Vietnam Journal of Mathematics (2022) 50:161–169
https://doi.org/10.1007/s10013-021-00481-z

ORIGINAL ARTICLE

A Note on Co-Harada Rings

Phan Dan[1] · **Banh Duc Dung**[2]

Received: 13 January 2020 / Accepted: 19 October 2020 / Published online: 2 March 2021

Abstract
In the early 1990s, Harada and Oshiro introduced extending and lifting properties for modules and, simultaneously, considered two new classes of artinian rings which contain quasi-Frobenius (QF-) rings and Nakayama rings: one is the class of right Harada rings and the other is the class of right co-Harada rings. Although QF-rings and Nakayama rings are left-right symmetric, Harada and co-Harada rings are not left-right symmetric. However, Oshiro showed that left Harada rings and right co-Harada rings are coinside. In this paper we provide many characterizations of right co-Harada rings and (right and left) co-Harada rings.

Keywords Harada rings · Co-Harada rings · Small modules · Nonsmall modules · Cosmall modules · Non-cosmall modules

Mathematics Subject Classification (2010) 16D70 · 16D80 · 16P20

1 Introduction

Throughout this paper all rings will be associative with identity and modules will be unital modules. For an R-module M we write M_R $({}_RM)$ to indicate that M is a right (left) R-module. By $J(M)$, $E(M)$, $Z(M)$ we denote the Jacobson radical, the injective hull and the singular submodule of M, respectively. We denote the set of primitive idempotents of R by $\mathrm{Pi}(R)$. A ring R is said to have enough idempotents if the identity element of R can be written as the sum of a finite number of orthogonal primitive idempotens of R.

Let R be a ring and M a right R-module. $N \leq M$ will mean N is a submodule of M. A submodule N of M is called *small in* M, denoted by $N \leq_{sm} M$, whenever for every submodule L of M, $N + L = M$ implies $L = M$. A non-zero submodule N of M is said to

✉ Banh Duc Dung
dungbd@hcmute.edu.vn

Phan Dan
danp@hiu.vn

[1] Faculty of General Education, Hong Bang International University, Ho Chi Minh City, Vietnam

[2] Faculty of Applied Sciences, HCMC University of Technology and Education, Ho Chi Minh City, Vietnam

be an *essential submodule* of M, denoted by $N \leq_e M$, if for every $0 \neq L \leq M$, $N \cap L \neq 0$. A non-zero module M is called *uniform* if $N \leq_e M$ for every non-zero submodule N of M.

A module M is said to be *small* if M is small in its injective hull. A (right) R-module M is called *non-small* if M is not a small submodule in its injective hull, which is equivalent to the fact that M is not a small submodule in any extension module of M (see [11, Proposition 1.1]). Dually, M is called a *non-cosmall* module, following [19], if M is a homomorphic image of a projective module P whose kernel is not essential in P, which is equivalent to the fact that if M is a homomorphic image of a module N, then the kernel is always not essential in N (see [11, Proposition 3.1]).

A module M is called an *extending module* (*CS module*) if every submodule of M is essential in a direct summand of M. A ring R is called right *CS* if R_R is an extending module. A module M is said to be a *local module* if M has a unique maximal submodule.

A ring R is called *right QF-3* if R_R has a direct summand eR (e is an idempotent of R) which is a faithful injective right ideal, and it is called *right QF-3*$^+$ if $E(R_R)$ is projective. A ring R is called *right QF-2* if it is a direct sum of right uniform ideals (as right R-modules). R is called *right* (*left*) *nonsingular* if the right (left) singular ideal of R is zero.

M is called *uniserial* if the set of submodules of M is linearly ordered and M is called *serial* if it is a direct sum of uniserial modules. A ring R is a *right* (*left*) *serial ring* if R_R ($_RR$) is a serial module, and R is called a *serial ring* if R are both right and left serial. A two-sided artinian serial ring is also called a *Nakayama ring*.

A submodule N of a module M is called a *waist* in M if either $N \leq X$ or $X \leq N$ is satisfied for any submodule X of M.

Definition 1 A module M is called *z-serial* if M satisfies three following conditions:

(1) M is uniform,
(2) $Z(M)$ is a waist in M,
(3) $M/Z(M)$ is uniserial.

