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ORIGINAL ARTICLE

Quantitative Unique Continuation for Second Order Elliptic Operators with Singular Coefficients

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Abstract
We establish some quantitative strong unique continuation properties for solution of $|Pu| \leq C_1|x|^{-1}|\nabla u| + C_0|x|^{-2}|u|$ where P is a second order elliptic operator. As in (Rev. Mat. Iberoam. 27: 475–491, 2011), our result is of quantitative nature but requires weaker conditions on the coefficients of P.

Keywords Carleman estimate · Doubling inequality · Strong unique continuation

Mathematics Subject Classification (2010) 35J15 · 35A02

1 Introduction

Let $Pu = \sum_{kl} a_{kl}(x)\partial_{kl}u$ be a second order elliptic operator in $\mathbb{R}^n$, where $A = (a_{kl})$ is a real, symmetric elliptic matrix. Aronszajn et al. showed in [5] that if A is continuous in $B_1 := \{x : |x| \leq 1\}$ and for some $\varepsilon > 0$,

$$|\nabla A(x)| \leq C|x|^{-1+\varepsilon} \qquad \text{for a.e. } x \in B_1, \tag{1}$$

then any u that satisfies

$$|Pu| \leq C|x|^{-1+\varepsilon}|\nabla u| + C|x|^{-2+\varepsilon}|u| \tag{2}$$

and vanishes to infinite order at 0 must vanish identically in B_1. In other words, (2) has the strong unique continuation property. Earlier results with stronger smoothness assumptions on A were obtained in [4] and [8].

Alinhac and Baouendi [2] proved the same result for complex-valued $A \in C^\infty$, provided that $A(0)$ is a multiple of a real positively definite matrix. The necessity of the assumption on $A(0)$ was shown by an example of Alinhac [1]. Subsequently, Hörmander [7] weakened the smoothness assumption on A to (1). The example of Pliš [14] shows that unique continuation may not hold if A is only assumed to belong to the Hölder space C^α with $0 < \alpha < 1$. In [14], A is even Lipschitz outside a hypersurface.

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In [10], Meshkov showed that instead of (2), it suffices to assume

$$|Pu| \leq C_1|x|^{-1}|\nabla u| + C_0|x|^{-2}|u|, \tag{3}$$

provided C_1 is sufficiently small, depending on P. This result was later reproved in [15] using slightly different Carleman estimates. Meshkov [11] outlined a possible way to construct an example showing the necessity of the smallness condition on C_1. Explicit examples were later given in [3] and [13]. Related work can be found in [6, 12, 16].

Subsequently, more quantitative properties of unique continuation for (3), namely polynomial lower bound and doubling property were proved by Lin, Nakamura and Wang in [9]. A key element in the proof of [9] is a three-ball inequality deduced from the Carleman estimates of [15]. The main result of this note is an improvement on those of [9] and [10], requiring a weaker condition on ∇A. As in [9], we use the same Carleman estimates from [10] and [15]. However, our proof is more direct as it does not use a three-ball inequality. We next state our result, whose proof is contained in Section 3, after some preparation in Section 2.

Theorem 1 *Let $A \in C(B_1)$ be a symmetric matrix function and suppose that there exist positive constants $\lambda, \delta, C_\delta$ so that*

$$\lambda|\xi|^2 \leq \Re\langle A(x)\xi, \xi\rangle \leq \lambda^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, x \in B_1,$$

and

$$|\nabla A(x)| \leq \frac{C_\delta}{|x||\log|x||^{2+\delta}} \quad \textit{for a.e. } x \in B_1. \tag{4}$$

Furthermore, assume that $A(0)$ is real and positively definite. Then there exist positive constants $R = R(n, \lambda, \delta, C_\delta)$ and $C^* = C^*(n, \lambda)$ such that if u satisfies

$$|Pu| \leq \frac{C_1}{|x|}|\nabla u| + \frac{C_0}{|x|^2}|u|, \tag{5}$$

with $C_1 < C^*$ then there exist $k, M_1, M_2 > 0$ depending on u such that for $0 < r < R$,

$$\int_{B_r} |u|^2 \geq M_1 r^k \tag{6}$$

and

$$\int_{B_{2r}} |u|^2 \leq M_2 \int_{B_r} |u|^2. \tag{7}$$

Here, $B_r := \{x : |x| \leq r\}$.

