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ORIGINAL ARTICLE

The Oseen–Frank Energy Functional on Manifolds

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Abstract
We observe that for a unit tangent vector field $u \in TM$ on a 3-dimensional Riemannian manifold M, there is a unique unit cotangent vector field $A \in T^*M$ associated to u such that we can define the curl of u by dA. Through a unit cotangent vector field $A \in T^*M$, we define the Oseen–Frank energy functional on 3-dimensional Riemannian manifolds. Moreover, we prove partial regularity of minimizers of the Oseen–Frank energy on 3-dimensional Riemannian manifolds.

Keywords The Oseen–Frank energy · Partial regularity

Mathematics Subject Classification (2010) 35J50 · 58J05

1 Introduction

A liquid crystal is a mesomorphic phase of a material which occurs between its liquid and solid phase. The material is composed of rod like molecules which display orientational order, unlike a liquid, but lacking the lattice structure of a solid. In their pioneering works, Oseen [36] and Frank [11] established the static mathematical continuum theory on nematic liquid crystals through a director u, which is the average direction of molecules [35]. There are a lot of analytical and computational issues in study of static equilibrium configurations.

Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with smooth boundary $\partial\Omega$. Set

$$H^1(\Omega; S^2) = \left\{u \in H^1(\Omega; \mathbb{R}^3) : \ |u| = 1 \text{ a.e. on } \Omega\right\}.$$

The Oseen–Frank energy associated to a director $u \in H^1(\Omega; S^2)$ is given by

$$E(u) = \int_\Omega W(u, \nabla u)\, dx,$$

Dedicated to Professor Jürgen Jost on the occasion of his 65th birthday.

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where $W(u, \nabla u)$ is the Oseen–Frank free energy density given by

$$W(u, \nabla u) = k_1(\operatorname{div} u)^2 + k_2(u \cdot \operatorname{curl} u)^2 + k_3|u \times \operatorname{curl} u|^2 + (k_2 + k_4)\left[\operatorname{tr}(\nabla u)^2 - (\operatorname{div} u)^2\right],$$

in which k_1, k_2, k_3 are the Frank constants for molecular distortion of splay, twist and bend respectively, and k_4 is the Frank constant for the surface energy (e.g. [35]).

Let $\gamma : \partial\Omega \to S^2$ be a given smooth boundary data. For any map $u \in H^1_\gamma(\Omega, S^2)$, the integral

$$\Phi(\gamma) = \frac{1}{2}\int_\Omega [\operatorname{tr}(\nabla u)^2 - (\operatorname{div} u)^2]dx$$

is a number depending on only γ (see [20]). Therefore, without loss of generality, we assume

$$W(u, \nabla u) := k|\nabla u|^2 + V(u, \nabla u), \tag{1.1}$$

where $k = \min\{k_1, k_2, k_3\} > 0$ and

$$V(u, \nabla u) = (k_1 - k)(\operatorname{div} u)^2 + (k_2 - k)(u \cdot \operatorname{curl} u)^2 + (k_3 - k)|u \times \operatorname{curl} u|^2.$$

An equilibrium configuration of liquid crystals corresponds to an extremal (critical point) of the functional E. The Euler–Lagrange equations associated with E in $H^1(\Omega; S^2)$ is

$$\left(\delta_{ik} - u^i u^k\right)\left(\nabla_\alpha W_{p^k_\alpha}(u, \nabla u) - W_{u^k}(u, \nabla u)\right) = 0 \tag{1.2}$$

for $i = 1, 2, 3$. Here and in the sequel, we adopt the Einstein summation convention and denote by δ_{ik} the Kronecker delta. In the special case of $k_1 = k_2 = k_3 = 1$ and $k_4 = 0$, the equation (1.2) is

$$\triangle u + |\nabla u|^2 u = 0 \ \text{ in } \Omega,$$

which is the equation of harmonic maps from Ω into S^2. In 1964, Eells and Sampson [8] introduced the study of harmonic maps between two Riemannian manifolds. There are many interesting results on harmonic maps (e.g. [7, 29]). In particular, Giaquinta–Giusti [13, 14] and Schoen–Uhlenbeck [33] proved partial regularity of minimizing harmonic maps. For further developments on harmonic maps, see [18, 24].

Numerical and experimental analysis on liquid crystals has shown that equilibrium configurations of the system (1.2) expect to have point and line singularities. In physics, it is called the one-constant approximation for the special case of $k_1 = k_2 = k_3 = 1$ and $k_4 = 0$ (e.g. [35, Section 2.2.1]). For this special case, Brezis, Coron and Leib [4] investigated the local behavior of isolated singularities of energy minimizing maps. Bethuel, Brezis and Coron [3] introduced a relaxed energy for harmonic maps and proved existence of infinitely many weak solutions of harmonic maps (see also [23]). Bethuel and Brezis [2] studied the regularity problem of minimizers of modified relaxed problems for harmonic maps. Using Cartesian currents, Giaquinta, Modica and Soucek [16] proved partial regularity of minimizers of the relaxed energy for harmonic maps. In the same spirit of Sacks–Uhlenbeck [34] and Uhlenbeck [38], Giaquinta, the author and Yin [15] proposed an approximation for the relaxed energy of the Dirichlet energy and proved partial regularity of a minimizer of the relaxed energy for harmonic maps without using Cartesian currents.

In the theory of liquid crystals, the Frank elastic constants k_1, k_2, k_3 in (1.1) are unequal in general. For an example, the work of Zwetkoff in 1937 was mentioned by Stewart in [35] that the Frank elastic constants for para-Azoxyanisole (PPA) at $T = 125°C$ are

$$k_1 = 9 \times 10^{-12}N, \quad k_2 = 5.8 \times 10^{-12}N, \quad k_3 = 19 \times 10^{-12}N.$$

For the general case of the unequal Frank constants k_1, k_2 and k_3, Hardt, Kinderlehrer and Lin [20, 21] proved that an energy minimizer u is smooth on some open subset $\Omega_0 \subset \Omega$

and moreover $\mathcal{H}^\beta(\Omega\backslash\Omega_0) = 0$ for some positive $\beta < 1$, where $\mathcal{H}^\beta$ is the Hausdorff measure. Almgren and Lieb [1] did more analysis on singularities of energy minimizing maps when k_1, k_2 and k_3 are close to k. Giaquinta, Modica and Soucek [17] studied the relaxed energy of the Oseen–Frank functional. As we pointed out before, harmonic maps have been extensively studied between two Riemannian manifolds (e.g. [7, 18, 29]), so it is interesting to generalize the Oseen–Frank energy on Riemannian manifolds.

In this paper, we investigate the Oseen–Frank energy functional on 3-dimensional Riemannian manifolds. Let (M, g) be a 3-dimensional Riemannian manifold (with possible boundary). In local coordinates around a point $x \in M$, a smooth Riemannian metric g can be represented by

$$g = g_{ij} dx_i \otimes dx_j,$$

where (g_{ij}) is a positive definite symmetric $n \times n$ matrix. Let $(g^{ij}) := (g_{ij})^{-1}$ be the inverse matrix of (g_{ij}) and the volume element of $(M; g)$ is

$$dv_g = \sqrt{|g|}dx \quad \text{with } |g| := \det(g_{ij}).$$

For a unit tangent vector field $u \in H^1(\Omega; TM)$, we write $u(x) = u^i(x)\frac{\partial}{\partial x^i}$ in local coordinates with the norm

$$|u(x)|^2 = g_{ij}u^i(x)u^j(x) = 1.$$

Although one can define $\operatorname{curl} u$ in the tangent space through a normal frame, there is no clear form of $\operatorname{curl} u$. As pointed out in [6, Chapter 3, p. 79], $\operatorname{curl} u$ associates to dA through an one-form A. Motivated by this observation, we have

Theorem 1.1 *For a unit tangent vector field $u \in TM$, there is a unique unit cotangent vector field $A \in T^*M$ associated to u such that*

$$|d^*A|^2 = |\operatorname{div} u|^2, \quad |\langle A, *dA\rangle|^2 = |u \cdot \operatorname{curl} u|^2, \quad |A \wedge *dA|^2 = |u \times \operatorname{curl} u|^2,$$

where $*$ is the Hodge star operator (e.g. [31]).