The following series of results from various sources is presented here in order to make it easier to refer to them later in the paper.

Lemma 1 ([7, Lemma 7.1]) *Every direct summand of an extending module is an extending module.*

Lemma 2 ([9, 18.23]) *Every local right* ($left$) *R-module over semiperfect ring R is isomorphic to a homomorphic image of* eR (Re) *for some* $e \in \mathrm{Pi}(R)$. *If R is right* ($left$) *QF-2 then every local right* ($left$) *R-module is either projective or singular.*

In [11] Harara has studied the following conditions:

$(*)_r$ Every non-small right R-module contains a non-zero injective submodule.
$(*)_r^*$ Every non-cosmall right R-module contains a non-zero projective direct summand.

He also gave a characterization of semiperfect rings with $(*)_r^*$ as follows:

Theorem 1 ([11, Theorem 3.6]) *Let R be semiperfect. Then* $(*)_r^*$ *holds if and only if there exists a set of primitive idempotents* $\{e_i\}$ *and of integers* $\{n_i\}$ *such that:*

(1) *$e_i R$ is injective,*
(2) *$e_i J^{t_i}$ is projective for $t_i \leq n_i$ and $e_i J^{n_i+1}$ is singular, and*

(3) *Every indecomposable projective module is isomorphic to some* $e_i J^{t_i}$.

In this case, every submodule $e_i B$ *in* $e_i R$ *either is contained in* $e_i J^{n_i+1}$ *or equal to some* $e_i J^{t_i}$, $t_i \leq n_i + 1$, *where* $J = J(R)$.

The following classes of rings have been defined by Oshiro [16]: A ring R is called a *right Harada ring* if it is right artinian and satisfies the condition $(*)_r$. Dually, a ring R is called a *right co-Harada ring* if it satisfies the condition $(*)_r^*$ and the ACC on right annihilators.

Huynh in [13] studied right co-Harada rings under the name right Σ-CS rings. Many results on onesided Harada (or co-Harada) rings are given in [1, 10] and [16].

In 1993, Vanaja ([21]) has generalized $(*)_r^*$ by considering the following condition

$(*)_{1,r}^*$ Every finitely generated non-cosmall right R-module contains a non-zero projective direct summand.

It is known ([21, Theorem 1.10]) that the following are equivalent for a semiperfect ring R: (1) R satisfies $(*)_{1,r}^*$; (2) $R_R^{(n)}$ is an extending module; and (3) Direct sum of any two indecomposable projective right R-modules is extending.

Lemma 3 ([9, Theorem 20.15]) *Every indecomposable injective and projective right* R-*module* M *is isomorphic to a summand of* R, *that is, there exists an idempotent* $e \in R$ *such that* $M \cong eR$.

Lemma 4 ([11, 19]) *The following statements holds for non-cosmall modules:*

(1) *An* R-*module* M *is non-cosmall if and only if* $M \neq Z(M)$;
(2) *If an* R-*module* M *contains a non-zero projective submodule, then it is non-cosmall.*

From the definition of non-cosmall modules and Lemma 3 we have

Lemma 5 *The following statements are equivalent for a ring* R *and a cardinal* α:

(1) $R_R^{(\alpha)} = \oplus_I R_R$ *is an extending module, where* $\mathrm{card}(I) = \alpha$.
(2) *Every* α-*generated right* R-*module* M *is a direct sum of a projective module and a singular module.*

The proof of the following lemma is straightforward and will be omitted.

Lemma 6 ([11]) *Let* R *be a ring.*

(1) *If* $\{X_i\}_{i=1}^n$ *is a set of small submodules of a right* R-*module* X, *then* $\sum_{i=1}^n X_i$ *is a small module,*
(2) *If* R *is right perfect, then a right* R-*module* M *is nonsmall if and only if there exists an element* $m \in M$ *such that module cyclic* mR *is nonsmall.*

Lemma 7 ([17]) *A ring* R *is a right (resp. left) Harada ring if and only if* R *is left (resp. right) co-Hadada.*

The following lemma is a special case of [2, Lemma 11].