We note that the fact that k, M_1, and M_2 depend on u is unavoidable, as the example of spherical harmonics shows. Note also that the properties (6) and (7) are stronger than the strong unique continuation property, as they imply that a solution u of (3) that vanishes to infinite order at 0 must vanish in a neighborhood of 0. Then by using [7, Theorem 2.4], it follows that u vanishes identically in B_1.

2 Preliminaries

We first state the two Carleman estimates that will be used in the proof. To simplify the notation, we assume the constants C_2 in Lemmas 1 and 2 below are the same. A proof of the first estimate can be found in [10]. (It was also reproved in [13] and [15].)

Lemma 1 ([10, Theorem 2]) *There exists* $C_2 > 0$ *depending only on* n *such that for any* $\tau \in \frac{1}{2} + \mathbb{N}$ *and* $u \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$,

$$\sum_{|\alpha| \le 2} \int \tau^{2-2|\alpha|} |x|^{-2\tau+2|\alpha|-n} |D^\alpha u|^2 \le C_2 \int |x|^{-2\tau+4-n} |\Delta u|^2.$$

The next estimate was proved in [15, Theorem 1.2] under the slightly stronger assumption (1) on ∇A. For the sake of completeness, we provide a quick proof.

Lemma 2 *Assume that* A *satisfies* (4) *with* $A(0) = Id$. *Let* $\varphi(x) = \frac{1}{2} |\log |x||^2$. *Then there exists* $R_0 \in (0, 1)$ *and positive constants* $\gamma_0 \ge 2$ *and* C_2 *depending only on* n, δ *and* C_δ *such that for* $\gamma \ge \gamma_0$ *and* $u \in C_0^\infty(B_{R_0} \backslash \{0\})$,

$$\gamma^3 \int |x|^{-n} |\log |x||^2 e^{2\gamma\varphi} u^2 + \gamma \int |x|^{-n+2} e^{2\gamma\varphi} |\nabla u|^2 \le C_2 \int |x|^{-n+4} e^{2\gamma\varphi} |Pu|^2.$$

For the proof of this lemma, we shall need the following elliptic estimate.

Lemma 3 *Suppose the assumptions of Lemma 2 hold. Then there exists* $R_0 \in (0, 1)$ *and positive constants* γ_0 *and* C *depending only on* n, δ *and* C_δ *such that for any* $\gamma > \gamma_0$ *and* $u \in C_0^\infty(B_{R_0} \backslash \{0\})$,

$$\begin{aligned}\int |x|^{-n+4} |\log |x||^{-2-\delta} e^{2\gamma\varphi} |\nabla^2 u|^2 \le\ & 2 \int |x|^{-n+4} |\log |x||^{-2-\delta} e^{2\gamma\varphi} |Pu|^2 \\ & + C\gamma^2 \int |x|^{-n+2} |\log |x||^{-\delta} e^{2\gamma\varphi} |\nabla u|^2. \qquad (8)\end{aligned}$$

Here $\nabla^2 u = (\partial_{ij} u)_{i,j=1}^n$ *is the Hessian of* u.

Proof Since $|A(x) - Id| \le C_\delta |\log |x||^{-1-\delta}$, by triangle inequality, it suffices to prove (8) with Δu in place of Pu on the right-hand side. By splitting u into real and imaginary parts, we can further assume u is real-valued. Integrating by parts twice gives

$$\int |x|^{-n+4} |\log |x||^{-2-\delta} e^{2\gamma\varphi} \partial_{ii} u \partial_{jj} u = \int |x|^{-n+4} |\log |x||^{-2-\delta} e^{2\gamma\varphi} |\partial_{ij} u|^2 + E,$$

where

$$\begin{aligned}|E| \le\ & C\gamma \int |x|^{-n+3} |\log |x||^{-1-\delta} e^{2\gamma\varphi} |\nabla^2 u| |\nabla u| \\ \le\ & \frac{1}{2n^2} \int |x|^{-n+4} |\log |x||^{-2-\delta} e^{2\gamma\varphi} |\nabla^2 u|^2 + C\gamma^2 \int |x|^{-n+2} |\log |x||^{-\delta} e^{2\gamma\varphi} |\nabla u|^2.\end{aligned}$$

