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ORIGINAL ARTICLE

Characterization of *n*-Jordan Multipliers

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Abstract
Let A be a Banach algebra and X be a (Banach) A-bimodule. A linear map $T : A \longrightarrow X$ is called an n-Jordan multiplier if $T(a^n) = aT(a^{n-1})$ for all $a \in A$. In this paper, among other things, we show that under special hypotheses every $(n+1)$-Jordan multiplier is an n-Jordan multiplier and vice versa.

Keywords n-Multiplier · n-Jordan multiplier · Unitary Banach A-module

Mathematics Subject Classification (2010) Primary 47B48 · Secondary 46L05 · 46H25

1 Introduction and Preliminaries

Let A be a Banach algebra and X be a (Banach) A-bimodule. A map $T : A \longrightarrow X$ is called *left multiplier* (*right multiplier*) if for all $a, b \in A$,

$$T(ab) = T(a)b \qquad (T(ab) = aT(b)),$$

and T is called a *multiplier* if it is both left and right multiplier. Also, T is called *left Jordan multiplier* (*right Jordan multiplier*) if for all $a \in A$,

$$T(a^2) = T(a)a \qquad (T(a^2) = aT(a)),$$

and T is called a *Jordan multiplier* if T is a left and a right Jordan multiplier.

The term multiplier was introduced by S. Helgason in [4]. Even nowadays some authors use the term centralizer instead of multiplier. This terminology was introduced by J.G. Wendel in [9]. The general theory of (centralizers) multipliers on Banach algebras has been developed by B.E. Johnson [5]. He proved that each multiplier $T : A \longrightarrow A$ on without order Banach algebra A is linear and continuous.

Recall that the Banach algebra A is called *without order*, if for all $x \in A$, $xA = \{0\}$ $(Ax = \{0\})$ implies $x = 0$.

Clearly, every left (right) multiplier is a left (right) Jordan multiplier, but the converse is not true in general, as was demonstrated in [3, Example 2.6]. Zalar [10] proved that every

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right (left) Jordan multiplier on a 2-torsion free semiprime ring is a right (left) multiplier. Another approach to the same result can be found in [8]. One may refer to the monograph [7] for the additional fundamental results in the theory of multipliers.

There exists another related concept, called (two-sided) multiplier. A map $T : A \longrightarrow X$ is said to be multiplier if for every $a, b \in A$,

$$aT(b) = T(a)b. \tag{1}$$

If T is both left and right multiplier, then T is a multiplier, according to (1), but the converse is, in general, false. The following example obtained by the author in [11].

Example 1.1 Let

$$A = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

and define $T : A \longrightarrow A$ by

$$T\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, for all $x, y \in A$, $T(x)y = xT(y)$, hence T is a multiplier as in (1), but it is not left (right) multiplier, because $T(x)y \neq T(xy) = 0$, in general.

We say that $w \in A$ is a left (right) separating point of A-bimodule X if the condition $wx = 0$ $(xw = 0)$ for $x \in X$ implies that $x = 0$.

If $w \in A$ is a left (right) separating point of A-bimodule X and $T : A \longrightarrow X$ satisfies in (1), then T is a left (right) multiplier. Indeed, let $aT(b) = T(a)b$, then for all $x \in A$,

$$x\big(T(ab)\big) = T(x)ab = xT(a)b = x\big(T(a)b\big).$$

In particular, $w(T(ab)) = w(T(a)b)$. Since w is a left separating point of X, we get $T(ab) = T(a)b$, for all $a, b \in A$, which means that T is a left multiplier. We can prove that T is a right multiplier in a similar way.