Lemma 8 *Let* U *be a uniform module, and suppose that* U *is not isomorphic to any of its proper submodules. Then* $\mathrm{End}(U)$ *is a local ring.*

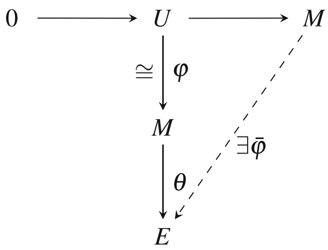
Proof Let $\alpha \in \mathrm{End}(U)$. Then $\mathrm{Ker}(\alpha) \cap \mathrm{Ker}(1-\alpha) = 0$ in U. Since U is uniform, it follows that either $\mathrm{Ker}(\alpha) = 0$ or $\mathrm{Ker}(1-\alpha) = 0$, respectively either α or $(1-\alpha)$ is monomorphic. Since U is not isomorphic to any of its proper submodules, one of two homomorphisms is isomorphic, as required. □

2 Co-Harada Rings

We start the section with a lemma which will be useful in the sequel.

Lemma 9 *Let M be a uniform cyclic module such that $E(M)/M$ has ACC on cyclic submodules. Then M is not isomorphic to any of its proper submodules, and hence* $\mathrm{End}(M)$ *is a local ring.*

Proof Let $E = E(M)$ and suppose that there is a proper submodule U of M and an isomorphism $\varphi : U \to M$. Consider the commutative diagram



where $\bar{\varphi}$ is an extension of φ. Let $\bar{\varphi}(eR) = U_1$ and $\bar{\varphi}(U_1) = U_2$, and $\bar{\varphi}(U_n) = U_{n+1}$ for any integer number n. Since E is uniform and φ is an isomorphism, it follows that $\bar{\varphi}$ is a monomorphism. So that, we get an infinite strictly ascending chain of cyclic modules $M \leq U_1 \leq U_2 \leq \cdots \leq U_n \leq \cdots$ in E, which is a contradiction, since E/M has ACC for cyclic submodules. Hence, M is not isomorphic to any of its proper submodules. Moreover, by Lemma 8, $\mathrm{End}(M)$ is local. □

The following theorem provides characterizations of semiperfect rings with $(*)_r^*$. This generalizes [15, Theorem 2.3] and [20, Theorem 3.8] (without assuming $E(eR)$ is a z-serial module for every $e \in \mathrm{Pi}(R)$).

Theorem 2 *Let R be a ring. The following statements are equivalent:*

(1) *R is a semiperfect ring with $(*)_r^*$.*
(2) *R is a ring with enough idempotents, eR is a waist in $E(eR)$ and $E(eR)/eR$ has ACC on cyclic submodules for every $e \in \mathrm{Pi}(R)$.*
(3) *R is a ring with enough idempotents, eR is a waist in $E(eR)$ and $eR/Z(eR)$ has finite length for any $e \in \mathrm{Pi}(R)$.*

Proof (1) ⇒ (2). It is obvious, see Theorem 1.

(2) ⇒ (1). Assume (2). Then $R_R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$, where $e_i \in \mathrm{Pi}(R)$ and e_iR is indecomposable. Let $e = e_i$. Since eR is a waist in $E(eR)$, it follows that eR is uniform, so that $\mathrm{End}(eR)$ is local by Lemma 9. Hence R is a semiperfect ring. By [20, Theorem 2.4],

R satisfies condition $(*)^*_{1,r}$. Moreover, every eR is not isomorphic to any of its proper submodules, by Lemma 9. So that R satisfies $(*)^*_r$, by [20, Lemma 3.1].

