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REVIEW ARTICLE

Boundary Value Problems for Dirac-Harmonic Maps and Their Heat Flows

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Abstract
Dirac-harmonic maps are critical points of an action functional that is motivated from the nonlinear σ-model of quantum field theory. They couple a harmonic map like field with a nonlinear spinor field. In this article, we shall discuss the latest progress on heat flow approaches for the existence of Dirac-harmonic maps under appropriate boundary conditions. Also, we discuss the refined blow-up analysis for two types of approximating Dirac-harmonic maps arising from those heat flow approaches.

Keywords Dirac-harmonic map · Dirac-harmonic map flow · α-Dirac-harmonic map · α-Dirac-harmonic map flow · Dirichlet boundary · Chiral boundary

Mathematics Subject Classification (2010) 53C43 · 58E20

1 Introduction

Motivated by the supersymmetric nonlinear sigma model from quantum field theory [11], Dirac-harmonic maps were introduced by Jost and his collaborators in [6]. They are natural generalizations of harmonic maps and harmonic spinors.

Let (M, g) be a Riemannian manifold and let (N, h) be a compact Riemannian manifold with dimension $n \geq 2$. Let ϕ be a smooth map from M to N. Denote ϕ^*TN the pull-back bundle of TN by ϕ and then we get the twisted bundle $\Sigma M \otimes \phi^*TN$. There is a natural metric $\langle \cdot, \cdot \rangle_{\Sigma M \otimes \phi^*TN}$ on $\Sigma M \otimes \phi^*TN$ induced from the metrics on ΣM and ϕ^*TN.

Dedicated to Professor Jürgen Jost's 65th birthday.

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Likewise, there is a natural connection $\widetilde{\nabla}$ on $\Sigma M \otimes \phi^*TN$ induced from the connections on ΣM and ϕ^*TN. Let ψ be a section of the bundle $\Sigma M \otimes \phi^*TN$. In local coordinates, it can be written as

$$\psi = \psi^i \otimes \partial_{y^i}(\phi),$$

where each ψ^i is a usual spinor on M and ∂_{y^i} is the nature local basis on N. Then $\widetilde{\nabla}$ becomes

$$\widetilde{\nabla}\psi = \nabla\psi^i \otimes \partial_{y^i}(\phi) + \left(\Gamma^i_{jk}\nabla\phi^j\right)\psi^k \otimes \partial_{y^i}(\phi),$$

where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of N. The Dirac operator along the map ϕ is defined by

$$\not{D}\psi := e_\gamma \cdot \widetilde{\nabla}_{e_\gamma}\psi.$$

We consider the following functional

$$L(\phi, \psi) = \frac{1}{2}\int_M \left(|d\phi|^2 + \langle\psi, \not{D}\psi\rangle_{\Sigma M \otimes \phi^* TN}\right) dM,$$

where $dM = dvol_g$.

Critical points (ϕ, ψ) of the above functional L are called Dirac-harmonic maps from M to N. In terms of local coordinates, the corresponding Euler–Lagrange equations are given by the following

$$\begin{aligned}\left(\Delta_g\phi^i + \Gamma^i_{jk} g^{\alpha\beta}\phi^j_\alpha\phi^k_\beta\right)\frac{\partial}{\partial y^i}(\phi(x)) &= R(\phi, \psi),\\ \not{D}\psi &= 0,\end{aligned}$$

where $\Delta_g := \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\beta}\left(\sqrt{g}g^{\beta\gamma}\frac{\partial}{\partial x^\gamma}\right)$ is the Laplacian operator with respect to the Riemannian metric g and $R(\phi, \psi)$ is defined by

$$R(\phi, \psi) = \frac{1}{2}R^m_{lij}(\phi(x))\left\langle\psi^i, \nabla\phi^l \cdot \psi^j\right\rangle \frac{\partial}{\partial y^m}(\phi(x)).$$

Here R^m_{lij} is the Riemann curvature tensor of the target manifold (N, h).

By Nash's embedding theorem, we embed N isometrically into some Euclidean space $\mathbb{R}^K$. Then, critical points (ϕ, ψ) of the functional L satisfy the following extrinsic Euler–Lagrange equations

$$\begin{aligned}\Delta_g\phi + A(\phi)(d\phi, d\phi) &= Re(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi); \psi)),\\ \not{\partial}_g\psi &= \mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi),\end{aligned}$$

where $\not{\partial}_g$ is the usual Dirac operator for the spinor bundle on (M, g), $A(\cdot, \cdot)$ is the second fundamental form of N in $\mathbb{R}^K$, and

$$\begin{aligned}\mathcal{A}\left(d\phi(e_\gamma), e_\gamma \cdot \psi\right) &:= \left(\nabla\phi^i \cdot \psi^j\right) \otimes A\left(\partial_{y^i}, \partial_{y^j}\right),\\ Re(P(\mathcal{A}(d\phi(e_\gamma), e_\gamma \cdot \psi); \psi)) &:= P\left(A\left(\partial_{y^l}, \partial_{y^j}\right); \partial_{y^i}\right) Re\left(\left\langle\psi^i, d\phi^l \cdot \psi^j\right\rangle\right).\end{aligned}$$

Here $P(\xi; \cdot)$ denotes the shape operator, defined by $\langle P(\xi; X), Y\rangle = \langle A(X, Y), \xi\rangle$ for $X, Y \in \Gamma(TN)$, and $Re(z)$ denotes the real part of $z \in \mathbb{C}$.

When the domain M is of dimension 2, the functional $L(\phi, \psi)$ is conformally invariant, see [6]. That is, for any conformal diffeomorphism $f : M \to M$, if we set

$$\widetilde{\phi} = \phi \circ f \qquad \text{and} \qquad \widetilde{\psi} = \lambda^{-1/2}\psi \circ f,$$

where the positive function $\lambda > 0$ is the conformal factor of the conformal map f, i.e. $f^*g = \lambda^2 g$, then there holds

$$L\left(\widetilde{\phi}, \widetilde{\psi}\right) = L(\phi, \psi).$$

Similarly to two dimensional harmonic maps, since the conformal group is noncompact, it makes the variational problem borderline cases of the Palais–Smale condition, and hence standard PDE methods can not be applied to get the existence of critical points.

To investigate the existence problem of Dirac-harmonic maps, another key difficulty arises from the fact that the action functional L is not bounded from below. Therefore, classical variational approaches developed for harmonic maps cannot be directly applied to study the existence of Dirac-harmonic maps. There have been several other approaches, for instance, see [1, 7, 22]. The methods used as well as the results obtained in those papers are rather different from the present ones discussed here. In [22], some explicit examples of non-trivial Dirac-harmonic maps were constructed, however they are rather special and cannot replace a general scheme for the existence problem. In [7], under the condition that the target manifold satisfies certain convexity assumption, a subharmonic function is constructed from a solution to which a maximum principle can be applied. Ammann and Ginoux [1] uses some powerful methods from index theory, however in a more constrained setting.

In this article, we shall discuss some approaches that seem to be most promising to us for addressing the general existence issue and propose some open problems related to them.

The first approach is a heat flow for Dirac-harmonic maps which was firstly introduced in [9]. This flow couples a parabolic second order system for the map part with a first order elliptic system for the spinor part. That is, the solution of the first order Dirac type equation is carried along a second order harmonic map type heat flow. When the spinor vanishes, this flow reduces to the classical harmonic map heat flow introduced in [13], see [15] for the case of domain manifolds with boundary. Of course, we are interested in the case when the spinor field is non-trivial. Then the Dirac type equation for the spinor can be considered as some side constraint which depends nonlinearly on the heat flow for the map.

The heat flow for Dirac-harmonic maps introduced in [9] is the following elliptic-parabolic system:

$$\begin{cases} \partial_t \phi = \Delta\phi + A(\phi)(d\phi, d\phi) - \mathcal{P}(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi) & \text{in } M \times [0, T); \\ \partial\!\!\!/ \psi = \mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi) & \text{in } M \times [0, T). \end{cases} \tag{1.1}$$

When the domain M is closed, for some given fixed map Φ, solutions to the Dirac type equation

$$\partial\!\!\!/ \psi = \mathcal{A}\left(d\phi(e_\alpha), e_\alpha \cdot \psi\right)$$

are in general not unique. In order to get the uniqueness of solution to the above Dirac type equation, in [9], the authors considered the case that the domain manifold M is compact and has non-empty smooth boundary ∂M. Then they imposed the following boundary-initial data for the flow (1.1)

$$\begin{cases} \phi(x, t) = \phi_0(x, t) & \text{on } M \times \{0\} \cup \partial M \times [0, T]; \\ \mathbf{B}\psi(x, t) = \mathbf{B}\psi_0(x, t) & \text{on } \partial M \times [0, T], \end{cases} \tag{1.2}$$

and proved the short-time existence and uniqueness. Later, the case of 1 dimensional domains, namely the heat flow for Dirac-geode.sics, was considered in [8].