Summing over i and j, we obtain the desired inequality. □

Proof of Lemma 2 In view of Lemma 3, to prove Lemma 2, it suffices to show

$$\begin{aligned}&\gamma^3 \int |x|^{-n} |\log |x||^2 e^{2\gamma\varphi} u^2 + \gamma \int |x|^{-n+2} e^{2\gamma\varphi} |\nabla u|^2 \\ &\le C_2 \int |x|^{-n+4} e^{2\gamma\varphi} |Pu|^2 + C_2 \gamma^{-1} \int |x|^{-n+4} |\log |x||^{-2-\delta} e^{2\gamma\varphi} |\nabla^2 u|^2. \qquad (9)\end{aligned}$$

Let $v = ue^{\gamma\varphi}$ then $e^{\gamma\varphi} Pu = P_\gamma v$ where

$$P_\gamma v = e^{\gamma\varphi} P(e^{-\gamma\varphi} v).$$

It is easy to see that (9) follows from

$$\gamma^3 \int |x|^{-n} |\log|x||^2 v^2 + \gamma \int |x|^{-n+2} |\nabla v|^2 \leq C_2 \int |x|^{-n+4} |P_\gamma v|^2 + C_2 \gamma^{-1} \int |x|^{-n+4} |\log|x||^{-2-\delta} |\nabla^2 v|^2. \tag{10}$$

Let $\omega = x/|x|$ and $t = \log|x|$ for $x \neq 0$, i.e., $x = e^t \omega$. Then

$$\partial_j = e^{-t}(\omega_j \partial_t + \Omega_j),$$

where Ω_j are vector fields on $\mathbb{S}^{n-1}$ satisfying

$$\sum_j \omega_j \Omega_j = 0, \quad \sum_j \Omega_j \omega_j = n-1, \quad \Omega_j^* = (n-1)\omega_j - \Omega_j.$$

Let $(D_0, \ldots, D_n) = (i\partial_t, i\Omega_1, \ldots, i\Omega_n)$. We denote by Dv the vector $(D_0 v, \ldots, D_n v)$ and by $D^2 v$ the matrix $D_j D_k v$, $0 \leq j, k \leq n$. Then (10) takes the form

$$\gamma^3 \int |v|^2 t^2 dt d\omega + \gamma \int |Dv|^2 dt d\omega \leq C_2 \int \left(|e^{2t} P_\gamma v|^2 + \gamma^{-1} t^{-2-\delta} |D^2 v|^2 \right) dt d\omega. \tag{11}$$

We have

$$\begin{aligned} Pu &= e^{-2t} \sum_{k,l=1}^n a_{kl}(e^t \omega)(\omega_k \partial_t - \omega_k + \Omega_k)(\omega_l \partial_t + \Omega_l) u \\ &= e^{-2t} \left[\partial_t^2 u + (n-2)\partial_t u + \Delta_\omega u + \sum_{j+|\alpha| \leq 2} C_{j,\alpha}(t,\omega) \partial_t^j \Omega^\alpha u \right], \end{aligned}$$

and consequently,

$$\begin{aligned} e^{2t} P_\gamma v &= (\partial_t - \gamma t)^2 v + (n-2)(\partial_t - \gamma t) v + \Delta_\omega v + \sum_{j+|\alpha| \leq 2} C_{j,\alpha}(t,\omega)(\partial_t - \gamma t)^j \Omega^\alpha v \\ &= \partial_t^2 v + \Delta_\omega v + [(n-2) - 2\gamma t]\partial_t v + [\gamma^2 t^2 - (n-2)\gamma t - \gamma] v \\ &\quad + \sum_{j+|\alpha| \leq 2} C_{j,\alpha}(t,\omega)(\partial_t - \gamma t)^j \Omega^\alpha v. \end{aligned}$$

Let

$$Qv = \partial_t^2 v + \Delta_\omega v - 2\gamma t \partial_t v + \left[\gamma^2 t^2 - 2\gamma \right] v + \sum_{j+|\alpha|=2} C_{j,\alpha}(t,\omega)(\partial_t - \gamma t)^j \Omega^\alpha v.$$