Let Ω be a domain in M. For a unit tangent vector field $u \in H^1(\Omega; TM)$, let $A \in H^1(\Omega; T^*M)$ be the unit cotangent vector field associated to u in Theorem 1.1 and denote by ∇A the covariant derivative of A (e.g. [31]). Then, we define the Oseen–Frank energy functional of A in Ω by

$$E(A; \Omega) = \int_\Omega W(A(x), \nabla A(x))\, dv_g, \tag{1.3}$$

where $W(A, \nabla A)$ is the Oseen–Frank energy density defined by

$$W(A, \nabla A) := k|\nabla A|^2 + V(A, d^*A, dA)$$

with $k = \min\{k_1, k_2, k_3\} > 0$, satisfying

$$V(A, d^*A, dA) = (k_1 - k)|d^*A|^2 + (k_2 - k)|\langle A, *dA\rangle|^2 + (k_3 - k)|A \wedge *dA|^2.$$

Denote $T^*_M(S^2) = \{A \in T^*M : |A| = 1\}$. Then $A \in H^1(\Omega, T^*_M(S^2))$ is a weak solution to the liquid crystal system if A satisfies the Euler–Lagrange equation

$$\begin{aligned} & k\left[\nabla^*\nabla A - |\nabla A|^2 A\right] + (k_1 - k)\left[dd^*A - \langle dd^*A, A\rangle A)\right] \\ & +(k_2 - k_3)\left[d^*(\langle A, *dA\rangle * A) - \langle d^*(\langle A, *dA\rangle * A), A\rangle A)\right] \\ & +(k_3 - k)\left[d^*dA - \langle d^*dA, A\rangle A\right] + (k_2 - k_3)\left[\langle A, *dA\rangle * dA - \langle A, *dA\rangle^2 A\right] = 0 \end{aligned} \tag{1.4}$$

in the sense of distribution (see details in Section 4).

Then we prove partial regularity of minimizers of the Oseen–Frank energy:

Theorem 1.2 *Let A be a minimizer of the Oseen–Frank energy functional (1.3) in $H^1_\gamma(\Omega; T^*_M(S^2))$, where γ is a given boundary value. Then, A is smooth in a set $\Omega_0 \subset \bar{\Omega}$ and $\mathcal{H}^\beta(\bar{\Omega}\backslash\Omega_0) = 0$ for some positive $\beta < 1$, where $\mathcal{H}^\beta$ is the Hausdorff measure.*

The idea of proofs of Theorem 1.2 is to modify an approach of the direct method in [22], which is based on a reverse Hölder inequality. The direct method for elliptic systems was extensively studied in [12] and [19]. Using the minimality, we prove a Caccioppoli's inequality and a reverse Hölder inequality. Since the principal term in (1.4) is complicated, the liquid crystal system is not a standard elliptic system, which is not discussed in [12] and [19].

We would like to outline some key ideas to handle the extra terms in (1.4). We choose a normal co-frame $\{\omega_i\}_{i=1}^3$ around $x_0 \in M$ such that $\omega_3 = \frac{A_{x_0,R}}{|A_{x_0,R}|}$. Set $\tilde{A} = \tilde{A}_1\omega_1 + \tilde{A}_2\omega_2$. Using the fact that $|A| = 1$, we can prove

$$|\nabla(A - \tilde{A})|^2 \le C\left(|A - A_{x_0,R}| + R^2 + R^{-1}\int_{B_R(x_0)} |\nabla A|^2\, dv_g\right)|\nabla A|^2.$$

We rewrite (1.4) into an equation on $\tilde{A}$ (see (4.4)). Then we can estimate the difficult terms in (1.4). Finally, we modify the freezed coefficient method from [12] to prove partial regularity.

Remark 1 Based on the static Oseen–Frank theory, Ericksen [9] and Leslie [32] proposed a hydrodynamic theory to describe the behavior of liquid crystal flows. Recently, there are a lot of progress about the Ericksen–Leslie system in $\mathbb{R}^3$ with unequal Frank constants k_1, k_2, k_3 (e.g. [10, 26–28]). Comparing with the result of Struwe [37] and Chen–Struwe [5] on the harmonic map flow between manifolds, it is interesting to study the Ericksen–Leslie system on manifolds for unequal Frank constants k_1, k_2, k_3.

Finally, I would like to dedicate this paper to Professor Jürgen Jost on the occasion of his 65th birthday. I met Jürgen first time at ETH-Zürich in 1994 when I was a postdoctoral fellow under supervision of Professor Michael Struwe. Through our collaboration [25], I learnt a lot of mathematics from Jürgen and Michael. Furthermore, I also learn a lot of knowledge on differential geometry from Jürgen's books [30, 31].

The paper is organized as follows. In Section 2, we outline geometric setting for the Oseen–Frank energy and prove Theorem 1.1. In Section 3, we prove the Caccioppoli inequality and the reverse Hölder inequality. In Section 4, we prove Theorem 1.2.

2 Geometric Setting for the Oseen–Frank Energy

Let M be a smooth Riemannian 3-manifold M equipped with a Riemannian metric g; i.e., for each tangent space T_xM, there is an inner product $\langle\cdot,\cdot\rangle$. In local coordinates,

$$g_{ij} := \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle.$$

For $X, Y, Z \in C^\infty(TM)$, the connection ∇ satisfies

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

The connection, which satisfies the above identity, is called Riemannian. In local coordinates, the Christoffel symbols are defined by

$$\Gamma^k_{ij} := \frac{1}{2} g^{kl}\left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

satisfying

$$\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) = \Gamma^k_{ij}\frac{\partial}{\partial x^k}, \quad \nabla_{\frac{\partial}{\partial x^i}}(dx^j) = -\Gamma^j_{ik} dx^k.$$

We recall that the curvature tensor of Levi-Civita connection R is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X, Y, Z \in C^\infty(TM)$. In local coordinates,

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l} = R^k_{lij}\frac{\partial}{\partial x^k}.$$

We set

$$R_{klij} := g_{km} R^m_{lij} = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right\rangle.$$

In local coordinates, we have

$$R^k_{lij} = \left(\frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}\right).$$

For each $x \in M$, let $u(x)$ be a unit tangent vector in $T_x M$. In local coordinates, we write $u(x) = u^i(x)\frac{\partial}{\partial x^i}$ with the norm $|u(x)|^2 = g_{ij} u^i(x) u^j(x) = 1$. It follows from [6, Chapter 4, pp. 114–115] that the absolute differential of u is defined by

$$\nabla u = (du^i + u^j \omega^i_j) \otimes \frac{\partial}{\partial x^i} = \left(\frac{\partial u^i}{\partial x_j} + u^k \Gamma^i_{kj}\right) dx^j \otimes \frac{\partial}{\partial x^i},$$

where ω^i_j is defined by $\omega^i_j = \Gamma^i_{jk} du^k$.