To investigate the long time behavior of the flow (1.1) and (1.2), in [18], the authors considered the case of a two dimensional domain and established a global weak solution of (1.1) which is unique and regular with the exception of at most finite singular times, which can be considered as an extension of the global weak solution to the two dimensional harmonic map heat flow introduced in [34]. As an application, we deduce some existence results for Dirac-harmonic maps where the spinor part is nontrivial. However, due to some technical reasons for the Dirac type equation for the spinor, in [18], we need to assume some extra initial-boundary constraint. In analogy to the two dimensional harmonic map flow with Dirichlet boundary condition studied in [3], we show that this initial-boundary constraint can be improved, see Theorem 2.6 in Section 2.

In order to get a general existence result in dimension 2, in [19], the authors developed a new scheme in geometric analysis, which is the second approach. Firstly, they improved a key estimate for the Dirac operator along a given map, see (2.5). Then, based on this improved estimate and inspired by the Sacks–Uhlenbeck approximation for harmonic maps in [33], in [19], the authors introduced the following functional:

$$L_\alpha(\phi, \psi) = \frac{1}{2}\int_M \left\{\left(1 + |d\phi|^2\right)^\alpha + \langle \psi, \not{D}\psi\rangle\right\} dM,$$

where $\alpha > 1$ is a constant. Critical points (ϕ, ψ) of the above functional L_α are called α-Dirac-harmonic maps, see [19].

Similarly to the action functional L for Dirac-harmonic maps, the new functional L_α is not bounded from below and classical variational methods can not be applied to get the existence of critical points, namely α-Dirac-harmonic maps. To overcome this issue, motivated by the Sacks–Uhlenbeck flow introduced in [17], the authors [19] introduced the heat flow for α-Dirac-harmonic maps:

$$\begin{cases} \partial_t \phi &= \Delta_g \phi + (\alpha - 1)\frac{\nabla_g |\nabla_g \phi|^2 \nabla_g \phi}{1+|\nabla_g \phi|^2} + A(d\phi, d\phi) - \frac{Re\left(P(\mathcal{A}\left(d\phi(e_\gamma), e_\gamma \cdot \psi\right); \psi)\right)}{\alpha\left(1+|\nabla_g \phi|^2\right)^{\alpha-1}}, \\ \not{\partial}_g \psi &= \mathcal{A}\left(d\phi(e_\gamma), e_\gamma \cdot \psi\right). \end{cases} \tag{1.3}$$

This is a new parabolic-elliptic system. By proving the global regular solution of (1.3) (for every fixed $\alpha > 1$ which is close to 1), one can get the existence of α-Dirac-harmonic maps which are the critical points of the functional L_α. Then by studying the limit for a sequence of α-Dirac-harmonic maps as α goes to 1, one successfully showed the general existence of Dirac-harmonic maps under suitable non-trivial boundary condition. This is the general existence scheme developed in [19].

Furthermore, when the domain is closed spin Riemannian manifold, the short time existence of the heat flow for Dirac-harmonic maps (1.1) is proved in [35] under some extra constant imposed for the spinor. Based on the notions and methods of α-Dirac-harmonic map and α-Dirac-harmonic map flow introduced in [19], and the techniques of handling the closed domain case developed in [35], some existence results for Dirac-harmonic maps from closed surfaces are recently obtained in [26, 27].

The rest of the article is organized as follows. In Section 2, two types of heat flow methods are discussed which yield some existence results of Dirac-harmonic maps. In Section 3, the refined blow-up analysis for approximating Dirac-harmonic maps arising from those two heat flows introduced in Section 2 are explored, such as (generalized) energy identity and (no) neck property. In the last section, we propose some problems related to these two heat flow approaches.

2 Heat Flow Method

In this section, we shall discuss two heat flow methods for the existence problem of Dirac-harmonic maps. Firstly, we need the following notations.

Notations Denote

$$E(\phi)=\frac{1}{2}\int_M |d\phi|^2 dM,\quad E(\psi)=\int_M |\psi|^4 dM,\quad E(\phi,\psi)=\int_M \left(|d\phi|^2+|\psi|^4\right) dM.$$

Denote

$$\Omega_s^t=\Omega\times[s,t],\quad M_s^t=M\times[s,t],\quad M^T=M\times[0,T]$$

and denote the standard Sobolev and Hölder spaces by

$$W_p^{2k,k}\left(M^T\right),\quad C^{2k,k,\alpha}\left(M^T\right) = C^{2k+\alpha,k+\frac{\alpha}{2}}\left(M^T\right),$$

$$C^{1,0,\alpha}\left(M^T\right) := C^{\alpha,\alpha/2}\left(M^T\right)\cap\left\{\sup_{0\le t\le T}\|u\|_{C^{1+\alpha}(M)}<\infty\right\}.$$

Finally,

$$V\left(M_s^t\right) := \left\{(\phi,\psi): M\times[s,t]\to N\times\left(\Sigma M\otimes\phi^{-1}TN\right)\ \Big|\ \sup_{s\le\sigma\le t}\|\nabla\phi\|_{L^2(M)}\right.$$
$$\left.+\sup_{s\le\sigma\le t}\|\psi\|_{W^{1,4/3}(M)}+\sup_{s\le\sigma\le t}\|\psi\|_{L^8(M)}+\int_{M_s^t}\left(|\partial_t\phi|^2+|\nabla^2\phi|^2\right)dMdt<\infty\right\}.$$

We recall some basic notions from spin geometry. Let M be a compact Riemann surface with smooth boundary ∂M, equipped with a Riemannian metric g and with a fixed spin structure, ΣM be the spinor bundle over M and $\langle\cdot,\cdot\rangle_{\Sigma M}$ be the natural Hermitian inner product on ΣM. Choosing a local orthonormal basis e_γ, $\gamma=1,2$ on M, the usual Dirac operator is defined as $\partial\!\!\!/ := e_\gamma\cdot\nabla_{e_\gamma}$, where ∇ is the spin connection on ΣM and $\cdot$ is the Clifford multiplication. This multiplication is skew-adjoint:

$$\langle X\cdot\psi,\varphi\rangle_{\Sigma M}=-\langle\psi,X\cdot\varphi\rangle_{\Sigma M}$$

for any $X\in\Gamma(TM)$, $\psi,\ \varphi\in\Gamma(\Sigma M)$. The usual Dirac operator $\partial\!\!\!/$ on a surface can be seen as the Cauchy–Riemann operator. Consider $\mathbb{R}^2$ equipped with the Euclidean metric dx^2+dy^2. Let $e_1=\frac{\partial}{\partial x}$ and $e_2=\frac{\partial}{\partial y}$ be the standard orthonormal frame. A spinor field on $\mathbb{R}^2$ is simply a map

$$\psi:\ \mathbb{R}^2\to\Delta_2=\mathbb{C}^2$$

and the action of e_1 and e_2 on spinors can be identified with multiplication with the following two matrices

$$e_1=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad e_2=\begin{pmatrix}0&i\\i&0\end{pmatrix}.$$

Let $\psi:=\begin{pmatrix}\psi_1\\\psi_2\end{pmatrix}:\mathbb{R}^2\to\mathbb{C}^2$ be a spinor field on $\mathbb{R}^2$, then the Dirac operator $\partial\!\!\!/$ is given by

$$\partial\!\!\!/\psi=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_1}{\partial x}\\\frac{\partial\psi_2}{\partial x}\end{pmatrix}+\begin{pmatrix}0&i\\i&0\end{pmatrix}\begin{pmatrix}\frac{\partial\psi_1}{\partial y}\\\frac{\partial\psi_2}{\partial y}\end{pmatrix}=2\begin{pmatrix}\frac{\partial\psi_2}{\partial\overline{z}}\\-\frac{\partial\psi_1}{\partial z}\end{pmatrix},$$

where

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\quad\frac{\partial}{\partial\overline{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

For more details on spin geometry and Dirac operators, we refer to [28].

2.1 Dirac-harmonic Map Flow

Let us recall the heat flow for Dirac-harmonic maps introduced in [9]. For a map $\phi \in C^{2,1,\alpha}(M \times (0,T]; N)$ and a spinor field $\psi \in C^{1,0,\alpha}(M \times (0,T]; \Sigma M \otimes \phi^* TN)$, we consider the following system

$$\begin{cases} \partial_t \phi = \tau(\phi) - \mathcal{P}(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi) & \text{in } M \times [0,T); \\ \not\partial \psi = \mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi) & \text{in } M \times [0,T), \end{cases} \tag{2.1}$$

where $\tau(\phi)$ is the tension field of ϕ. The above elliptic-parabolic system (2.1) is called the *Dirac-harmonic map flow*. Moreover, we impose the following boundary-initial data

$$\begin{cases} \phi(x,t) = \phi_0(x,t) & \text{on } M \times \{0\} \cup \partial M \times [0,T]; \\ \mathbf{B}\psi(x,t) = \mathbf{B}\psi_0(x,t) & \text{on } \partial M \times [0,T], \end{cases} \tag{2.2}$$

where $\phi_0 \in C^{2,1,\alpha}(M \times \{0\} \cup \partial M \times [0,T]; N)$, $\psi_0 \in C^{1,0,\alpha}\left(\partial M \times [0,T]; \Sigma M \otimes \phi^{-1}TN\right)$ and $\mathbf{B} = \mathbf{B}^{\pm}$ is the chiral boundary operator defined as follows:

$$\mathbf{B}^{\pm} : L^2\left(\partial M, \Sigma M \otimes \phi^{-1}TN|_{\partial M}\right) \to L^2\left(\partial M, \Sigma M \otimes \phi^{-1}TN|_{\partial M}\right)$$
$$\psi \mapsto \frac{1}{2}\left(Id \pm \overrightarrow{n} \cdot G\right) \cdot \psi,$$

where $\overrightarrow{n}$ is the outward unit normal vector field on ∂M, $G = ie_1 \cdot e_2$ is the chiral operator defined using a local orthonormal frame $\{e_\alpha\}_{\alpha=1}^2$ on M and satisfying the following properties:

$$G^2 = \mathrm{Id}, \quad G^* = G, \quad \nabla G = 0, \quad G \cdot X = -X \cdot G,$$

for any $X \in \Gamma(TM)$. The classical chiral boundary operator for usual spinors was firstly introduced in [14], see also [2, 16] for more abstract settings. In [10], the notion of chiral boundary operator was firstly extended to spinor fields along a map in order to propose the boundary value problems for Dirac-harmonic maps. The chiral boundary condition is also used in the study of boundary value problems for the super-Liouville system [23–25].