Since by (4), $C_{j,\alpha}$ are bounded, it follows that

$$|e^{2t} P_\gamma v - Qv| \leq C(\gamma |tv| + |Dv|).$$

Thus, by triangle inequality, it suffices to prove (11) with Qv in place of $e^{2t} P_\gamma v$ on the right-hand side. The last term of Qv can be written as

$$\sum_{j+|\alpha|=2} C_{j,\alpha}(t,\omega)(\partial_t - \gamma t)^j \Omega^\alpha v = \sum_{|\alpha| \leq 2} (V_{\alpha,0} + i V_{\alpha,1})(t,\omega)(\gamma t)^{2-|\alpha|} D^\alpha v,$$

where the real-valued functions $V_{\alpha,k}$'s are linear combinations of $C_{j,\alpha}$'s.

Let

$$Mv = \partial_t^2 v + \Delta_\omega v + \gamma^2 t^2 v + \sum_{|\alpha| \leq 2} V_{\alpha,0}(t,\omega)(\gamma t)^{2-|\alpha|} D^\alpha v$$

and

$$Nv = -2\gamma t\partial_t v - 2\gamma v + \sum_{|\alpha|\le 2} iV_{\alpha,1}(t,\omega)(\gamma t)^{2-|\alpha|}D^\alpha v.$$

Then $\|Qv\|_{L^2}^2 = \|Mv + Nv\|_{L^2}^2 \ge 2\Re\langle Mv, Nv\rangle$. The right-hand side consists of the following terms:

$$\begin{aligned} T_1 &= 2\Re\langle \partial_t^2 v, -2\gamma t\partial_t v - 2\gamma v\rangle = 6\gamma\|\partial_t v\|^2,\\ T_2 &= 2\Re\langle \Delta_\omega v, -2\gamma t\partial_t v - 2\gamma v\rangle = 2\gamma\|\Omega v\|^2,\\ T_3 &= 2\Re\langle \gamma^2 t^2 v, -2\gamma t\partial_t v - 2\gamma v\rangle = 2\gamma^3\|tv\|^2,\\ T_4 &= 2\Re\left\langle \sum_{|\alpha|\le 2} V_{\alpha,0}(t,\omega)(\gamma t)^{2-|\alpha|}D^\alpha v, -2\gamma v\right\rangle\\ &\ge -\frac{1}{2}\gamma^3\|tv\|^2 - C\sum_{|\alpha|\le 2}\gamma^{3-2|\alpha|}\left\|t^{-\delta-|\alpha|}D^\alpha v\right\|^2. \end{aligned}$$

Here we have used Cauchy–Schwarz at the last line. The remaining terms have the form

$$\Im\left\langle W_{\alpha,\beta}(t,\omega)(\gamma t)^{4-|\alpha|-|\beta|}D^\alpha v, D^\beta v\right\rangle,$$

where $W_{\alpha,\beta}$ are real-valued function satisfying $|W_{\alpha,\beta}| + |DW_{\alpha,\beta}| = O(|t|^{-1-\delta})$. Integrating by parts $|\alpha| + |\beta|$ times gives

$$\begin{aligned} \Im\left\langle W_{\alpha,\beta}(\gamma t)^{4-|\alpha|-|\beta|}D^\alpha v, D^\beta v\right\rangle &= \Im\left\langle W_{\alpha,\beta}(\gamma t)^{4-|\alpha|-|\beta|}D^\beta v, D^\alpha v\right\rangle\\ &\quad + \sum_{\substack{|\alpha'|+|\beta'|\\ <|\alpha|+|\beta|}} \Im\left\langle Z_{\alpha',\beta'}(\gamma t)^{4-|\alpha|-|\beta|}D^{\alpha'} v, D^{\beta'} v\right\rangle. \end{aligned}$$