The divergence of the vector $u(x) = u^i(x)\frac{\partial}{\partial x^i}$ (e.g. [31]) is defined as

$$\operatorname{div} u = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^j}\left(\sqrt{|g|} u^j\right). \tag{2.1}$$

For a unit vector $u(x) = u^i(x)\frac{\partial}{\partial x^i} \in TM$, there is a unique corresponding cotangent vector A in T^*M defined by

$$A(x) = A^i(x)\, dx^i = g_{ij} u^j dx^i \tag{2.2}$$

satisfying

$$|A|^2 = \left\langle g_{ij} u^j dx^i, g_{kl} u^l dx^k\right\rangle = g_{ij} g_{kl} g^{ik} u^l u^j = g_{ij} u^i u^j = 1.$$

Then the absolute differential of A (e.g. [6, Chapter 4]) is given by

$$\nabla A = (dA^i - A^j \omega^j_i) \otimes du^i = \left(\frac{\partial A^i}{\partial x^j} - A^k \Gamma^k_{ij}\right) du^j \otimes du^i.$$

Moreover,

$$dA = dA^j \wedge dx^j = \frac{1}{2}\left(\frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x_j}\right) dx^i \wedge dx^j = \sum_{i<j}\left(\frac{\partial A^j}{\partial x_i} - \frac{\partial A^i}{\partial x_j}\right) dx^i \wedge dx^j.$$

It implies

$$|dA|^2 = \frac{1}{4} g^{ik} g^{jl}\left(\frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x^j}\right)\left(\frac{\partial A^l}{\partial x^k} - \frac{\partial A^k}{\partial x^l}\right).$$

Let d be the exterior derivative given by

$$d : \Omega^k(TM) \to \Omega^{k+1}(TM) \quad \text{for an integer } k \geq 0$$

and the adjoint operator

$$d^* : \Omega^{k+1}(TM) \to \Omega^k(TM)$$

satisfying the property

$$\int_M \langle da, b\rangle dv = \int_M \langle a, d^* b\rangle dv.$$

Let $*$ be the Hoge star operator (e.g. [31]) by

$$* : \Lambda^k(T_x^* M) \to \Lambda^{3-k}(T_x^* M).$$

Using the star operator $*$, we have

$$*dA = \sum_{i<j}\left(\frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x^j}\right) * (dx^i \wedge dx^j) \in \Lambda^1(T_x^* M).$$

Then

$$\langle A, *dA\rangle = A^1\left(\frac{\partial A^3}{\partial x_2} - \frac{\partial A^2}{\partial x_3}\right)\langle dx_1, *(dx^2 \wedge dx^3)\rangle + A^2\left(\frac{\partial A^3}{\partial x_2} - \frac{\partial A^2}{\partial x_3}\right)\langle dx_2, *(dx^i \wedge dx^j)\rangle$$
$$+ A^3\left(\frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}\right)\langle dx^3, *(dx^1 \wedge dx^2)\rangle.$$

Now we complete a proof of Theorem 1.1.

Proof Through the dual operator d^*, we have

$$\int_M \langle d^* A, f\rangle dv_g = \int_M \langle A, df\rangle dv_g$$

for a smooth function f with compact support in M. Then we have in local coordinates

$$d^* A = -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^j}\left(\sqrt{|g|} g^{ij} A^i\right).$$

It can be checked that $d^* A = -\operatorname{div} u$ through (2.1)–(2.2).

In normal coordinates at each fixed $x_0 \in M$ (e.g. [31, p. 21]), we have

$$g_{ij}(x_0) = \delta_{ij}, \quad \Gamma^i_{jk}(x_0) = 0, \quad \frac{\partial g_{ij}}{\partial x_k}(x_0) = 0$$

for i, j, k. At each $x_0 \in M$, we can write $u(x_0) = u^i(x_0)e_i \in S^2 \subset T_{x_0}M = \mathbb{R}^3$ with a normal frame $\{e_i = \frac{\partial}{\partial x_i}\}$. Let $\{\omega_i = dx_i\}$ be a normal co-frame at x_0. Then at x_0, we have

$$A = A^1 dx_1 + A^2 dx_2 + A^3 dx_3 = u^1\omega_1 + u^2\omega_2 + u^3\omega_3.$$

At each x_0, we have

$$dA = \left(\frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3}\right)\omega_2 \wedge \omega_3 + \left(\frac{\partial A^3}{\partial x_1} - \frac{\partial A^1}{\partial x_3}\right)\omega_1 \wedge \omega_3 + \left(\frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2}\right)\omega_1 \wedge \omega_2$$

and

$$*dA = \left(\frac{\partial A^3}{\partial x_2} - \frac{\partial A^2}{\partial x_3}\right)\omega_1 + \left(\frac{\partial A^1}{\partial x_3} - \frac{\partial A^3}{\partial x_1}\right)\omega_2 + \left(\frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2}\right)\omega_3.$$

Using the formula (2.2.1) of Chapter 2 in [31] with $|A| = 1$, we have

$$\begin{aligned} |*dA|^2 &= |\mathrm{curl}\, u|^2, \quad |\langle A, *dA\rangle|^2 = |u \cdot \mathrm{curl}\, u|^2, \\ |A \wedge *dA|^2 &= |A|^2 |*dA|^2 - |\langle A, *dA\rangle|^2 = |u \times \mathrm{curl}\, u|^2. \end{aligned}$$

This proves our claim. □

3 Caccioppoli's Inequality and the Reverse Hölder Inequality

In the section, we will follow the approach in [22] for proving Caccioppoli's inequality and a reverse Hölder inequality of energy minimizers.

At first, we generalize Hardt–Lin's extension Lemma in [21]:

Lemma 3.1 *Let $B_R(x_0)$ be a geodesic ball in M for all $R \le R_0$ with some $R_0 > 0$. For any $v \in H^1(B_R(x_0); T^*M)$ with $|v| = 1$ on $\partial B_R(x_0)$, there exists an one form $w \in H^1(B_R(x_0); T^*_M(S^2))$ such that*

$$w = v \quad \text{on } \partial B_R(x_0)$$

and

$$\int_{B_R(x_0)} |\nabla w|^2 \, dv_g \le C \int_{B_R(x_0)} (|\nabla v|^2 + |x - x_0|^2) \, dv_g$$

for a constant C independent of v, w and R.

Proof We modify a proof in the Appendix of [21]. At a fixed $x_0 \in M$, there is normal coordinates (e.g. [31, p. 21]) in $B_{R_0}(x_0)$ such that

$$g_{ij}(x_0) = \delta_{ij}, \qquad \left|\frac{\partial g_{ij}}{\partial x_k}(x)\right| \le C|x - x_0|$$

for i, j, k. In the coordinate at x_0, we write

$$v = v^i(x) dx^i.$$

Let $\tilde{a} = a^i dx^i \in T^*M$ be the one form corresponding to the constant $a = (a^1, a^2, a^3) \in \mathbb{R}^3$ with $|\tilde{a}| \le \frac{1}{2}$. Then we consider a one form

$$w_a(x) = \frac{v(x) - \tilde{a}}{|v(x) - \tilde{a}|}, \qquad x \in \Omega.$$

Then at $x = x_0$

$$\nabla w_a = \frac{\nabla(v - \tilde{a})}{|v - \tilde{a}|} - \frac{(v - \tilde{a})\langle v - \tilde{a}, \nabla v\rangle}{|v - \tilde{a}|^3}.$$

Integrating over Ω with respect to x and over $B_{1/2}$ with respect to a, we obtain

$$\int_{B_{1/2}}\int_{B_R(x_0)} |\nabla w_a|^2\, dv_g\, d\tilde a = \int_{B_R(x_0)}\int_{B_{1/2}} |\nabla w_a|^2\, d\tilde a\, dv_g$$
$$\le C\int_{B_R(x_0)} |\nabla(v-\tilde a)|^2\, dv_g \le C\int_{B_R(x_0)} |\nabla v|^2+|x-x_0|^2\, dv_g$$

due to the fact that

$$\int_{B_{1/2}} |v-\tilde a|^{-2}\, d\tilde a \le K,$$

where K is a positive constant. Hence there exists a point a_0 with $|a_0|\le \frac{1}{2}$ such that

$$\int_{B_R(x_0)} |\nabla w_{a_0}|^2\, dx \le C\int_{B_R(x_0)} |\nabla B|^2 + |x-x_0|^2\, dv_g. \tag{3.1}$$

For any $a\in T^*M$ with $|a|\le \frac{1}{2}$, we define Π_a to be a C^1-bilipshitz diffeomorphism of $T^*_M(S^2)$ onto itself by

$$\Pi_a(\xi) = \frac{\xi - a}{|\xi - a|}.$$

Indeed,

$$\Pi_a^{-1}(\eta) = a + \left([(a\cdot\eta)^2 + (1-|a|^2)]^{1/2} - a\cdot\eta\right)\eta$$

and

$$|\nabla\Pi_a^{-1}(\eta)| \le \Lambda$$

for a uniform constant Λ independently of a with $|a|\le\frac{1}{2}$.