In fact, in the boundary-initial data (2.2), one can also take $\mathbf{B}$ to be the MIT bag boundary operator $\mathbf{B}^{\pm}_{MIT}$ or the J-boundary operator $\mathbf{B}^{\pm}_{J}$ as considered in [9, 10]. For the sake of convenience, in the sequel, we shall only consider the case of chiral boundary conditions and omit the discussion of the other two types of boundary conditions, as the arguments for them are the same.

In [9], a short time existence and uniqueness result for the flow (2.1) and (2.2) was obtained:

Theorem 2.1 [9, Theorem 1.3] *Let M^m $(m \geq 2)$ be a compact spin Riemannian manifold with smooth boundary ∂M, N be a compact Riemannian manifold. Suppose that*

$$\phi_0 \in \cap_{T>0} C^{2,1,\alpha}(M \times [0,T]; N)$$

and

$$\psi_0 \in \cap_{T>0} C^{1,0,\alpha}\left(\partial M \times [0,T]; \Sigma M \otimes \phi_0^* TN\right)$$

for some $0 < \alpha < 1$*, then the problem consisting of* (2.1) *and* (2.2) *admits a unique solution*

$$\phi \in \cap_{0<t<s<T_1} C^{2,1,\alpha}(M \times [t,s]) \cap C^0(M \times [0,T_1], N)$$

and

$$\psi \in \cap_{0<t<s<T_1} C^{1,0,\alpha}(M \times [t,s]) \cap C^{1,0,0}\left(M \times [0,T_1]; \Sigma M \otimes \phi^* TN\right)$$

for some time $T_1 > 0$ *which is characterized by*

$$\limsup_{t \nearrow T_1} \|\nabla \phi(\cdot, t)\|_{C^0(M)} = \infty.$$

In order to solve the Dirichlet-chiral problem for Dirac-harmonic maps from a two dimensional domain, [18] later studied the existence of a global weak solution of the Dirac-harmonic map flow. Before stating the next theorem, we give some definitions.

Define a constant $\Lambda = \Lambda(M, N)$.

$$\Lambda := \sup \Big\{ \widetilde{\Lambda} \in [0, \infty] : \text{For any } (\phi, \psi) \in W^{1,2}(M, N) \times W^{1,4/3}\left(M, \Sigma M \otimes \mathbb{R}^N\right), \text{ if } E(\phi) \le \widetilde{\Lambda}^2, \\ \text{then } \|\psi\|_{W^{1,4/3}(M)} \le C\left(M, N, \widetilde{\Lambda}\right)\left(\|\not{D}_\phi \psi\|_{L^{4/3}(M)} + \|\mathbf{B}\psi\|_{W^{1/4,4/3}(\partial M)}\right) \Big\}. \quad (2.3)$$

We remark that, in the above definition (2.3), if we considered $\phi \in W^{1,p}(M, N)$ with $p > 2$ and replaced $E(\phi)$ with $\|\phi\|_{W^{1,p}}$, then the corresponding constant Λ would be ∞ (see [18, Lemma 2.6] or [9, Theorem 1.1]). However, in the critical case of $\phi \in W^{1,2}(M, N)$, for general M and N, we do not know whether Λ is ∞ or not. It would be interesting to know how large the constant Λ can be.

In fact, what we know is that the constant Λ defined above has a positive lower bound (see [18, Lemma 2.9]). More precisely, we have

$$\Lambda \ge \frac{1}{2\Lambda_1 \cdot \Lambda_2 \cdot \Lambda_3} > 0,$$

where $\Lambda_1 = \Lambda_1(M, N) > 0$ (see [18, Lemma 2.7]) is the elliptic estimate constant for the usual Dirac operator $\not{\partial}$:

$$\|\psi\|_{W^{1,4/3}(M)} \le \Lambda_1 \left(\|\not{\partial}\psi\|_{L^{4/3}(M)} + \|\mathbf{B}\psi\|_{W^{1/4,4/3}(\partial M)}\right) \quad \forall\, \psi \in W^{1,4/3}\left(M, \Sigma M \otimes \mathbb{R}^N\right).$$

$\Lambda_2 = \Lambda_2(M, N) > 0$ is the following Sobolev embedding constant:

$$\|f\|_{L^4(M)} \le \Lambda_2 \|f\|_{W^{1,4/3}(M)} \quad \forall\, f \in W^{1,4/3}\left(M, \mathbb{R}^N\right)$$

and $\Lambda_3 > 0$ denotes any upper bound of the L^∞-norm $\|A\|_{L^\infty(N)}$ of the spinorial extension of the second fundamental form $\mathcal{A}$:

$$|\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi)| \le \sqrt{2}\|A\|_{L^\infty(N)}|d\phi||\psi|$$

for any $(\phi, \psi) \in W^{1,2}(M, N) \times W^{1,4/3}\left(\Sigma M \otimes \phi^{-1}TN\right)$. It is easy to see that if N is some compact region in the Euclidean space $\mathbb{R}^K$, then $\Lambda_3 > 0$ can be chosen to be arbitrary small and hence the constant Λ can be ∞. However, this is a trivial case, since then the maps ϕ become harmonic functions and the spinor fields ψ become harmonic spinors.

Now, we state the main theorem in [18] that

Theorem 2.2 [18, Theorem 1.2] *Let M be a compact Riemann spin surface with smooth boundary ∂M and let $N \subset \mathbb{R}^N$ be a compact Riemannian manifold. Suppose $\phi_0 \in H^1(M, N)$, $\varphi \in C^{2+\alpha}(\partial M, N)$, $\psi_0 \in C^{1+\alpha}(\partial M, \Sigma M \otimes \varphi^* TN)$ and satisfy the following boundary-initial constraint:*

$$E(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)} < \Lambda^2,$$

where $\Lambda = \Lambda(M, N) > 0$ is the constant defined in (2.3)*. Then there exists a global weak solution of* (2.1) *with the boundary-initial data* (2.2)*, which is defined in $M \times [0, \infty)$ and* *satisfies*

$$E(\pi(t)) + \int_{M^t} |\partial_t \phi|^2 dMdt \le E(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)} \quad \forall\, t \ge 0,$$

$$E(\phi(t)) + \frac{1}{2}\int_{\partial M} \left\langle \overrightarrow{n} \cdot \mathbf{B}\psi_0, \psi \right\rangle (t) \le E(\phi(s)) + \frac{1}{2}\int_{\partial M} \left\langle \overrightarrow{n} \cdot \mathbf{B}\psi_0, \psi \right\rangle (s) \quad \forall\, 0 \le s \le t < \infty.$$

Moreover, there exists an integer $K > 0$ depending only on M, N, $E(\phi_0)$, $\|\varphi\|_{C^{2+\alpha}(\partial M)}$ and $\|\mathbf{B}\psi_0\|_{C^{1+\alpha}(\partial M)}$ and there exist finitely many singular times $\{T_k\}$, $1 \le k \le K$, satisfying

$$\phi \in C^{2,1,\alpha}_{loc}\left(M \times \left((0,\infty) \setminus \{T_k\}_{k=1}^K\right)\right) \quad \textit{and} \quad \psi \in C^{1,0,\alpha}_{loc}\left(M \times \left((0,\infty) \setminus \{T_k\}_{k=1}^K\right)\right).$$

These singular times are characterized by the condition

$$\limsup_{\substack{x\in M \\ t \nearrow T_k}} E\left(\phi(t); B^M_R(x)\right) > \overline{\varepsilon} \quad \textit{for all} \;\; R > 0,$$

where $\overline{\varepsilon} > 0$ is a constant depending only on M, N, $E(\phi_0)$, $\|\varphi\|_{C^{2+\alpha}(\partial M)}$, $\|\mathbf{B}\psi_0\|_{C^{1+\alpha}(\partial M)}$ and $B^M_R(x)$ is the geodesic ball in M with center x and radius R.

Moreover, we show that, at each singular time $\{T_k\}$, that is, when the energy of the map concentrates, after some suitable space-time rescaling, a bubble, namely, a nontrivial Dirac-harmonic map from 2-sphere to N can split off.