Let A be a unital Banach algebra with unit e_A. An A-bimodule X is called *unitary* if $e_A x = xe_A = x$ for all $x \in X$. For example, A^* is a unitary A-bimodule with the following actions.

$$a \cdot f(x) := f(xa), \quad f \cdot a(x) := f(ax), \qquad a, x \in A,\ f \in A^*.$$

Definition 1.2 Let A be a Banach algebra, X be a left A-module and let $T : A \longrightarrow X$ be a linear map. Then T is called *right n-multiplier* if

$$T(a_1 a_2 \dots a_n) = a_1 T(a_2 \dots a_n),$$

for all $a_1, a_2, \dots, a_n \in A$. Moreover, T is called *right n-Jordan multiplier* if for all $a \in A$,

$$T(a^n) = aT(a^{n-1}).$$

The *left n-multiplier* (*left n-Jordan multiplier*) and *n-multiplier* (*n-Jordan multiplier*) can be defined analogously.

Note: Since all results which are true for right multipliers have obvious analogue statements for left multipliers, we will focus in the sequel just on the right versions. We will also drop the prefix right for simplicity.

The concept of n-multiplier was introduced and studied by Laali and Fozouni [6], where some interesting results related to these maps were obtained. The notion of n-Jordan multiplier was introduced in [2]. Following [6] and [2], let $\mathrm{Mul}_n(A, X)$ and $\mathrm{JMul}_n(A, X)$ be the set of all n-multipliers and n-Jordan multipliers from a Banach algebra A into its module X.

It is clear that every n-multiplier is an $(n + 1)$-multiplier, while on the other hand it was shown in [6, Theorem 2], that in the general case $\mathrm{Mul}_n(A, X) \subsetneq \mathrm{Mul}_{n+1}(A, X)$. Moreover, by [6, Theorem 3] if A is an essential Banach algebra, then $\mathrm{Mul}_n(A, X) = \mathrm{Mul}_{n+1}(A, X)$.

Obviously,

$$\mathrm{Mul}_n(A, X) \subseteq \mathrm{JMul}_n(A, X),$$

and this inclusion may be strict by [2, Proposition 6]. See also [3, Example 2.6] for $n = 2$. Next we establish another example for the case $n = 3$.

Example 1.3 Let

$$A = \left\{ \begin{bmatrix} 0 & a & x & c \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix} : \ a, b, c, x, y \in \mathbb{R} \right\}$$

and define $T : A \longrightarrow A$ via

$$T\left(\begin{bmatrix} 0 & a & x & c \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & y & c \\ 0 & 0 & b & x \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, for all $u \in A$, $T(u^2) = u^2$ and $T(u^3) = u^3$. Therefore T is a 3-Jordan multiplier, but T is not 3-multiplier.

Note that the inclusion $\mathrm{JMul}_{n+1}(A, X) \subseteq \mathrm{JMul}_n(A, X)$ is not true, in general. The next example illustrates this fact.

Example 1.4 Let

$$A = \left\{ \begin{bmatrix} 0 & x & a & b \\ 0 & 0 & y & c \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{bmatrix} : \ x, y, z, a, b, c \in \mathbb{R} \right\}$$

and define $T : A \longrightarrow A$ via

$$T\left(\begin{bmatrix} 0 & x & a & b \\ 0 & 0 & y & c \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & z & c & 0 \\ 0 & 0 & y & a \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, $T(u^n) = uT(u^{n-1})$ for all $u \in A$ and for $n \geq 4$. So, T is an n-Jordan multiplier, but $T(u^3) \neq uT(u^2)$ for all $u \in A$, where $x, y, z \neq 0$. Hence, T is not 3-Jordan multiplier.

Each Jordan multiplier is an n-Jordan multiplier (see Lemma 2.1 below), but an analogues of this fact need not hold for $n \geq 3$. That is, in general every n-Jordan multiplier is not m-Jordan multiplier, where $m > n \geq 3$.

Also by Example 1.4, some $(n + 1)$-Jordan multipliers fail to be n-Jordan multipliers. Therefore neither $(n + 1)$-Jordan multipliers are necessarily n-Jordan multipliers nor n-Jordan multipliers are automatically $(n+1)$-Jordan multipliers. Now the following questions can be raised.