Therefore, taking

$$w = \Pi_{a_0}^{-1}\circ w_{a_0},$$

we have

$$|\nabla w| \le C(\Lambda)|\nabla w_{a_0}|. \tag{3.2}$$

Our claim follows from (3.1) and (3.2). $\square$

We recall Lemma 3.1 in Chapter V of [12]:

Lemma 3.2 *Let $f(t)$ be a nonnegative bounded function defined in $[r_0, r_1]$, $r_0 \ge 0$. Suppose that for any two t, s with $r_0\le t< s\le r_1$ we have*

$$f(t) \le \left[C(s-t)^\alpha + B\right] + \theta f(s), \tag{3.3}$$

where C, B, α, θ are nonnegative constants with $0\le\theta<1$. Then all ρ, R with $r_0\le\rho< R\le r_1$ we have

$$f(\rho) \le C\left[(R-\rho)^\alpha + B\right].$$

Using Lemma 3.1 and Lemma 3.2, we prove

Lemma 3.3 (Caccioppoli's inequality) *Let $x_0\in\Omega$ and $R_0>0$ such that $B_{R_0}(x_0)\subset\Omega$. Let A be a minimizer of E in $H^1_\gamma(\Omega, T^*_M(S^2))$. Then for any $R\le R_0$, we have*

$$\int_{B_{R/2}(x_0)} |\nabla A|^2\, dv_g \le CR^{-2}\int_{B_R(x_0)} |A - A_{x_0,R}|^2\, dv_g + CR^5, \tag{3.4}$$

where $A_{x_0,R} := \sum_{i=1}^3 A^i_{x_0,R}dx^i$ is denoted by

$$A^i_{x_0,R} = \frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} A^i\, dv_g. \tag{3.5}$$

Proof Let $v \in H^1_\gamma(B_{R_0}(x_0), T^*M)$ with $v = A$ on $\partial B_{R_0}(x_0)$ for a $R_0 > 0$. By Lemma 3.1, there is a $w \in H^1(B_{R_0}(x_0); T^*_M(S^2))$ with $w = A$ on $\partial B_{R_0}(x_0)$ such that

$$\int_{B_{R_0}(x_0)} |\nabla w|^2\, dv_g \le C \int_{B_{R_0}(x_0)} |\nabla v|^2 + |x - x_0|^2\, dv_g.$$

Since A is a minimizer of E in $H^1_\gamma(\Omega, T^*_M(S^2))$ and $w = A$ on $\partial B_{R_0}(x_0)$, we have

$$\begin{aligned} k \int_{B_{R_0}(x_0)} |\nabla A|^2\, dv_g &\le \int_{B_{R_0}(x_0)} W(A, \nabla A)\, dx \\ &\le \int_{B_{R_0}(x_0)} W(w, \nabla w)\, dv_g \le C \int_{B_{R_0}(x_0)} |\nabla v|^2 + |x - x_0|^2\, dv_g \end{aligned}$$

for any $v \in H^1_A(B_{R_0}(x_0))$.

For any two positive t, s with $\frac{R}{2} \le t < s \le R$, we choose a cut-off function $\eta \in C_0^\infty(B_s)$ such that $0 \le \eta \le 1$ with $\eta \equiv 1$ in B_t and $|\nabla \eta| \le \frac{C}{s-t}$. Taking $v = A - \eta(A - A_{x_0,R})$, we see

$$\nabla v = (1 - \eta)\nabla(A - A_{x_0,R}) - \nabla\eta(A - A_{x_0,R}).$$

It follows from (3.3) that

$$\int_{B_s} |\nabla A|^2\, dx \le C \int_{B_s} |\nabla v|^2 + |x - x_0|^2\, dv_g.$$

Then

$$\int_{B_s} |\nabla A|^2\, dv_g \le C_1 \int_{B_s \setminus B_t} |\nabla A|^2\, dv_g + C_1 R^5 + C_1 (s-t)^{-2} \int_{B_R} |A - A_{x_0,R}|^2\, dv_g.$$

By the standard filling hole trick, there exists a positive $\theta = \frac{C_1}{1+C_1} < 1$ such that

$$\int_{B_t} |\nabla A|^2\, dv_g \le \theta \int_{B_s} |\nabla A|^2\, dv_g + C R^5 + C(s-t)^{-2} \int_{B_R} |A - A_{x_0,R}|^2\, dv_g$$

for all t, s with $\frac{R}{2} \le t < s \le R$. It implies from using Lemma 3.2 that

$$\int_{B_{R/2}} |\nabla A|^2\, dv_g \le C R^{-2} \int_{B_R} |A - A_{x_0,R}|^2\, dv_g + C R^5.$$

This proves our claim. $\square$

By applying the Sobolev–Poincare inequality to (3.4), we have

$$\begin{aligned} \left(\fint_{B_{R/2}(x_0)} |\nabla A|^2\, dv_g \right)^{1/2} &\le \frac{C}{R} \left(\fint_{B_R(x_0)} |A - A_R|^2\, dv_g \right)^{1/2} + CR \\ &\le C \left(\fint_{B_R(x_0)} |\nabla A|^{\frac{6}{5}}\, dv_g \right)^{\frac{5}{6}} + CR \end{aligned}$$

for any $x_0 \in \Omega$ and any $R > 0$ with $B_R(x_0) \subset B_{\frac{R_0}{2}}(x_0) \subset \Omega$ for some $R_0 > 0$. Then

$$\left(\fint_{B_{R/2}(x_0)} (|\nabla A| + R)^2\, dv_g \right)^{1/2} \le C \left(\fint_{B_R(x_0)} (|\nabla A| + R)^{\frac{6}{5}}\, dv_g \right)^{\frac{5}{6}}.$$

Using the above result, we use the standard trick (see Proposition 1.1 of [12]; pp. 122–123) to obtain that there exists an exponent $q > 2$ such that for all $x_0 \in \Omega$ and $R \le R_0$, we have

$$\left(\fint_{B_{R/2}(x_0)} (|\nabla A| + R)^q \, dx \right)^{1/q} \le C \left(\fint_{B_R(x_0)} (|\nabla A| + R)^2 \, dx \right)^{1/2}, \tag{3.6}$$

where C is a constant independent of A. Equation (3.6) is called a reverse Hölder inequality.

4 Partial Regularity of Weal Solution of Liquid Crystal Systems

In the section, we will modify an approach in [22] to prove partial regularity of weak solutions having Caccioppoli's inequality (see Lemma 3.3).