Theorem 2.3 [18, Theorem 1.3] *Let (ϕ, ψ) be a solution to* (2.1) *with the boundary-initial data* (2.2) *from Theorem 2.2. Suppose T_1 is a singular time, i.e.,*

$$\limsup_{\substack{x\in M \\ t \nearrow T_1}} E\left(\phi(t); B^M_R(x)\right) > \overline{\varepsilon} \quad \textit{for all} \;\; R > 0.$$

There exist sequences $t_i \nearrow T_1$, $x_i \to x_0 \in M$, $r_i \to 0$ and a nontrivial Dirac-harmonic map $\left(\widetilde{\phi}, \widetilde{\psi}\right) : \mathbb{R}^2 \to N \times \left(\Sigma\mathbb{R}^2 \otimes \widetilde{\phi}^* TN\right)$, such that

(1) *if $x_0 \in M \setminus \partial M$, then as $i \to \infty$,*

$$\phi_i(x) := \phi(x_i + r_i x, t_i) \to \widetilde{\phi}(x) \qquad \textit{in } C^1_{loc}\left(\mathbb{R}^2\right) \quad \textit{and}$$
$$\psi_i(x) := \sqrt{r_i}\,\psi(x_i + r_i x, t_i) \to \widetilde{\psi}(x) \quad \textit{in } C^1_{loc}\left(\mathbb{R}^2\right).$$

$\left(\widetilde{\phi}, \widetilde{\psi}\right)$ has finite energy and conformally extends to a smooth Dirac-harmonic sphere.

(2) *if $x_0 \in \partial M$, then $\frac{dist(x_i, \partial M)}{r_i} \to \infty$ and the same bubbling statement as in* (1) *holds.*

We remark that in the above Theorem 2.3, for a boundary blow-up point, the case that $\frac{dist(x_i,\partial M)}{r_i}$ is uniformly bounded cannot occur (see [18, Theorem 1.4]).

With the help of the above theorems, we can now present some existence results for Dirac-harmonic maps from surfaces with boundary.

Theorem 2.4 [18, Theorem 1.5] *Let (ϕ, ψ) be a solution to* (2.1) *with the boundary-initial data* (2.2) *as obtained in Theorem 2.2 and defined in* $[0, \infty)$. *Then there exists a sequence $t_i \nearrow \infty$ such that $(\phi(\cdot, t_i), \psi(\cdot, t_i))$ converges weakly in $W^{1,2}(M) \times W^{1,4/3}(M)$ to a Dirac-harmonic map*

$$(\phi_\infty, \psi_\infty) \in C^{2+\alpha}(M, N) \times C^{1+\alpha}\left(M, \Sigma M \otimes \phi^*_\infty TN\right)$$

with boundary data $\phi_\infty|_{\partial M} = \varphi$ *and* $\mathbf{B}\psi_\infty|_{\partial M} = \mathbf{B}\psi_0$.

Furthermore, if we assume that the boundary-initial data are small enough, then the map part of the limiting Dirac-harmonic map $(\phi_\infty, \psi_\infty)$ obtained in the above theorem has to be homotopic to the initial map ϕ_0.

Corollary 2.5 [18, Corollary 1.6] *We define a constant* $\varepsilon_0 = \varepsilon_0(N) > 0$*:*

$$\varepsilon_0 := \inf\left\{E(\phi) \mid (\phi, \psi) : S^2 \to N \text{ is a nontrivial smooth Dirac-harmonic map}\right\}.$$

For any $\phi_0 \in H^1(M, N) \cap C^0(M, N)$*,* $\varphi \in C^{2+\alpha}(\partial M, N)$*,* $\psi_0 \in C^{1+\alpha}(\partial M, \Sigma M \otimes \varphi^* TN)$*, if*

$$E(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)} < \min\left\{\Lambda^2, \varepsilon_0\right\}, \tag{2.4}$$

where $\Lambda > 0$ *is defined in* (2.3)*, there exists a Dirac-harmonic map* $(\phi, \psi) : M \to N$ *with* ϕ *lying in the same homotopy class as* ϕ_0.

In fact, from the proof of Corollary 1.6 in [18], it is not hard to see that the upper bound in (2.4) can be improved. Here, we state as a new theorem.

Theorem 2.6 *We define two constants* $\varepsilon_0 = \varepsilon_0(N) > 0$ *and* $\varepsilon_1 = \varepsilon_1(M, N)$*:*

$$\begin{aligned}\varepsilon_0 &:= \inf\left\{E(\phi) \mid (\phi, \psi) : S^2 \to N \text{ is a nontrivial smooth Dirac-harmonic map}\right\},\\ \varepsilon_1 &:= \inf\left\{E(u) \mid u \in W^{1,2}(M, N),\ u|_{\partial M} = \phi_0\right\}.\end{aligned}$$

For any $\phi_0 \in H^1(M, N) \cap C^0(M, N)$*,* $\varphi \in C^{2+\alpha}(\partial M, N)$*,* $\psi_0 \in C^{1+\alpha}(\partial M, \Sigma M \otimes \varphi^* TN)$*, if*

$$E(\phi_0) + \sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)} < \min\left\{\Lambda^2, \varepsilon_0 + \varepsilon_1\right\},$$

where $\Lambda > 0$ *is defined in* (2.3)*, there exists a Dirac-harmonic map* $(\phi, \psi) : M \to N$ *with* ϕ *lying in the same homotopy class as* ϕ_0.

We remark that the constant $\varepsilon_1 = \varepsilon_1(M, N)$ is positive when the given boundary map ϕ_0 is not a constant map.

Proof It is sufficient to prove that no blow-up will occur along the flow. In fact, if the flow blows up at some singular time $T \leq \infty$, then there exists at one singularity (x_0, T). By the proof of Theorem 1.2 and Theorem 1.5 in [18], we know there exists a weak limit $(\phi(x, T), \psi(x, T))$ of $(\phi(x, t_i), \psi(x, t_i))$ in the sense of $W^{1,2}(M) \times W^{1,\frac{4}{3}}(M)$ as $t_i \to T$. By Theorem 2.3, some nontrivial Dirac-harmonic spheres appear. Assume $\left(\widetilde{\phi}, \widetilde{\psi}\right)$ is one, then by Theorem 2.3, we have

$$\begin{aligned}E(\phi(\cdot,T)) &= \lim_{R\to 0} E\left(\phi(\cdot,T), M\setminus B_R^M(x_0)\right)\\ &\le \lim_{R\to 0}\liminf_{t\nearrow T} E\left(\phi(\cdot,t), M\setminus B_R^M(x_0)\right)\\ &= \lim_{R\to 0}\liminf_{t\nearrow T}\left(E(\phi(\cdot,t)) - E(\phi(\cdot,t), B_R^M(x_0))\right)\\ &\le \liminf_{t\nearrow T} E(\phi(\cdot,t)) - \lim_{R\to 0}\limsup_{t\nearrow T} E\left(\phi(\cdot,t), B_R^M(x_0)\right)\\ &\le \liminf_{t\nearrow T} E(\phi(\cdot,t)) - E\left(\widetilde{\phi}\right).\end{aligned}$$

However, by Lemma 3.2 in [18], we have

$$\begin{aligned}\varepsilon_0+\varepsilon_1 \le E\left(\widetilde{\phi}\right)+E(\phi(T)) &\le \limsup_{t\to T} E(\phi) \le E(\phi_0)+\sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}\\ &< \min\left\{\Lambda^2, \varepsilon_0+\varepsilon_1\right\}.\end{aligned}$$

This is a contradiction which finishes the proof. □

2.2 α-Dirac-harmonia Map Flow

We note that a technical difficulty for the long time behavior of the Dirac-harmonic map flow stems from the fact that along the Dirac-harmonic map flow considered in Theorem 2.2, we only have that the energy of the map ϕ is uniformly bounded, i.e.,

$$E(\phi(\cdot,t)) = \int_M |\nabla\phi(\cdot,t)|^2 dM \le C < +\infty.$$

However, the Dirac type equation for the spinor ψ does not provide good control of the energy of the spinor field

$$E(\psi(\cdot,t)) = \int_M |\psi(\cdot,t)|^4 dM,$$

as time t approaches the first singular time $T_1 > 0$, even for the L^1-norm. This is the main difficulty and why we need to impose the additional boundary-initial constraint in Theorem 2.2 in order to obtain a global weak solution to the Dirac-harmonic map flow and prove some existence results by letting time t goes to infinity.

The general question we are interested in, however, is the following

Question Does there exist a Dirac-harmonic map from a compact Riemann surface with boundary to a compact Riemannian manifold with general Dirichlet-chiral boundary data?