(3) $\Rightarrow$ (1) Assume (3). Then $R_R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$, where $e_i \in \mathrm{Pi}(R)$ and e_iR is indecomposable. Let $e = e_i$. It follows that eR is uniform, since eR is a waist in $E(eR)$. If $Z(eR) = 0$ then $\mathrm{End}(eR)$ is local, since eR has finite length. If eR is injective then $\mathrm{End}(eR)$ is local, since eR is uniform. We consider the case $W = Z(eR) \neq 0$ and $E = E(eR) \neq eR$. By (3), $\mathrm{length}(eR/W) = n > 0$. Using the same argument as in the proof of Lemma 9, we get an infinite strictly ascending chain of cyclic modules $eR \leq U_1 \leq U_2 \leq \cdots \leq U_n \leq \cdots$ in E, where $eR \cong U_i$ for $i = 1, 2, \ldots$. Since $Z(E) \neq E$ and eR is a waist in E, $Z(U_i) \neq eR$, so that $Z(U_i) = Z(eR) = W$. Then eR/W is not a module with finite length, a contradiction. It follows that eR is not isomorphic to any of its proper submodules. Hence $\mathrm{End}(eR)$ is local, by Lemma 8. Therefore R is a semiperfect ring. Since eR is a waist in $E(eR)$ for every $e \in \mathrm{Pi}(R)$, R satisfies $(*)^*_{1,r}$, by [20, Theorem 2.4]. Since every eR is not isomorphic to any of its proper submodules, it follows R satisfies $(*)^*_r$, by [20, Lemma 3.1]. □

A well-known result of C. Faith [8] asserts that a right self-injective ring is QF if and only if R has ACC (or DCC) on right annihilators. Next we provide some similar characterizations of right co-Harada rings.

Theorem 3 *Let R be a ring satisfying the ACC (or DCC) on right annihilators. Then the following statements are equivalent:*

(1) *R is a right co-Harada ring;*
(2) *$R_R^{(2)}$ is an extending module and $E(R_R)/R$ has ACC on cyclic submodules.*
(3) *For every $e \in \mathrm{Pi}(R)$, eR is a waist in $E(eR)$ and $E(eR)/eR$ has ACC on cyclic submodules.*
(4) *For every $e \in \mathrm{Pi}(R)$, eR is a waist in $E(eR)$, $E(eR)$ is a projective module and $eR/Z(eR)$ has ACC on cyclic submodules.*

Proof (1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). Assume (2). It is easy to see that R is a direct sum of indecomposable right ideals if R satisfies the ACC (or DCC) on right annihilators. Since $R_R^{(2)}$ is extending, so is R_R. Hence R is a direct sum of uniform right ideals. Let $R_R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$, where $\{e_i\}_{i=1}^n$ is a set of primitive idempotents of R and each e_iR is uniform. Let $e = e_i$ and $E = E(e_iR)$. It is easy to see that E/eR has ACC for cyclic submodules, since it is isomorphic to a direct summand of $E(R_R)/R$. If eR is injective then $\mathrm{End}(eR)$ is local since eR is uniform. Assume eR is not injective. It follows from Lemma 9 that $\mathrm{End}(eR)$ is local. So that R is semiperfect. Since R_R^2 is extending, it follows from [20, Lemma 12.8] that eR is a waist in $E(eR)$. Therefore (3) holds.

(3) $\Rightarrow$ (1). Assume (3). If R has the ACC (or DCC) on right annihilators then R is a ring with enough idempotents. It follows from (3) and Theorem 2 that R is a semiperfect ring with $(*)^*_r$. Hence R is a right co-Harada ring, by [6, Corollary 3.8].

(1) $\Rightarrow$ (4). It is clear, since R is an artinian ring, and R satisfies $(*)^*_r$.

(4) $\Rightarrow$ (3). Let $f \in \mathrm{Pi}(R)$ be any element. Using the argument as above, it follows that every $E(fR)$ is uniform and projective. So that $E(fR) \cong eR$ for some $e \in \mathrm{Pi}(R)$, by Lemma 3. Since fR is a waist in eR, then $Z(eR) \leq fR$. It implies that $Z(eR) = Z(fR)$. So that $eR/Z(fR)$ has ACC on cyclic submodules, by (4). It follows that $E(fR)/fR$ has ACC on cyclic submodules.

The proof of the theorem is complete. □

The main result in [5] is a special case of the following result.

Proposition 1 *Let R be a ring satisfying the ACC (or DCC) on right annihilators. Then the following statements are equivalent:*

(1) *R is a right co-Harada ring.*
(2) *R is a right perfect ring and $R_R^{(2)}$ is an extending module.*
(3) *R is a left perfect ring and $R_R^{(2)}$ is an extending module.*

Proof (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (1). Assume R is a right perfect ring. By [14], every right R-module has ACC on cyclic modules. By (2) $\Rightarrow$ (1) in Theorem 3, it follows (1).