For a smooth one-form $\phi \in C_0^\infty(\Omega)$, we consider a variation

$$A_t(x) = \frac{A + t\phi}{|A(x) + t\phi(x)|} = \frac{A + t\phi}{(1 + 2t\langle A, \phi\rangle + t^2|\phi|^2)^{1/2}}.$$

We calculate

$$\frac{dA_t}{dt} = \frac{\phi}{|A(x) + t\phi(x)|} - \frac{(A + t\phi)(\langle A, \phi\rangle + t|\phi|^2)}{(1 + 2t\langle A, \phi\rangle + t^2|\phi|^2)^{3/2}}.$$

Note $|A| = 1$. Then

$$\left.\frac{dA_t}{dt}\right|_{t=0} = \phi - A\langle A, \phi\rangle, \quad \left.\frac{d\nabla A_t}{dt}\right|_{t=0} = \nabla\phi - \nabla(A\langle A, \phi\rangle).$$

Note that

$$d^* A_t = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^j} \left(\sqrt{|g|} g^{ij} A_t^i \right).$$

Using the fact that $|A| = 1$, we have

$$\left.\frac{d}{dt} d^* A_t\right|_{t=0} = d^*(\phi - A\langle A, \phi\rangle)$$

and

$$\left.\frac{d}{dt} dA_t\right|_{t=0} = d(\phi - A\langle A, \phi\rangle).$$

To derive the Euler–Lagrange equation, we compute

$$\left.\frac{d}{dt} \int_\Omega W(A_t, \nabla A_t) \, dx\right|_{t=0} = 0.$$

Using $|A|^2 = 1$, we have

$$\begin{aligned}
&\int_\Omega k\langle \nabla A, \nabla\phi - \nabla A\langle A, \phi\rangle\rangle + (k_1 - k)\langle d^* A, d^*[\phi - A\langle A, \phi\rangle]\rangle \\
&\quad + (k_2 - k)\langle A, *dA\rangle\langle *A, d(\phi - A\langle A, \phi\rangle)\rangle \\
&\quad + (k_2 - k)\langle A, *dA\rangle\langle *dA, \phi - A\langle A, \phi\rangle\rangle \\
&\quad + (k_3 - k)\langle A \wedge *dA, A \wedge *[d(\phi - A\langle A, \phi\rangle)]\rangle \\
&\quad + (k_3 - k)\langle A \wedge *dA, (\phi - A\langle A, \phi\rangle) \wedge *dA\rangle dv_g = 0
\end{aligned}$$

for any one-form $\phi \in C_0^\infty(\Omega)$.

Using the formula (2.2.1) of Chapter 2 in [31] and that $|A| = 1$, we have

$$\begin{aligned}
&\langle A \wedge *dA, A \wedge *[d\phi - d(A\langle A, \phi\rangle)]\rangle \\
&\quad = |A|^2 \langle *dA, *[d\phi - d(A\langle A, \phi\rangle)]\rangle - \langle A, *[d\phi - d(A\langle A, \phi\rangle)]\rangle\langle *dA, A\rangle \\
&\quad = \langle dA, d[\phi - A\langle A, \phi\rangle]\rangle - \langle\langle *dA, A\rangle * A, d(\phi - A\langle A, \phi\rangle)\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\langle A \wedge *dA, (\phi - A\langle A, \phi\rangle) \wedge *dA\rangle \\
&\quad = \langle A, \phi - A\langle A, \phi\rangle\rangle | * dA|^2 - \langle A, *dA\rangle\langle *dA, \phi - A\langle A, \phi\rangle\rangle \\
&\quad = -\langle A, *dA\rangle\langle *dA, \phi - A\langle A, \phi\rangle\rangle.
\end{aligned}$$

Therefore, $A \in H^1(\Omega, T^*_M(S^2))$ is said to be a weak solution to the liquid crystal system if A satisfies (1.4) in weak sense, i.e.,

$$\begin{aligned}
\int_\Omega & k\langle \nabla A, \nabla\phi - \nabla A\langle A, \phi\rangle\rangle + (k_1 - k)\langle d^* A, d^*[\phi - A\langle A, \phi\rangle]\rangle \\
&+ (k_2 - k_3)\langle A, *dA\rangle\langle *A, d(\phi - A\langle A, \phi\rangle)\rangle \\
&+ (k_2 - k_3)\langle A, *dA\rangle\langle *dA, \phi - A\langle A, \phi\rangle\rangle \\
&+ (k_3 - k)\langle dA, d[\phi - A\langle A, \phi\rangle]\rangle dv_g = 0
\end{aligned}$$

for any one-form $\phi \in C_0^\infty(\Omega)$.

Now we prove

Theorem 4.1 *Let $A \in H^1(\Omega, T^*_M(S^2))$ be any weak solution of (1.4) and assume that A has the Caccioppoli inequality. Then A is smooth in an open set $\Omega_0 \subset \Omega$ and $\mathcal{H}^\beta(\Omega\backslash\Omega_0) = 0$ for some positive $\beta < 1$.*

Proof Let x_0 be a point in Ω with $B_{R_0}(x_0) \subset \Omega$ with $R_0 \le \frac{1}{2}\mathrm{dist}(x_0, \partial\Omega)$. For any R, we denote $A_{x_0,R} := \sum_{i=1}^3 A^i_{x_0,R} dx^i$ as in (3.5). In normal coordinates around a point $x_0 \in M$ (e.g. [31, p. 21]), we have

$$g_{ij}(x_0) = \delta_{ij}, \quad \left|\frac{\partial g_{ij}}{\partial x_k}(x)\right| \le C|x - x_0|, \quad |g_{ij}(x) - \delta_{ij}| \le C|x - x_0|^2.$$

By the Sobolev–Poincare inequality, we have

$$\begin{aligned}
|1 - |A_{x_0,R}|^2| &\le \fint_{B_R(x_0)} |A - A_{x_0,R}|^2\, dv_g + C|x - x_0|^2 \\
&\le CR^{-1}\int_{B_R(x_0)} |\nabla A|^2\, dv_g + CR^2.
\end{aligned} \tag{4.1}$$

If $R^{-1}\int_{B_R(x_0)} |\nabla A|^2\, dv_g$ and R are sufficiently small, it can be seen from (4.1) that $|A_{x_0,R}| \ne 0$. We set $\omega_3 = \frac{A_{x_0,R}}{|A_{x_0,R}|}$. Then it follows from (4.1) that

$$\begin{aligned}
|\omega_3 - A_{x_0,R}| &= |1 - |A_{x_0,R}|| \le |1 - |A_{x_0,R}|^2| \\
&\le CR^{-1}\int_{B_R(x_0)} |\nabla A|^2\, dv_g + CR^2.
\end{aligned}$$

Then we have

$$\begin{aligned}|A-\omega_3| &\le |A-A_{x_0,R}|+|A_{x_0,R}-\omega_3|\\ &\le |A-A_{x_0,R}|+CR^{-1}\int_{B_R(x_0)}|\nabla A|^2\,dv_g+CR^2.\end{aligned}\tag{4.2}$$

At each point $x\in B_{R_0}(x_0)$ with a sufficiently small $R_0>0$, there exists an normal frame field $e_j=\frac{\partial}{\partial x^j}$ in T_M (e.g. [6, 31]). Assume that $\{\omega_i\}_{i=1}^3$ is a normal co-frame field in T^*M in $B_{R_0}(x_0)$, where $\omega_3=\frac{A_{x_0,R}}{|A_{x_0,R}|}$. Then we write

$$\begin{aligned}A(x) &= \langle A,\omega_1\rangle\omega_1+\langle A,\omega_2\rangle\omega_2+\langle A,\omega_3\rangle\omega_3\\ &:= \tilde A_1\omega_1+\tilde A_2\omega_2+\tilde A_3\omega_3.\end{aligned}$$

Using $|A(x)|^2=1$, we have

$$\nabla\tilde A_3=-\tilde A_1\nabla\tilde A_1-\tilde A_2\nabla\tilde A_2+(1-\tilde A_3)\nabla\tilde A_3.$$

It follows from (4.2) and using Cauchy's inequality that

$$\begin{aligned}|\nabla\tilde A_3|^2 &\le (1-|\tilde A_3|^2+(1-\tilde A_3)^2)|\nabla A|^2=2(1-\tilde A_3)|\nabla A|^2\\ &= 2(|A|^2-\langle A,\omega_3\rangle)|\nabla A|^2\le 2|A-\omega_3||\nabla A|^2\\ &\le C\left(|A-A_{x_0,R}|+R^2+R^{-1}\int_{B_R(x_0)}|\nabla A|^2\,dv_g\right)|\nabla A|^2.\end{aligned}\tag{4.3}$$