To investigate this issue, in [19], we introduced a new parabolic-elliptic system and gave an affirmative answer to this question. In our new approach, one crucial observation is the following key estimate for the Dirac operator $\not{D}$ along a given map ϕ (see [19, Lemma 3.4]):

Key Estimate Let $\phi \in W^{1,q}(M,N)$ for some $q > 2$ and $\psi \in W^{1,p}(M, \Sigma M\otimes\phi^*TN)$ for some $1 < p < 2$, then there holds

$$\|\psi\|_{W^{1,p}(M)} \le C\left(p, M, N, \|\nabla\phi\|_{L^q(M)}\right)\left(\|\not{D}\psi\|_{L^p(M)} + \|\mathbf{B}\psi\|_{W^{1-1/p,p}(\partial M)}\right). \quad (2.5)$$

We remark that the above estimate has two key properties. The first one is that the positive constant $C = C(p, M, N, \|\nabla\phi\|_{L^q(M)}) > 0$ depends on the norm $\|\nabla\phi\|_{L^q(M)}$ with $q > 2$ of the map ϕ. The second one is that the two numbers $q > 2$ and $1 < p < 2$ are independent of each other. In fact, such kind of estimate holds true for more general Dirac type systems (see [19, Lemma 3.3]) which should be useful in other problems.

Note that the key estimate for the Dirac operator $\not{D}$ along a given map in (2.5) requires that the map ϕ lies in $W^{1,q}(M, N)$ for some $q > 2$. Inspired by this fact and the well known Sacks–Uhlenbeck's approximation, in [19], we introduced the following functional

$$L_\alpha(\phi, \psi) = \frac{1}{2}\int_M \left\{\left(1 + |d\phi|^2\right)^\alpha + \langle\psi, \not{D}\psi\rangle\right\} dM,$$

where $\alpha > 1$ is a constant. Critical points $(\phi_\alpha, \psi_\alpha)$ of the above functional L_α are called *α-Dirac-harmonic maps* from M to N. When the spinor field is vanishing, the above functional reduces to Sacks–Uhlenbeck's approximation for harmonic maps in [33].

By a direct computation, one can verify that critical points $(\phi_\alpha, \psi_\alpha)$ of the new functional L_α satisfy the following Euler–Lagrange equations (see [19, Lemma 3.2]):

$$\begin{aligned} \Delta_g\phi &= -(\alpha - 1)\frac{\nabla_g|\nabla_g\phi|^2\nabla_g\phi}{1 + |\nabla_g\phi|^2} - A(d\phi, d\phi) + \frac{Re\left(P(\mathcal{A}\left(d\phi(e_\gamma), e_\gamma\cdot\psi\right); \psi)\right)}{\alpha(1 + |\nabla_g\phi|^2)^{\alpha-1}},\\ \not{\partial}_g\psi &= \mathcal{A}\left(d\phi(e_\gamma), e_\gamma\cdot\psi\right). \end{aligned}$$

One crucial step in our new scheme is to get the existence result of Dirac-harmonic maps through studying the limit behaviour of a sequence of α-Dirac-harmonic maps as $\alpha \searrow 1$.[1] Suppose that there exists a sequence of α-Dirac-harmonic maps $(\phi_\alpha, \psi_\alpha)$ with

$$E_\alpha(\phi_\alpha) := \int_M \left(1 + |d\phi_\alpha|^2\right)^\alpha dM \le \Lambda < \infty,$$

then the key estimate (2.5) implies the following uniform control of the spinors:

$$\|\psi_\alpha\|_{W^{1,p}(M)} \text{ with } 1 < p < 2, \text{ is uniformly bounded as } \alpha \searrow 1.$$

Thus, we can do the blow-up analysis and we will show that the weak limit is just the desired Dirac-harmonic map. In the case of a two dimensional domain surface, this approach is better than the Dirac-harmonic map flow [9, 18], and therefore, here lies the advantage of considering α-Dirac-harmonic maps.

The remaining task then is to show the existence of such a sequence of α-Dirac-harmonic maps. In fact, this is one crucial step in our new scheme. Since the second term of the functional L_α is not bounded from below, classical Ljusternik–Schnirelman theory may not be applied here to obtain critical points of L_α. Therefore, we need to develop a new method to proceed with our scheme.

In [19], we introduced the following new parabolic-elliptic system:

$$\partial_t\phi = \Delta_g\phi + (\alpha - 1)\frac{\nabla_g|\nabla_g\phi|^2\nabla_g\phi}{1 + |\nabla_g\phi|^2} + A(d\phi, d\phi) - \frac{Re\left(P(\mathcal{A}\left(d\phi(e_\gamma), e_\gamma\cdot\psi\right); \psi)\right)}{\alpha\left(1 + |\nabla_g\phi|^2\right)^{\alpha-1}}, \quad (2.6)$$

$$\not{\partial}_g\psi = \mathcal{A}\left(d\phi(e_\gamma), e_\gamma\cdot\psi\right), \quad (2.7)$$

[1] Here and in the sequel, for simplicity of notations, when talking about a sequence of $(\phi_\alpha, \psi_\alpha)$ for $\alpha \searrow 1$, we mean the sequence of $(\phi_{\alpha_k}, \psi_{\alpha_k})$ for a given sequence of $\alpha_k \searrow 1$.

with the following boundary-initial data:

$$\begin{cases} \phi(x,t) = \varphi(x) & \text{on } \partial M \times [0,T]; \\ \phi(x,0) = \phi_0(x) & \text{in } M; \\ \mathbf{B}\psi(x,t) = \mathbf{B}\psi_0(x) & \text{on } \partial M \times [0,T]; \\ \phi_0(x) = \varphi(x) & \text{on } \partial M. \end{cases} \tag{2.8}$$

The new system (2.6)–(2.7) are called the *α-Dirac-harmonic map flow*.

Now, we state a result about the global existence of the α-Dirac-harmonic map flow with a Dirichlet-chiral boundary condition.

Theorem 2.7 [19, Theorem 2.1] *Let M be a compact spin Riemann surface with smooth boundary ∂M and let $N \subset \mathbb{R}^K$ be a compact Riemannian manifold. Suppose*

$$1 < \alpha < 1 + \varepsilon_1,$$

where $\varepsilon_1 > 0$ is the positive constants depending only on M and N. Then for any $\phi_0 \in C^{2+\lambda}(M, N)$, $\varphi \in C^{2+\lambda}(\partial M, N)$, $\psi_0 \in C^{1+\lambda}(\partial M, \Sigma M \otimes \varphi^* TN)$ where $0 < \lambda < 1$ is a constant, there exists a unique global solution

$$\phi \in C^{2+\lambda, 1+\frac{\lambda}{2}}_{loc}(M \times [0,\infty), N)$$

and

$$\psi \in C^{\lambda, \frac{\lambda}{2}}_{loc}(M \times [0,\infty), \Sigma M \otimes \phi^* TN) \cap L^\infty\left([0,\infty), \|\psi(\cdot,t)\|_{C^{1+\lambda}(M)}\right)$$

to the problem (2.6)–(2.7) with boundary-initial data (2.8), satisfying

$$E_\alpha(\phi(t)) \le E_\alpha(\phi_0) + 2\sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}$$

and

$$\|\psi(\cdot,t)\|_{W^{1,p}(M)} \le C\left(p, M, N, E_\alpha(\phi_0) + 2\sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}\right),$$

where $1 < p < 2$.

Moreover, there exist a time sequence $t_i \to \infty$ and an α-Dirac-harmonic map

$$(\phi_\alpha, \psi_\alpha) \in C^{2+\lambda}(M, N) \times C^{1+\lambda}\left(M, \Sigma M \otimes \phi_\alpha^* TN\right)$$

with the boundary data

$$(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0),$$

such that $(\phi(\cdot,t_i), \psi(\cdot,t_i))$ converges to $(\phi_\alpha, \psi_\alpha)$ in $C^2(M) \times C^1(M)$.

When the spinor field is vanishing and the domain is a closed surface, our flow reduces to the so called Sacks–Uhlenbeck flow studied in [17].

By Theorem 2.7, for any $\alpha > 1$ sufficiently close to 1, there exists an α-Dirac-harmonic map $(\phi_\alpha, \psi_\alpha) \in C^{2+\lambda}(M, N) \times C^{1+\lambda}(M, \Sigma M \otimes \phi_\alpha^* TN)$ with the Dirichlet-chiral boundary condition $(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$ and satisfies the following two properties

$$E_\alpha(\phi_\alpha) \le E_\alpha(\phi_0) + 2\sqrt{2}\,\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}$$

and

$$\|\psi_\alpha\|_{W^{1,p}(M)} \le C\left(p, M, N, E_\alpha(\phi_0) + 2\sqrt{2}\|\mathbf{B}\psi_0\|^2_{L^2(\partial M)}\right)$$

for any $1 < p < 2$. With this result in hand, we are able to prove the existence of Dirac-harmonic maps by applying the blow-up analysis.

Generally, we have the following existence theorem of Dirac-harmonic maps corresponding to the previous **Question**.