Under which conditions for a Banach algebra A is any $(n+1)$-Jordan multiplier $T : A \longrightarrow X$ automatically an n-Jordan multiplier and vice versa?

Moreover, when is any n-Jordan multiplier automatically an n-multiplier?

In this paper, we investigate this question and prove that under suitable conditions concepts of $(n+1)$-Jordan multiplier, n-Jordan multiplier and Jordan multiplier are equivalent.

2 *n*-Multiplier and *n*-Jordan Multiplier

We commence with the following lemma.

Lemma 2.1 *Let A be a Banach algebra, X be a left A-module and let $T : A \longrightarrow X$ be a linear and Jordan multiplier. Then T is an n-Jordan multiplier, for $n \geq 2$.*

Proof Assume that T is a Jordan multiplier, then

$$T(a^2) = aT(a), \qquad a \in A. \tag{2}$$

Replacing a by $a+b$ we get

$$T(ab + ba) = aT(b) + bT(a), \tag{3}$$

for all $a, b \in A$. Interchanging b by a^2 in (3), we obtain

$$2T(a^3) = aT(a^2) + a^2T(a). \tag{4}$$

It follows from (2) and (4) that $T(a^3) = aT(a^2)$. Hence T is a 3-Jordan multiplier.

Now let the result has been established for all $3 \leq k \leq n$. Hence

$$T(a^k) = aT(a^{k-1}), \tag{5}$$

for all $3 \leq k \leq n$. Replacing b by a^k in (3), we obtain

$$2T(a^{k+1}) = aT(a^k) + a^kT(a). \tag{6}$$

From (5), we get

$$a^kT(a) = a^{k-1}aT(a) = a^{k-1}T(a^2) = a^{k-2}aT(a^2) = a^{k-2}T(a^3) = \cdots = aT(a^k). \tag{7}$$

Consequently, by (6) and (7), T is an $(k+1)$-Jordan multiplier and the result follows. □

Example 2.2 Let A and T be as in Example 1.1 or Example 1.3. Then $T(u^3) = uT(u^2)$ for all $u \in A$, and so T is a 3-Jordan multiplier, but T is not a Jordan multiplier. Thus, the converse of above lemma is false in general.

Theorem 2.3 *Let A be a unital Banach algebra, and X be a unitary Banach left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map. If $T(ab) = aT(b)$ for all $a, b \in A$ with $ab = e_A$, then T is an n-Jordan multiplier.*

Proof Let $a \in A$ be arbitrary. For $\lambda \in \mathbb{C}$, with $|\lambda| < 1/\|a\|$, $e_A - \lambda a$ is invertible and $(e_A - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$. Then

$$\begin{aligned} T(e_A) &= T((e_A - \lambda a)(e_A - \lambda a)^{-1} e_A) \\ &= (e_A - \lambda a) T((e_A - \lambda a)^{-1} e_A) \\ &= (e_A - \lambda a) T\left(\sum_{n=0}^{\infty} \lambda^n a^n e_A\right) \\ &= e_A T(e_A) + e_A T\left(\sum_{n=1}^{\infty} \lambda^n a^n e_A\right) - \lambda a T\left(\sum_{n=0}^{\infty} \lambda^n a^n e_A\right) \\ &= T(e_A) + \sum_{n=1}^{\infty} \lambda^n T(a^n e_A) - \lambda a \sum_{n=0}^{\infty} \lambda^n T(a^n e_A). \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \lambda^n [T(a^n) - aT(a^{n-1})] = 0,$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| < 1/\|a\|$. Consequently, $T(a^n) = aT(a^{n-1})$ for all $n \in \mathbb{N}$. This finishes the proof. □

As a consequence of Theorem 2.3, we have the next result.