Set $\tilde A=\tilde A_1\omega_1+\tilde A_2\omega_2$. Then we re-write (1.4) into the following equation:

$$\begin{aligned}&k[\nabla^*\nabla\tilde A]+(k_1-k)dd^*\tilde A+(k_2-k_3)d^*(\langle A,*d\tilde A\rangle * A)+(k_3-k)d^*d\tilde A\\ &\quad=-k\big[\nabla^*\nabla(A-\tilde A)-|\nabla A|^2A\big]-(k_1-k)\big[dd^*(A-\tilde A)-\langle dd^*A,A\rangle A\big]\\ &\qquad-(k_2-k_3)\big[d^*(\langle A,*dA\rangle * A)-\langle d^*(\langle A,*dA\rangle * A),A\rangle A\big]\\ &\qquad-(k_3-k)\big[d^*d(A-\tilde A)-\langle d^*dA,A\rangle A\big]\\ &\qquad-(k_2-k_3)\big[\langle A,*dA\rangle * dA-\langle A,*dA\rangle^2A\big].\end{aligned}\tag{4.4}$$

For each $R\le R_0$, let $\tilde v=\tilde v_1\omega_1+\tilde v_2\omega_2\in H^1(B_R(x_0))$ be the solution of

$$\begin{aligned}&k\nabla^*\nabla v+(k_1-k)\big[dd^*v-\langle dd^*v,\omega_3\rangle\omega_3\big]\\ &+(k_2-k_3)\big[d^*(\langle\omega_3,*dv\rangle*\omega_3)-\langle d^*(\langle\omega_3,*dv\rangle*\omega_3),\omega_3\rangle\omega_3\big]\\ &+(k_3-k)\big[d^*dv-\langle d^*dv,\omega_3\rangle\omega_3\big]=0\end{aligned}\tag{4.5}$$

with boundary value $v|_{\partial B_R(x_0)}=\tilde A_1\omega_1+\tilde A_2\omega_2|_{\partial B_R(x_0)}$. Note that (4.5) is a strong elliptic linear system on (v_1,v_2). Then for every $\rho<R$, we have (e.g. [12])

$$\int_{B_\rho(x_0)}|\nabla\tilde v|^2\,dx\le C\left(\frac{\rho}{R}\right)^3\int_{B_R(x_0)}|\nabla\tilde v|^2\,dx.\tag{4.6}$$

Moreover, using the maximum principle of a linear elliptic system (e.g. Proposition 2.3 of Chapter III in [12]) and $|A|=1$, $|\tilde v|\le C$ in $B_R(x_0)$ for some positive constant C.

Choosing $\phi=\tilde w=v-\tilde A=\tilde v_1\omega_1+\tilde v_2\omega_2-\tilde A_1\omega_1-\tilde A_2\omega_2$ as test function in (4.5), we have

$$\begin{aligned}&\int_{B_R(x_0)}k\langle\nabla v,\nabla\tilde w\rangle+(k_1-k)\langle d^*v,d^*\tilde w\rangle+(k_2-k)\langle\omega_3,*dv\rangle\langle*\omega_3,d\tilde w\rangle\\ &+(k_3-k)\langle\omega_3\wedge*dv,\omega_3\wedge*d\tilde w\rangle dv_g=0.\end{aligned}\tag{4.7}$$

Here we use the fact that $\langle \phi, \omega_3 \rangle = 0$ and

$$\langle \omega_3 \wedge *dv, \omega_3 \wedge *d\tilde{w} \rangle = \langle dv, d\tilde{w} \rangle - \langle \omega_3, *dv \rangle \langle \omega_3, *d\tilde{w} \rangle.$$

Multiplying (4.4) by $\tilde{w} = v - \tilde{A} = \tilde{v}_1\omega_1 + \tilde{v}_2\omega_2 - \tilde{A}_1\omega_1 - \tilde{A}_2\omega_2$, we have

$$\begin{aligned}
&\int_{B_R(x_0)} k\langle \nabla \tilde{A}, \nabla \tilde{w} \rangle + (k_1 - k)\langle d^*\tilde{A}, d^*\tilde{w} \rangle + (k_2 - k)\langle A, *d\tilde{A} \rangle \langle *A, d\tilde{w} \rangle \\
&\qquad + (k_3 - k)\langle A \wedge *d\tilde{A}, A \wedge *d\tilde{w} \rangle dv_g \\
&\qquad = \int_{B_R(x_0)} k\langle \nabla (A - \tilde{A}), \nabla \tilde{w} \rangle - k|\nabla A|^2 \langle A, \tilde{w} \rangle + (k_1 - k)\langle d^*(A - \tilde{A}), d^*\tilde{w} \rangle \\
&\qquad\quad - (k_1 - k)\langle d^*A, d^*[A\langle A, \tilde{w} \rangle] \rangle + (k_2 - k_3)\langle A, *d(A - \tilde{A}) \rangle \langle *A, d\tilde{w} \rangle \\
&\qquad\quad - (k_2 - k_3)\langle A, *dA \rangle \langle *A, d(A\langle A, \tilde{w} \rangle) \rangle + (k_3 - k)\langle d(A - \tilde{A}), d\tilde{w} \rangle \\
&\qquad\quad + (k_3 - k)\langle dA, d[A\langle A, \tilde{w} \rangle] \rangle dv_g.
\end{aligned} \tag{4.8}$$

Combining (4.8) with (4.7), we obtain

$$\begin{aligned}
&\int_{B_R(x_0)} k|\nabla \tilde{w}|^2 + (k_1 - k)|d^*\tilde{w}|^2 + (k_2 - k)|\langle \omega_3, *d\tilde{w} \rangle|^2 + (k_3 - k)|\omega_3 \wedge *d\tilde{w}|^2 \, dv_g \\
&\quad = \int_{B_R(x_0)} (k_2 - k)\big[\langle \omega_3, *d\tilde{A} \rangle \langle *\omega_3, d\tilde{w} \rangle - \langle A, *d\tilde{A} \rangle \langle *A, d\tilde{w} \rangle\big] \\
&\quad + (k_3 - k)\big[\langle A \wedge *d\tilde{A}, A \wedge *d\tilde{w} \rangle - (k_3 - k)\langle A \wedge *d\tilde{A}, A \wedge *d\tilde{w} \rangle\big] \\
&\quad + k\langle \nabla (A - \tilde{A}), \nabla \tilde{w} \rangle - k|\nabla A|^2 \langle A, \tilde{w} \rangle + (k_1 - k)\langle d^*(A - \tilde{A}), d^*\tilde{w} \rangle \\
&\quad - (k_1 - k)\langle d^*A, d^*[A\langle A, \tilde{w} \rangle] \rangle + (k_2 - k_3)\langle A, *d(A - \tilde{A}) \rangle \langle *A, d\tilde{w} \rangle \\
&\quad - (k_2 - k_3)\langle A, *dA \rangle \langle *A, d(A\langle A, \tilde{w} \rangle) \rangle + (k_3 - k)\langle d(A - \tilde{A}), d\tilde{w} \rangle \\
&\quad + (k_3 - k)\langle dA, d[A\langle A, \tilde{w} \rangle] \rangle dv_g.
\end{aligned} \tag{4.9}$$

Since $\tilde{w} = \tilde{v}_1\omega_1 + \tilde{v}_2\omega_2 - \tilde{A}_1\omega_1 - \tilde{A}_2\omega_2$, then $\langle \tilde{w}, \omega_3 \rangle = 0$, which implies

$$\langle A, \tilde{w} \rangle = \langle A - \omega_3, w \rangle.$$

To hand those difficult terms on the right-hand side of (4.9), we have

$$\begin{aligned}
d[\langle A, \tilde{w} \rangle A] &= d\langle A - \omega_3, \tilde{w} \rangle \wedge A + \langle A - \omega_3, \tilde{w} \rangle dA \\
&= (\langle \nabla A, \tilde{w} \rangle + \langle A - \omega_3, \nabla \tilde{w} \rangle) \wedge A + \langle A - \omega_3, \tilde{w} \rangle dA
\end{aligned}$$

and

$$\begin{aligned}
d^*[\langle A, \tilde{w} \rangle A] &= *d*[\langle A, \tilde{w} \rangle A] \\
&= *(\langle \nabla A, \tilde{w} \rangle + \langle A - \omega_3, \nabla \tilde{w} \rangle) \wedge A + \langle A - \omega_3, \tilde{w} \rangle d^*A,
\end{aligned}$$

where we used the fact that $\nabla \omega_3 = 0$.