Theorem 2.8 [19, Theorem 2.2] *Let $(\phi_\alpha, \psi_\alpha) : M \to N$ be a sequence of α-Dirac-harmonic maps with Dirichlet-chiral boundary condition $(\phi_\alpha, \mathbf{B}\psi_\alpha)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$ and with uniformly bounded energy*

$$E_\alpha(\phi_\alpha) + \|\psi_\alpha\|_{L^4(M)} \le \Lambda.$$

Denoting $E(\phi_\alpha; \Omega) := \int_\Omega |\nabla\phi_\alpha|^2 dvol_g$, $\Omega \subset M$ and the energy concentration set

$$\mathbf{S} := \left\{ x \in M \mid \liminf_{\alpha\to 1} E(\phi_\alpha; B_r^M(x)) \ge \frac{\varepsilon_0}{2} \ \textit{for all}\ r > 0 \right\},$$

where ε_0 is the positive constant depending only on M, N, $B_r^M(x)$ is the geodesic ball in M with center point x and radius r, then $\mathbf{S}$ is a finite set. Moreover, after selection of a subsequence of $(\phi_\alpha, \psi_\alpha)$ (without changing notation), there exists a Dirac-harmonic map

$$(\phi, \psi) \in C^{2+\lambda}(M, N) \times C^{1+\lambda}(M, \Sigma M \otimes \phi^* TN)$$

with Dirichlet-chiral boundary data $(\phi, \mathbf{B}\psi)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$, such that

$$(\phi_\alpha, \psi_\alpha) \to (\phi, \psi) \quad \textit{in}\ C^2_{loc}(M \setminus \mathbf{S}) \times C^1_{loc}(M \setminus \mathbf{S}).$$

We remark that since we can impose nontrivial boundary conditions for both the map and the spinor field, we shall obtain Dirac-harmonic maps with nontrivial map part as well as nontrivial spinor part.

Moreover, similarly to the Dirac-harmonic map flow, we show that at each singular point x_0, where the energy of the map concentrates, after suitable rescaling, a bubble, namely, a nontrivial Dirac-harmonic sphere splits off.

Theorem 2.9 [19, Theorem 2.4] *Under the same assumption as in Theorem 2.8, suppose $x_0 \in \mathbf{S}$ is an energy concentration point, i.e.,*

$$\liminf_{\alpha\to 1} E\left(\phi_\alpha; B_r^M(x_0)\right) \ge \frac{\varepsilon_0}{2} \quad \textit{for all}\ r > 0.$$

Then,

(1) *if $x_0 \in M \setminus \partial M$, there exist a subsequence of $(\phi_\alpha, \psi_\alpha)$ (still denoted by $(\phi_\alpha, \psi_\alpha)$) and sequences $x_\alpha \to x_0$, $\lambda_\alpha \to 0$ and a nontrivial Dirac-harmonic map $(\sigma, \xi) : \mathbb{R}^2 \to N$, such that as $\alpha \to 1$,*[2]

$$\left(\phi_\alpha(x_\alpha + \lambda_\alpha x), \lambda_\alpha^{\alpha-1}\sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha + \lambda_\alpha x)\right) \to (\sigma(x), \xi(x)) \quad \textit{in}\ C^1_{loc}\left(\mathbb{R}^2\right) \times C^0_{loc}\left(\mathbb{R}^2\right).$$

(σ, ξ) has finite energy and conformally extends to a smooth Dirac-harmonic sphere.[3]

(2) *if $x_0 \in \partial M$, then $\frac{dist(x_\alpha, \partial M)}{\lambda_\alpha} \to \infty$ and the same bubbling statement as in (1) holds.*

Furthermore, we can show that the bubbles in the above Theorem 2.9 are in fact just harmonic spheres [19].

So far, we have answered the **Question** about the existence of Dirac-harmonic maps with given Dirichlet-chiral boundary data. It is natural to ask whether the map component ϕ of the limit Dirac-harmonic map stays in the same homotopy class as ϕ_0.

---

[2]Compared to the usual rescaling, i.e. $\left(\phi_\alpha(x_\alpha + \lambda_\alpha x), \sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha + \lambda_\alpha x)\right)$, for a blow-up sequence of Dirac-harmonic maps given in [5], here the additional factor $\lambda_\alpha^{\alpha-1}$ comes from the fact that α-Dirac-harmonic maps are not conformally invariant.

[3]Here we have used the fact that the unique spin structure on $\mathbb{S}^2 \setminus \{p\}$ extends to the unique spin structure on $\mathbb{S}^2$ and so does the associated spinor bundle.

Here we give a positive answer under some natural condition as in the harmonic map case.

Theorem 2.10 [19, Theorem 2.5] *Let M be a compact spin Riemann surface with smooth boundary ∂M and let $N \subset \mathbb{R}^K$ be a compact Riemannian manifold. For any $\phi_0 \in C^{2+\lambda}(M, N)$, $\varphi \in C^{2+\lambda}(\partial M, N)$, $\psi_0 \in C^{1+\lambda}(\partial M, \Sigma M \otimes \varphi^* TN)$ where $\phi_0|_{\partial M} = \varphi$ and $0 < \lambda < 1$ is a constant, if (N, h) dose not admit any nontrivial harmonic sphere, then there exists a Dirac-harmonic map*

$$(\phi, \psi) \in C^{2+\lambda}(M, N) \times C^{1+\lambda}(M, \Sigma M \otimes \phi^* TN)$$

with Dirichlet-chiral boundary data $(\phi, \mathbf{B}\psi)|_{\partial M} = (\varphi, \mathbf{B}\psi_0)$ such that the map component ϕ is in the same homotopy class as ϕ_0.

3 Refined Blow-up Analysis

In this section, we will study some refined blow-up analysis for sequences of approximating Dirac-harmonic maps arising from the two approaches discussed in Section 2.

3.1 Approximate Dirac-harmonic Maps

In order to investigate the blow-up picture near a singularity of the Dirac-harmonic map flow, we shall first define approximate Dirac-harmonic maps. Denote

$$W^{2,2}(M, N) := \left\{\phi \in W^{2,2}\left(M, \mathbb{R}^K\right) \text{ with } \phi(x) \in N \text{ for a.e. } x \in M\right\},$$
$$W^{1,4/3}(M, \Sigma M \otimes \phi^\star TN) := \left\{\psi \in W^{1,4/3}\left(M, \Sigma M \otimes \mathbb{R}^K\right) \text{ with } \psi(x) \in \Sigma M \otimes \phi^\star TN \text{ for a.e. } x \in M\right\}.$$

In this part, we want to consider pairs (ϕ, ψ) that satisfy the Euler–Lagrange equations for Dirac-harmonic maps up to some error term in L^1. Here is the precise definition.

Definition 3.1 $(\phi, \psi) \in W^{2,2}(M, N) \times W^{1,\frac{4}{3}}(M, \Sigma M \times \phi^\star TN)$ is called an approximate Dirac-harmonic map if there exists a pair $(\tau(\phi, \psi), h(\phi, \psi)) \in L^1(M)$ such that

$$\tau(\phi, \psi) = \Delta\phi + A(d\phi, d\phi) - Re\left(P(\mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi); \psi)\right),$$
$$h(\phi, \psi) = \not{\partial}\psi - \mathcal{A}(d\phi(e_\alpha), e_\alpha \cdot \psi).$$

We remark that such approximate Dirac-harmonic map appear in the Dirac-harmonic map flow.

Thus, a pair of field (ϕ, ψ) is a Dirac-harmonic map if and only if $\tau(\phi, \psi) = h(\phi, \psi) = 0$. In the sequel, we shall assume that the error terms are in spaces smaller than L^1.

Theorem 3.2 ([20, Theorem 1.2] and [21, Theorem 1.1]) *Consider a sequence of approximate Dirac-harmonic maps $(\phi_n, \psi_n) \in C^2(M, N) \times C^1(M, \Sigma M \otimes \phi^* TN)$ from a compact Riemann surface M with smooth boundary ∂M to a compact Riemannian manifold N satisfying*

$$E(\phi_n, \psi_n) + \|\tau(\phi_n, \psi_n)\|_{L^2} + \|h(\phi_n, \psi_n)\|_{L^4} \le \Lambda,$$

and with boundary data

$$\phi_n|_{\partial M} = \varphi, \quad \mathbf{B}\psi_n|_{\partial M} = \mathbf{B}\chi,$$

where $\varphi \in C^{2+\alpha}(\partial M, N)$*,* $\chi \in C^{1+\alpha}(\partial M, \Sigma M \otimes \phi^*TN)$ *for some* $0 < \alpha < 1$*. We assume that* $(\phi_n, \psi_n) \rightharpoonup (\phi, \psi)$ *weakly in* $W^{1,2}(M, N) \times L^4(M, \Sigma M \otimes \phi^*TN)$*. Define the blow-up set*

$$\mathcal{S} := \cap_{r>0} \left\{ x \in M \;\middle|\; \liminf_{n\to\infty} \int_{D(x,r)} \left(|d\phi_n|^2 + |\psi_n|^4 \right) \geq \bar{\varepsilon} \right\},$$

where $\bar{\varepsilon} > 0$ *is the constant depending only on* M, N*. Then* $\mathcal{S}$ *is a (possibly empty) finite set* $\{p_1, \ldots, p_q, \ldots, p_I\}$*, where* $1 \leq q \leq I$*,* $\{p_1, \ldots, p_q\} \in M \setminus \partial M$*,* $\{p_{q+1}, \ldots, p_I\} \in \partial M$*. Moreover, a subsequence, still denoted by* $\{(\phi_k, \psi_k)\}$*, converges weakly in* $W^{2,2}_{loc}(M \setminus \mathcal{S}) \times W^{1,2}_{loc}(M \setminus \mathcal{S})$ *to* (ϕ, ψ) *and for each* $i = 1, \ldots, I$*, there is a finite set of Dirac-harmonic spheres* $\left(\sigma_i^l, \xi_i^l\right) : S^2 \to N$*,* $l = 1, \ldots, L_i$*, such that*

$$\lim_{n\to\infty} E(\phi_n) = E(\phi) + \sum_{i=1}^{I} \sum_{l=1}^{L_i} E\left(\sigma_i^l\right),$$

$$\lim_{n\to\infty} E(\psi_n) = E(\psi) + \sum_{i=1}^{I} \sum_{l=1}^{L_i} E\left(\xi_i^l\right),$$

and the image $\phi(M \setminus \partial M) \cup \bigcup_{i=1}^{q} \bigcup_{l=1}^{L_i} \left(\sigma_i^l\left(S^2\right)\right)$ *is a connected set.*

When (ϕ_n, ψ_n) are Dirac-harmonic maps, namely, all the error terms are vanishing, the corresponding blow-up analysis including energy identity and no neck property were proved in [5, 30, 36].