Corollary 2.4 *Let A be a unital Banach algebra, X be a unitary Banach left A-module and let $T : A \longrightarrow X$ be a continuous linear map. If $a \in Inv(A)$ and $T(aa^{-1}) = aT(a^{-1})$, then T is an n-Jordan multiplier.*

The following example shows that the condition $T(ab) = aT(b)$ for all $a, b \in A$ with $ab = e_A$ in Theorem 2.3 is essential.

Example 2.5 Let

$$A = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Then A is a unital Banach algebra and it is a unitary Banach A-bimodule. Define a continuous linear map $T : A \longrightarrow A$ by

$$T\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} -b & 0 \\ 0 & -a \end{bmatrix}.$$

Set

$$x = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $xy = e_A$, but $T(xy) \neq xT(y)$. Therefore, T is not an n-Jordan multiplier, because the condition $T(xy) = xT(y)$, for all $x, y \in A$ with $xy = e_A$, does not hold.

Theorem 2.6 *Let A be a unital Banach algebra, and X be a unitary Banach left A-module. Let $T : A \longrightarrow X$ be a continuous linear map. If for an idempotent $p \in A$, $T(ab) = aT(b)$ for all $a, b \in A$ with $ab = p$, then T is an n-Jordan multiplier on pAp.*

Proof It is easy to see that pAp is a unital closed subalgebra of A with unit p. Let $a \in A$ be arbitrary. For $\lambda \in \mathbb{C}$, with $|\lambda| < 1/\|pap\|$, the elements p and $(p - \lambda pap)$ are invertible and $(p - \lambda pap)^{-1} = \sum_{n=0}^{\infty} \lambda^n (pap)^n$. Then

$$\begin{aligned} T(p) &= T((p - \lambda pap)(p - \lambda pap)^{-1} p) \\ &= (p - \lambda pap) T((p - \lambda pap)^{-1} p) \\ &= (p - \lambda pap) T\left(\sum_{n=0}^{\infty} \lambda^n (pap)^n p\right) \\ &= pT(p) + pT\left(\sum_{n=1}^{\infty} \lambda^n (pap)^n p\right) - \lambda(pap) T\left(\sum_{n=0}^{\infty} \lambda^n (pap)^n p\right) \\ &= T(p) + p \sum_{n=1}^{\infty} \lambda^n T((pap)^n) - \lambda(pap) \sum_{n=0}^{\infty} \lambda^n T((pap)^n). \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \lambda^n [pT((pap)^n) - (pap)T((pap)^{n-1})] = 0,$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| < 1/\|pap\|$. Therefore

$$pT((pap)^n) = (pap)T((pap)^{n-1}). \tag{8}$$

Multiplying p^{-1} from the right in (8) and using $p^2 = p$, we arrive at

$$T((pap)^n) = (pap)T((pap)^{n-1}).$$

Consequently, T is an n-Jordan multiplier on pAp. □

Theorem 2.7 *Let $n \in \{2, 3\}$ be fixed, A be a unital Banach algebra and X be a unitary left A-module. Then every $(n+1)$-Jordan multiplier $T : A \longrightarrow X$ is an n-Jordan multiplier.*

Proof Let $n = 2$ and T be a 3-Jordan multiplier. Then $T(a^3) = aT(a^2)$ for all $a \in A$. Replacing a by $a + e_A$ we get

$$3T(a^2 + a) = aT(e_A) + 2aT(a) + e_A T(a^2) + 2e_A T(a). \tag{9}$$

Interchanging a by $-a$ in (9), we obtain

$$3T(a^2 - a) = -aT(e_A) + 2aT(a) + e_A T(a^2) - 2e_A T(a), \tag{10}$$

for all $a \in A$. It follows from (9) and (10) that

$$3T(a^2) = 2aT(a) + e_A T(a^2).$$

Since X is unitary, we get $T(a^2) = aT(a)$. Thus, T is a Jordan multiplier.