(3) $\Rightarrow$ (1). Assume R is a left perfect ring. Let $e \in \mathrm{Pi}(R)$. It easy to see that eR is a waist in $E(eR)$ and $eR/Z(eR)$ has finite length. It follows that R satisfies $(*)_r^*$, by (3) $\Rightarrow$ (1) in Theorem 3, it follows (1). □

Corollary 1 *Let R be a right nonsingular ring. Then the following statements are equivalent:*

(1) *R is a right co-Harada ring.*
(2) *R is a right perfect ring and $R_R^{(2)}$ is an extending module.*
(3) *R is a left perfect ring and $R_R^{(2)}$ is an extending module.*
(4) *R is Morita equivalent to a finite direct sum of upper triangular matrix rings over division rings.*

Proof It is shown by [3, Lemma 1.14] that if R is a ring with finite right Goldie then R has ACC and DCC on right annihilators. From Proposition 1 it easy to see the equivalence of (1), (2), (3). It follows from [16] that (1) $\Leftrightarrow$ (4). □

It is also shown by Oshiro [16] that a ring R is a left Harada ring if and only if it is a right co-Harada ring, however a right Harada ring need not to be right co-Harada (see also Oshiro [16]).

Example 1 (Oshiro, [16]) There exists a right co-Harada ring which is not a right Harada ring. Consider the local QF ring $Q = K[x, y]/(x^2, y^2)$, in which K is a field.

Put

$$J = J(Q), \quad S = \mathrm{Soc}(Q_Q) \quad (= \mathrm{Soc}({}_Q Q)),$$
$$\bar{Q} = Q/S = \{\bar{a} \mid \bar{a} = a + S \ \forall a \in Q\}.$$

Defined V, W by

$$V = \begin{bmatrix} Q & Q \\ J & Q \end{bmatrix}, \quad W = \begin{bmatrix} Q & \bar{Q} \\ J & \bar{Q} \end{bmatrix}.$$

Then:

(a) V is a Harada and co-Harada ring (right and left).

(b) W is right co-Harada and left Harada. However, W is neither left co-Harada nor right Harada.

Finally we provide some characterizations of Harada rings (co-Harada rings) via perfect rings.

Theorem 4 *Let R be a ring. Then the following statements are equivalent:*

(1) *R is a Harada ring;*
(2) *R is a co-Harada ring;*
(3) *R is a perfect ring and $R_R^{(2)}$ and ${}_RR^{(2)}$ are extending modules;*
(4) *R is a perfect ring and every 2-generated right (or left) module M has a decomposition $M = P \oplus S_1 \oplus S_2$, where P is projective, S_1 is injective singular, S_2 is small singular;*
(5) *R is a perfect ring, eR is a waist in $E(eR)$ and Re is a waist in $E(Re)$, for any $e \in \mathrm{Pi}(R)$;*
(6) *R is a perfect ring and $M_2(R)$ is a CS ring.*

Proof (1) $\Leftrightarrow$ (2). It follows from a result due to Oshiro [17] (see, Lemma 7). We shall prove that (2) $\Leftrightarrow$ (4), (2) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (6).

(2) $\Rightarrow$ (4). Assume R is a co-Harada ring. Let M be a 2-generated right R-module. Since R is a right co-Harada ring, M has a decomposition $M = P \oplus S$, where P is projective and S is singular. Since R is a left co-H ring, it follows that R is a right Harada ring, by Lemma 7. Then, S has a decomposition $S = S_1 \oplus S_2$, where S_1 is injective and S_2 is small. Hence $M = P \oplus S_1 \oplus S_2$, where P is projective, S_1 is injective and singular and S_2 is small and singular.