Then, using Young's inequality, we have

$$\begin{aligned}
&\int_{B_R(x_0)} k|\nabla \tilde{w}|^2 + (k_1 - k)|d^*\tilde{w}|^2 + (k_2 - k)|\langle \omega_3, *d\tilde{w} \rangle|^2 + (k_3 - k)|\omega_3 \wedge *d\tilde{w}|^2 \, dv_g \\
&\quad \le \frac{k}{2} \int_{B_R(x_0)} |\nabla w|^2 \, dv_g + C \int_{B_R(x_0)} |\nabla A^3|^2 + |\nabla A|^2(|A - \omega_3|^2 + |w|^2 + R^2 + |w|) \, dv_g.
\end{aligned} \tag{4.10}$$

Applying (4.2) and (4.3) to (4.10), we have

$$\int_{B_R(x_0)} |\nabla \tilde{w}|^2\, dx \le C \int_{B_R(x_0)} |\nabla A|^2 (|A - A_{x_0,R}|^2 + R^2 + |w|)\, dv_g + C R^{-1} \int_{B_R(x_0)} |\nabla A|^2\, dv_g \int_{B_R(x_0)} |\nabla A|^2\, dv_g. \tag{4.11}$$

By (4.5), we have

$$\int_{B_R(x_0)} |\nabla \tilde{v}|^2\, dx \le C \int_{B_R(x_0)} |\nabla A|^2\, dx.$$

Then it follows from (4.6), (4.3) and (4.11) that

$$\begin{aligned}\int_{B_\rho(x_0)} |\nabla A|^2 dx &\le C \left(\frac{\rho}{R}\right)^3 \int_{B_R(x_0)} |\nabla A|^2\, dx + C \int_{B_R(x_0)} |\nabla \tilde{w}|^2\, dx + C \int_{B_R(x_0)} |\nabla A^3|^2\, dx \\ &\le C \left[\left(\frac{\rho}{R}\right)^3 + R^2 + \frac{1}{R} \int_{B_R(x_0)} |\nabla A|^2\, dx \right] \int_{B_R(x_0)} |\nabla A|^2\, dx \\ &\quad + C \int_{B_R(x_0)} (|A - A_{x_0,R}| + |\tilde{w}|) |\nabla A|^2\, dx.\end{aligned}$$

By the reverse Hölder inequality (3.6) and the Sobolev inequality, we obtain

$$\begin{aligned}\int_{B_R(x_0)} |\tilde{w}| |\nabla A|^2\, dx &\le \left(\int_{B_R(x_0)} |\nabla A|^q \right)^{\frac{2}{q}} \left(\int_{B_R(x_0)} |\tilde{w}|^{\frac{q}{q-2}}\, dx \right)^{\frac{q-2}{q}} \\ &\le C R^{\frac{3(2-q)}{q}} \int_{B_{2R}(x_0)} (|\nabla A|^2 + R^2)\, dx \left(\int_{B_R(x_0)} |\tilde{w}|^2\, dx \right)^{\frac{q-2}{q}} \\ &\le C \left(R^{-1} \int_{B_R(x_0)} |\nabla A|^2 \right)^{\frac{q-2}{q}} \int_{B_{2R}(x_0)} (|\nabla A|^2 + R^2)\, dx\end{aligned}$$

for some $q > 2$.

By a similar argument, it follows from using the Hölder inequality and the Sobolev–Poincare inequality that

$$\int_{B_R(x_0)} |A - A_{x_0,R}| |\nabla A|^2\, dx \le C \left(R^{-1} \int_{B_R(x_0)} |\nabla A|^2 \right)^{\frac{q-2}{q}} \int_{B_{2R}(x_0)} (|\nabla A|^2 + R^2)\, dx.$$

Then for every ρ and R with $0 < \rho < R \le R_0 < \frac{1}{2}\mathrm{dist}(x_0, \partial\Omega)$, we have

$$\int_{B_\rho(x_0)} |\nabla A|^2 \le C \left[\left(\frac{\rho}{R}\right)^3 + R^2 + \frac{1}{R} \int_{B_R(x_0)} |\nabla A|^2\, dx \right] \int_{B_{2R}(x_0)} |\nabla A|^2\, dx + C \left(R^{-1} \int_{B_R(x_0)} |\nabla A|^2 \right)^{\frac{q-2}{q}} \int_{B_{2R}(x_0)} (|\nabla A|^2 + R^2)\, dx. \tag{4.12}$$

By (4.12), it implies from the standard method [12] that A is Hölder continuous in $\alpha < 1$ inside $\Omega \backslash \Sigma$, where

$$\Sigma = \left\{ x \in \Omega :\ \liminf_{R \to 0^+} R^{-1} \int_{B_R(x)} |\nabla A|^2\, dx > 0 \right\}.$$

For completeness, we give a detailed proof here. For any $x_0 \in \Omega\backslash\Sigma$, there is a sufficiently small R_0 such that $B_{R_0}(x_0) \subset \Omega\backslash\Sigma$. For each $R \le R_0$, set

$$\phi(x_0, R) = \frac{1}{R}\int_{B_R(x_0)} (|\nabla A|^2 + R^2)\, dx = \frac{1}{R}\int_{B_R(x_0)} |\nabla A|^2\, dv_g + R|B_R(x_0)|,$$
$$\xi(x_0, R) = \frac{1}{R}\int_{B_R(x_0)} |\nabla A|^2\, dx + \left(R^{-1}\int_{B_R(x_0)} |\nabla A|^2\right)^{\frac{q-2}{q}}.$$

Choosing $\rho = 2rR$ in (4.12) with $0 < r < 1$, we have

$$\phi(x_0, r2R) \le C_1\left[r^2(1 + \xi(x_0, R)r^{-3}) + R^2\right]\phi(x_0, 2R) + 4r^2R^2|B_{2rR}(x_0)|. \tag{4.13}$$

For some α with $0 < \alpha < 1$, we choose r such that

$$2(C_1 + 1)r^{2-2\alpha} = 1. \tag{4.14}$$

For the above r, there are a sufficiently small ε_1 and R with $2R \le R_0$ such that

$$\xi(x_0, R) = \frac{1}{R}\int_{B_R(x_0)} |\nabla A|^2\, dx + \left(\frac{1}{R}\int_{B_R(x_0)} |\nabla A|^2\right)^{\frac{q-2}{q}} < \varepsilon_1, \tag{4.15}$$

which implies

$$\xi(x_0, R)r^{-3} \le 2\varepsilon_1 r^{-3} \le 1. \tag{4.16}$$

Assume that $\phi(x_0, 2R) < \varepsilon_0$ for $2R \le R_0$. It implies that $\xi(x_0, R) < \varepsilon_1$ for a sufficiently small ε_0. Then it follows from (4.13)–(4.16) that

$$\phi(x_0, r2R) \le r^{2\alpha}\phi(x_0, 2R)$$

implying

$$\phi(x_0, r^l 2R) \le r^{2l\alpha}\phi(x_0, 2R) < \varepsilon_0$$

for all number l.