Now let $n = 3$ and $T(a^4) = aT(a^3)$ for all $a \in A$. Replacing a by $a + e_A$ and since X is unitary, we arrive at

$$3T(a^3) + 3T(a^2) + T(a) = 3aT(a^2) + 3aT(a) + aT(e_A). \tag{11}$$

Switching a by $-a$ in (11) and plus the result by (11), we obtain

$$T(a^2) = aT(a). \tag{12}$$

By (11) and (12),

$$3T(a^3) + T(a) = 3aT(a^2) + aT(e_A), \qquad a \in A. \tag{13}$$

Replacing a by $a+e_A$ in (13) and simplify the result to get

$$9T(a^2)+9T(a)=6aT(a)+3T(a^2)+6T(a)+3aT(e_A). \tag{14}$$

From (12) and (14), we deduce that

$$T(a)=aT(e_A), \qquad a\in A. \tag{15}$$

It follows from (13) and (15) that $T(a^3)=aT(a^2)$ and so T is a 3-Jordan multiplier. □

Next by using the Vandermonde matrix, we show that Theorem 2.7 is also valid for all $n\in\mathbb{N}$.

Theorem 2.8 *Let A be a unital Banach algebra, X be a unitary Banach left A-module. Then every $(n+1)$-Jordan multiplier $T:A\longrightarrow X$ is an n-Jordan multiplier.*

Proof We firstly have

$$T((a+ke_A)^{n+1})=(a+ke_A)T(a+ke_A)^n, \tag{16}$$

for all $a\in A$, where k is an integer with $1\le k\le n$. It follows from the equality (16) and assumption that

$$\sum_{i=1}^{n}k^{n+1-i}\binom{n+1}{i}T(a^i)=\sum_{i=1}^{n}k^{n+1-i}\binom{n}{n+1-i}aT(a^{i-1})+\sum_{i=1}^{n}k^{n+1-i}\binom{n}{i}T(a^i), \tag{17}$$

for all $1\le k\le n$ and $a\in A$ where $\binom{n}{i}=\frac{n!}{i!(n-i)!}$. We can rewrite the equalities in (17) as $MX_i(a)=MY_i(a)+MZ_i(a)$, where

$$M=\begin{bmatrix}1 & 1 & \cdots & 1\\ 2^n & 2^{n-1} & \cdots & 2\\ 3^n & 3^{n-1} & \cdots & 3\\ \cdots & \cdots & \cdots & \cdots\\ n^n & n^{n-1} & \cdots & n\end{bmatrix}$$

and

$$X_i(a)=\binom{n+1}{i}T(a^i), \quad Y_i(a)=\binom{n}{n+1-i}aT(a^{i-1}), \quad Z_i(a)=\binom{n}{i}T(a^i).$$

It is shown in [1, Lemma 2.1] that the square matrix M is invertible. This implies that $X_i(a)=Y_i(a)+Z_i(a)$ for all $1\le i\le n$ and all $a\in A$. In particular, $X_n(a)=Y_n(a)+Z_n(a)$. Therefore

$$(n+1)T(a^n)=naT(a^{n-1})+T(a^n).$$

Consequently, $T(a^n)=aT(a^{n-1})$ and hence T is an n-Jordan multiplier. □

From Lemma 2.1 and Theorem 2.8 we get the following result.

Corollary 2.9 *Let A be a unital Banach algebra and X be a unitary Banach left A-module. Suppose that $T:A\longrightarrow X$ is a linear map. Then the following conditions are equivalent.*

(i) $T(a)=aT(e_A)$ *for all* $a\in A$.
(ii) T *is a Jordan multiplier.*
(iii) T *is an n-Jordan multiplier.*
(iv) T *is an $(n+1)$-Jordan multiplier.*

As a consequence of preceding corollary and Theorem 2.3, we get the following result.

Corollary 2.10 *Let A be a unital Banach algebra and X be a unitary Banach left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map. If $T(ab) = aT(b)$ for all $a, b \in A$ with $ab = e_A$, then T is an n-multiplier.*

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