(4) $\Rightarrow$ (2). Assume (4). Let M be any 2-generated right R-module. Then, M has a decomposition $M = P \oplus S_1 \oplus S_2$, where P is projective, S_1 is injective and singular and S_2 is small and singular. It means that $M = P \oplus (S_1 \oplus S_2)$, where P is projective and $S_1 \oplus S_2$ is singular. This shows that every 2-generated right R-module is a direct sum of a projective module and a singular module. By Lemma 5, it implies that $R_R^{(2)}$ is an extending module. Since R is perfect, it follows from [20, Proposition 3.4] that R satisfies the condition $(*)^*_{1,r}$. It is easy to see from Theorem 1 that every indecomposable projective right R-module is either injective or small. Let $P = P_1 \oplus P_2 \oplus \cdots \oplus P_k$. Let $I = \{1, 2, \ldots, k\}$. Let $I_1 = \{i \in I \mid P_i \text{ is injective}\}$ and $I_2 = \{j \in I \mid P_j \text{ is small}\}$. It is easy to see that $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$. Whence, we have

$$
\begin{aligned}
M = P \oplus (S_1 \oplus S_2) &= \left(\oplus_{i\in I_1} P_i\right) \oplus \left(\oplus_{j\in I_2} P_j\right) \oplus (S_1 \oplus S_2) \\
&= \left(\left(\oplus_{i\in I_1} P_i\right) \oplus S_1\right) \oplus \left(\left(\oplus_{j\in I_2} P_j\right) \oplus S_2\right),
\end{aligned}
$$

where $(\oplus_{i\in I_1} P_i \oplus S_1)$ is injective, and $(\oplus_{j\in I_2} P_j \oplus S_2)$ is small by Lemma 6. Hence every 2-generated (and hence every cyclic) right R-module is a direct sum of an injective module and a small module. Next, we shall show that R satisfies the condition $(*)_r$. Let X be a nonsmall right R-module. Since R is perfect, it follows from Lemma 6 that X contains a cyclic nonsmall module N. Then, $N = N_1 + N_2$, where N_1 is a nonzero injective module and N_2 is a small module. So, X contains nonzero injective module N_1. Therefore, R satisfies the condition $(*)_r$. Since R is a perfect ring and R satisfies the condition $(*)_r$, R is right artinian by [12]. Since R is right artinian and R satisfies the condition $(*)_r$, R is a right Harada ring and so R is a left co-Harada ring. It remains to show that R is a right co-Harada ring. However this is clear since R is right artinian and R satisfies the condition $(*)^*_{1,r}$. Hence R is a co-Harada ring.

(2) $\Rightarrow$ (5). Suppose that R is a co-Harada ring. Then R is an artinian ring (see [17, 18]). It follows from [20, Theorem 3.11] that eR is a waist in $E(eR)$ and Re is a waist in $E(Re)$.

(5) $\Rightarrow$ (3). Assume (5). Since R is perfect and eR (resp. Re) is a waist in $E(Re)$ (resp. $E(Re)$), it follows that $R^{(2)}_R$ (resp. $_RR^{(2)}$) is an extending module, by [20, Theorem 2.4]. We have (3).

(3) $\Rightarrow$ (2). Assume (3). If R is a left (resp. right) perfect ring and $R^{(2)}_R$ (resp. $_RR^{(2)}$) is an extending module, the condition $(*)^*_r$ (resp. $(*)^*_l$) holds, by [20, Proposition 3.4]. Furthermore, R is a perfect (two-sided) QF-3^+ ring, by [5]. Since R_R (resp. $_RR$) is isomorphic to a direct sum of modules of the form e_iR (resp. Re_i), we obtain a faithful injective right (resp. left) ideal by letting eR (resp. Re) be the sum of one of each isomorphism type of these e_iR (resp. Re_i). It implies that R is a right and left QF-3 ring. So, R is a semiprimary QF-3 ring. Hence R satisfies the ACC on right (and left) annihilators, by [4, Theorem 1.3]. Therefore R is a co-Harada ring.

(2) $\Rightarrow$ (6). Since R is right and left co-Harada, then R is two-sided artinian and $R^{(2)}_R$, $_RR^{(2)}$ are extending modules. By [7, Lemma 12.8], the ring $M_2(R)$ is a right and left CS ring.

(6) $\Rightarrow$ (2). Since $M_2(R)$ is a right and left CS ring, then $R^{(2)}_R$ and $_RR^{(2)}$ are extending modules, by [7, Lemma 12.8]. Since R is a perfect ring, we have (3); hence (2) holds.

The proof of the theorem is complete. □

Acknowledgements This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2018.02.

The authors would like to thank the referees for their useful suggestions.

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