As an application of Theorem 3.2, we shall study the asymptotic behavior at the infinite time for the Dirac-harmonic map flow in dimension 2.

Theorem 3.3 ([20, Theorem 1.4] and [21, Theorem 1.3]) *Let* M *be a compact spin Riemann surface with smooth boundary* ∂M*. Let* $\phi_0 \in H^1(M, N)$*,* $\varphi \in C^{2+\alpha}(\partial M, N)$*,* $\chi \in C^{1+\alpha}(\partial M, \Sigma M \otimes \varphi^*TN)$*. Let* $(\phi, \psi) : M \times [0, \infty) \to N \times (\Sigma M \otimes \phi^*TN)$ *be a global weak solution of* (2.1) *and* (2.2)*, which has finitely many singular times. Then there exist* $t_n \uparrow \infty$*, a Dirac-harmonic map* $(\phi_\infty, \psi_\infty) \in C^{2+\alpha}(M, N) \times C^{1+\alpha}(M, \Sigma M \otimes \phi_\infty^* TN)$ *with boundary data* $\phi_\infty|_{\partial M} = \varphi$ *and* $\mathbf{B}\psi_\infty|_{\partial M} = \mathbf{B}\chi$*, nonnegative integer* I *and a possibly empty set with at most finitely many points* $\{p_1, \ldots, p_q, \ldots, p_I\} \subset M$*, where* $1 \leq q \leq I$*,* $\{p_1, \ldots, p_q\} \in M \setminus \partial M$*,* $\{p_{q+1}, \ldots, p_I\} \in \partial M$ *such that*

(1) $(\phi_n, \psi_n) := (\phi(\cdot, t_n), \psi(\cdot, t_n)) \rightharpoonup (\phi_\infty, \psi_\infty)$ *in* $W^{1,2}(M, N) \times L^4(M, \Sigma M \times \phi_\infty^* TN)$*;*
(2) $(\phi_n, \psi_n) \to (\phi_\infty, \psi_\infty)$ *in* $W^{1,2}_{loc}(M \setminus \{p_1, \ldots, p_I\}) \times L^4_{loc}(M \setminus \{p_1, \ldots, p_I\})$*;*
(3) *For* $1 \leq i \leq I$*, there exist a positive integer* L_i *and* L_i *nontrivial Dirac-harmonic spheres* $\left(\sigma_i^l, \xi_i^l\right) : S^2 \to N$*,* $i = 1, \ldots, I$*;* $l = 1, \ldots, L_i$ *such that*

$$\lim_{n\to\infty} E(\phi_n) = E(\phi_\infty) + \sum_{i=1}^{I} \sum_{l=1}^{L_i} E\left(\sigma_i^l\right),$$

$$\lim_{n\to\infty} E(\psi_n) = E(\psi_\infty) + \sum_{i=1}^{I} \sum_{l=1}^{L_i} E\left(\xi_i^l\right),$$

and the image $\phi_\infty(M \setminus \partial M) \cup \bigcup_{i=1}^{q} \bigcup_{l=1}^{L_i} \left(\sigma_i^l\left(S^2\right)\right)$ *is a connected set.*

3.2 α-Dirac-harmonic Maps

In this part, we shall study the limit behavior for a sequence of α-Dirac-harmonic maps as $\alpha \searrow 1$.

Since in general, multiple bubbles can split off at a blow-up point and the functional L_α is not conformally invariant, to better understand the multiple bubbling behavior for α-Dirac-harmonic maps, we shall consider the following more general α-energy functionals[4]

$$L_{\alpha,\sigma_\alpha}(\phi,\psi) = \frac{1}{2}\int_{D_1(0)} \left\{\left(\sigma_\alpha + |\nabla_{g_\alpha}\phi|^2\right)^\alpha + \sigma_\alpha^{1-\alpha}\langle\psi, \not{D}\psi\rangle\right\} dvol_{g_\alpha}, \quad \alpha > 1,$$

where $g_\alpha = e^{\varphi_\alpha}\left(\left(dx^1\right)^2 + \left(dx^2\right)^2\right)$, $\varphi_\alpha \in C^\infty(D_1)$, $\varphi_\alpha(0) = 0$, φ_α converges smoothly to $\varphi_0 \in C^\infty(D_1)$ and $\sigma_\alpha > 0$ is a constant.

Critical points of L_{α,σ_α} are called *general α-Dirac-harmonic maps*, and they satisfy the following Euler–Lagrange equations

$$\begin{aligned} \Delta_{g_\alpha}\phi &= -(\alpha-1)\frac{\nabla_{g_\alpha}|\nabla_{g_\alpha}\phi|^2\nabla_{g_\alpha}\phi}{\sigma_\alpha + |\nabla_{g_\alpha}\phi|^2} - A(d\phi, d\phi) + \frac{Re\left(P(\mathcal{A}\left(d\phi(e_\gamma), e_\gamma\cdot\psi\right);\psi)\right)}{\alpha\left(\sigma_\alpha + |\nabla_{g_\alpha}\phi|^2\right)^{\alpha-1}}, \\ \not{\partial}_{g_\alpha}\psi &= \mathcal{A}\left(d\phi(e_\gamma), e_\gamma\cdot\psi\right). \end{aligned}$$

Before presenting further results, we shall first give a general description of the blow-up procedure and the bubbling phenomena for general α-Dirac-harmonic maps. We shall follow the general scheme as in the case of α-harmonic maps [29, 33].

Denote

$$E_{\alpha,\sigma_\alpha}(\phi) = \int_M \left(\sigma_\alpha + |d\phi|^2\right)^\alpha dM, \quad E_\alpha(\phi) = \int_M \left(1 + |d\phi|^2\right)^\alpha dM.$$

Consider a sequence of general α-Dirac-harmonic maps $\{(\phi_\alpha, \psi_\alpha)\} : M \to N$ with Dirichlet-chiral boundary data $(\phi_\alpha, \mathbf{B}\psi_\alpha)|_M = (\varphi, \mathbf{B}\psi_0)$ and with $\sigma_\alpha > 0$ satisfying

$$0 < \beta_0 \le \liminf_{\alpha\searrow 1} \sigma_\alpha^{\alpha-1} \le 1$$

for some $\beta_0 > 0$ and with uniformly bounded energy

$$E_{\alpha,\sigma_\alpha}(\phi_\alpha) + E(\psi_\alpha) \le \Lambda.$$

From Theorems 2.8 and 2.9, we know that, by passing to a subsequence, $(\phi_\alpha, \psi_\alpha)$ converges strongly to some limit Dirac-harmonic map $(\phi, \psi) : M \to N$ with Dirichlet-chiral boundary data $(\phi, \mathbf{B}\psi)|_M = (\varphi, \mathbf{B}\psi_0)$, away from at most finitely many blow-up points $\mathbf{S} = \{x_i\}_{i=1}^I$, as $\alpha \searrow 1$. Moreover, we show that at each blow-up point where the energy of the map concentrates, after suitable rescaling, a bubble, namely, a nontrivial Dirac-harmonic sphere can split off.

More precisely, for a fixed blow-up point x_i, $1 \le i \le I$, we may assume there are k_i bubbles occurring at this point, i.e., there are a sequence of points $\left\{x_\alpha^{ij}\right\}$, $j = 1, \ldots, k_i$, and a sequence of positive numbers $\left\{\lambda_\alpha^{ij}\right\}$ with $x_\alpha^{ij} \to x_i$, $\lambda_\alpha^{ij} \to 0$ as $\alpha \searrow 1$ and one of the following two alternatives holds true: if $1 \le j_1, j_2 \le k_i$ and $j_1 \ne j_2$,

[4]It is easy to check that a rescaled α-Dirac-harmonic map, e.g. $\left(\phi_\alpha(\lambda_\alpha x), \lambda_\alpha^{\alpha-1}\sqrt{\lambda_\alpha}\psi_\alpha(r_\alpha x)\right)$ is locally a critical point of this functional, we refer to Section 5 in [19] for details. We refer to the beginning of Section 2 in [29] for the analogous case of α-harmonic maps.

(A1) for any fixed $R > 0$, $B^M_{R\lambda_\alpha^{ij_1}}\left(x_\alpha^{ij_1}\right) \cap B^M_{R\lambda_\alpha^{ij_2}}\left(x_\alpha^{ij_2}\right) = \emptyset$, whenever $\alpha > 1$ is sufficiently close to 1.