We conclude that if $\phi(x_0, 2R) < \varepsilon_0$ for some $2R < R_0$, then

$$\phi(x_0, r^l 2R) \le r^{2l\alpha}\varepsilon_0.$$

This implies that for any $\rho < 2R \le R_0$, we have

$$\phi(x_0, \rho) \le C\left(\frac{\rho}{2R}\right)^{2\alpha}.$$

Hence, A belongs to $C^\alpha_{loc}(\Omega\backslash\Sigma)$ for some $\alpha < 1$. In fact, using (4.12), A belongs to $C^\alpha_{loc}(\Omega\backslash\Sigma)$ for any $\alpha < 1$. Repeating the same argument in Theorem 1.5 of Chapter IV of [12], we can prove that $\nabla\tilde{A}$ are Hölder continuous inside $\Omega\backslash\Sigma$, so is ∇A. Then it can be proved by the standard theory that A is smooth in $\Omega\backslash\Sigma$. By the reverse Hölder inequality, $A \in W^{1,q}_{loc}(\Omega)$ for some $q > 2$. Therefore $\mathcal{H}^\beta_{loc}(\Sigma) = 0$ for some positive $\beta < 1$. □

Theorem 1.2 is a consequence of Theorem 4.1 by using Lemma 3.3.

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References

1. Almgren, F.J., Lieb, E.H.: Singularities of energy minimizing maps from the ball to the sphere: Examples, counterexamples, and bounds. Ann. Math. **128**, 483–530 (1988)
2. Bethuel, F., Brezis, H.: Regularity of minimizers of relaxed problems for harmonic maps. J. Funct. Anal. **101**, 145–161 (1991)
3. Bethuel, F., Brezis, H., Coron, J.M.: Relaxed energies for harmonic maps. In: Berestycki, H., Coron, J.-M., Ekeland, I. (eds.) Variational Methods. Progress in Nonlinear Differential Equations and Their Applications, vol. 4, pp. 37–52. Birkhäuser, Basel (1990)
4. Brezis, H., Coron, J.-M., Lieb, E.H.: Harmonic maps with defects. Commun. Math. Phys. **107**, 649–705 (1986)
5. Chen, Y., Struwe, M.: Existence and partial regular results for the heat flow for harmonic maps. Math. Z. **201**, 83–103 (1989)
6. Chern, S.S., Chen, W.H., Lam, K.S.: Lectures on Differential Geometry. Series on University Mathematics, vol. 1. World Scientific, Singapore (1998)
7. Eells, J., Lemaire, L.: A report on harmonic maps. Bull. Lond. Math. Soc. **10**, 1–68 (1978)
8. Eells, J., Sampson, J.H.: Harmonic mappings of Riemannian manifolds. Amer. J. Math. **86**, 109–160 (1964)
9. Ericksen, J.L.: Conservation laws for liquid crystals. Trans. Soc. Rheol. **5**, 23–34 (1961)
10. Feng, Z., Hong, M.-C., Mei, Y.: Convergence of the Ginzburg-Landau approximation for the Ericksen-Leslie system. SIAM J. Math. Anal. **52**, 481–523 (2020)
11. Frank, F.C.: On theory of liquid crystals. Discuss. Faraday Soc. **25**, 19–28 (1958)
12. Giaquinta, M.: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. Princeton University Press, Princeton (1983)
13. Giaquinta, M., Giusti, E.: On the regularity of the minima of variational integrals. Acta Math. **148**, 31–46 (1982)
14. Giaquinta, M., Giusti, E.: The singular set of the minima of certain quadratic functionals. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **11**, 45–55 (1984)
15. Giaquinta, M., Hong, M.-C., Yin, H.: A new approximation of relaxed energies for harmonic maps and the Faddeev model. Calc. Var. Partial Differ. Equ. **41**, 45–69 (2011)
16. Giaquinta, M., Modica, G., Souček, J.: The Dirichlet energy of mappings with values into the sphere. Manuscripta Math. **65**, 489–507 (1989)
17. Giaquinta, M., Modica, G., Souček, J.: Liquid crystals: Relaxed energies, dipoles, singular lines and singular points. Ann. Scuola Norm. Sup. Pisa **17**, 415–437 (1990)
18. Giaquinta, M., Modica, G., Souček, J.: Cartesian Currents in the Calculus of Variations, Part II: Variational Integrals. A Series of Modern Surveys in Mathematics, vol. 38. Springer, Berlin (1998)
19. Giusti, E.: Direct Methods in the Calculus of Variations. World Scientific, Singapore (2003)
20. Hardt, R., Kinderlehrer, D., Lin, F.-H.: Existence and partial regularity of static liquid crystal configurations. Commun. Math. Phys. **105**, 547–570 (1986)
21. Hardt, R., Kinderlehrer, D., Lin, F.-H.: Stable defects of minimizers of constrained variational principles. Ann. Inst. Henri Poincaré, Non Linear Anal. **5**, 297–322 (1988)
22. Hong, M.-C.: Partial regularity of weak solutions of the liquid crystal equilibrium system. Indiana Univ. Math. J. **53**, 1401–1414 (2004)
23. Hong, M.-C.: Existence of infinitely many equilibrium configurations of the liquid crystal system prescribing the same nonconstant boundary value. Pac. J. Math. **232**, 177–206 (2007)
24. Hong, M.-C.: Some results on harmonic maps. Bull. Inst. Math. Acad. Sin. (N.S.) **9**, 187–221 (2014)
25. Hong, M.-C., Jost, J., Struwe, M.: Asymptotic limits of a Ginzburg-Landau type functional. In: Jost, J. (ed.) Geometric Analysis and the Calculus of Variations for Stefan Hildebrandt, pp. 99–123. International Press (1996)
26. Hong, M.-C., Li, J., Xin, Z.: Blow-up criteria of strong solutions to the Ericksen-Leslie system in $\mathbb{R}^3$. Commun. Partial Differ. Equ. **39**, 1284–1328 (2014)
27. Hong, M.-C., Mei, Y.: Well-posedness of the Ericksen–Leslie system with the Oseen–Frank energy in $L^3_{uloc}(\mathbb{R}^3)$. Calc. Var. Partial Differ. Equ. **58**, 3 (2019)
28. Hong, M.-C., Xin, Z.: Global existence of solutions of the liquid crystal flow for the Oseen–Frank model in $\mathbb{R}^2$. Adv. Math. **231**, 1364–1400 (2012)
29. Jost, J.: Harmonic Maps Between Surfaces. Lecture Notes in Mathematics, vol. 1062. Springer, Berlin (1984)
30. Jost, J.: Nonlinear Methods in Riemannian and Kählerian Geometry: Delivered at the German Mathematical Society Seminar in Düsseldorf in June, vol. 1986. Birkhäuser, Basel (1988)

31. Jost, J.: Riemannian Geometry and Geometric Analysis, 4th edn. Springer, Cham (2005)
32. Leslie, F.M.: Some constitutive equations for liquid crystals. Arch. Rational Mech. Anal. **28**, 265–283 (1968)
33. Sacks, J., Uhlenbeck, K.: The existence of minimal immersions of 2-spheres. Ann. Math. (2) **113**, 1–24 (1981)
34. Schoen, R., Uhlenbeck, K.: A regularity theory for harmonic maps. J. Differ. Geom. **17**, 307–335 (1982)
35. Stewart, I.W.: The Static and Dynamic Continuum Theory of Liquid Crystals: A Mathematical Introduction. Taylor and Francis, London (2004)
36. Oseen, C.W.: The theory of liquid crystals. Trans. Faraday Soc. **29**, 833–899 (1933)
37. Struwe, M.: On the evolution of harmonic maps of Riemannian surfaces. Commun. Math. Helv. **60**, 558–581 (1985)
38. Uhlenbeck, K.: Morse theory by perturbation methods with applications to harmonic maps. Trans. Amer. Math. Soc. **267**, 569–583 (1981)