A, \end{aligned}$$

and

$$LR = L_A M_1 M_2 R_A = L_A R_A = A.$$

This implies that

$$A = LR, \quad A\mathscr{R}^* L_A \mathscr{R}^* L \quad \text{and} \quad A\mathscr{L}^* R_A \mathscr{L}^* R,$$

and so LR is a space decomposition of A. □

Corollary 4.7 *If LR is a space decomposition of a regular matrix A, then both L and R are regular.*

Proof Let A be an $m \times n$ regular matrix of rank r. Suppose that LR is a space decomposition of A. By Proposition 4.6 it follows that there exist matrices M_1 and M_2 such that $L = P\begin{bmatrix} F \\ VF \end{bmatrix} M_1$, $R = M_2 \begin{bmatrix} FM & FC \end{bmatrix} Q$ and $FM_1M_2 = M_1M_2F = F$, where P and Q are permutation matrices, M is a monomial matrix. Thus we have

$$L_G = M_2 \begin{bmatrix} I_r & O \end{bmatrix} P^T \quad \text{and} \quad R_G = Q^T \begin{bmatrix} M^{-1} \\ O \end{bmatrix} M_1$$

are g-inverses of L and R, respectively, and so both L and R are regular. □

5 Other Type of g-Inverses

Notice that G is a $\{1, 3\}$-g-inverse of A if and only if $AGA = A$ and $(AG)^T = AG$.

Theorem 5.1 *Let A be an $m \times n$ matrix. The following statements are equivalent:*

(i) *A has a* $\{1, 3\}$-g-*inverse.*

(ii) *There exists an $m \times m$ permutation matrix P and an $n \times n$ permutation matrix Q such that*

$$A = P \begin{bmatrix} SM & SZ \\ VSM & VSZ \end{bmatrix} Q,$$

where S is a symmetric nonsingular idempotent matrix, M is a diagonal monomial matrix and V, Z are matrices such that $V^T V \leq S$.

(iii) $A\mathscr{L}^* A^T A$.

Proof (i) $\Longrightarrow$ (ii). Let an $n \times m$ matrix G be a $\{1, 3\}$-g-inverse of A. Then $A\mathscr{R}^* AG$ and AG is a symmetric idempotent matrix by Proposition 2.6. Let the rank of AG is r. It follows from Corollary 3.9 that there exists an $m \times m$ permutation matrix P such that $AG = P \begin{bmatrix} S & SV^T \\ VS & VSV^T \end{bmatrix} P^T$, and so

$$AGP = P \begin{bmatrix} S & SV^T \\ VS & VSV^T \end{bmatrix},$$

where S is a symmetric nonsingular idempotent matrix, V is a matrix such that $V^T V \leq S$. Notice that the set of the first r columns of AGP is a basis of $\mathrm{Col}(AGP)$. Since $A\mathscr{R}^* AG\mathscr{R}^* AGP$, it follows from Lemma 2.2 and Proposition 2.4 that there exists an $n \times n$ permutation matrix Q such that $A = AGP \begin{bmatrix} M & X \\ O & Y \end{bmatrix} Q$, where M is a diagonal monomial matrix. Thus

$$A = P \begin{bmatrix} S & SV^T \\ VS & VSV^T \end{bmatrix} \begin{bmatrix} M & X \\ O & Y \end{bmatrix} Q = P \begin{bmatrix} SM & SZ \\ VSM & VSZ \end{bmatrix} Q,$$

where $Z = X + V^T Y$.

(ii) $\Longrightarrow$ (iii). Now assume that (ii) holds. Since P is a permutation matrix, N is a symmetric idempotent matrix, M is a diagonal matrix and V is a matrix such that $V^T V \leq N$, it follows that

$$A^T A = Q^T \begin{bmatrix} MSM & MSZ \\ Z^T SM & Z^T SZ \end{bmatrix} Q.$$

Hence, since Q is a permutation matrix and M is a monomial matrix,

$$A = P \begin{bmatrix} M^{-1} & O \\ VM^{-1} & O \end{bmatrix} QA^T A.$$

Thus we have that $A\mathscr{L}^* A^T A$.

(iii) $\Longrightarrow$ (i). If $A\mathscr{L}^* A^T A$, then there exists an $m \times n$ matrix G, such that $A = GA^T A$. This implies that

$$AG^T A = (GA^T A)G^T A = G(A^T AG^T)A = G(GA^T A)^T A = GA^T A = A.$$

We also have

$$(AG^T)^T = (GA^T AG^T)^T = GA^T AG^T = AG^T.$$

Therefore, G^T is a $\{1, 3\}$-g-inverse of A. □

In Theorem 5.1(ii), we can easily check that $Q^T \begin{bmatrix} M^{-1} & M^{-1}V^T \\ O & O \end{bmatrix} P^T$ is a $\{1, 3\}$-g-inverse of A.

Similarly, we have the following result.

Proposition 5.2 *Let A be an $m \times n$ matrix. The following statements are equivalent:*

(i) *A has a* $\{1,4\}$-g-*inverse.*
(ii) *There exists an* $m \times m$ *permutation matrix* P *and an* $n \times n$ *permutation matrix* Q *such that*

$$A = P\begin{bmatrix} SM & SMU \\ WS & WSU \end{bmatrix} Q,$$

where S *is a symmetric nonsingular idempotent matrix,* M *is a diagonal monomial matrix and* U, W *are matrices such that* $UU^T \leq S$.
(iii) $A\mathscr{R}^* AA^T$.

In Proposition 5.2(ii), we can easily check that $Q^T \begin{bmatrix} M^{-1} & O \\ U^T M^{-1} & O \end{bmatrix} P^T$ is a $\{1,4\}$-g-inverse of A.

In the following result, we characterize matrices having Moore–Penrose inverses. The proof depends on the above two theorems, and we omit the proof:

Corollary 5.3 *Let* A *be an* $m \times n$ *matrix. The following statements are equivalent:*

(i) *A has a Moore–Penrose inverse.*
(ii) *There exists an* $m \times m$ *permutation matrix* P *and an* $n \times n$ *permutation matrix* Q *such that*

$$A = P\begin{bmatrix} SM & SMU \\ VSM & VSMU \end{bmatrix} Q,$$

where S *is a symmetric nonsingular idempotent matrix,* M *is a diagonal monomial matrix and* V, U *are matrices such that* $V^T V \leq S$ *and* $UU^T \leq S$.
(iii) $A\mathscr{L}^* A^T A$ *and* $A\mathscr{R}^* AA^T$.

In Corollary 5.3(ii), we can easily check that $Q^T \begin{bmatrix} SM^{-1} & SM^{-1}V^T \\ U^T SM^{-1} & U^T SM^{-1} V^T \end{bmatrix} P^T$ is a Moore–Penrose inverse of A.

Acknowledgements This work is supported by the National Natural Science Foundation of China (Grant No.11861045) and the Hongliu Foundation of First-class Disciplines of Lanzhou University of Technology, China.

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