(A2) $\frac{\lambda_\alpha^{ij_1}}{\lambda_\alpha^{ij_2}} + \frac{\lambda_\alpha^{ij_2}}{\lambda_\alpha^{ij_1}} = \infty$ as $\alpha \searrow 1$.

Moreover, one can show that the following two rescaled fields[5]

$$\sigma_\alpha^{ij} := \phi_\alpha\left(x_\alpha^{ij} + \lambda_\alpha^{ij} x\right), \quad \xi_\alpha^{ij} := \left(\lambda_\alpha^{ij}\right)^{\alpha-1}\sqrt{\lambda_\alpha^{ij}}\psi_\alpha\left(x_\alpha^{ij} + \lambda_\alpha^{ij} x\right)$$

converge in $C^k_{loc}\left(\mathbb{R}^2 \setminus \left\{p_1^{ij}, \dots, p_{s_j}^{ij}\right\}\right)$ to a nontrivial Dirac-harmonic map $\left(\sigma^{ij}, \xi^{ij}\right)$ defined on $\mathbb{R}^2$, which can be conformally extended to a nontrivial Dirac-harmonic map from $\mathbb{S}^2$. See the beginning of Section 7 in [19].

We define two types of quantities as follows

$$\mu_{ij} = \liminf_{\alpha \searrow 1}\left(\lambda_\alpha^{ij}\right)^{2-2\alpha}, \qquad \nu_{ij} = \liminf_{\alpha \searrow 1}\left(\lambda_\alpha^{ij}\right)^{-\sqrt{\alpha-1}}. \tag{3.1}$$

It is easy to check that $\nu_{ij} \in [1, \infty]$. Also, we can see that there exists a positive constant $\mu_{max} \geq 1$ such that $\mu_{ij} \in [1, \mu_{max}]$. In fact, for the sake of simplicity, we may assume that there is only one blow-up point which is denoted by $x \in M$, and there are k_1 bubbles occurring at this point, i.e., there are a sequence of points $\left\{x_\alpha^j\right\}$ and a sequence of positive numbers $\left\{\lambda_\alpha^j\right\}$, $1 \leq j \leq k_1$ satisfying (A1) or (A2). Without loss of generality, we may assume λ_α^1 is the smallest one, i.e.

$$\frac{\lambda_\alpha^1}{\lambda_\alpha^j} \leq C < \infty$$

for all $j = 2, \dots, k_1$ as $\alpha \searrow 1$. Then we need to show

$$\mu_1 = \liminf_{\alpha \searrow 1}\left(\lambda_\alpha^1\right)^{2-2\alpha} \leq \mu_{max}.$$

By applying the blow-up argument for general α-Dirac-harmonic maps (see [19, Section 7] for more details), we get

$$\left(\sigma_\alpha^1, \xi_\alpha^1\right) := \left(\phi_\alpha\left(x_\alpha^1 + \lambda_\alpha^1 x\right), \left(\lambda_\alpha^1\right)^{\alpha-1}\sqrt{\lambda_\alpha^1}\psi_\alpha\left(x_\alpha^1 + \lambda_\alpha^1 x\right)\right) \to \left(\sigma^1, \xi^1\right) \quad \text{in } C^k_{loc}\left(\mathbb{R}^2\right),$$

where $\left(\sigma^1, \xi^1\right)$ can be conformally extended to a nontrivial Dirac-harmonic sphere. Therefore, we have

$$\begin{aligned}\Lambda &\geq \lim_{R\to\infty}\lim_{\alpha\searrow 1}\int_{D_{\lambda_\alpha^1 R}(x_\alpha^1)} |\nabla_{g_\alpha}\phi_\alpha|^{2\alpha} dvol_{g_\alpha} \\ &= \lim_{R\to\infty}\lim_{\alpha\searrow 1}\left(\lambda_\alpha^1\right)^{2-2\alpha}\int_{D_R(0)} |\nabla_{g_\alpha}\sigma_\alpha^1|^{2\alpha} dvol_{g_\alpha(x_\alpha^1+\lambda_\alpha^1 x)} \\ &= \lim_{R\to\infty}\mu_1\int_{D_R(0)} |\nabla\sigma^1|^2 dx = \mu_1 E\left(\sigma^1\right).\end{aligned}$$

[5] Let us explain the transformation of the spinor part. In fact, it can be seen as a linear transformation (i.e. $\lambda_\alpha^{\alpha-1}\psi_\alpha$) composed with a conformal transformation (i.e. $\sqrt{\lambda_\alpha}\psi_\alpha(x_\alpha + \lambda_\alpha x)$). Since α-Dirac-harmonic maps are not conformally invariant, to get unified bubble equations, we need an additional factor $\lambda_\alpha^{\alpha-1}$ in the scaling.

By the energy gap theorem for nontrivial Dirac-harmonic spheres (see [19, Lemma 6.2]), we have

$$\mu_1 \leq \frac{\Lambda}{E\left(\sigma^1\right)} \leq \frac{\Lambda}{\varepsilon_4},$$

where $\varepsilon_4 = \varepsilon_4(N) > 0$ is a positive constant.

Now, we are able to state a result about generalized energy identities for a sequence of α-Dirac-harmonic maps that blows up at interior points.

Theorem 3.4 [19, Theorem 2.6] *Under the assumptions of Theorem 2.8, if we assume that* $\mathbf{S} \cap \partial M = \emptyset$*, i.e., all the blow-up points are interior points, then there are at most finitely many bubbles: a finite set of Dirac-harmonic spheres* $\left(\sigma_i^l, \xi_i^l\right) : S^2 \to N$*,* $l = 1, \ldots, l_i$*, where* $l_i \geq 1$*,* $i = 1, \ldots, I$*, such that, the following generalized energy identities hold:*

$$\lim_{k\to\infty} E_{\alpha_k}(\phi_{\alpha_k}) = E(\phi) + |M| + \sum_{i=1}^{I}\sum_{l=1}^{l_i} \mu_{il}^2 E\left(\sigma_i^l\right),$$

$$\lim_{k\to\infty} E(\psi_{\alpha_k}) = E(\psi) + \sum_{i=1}^{I}\sum_{l=1}^{l_i} \mu_{il}^2 E\left(\xi_i^l\right),$$

where the quantities $\mu_{il} \geq 1$ *are defined as in* (3.1).

Furthermore, we can show that the map parts of the α-Dirac-harmonic necks appearing in the interior blow-up process are converging to geodesics in the target manifold N and then derive the length formula of these neck geodesics.

Theorem 3.5 [19, Theorem 2.8] *Under the same assumptions as in Theorem 3.4, let* $x_1 \in \mathbf{S}$ *be an interior blow-up point. For simplicity, assume that there is only one bubble in* $B_r^M(x_1) \subset M$ *for some* $r > 0$*, for the sequence* $\left\{\left(\phi_{\alpha_k}, \psi_{\alpha_k}\right)\right\}$*, denoted by* $\left(\sigma^1, \xi^1\right)$*, which is a Dirac-harmonic sphere. Let*

$$\nu^1 = \liminf_{\alpha\searrow 1}\left(\lambda_\alpha^1\right)^{-\sqrt{\alpha-1}}.$$

Then, by passing to subsequences, the map part of the Dirac-harmonic neck appearing during the blow-up process converges to a geodesic in the target manifold N. Moreover, we have the following alternatives:

(1) *when* $\nu^1 = 1$*, the set* $\phi\left(B_r^M(x_1)\right) \cup \sigma^1\left(S^2\right)$ *is a connected set in the target* N*;*

(2) *when* $\nu^1 \in (1, \infty)$*, then the set* $\phi\left(B_r^M(x_1)\right)$ *and* $\sigma^1\left(S^2\right)$ *are connected by a geodesic of length*

$$L = \sqrt{\frac{E\left(\sigma^1\right)}{\pi}}\log \nu^1;$$

(3) *when* $\nu^1 = \infty$*, the map part of the Dirac-harmonic neck contains at least an infinite length curve which is a geodesic in* N*;*

4 Some Problems

Finally, we shall propose some problems related to the two approaches discussed in this article.

Problem 1 What is the blow-up behavior at the finite singular time of Dirac-harmonic map flow? Is the blow-up set finite? Can we also get an energy identity at the finite singular time?

For the harmonic map flow, Eells–Sampson [13] established a global smooth solution under certain curvature conditions of the target manifold. When the domains are of dimension two, Struwe [34] proved the existence of a global weak solution which is regular except finite singular points. Chang–Ding–Ye [4] constructed an example to show that harmonic map flow admits a finite singular time singularity. The refined blow-up analysis of two dimensional harmonic map flow at the finite or infinite singular time, namely energy identity and no neck property, were explored in [12, 31, 32] etc.

Problem 2 What is the limit of the α-Dirac harmonic map flow constructed in Theorem 2.7 as $\alpha \searrow 1$? Is the limit a weak solution of Dirac-harmonic map flow?

When the spinor is vanishing, α-Dirac harmonic map flow reduces to the Sacks–Uhlenbeck flow introduced in [17]. By studying the limit as $\alpha \searrow 1$, the existence of a weak solution of harmonic map flow was proved in [17].

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