Hence,

$$\Im\left\langle W_{\alpha,\beta}(\gamma t)^{4-|\alpha|-|\beta|}D^\alpha v, D^\beta v\right\rangle = \frac{1}{2}\sum_{\substack{|\alpha'|+|\beta'|\\ <|\alpha|+|\beta|}} \Im\left\langle Z_{\alpha',\beta'}(\gamma t)^{4-|\alpha|-|\beta|}D^{\alpha'} v, D^{\beta'} v\right\rangle.$$

Here, $Z_{\alpha',\beta'} = O(|t|^{-1-\delta})$. Applying Cauchy–Schwarz, we see that the right-hand side is bounded in absolute value by

$$C\sum_{|\alpha|\le 2}\gamma^{3-2|\alpha|}\|t^{\frac{2-\delta}{2}-|\alpha|}D^\alpha v\|^2.$$

Summing up all the terms, we obtain

$$2\Re\langle Mv, Nv\rangle \ge \gamma^3\|tv\|^2 + \gamma\|Dv\|^2 - C\gamma^{-1}\|t^{-1-\frac{\delta}{2}}D^2 v\|^2.$$

This gives the desired inequality (11). □

We will also need the following Caccioppoli type estimate. Since the proof follows standard arguments, we will skip it. For $0 < a < b$ let $\mathcal{A}(a,b) = \{x : a \le |x| \le b\}$.

Lemma 4 *There exist $C = C(n,\lambda) > 0$ such that if $C_3 = C(1 + C_0 + C_1^2 + C_\delta^2)$ then for any u satisfying (5) and $0 < r < \frac{1}{2}$,*

$$\int_{\mathcal{A}(5r/4,7r/4)} |x|^{-n+2}|\nabla u|^2 dx \le C_3\int_{\mathcal{A}(r,2r)} |x|^{-n}|u|^2 dx.$$

3 Proof of Theorem 1

In this proof, C denotes a constant depending only on n and λ, whose value may change from line to line. By a change of variable which may change the values of R and C^* by a factor of λ, we can assume $A(0) = \mathrm{Id}$. Under this additional assumption, we will prove Theorem 1 with $C^* = \frac{1}{4\sqrt{C_2}}$. By using [7, Theorem 2.4], it suffices to consider the case u does not vanish identically on any balls in B_1.

For $0 < r < R_0/8$ where R_0 is the constant which appears in Lemma 2, let ζ be a smooth cut-off function satisfying $\chi_{\mathcal{A}(7r/4, 5R_0/8)} \leq \zeta \leq \chi_{\mathcal{A}(5r/4, 7R_0/8)}$ and $|\partial^\alpha \zeta(x)| \leq 10|x|^{-|\alpha|}$, $\forall x \in \mathbb{R}^n$ and $|\alpha| \leq 2$. Here, χ_E denotes the characteristic function of the set E.

Let $v = \zeta u$ and $E = \mathcal{A}(5r/4, 7r/4) \cup \mathcal{A}(5R_0/8, 7R_0/8)$. Then

$$\begin{aligned} |Pv| &= |\zeta Pu + 2a_{jk}(\partial_j \zeta)(\partial_k u) + a_{jk}\partial_{jk}\zeta u| \\ &\leq \zeta \left(C_1|x|^{-1}|\nabla u| + C_0|x|^{-2}|u|\right) + C\left(|x|^{-1}|\nabla u| + |x|^{-2}|u|\right)\chi_E \\ &\leq C_1|x|^{-1}|\nabla v| + C_0|x|^{-2}|v| + C\left(|x|^{-1}|\nabla u| + |x|^{-2}|u|\right)\chi_E. \end{aligned}$$

Applying Lemma 2 to v and using the above inequality, we have

$$\begin{aligned} &\gamma^3 \int |x|^{-n} e^{2\gamma\varphi}|v|^2 dx + \gamma \int |x|^{-n+2} e^{2\gamma\varphi}|\nabla v|^2 dx \\ &\leq C_2 \int |x|^{-n+4} e^{2\gamma\varphi}|Pv|^2 dx \\ &\leq 4C_2 \int |x|^{-n+4} e^{2\gamma\varphi}\left(C_0^2|x|^{-4}|v|^2 + C_1^2|x|^{-2}|\nabla v|^2\right) dx \\ &\quad +4C_2C^2 \int_E |x|^{-n+4} e^{2\gamma\varphi}\left(|x|^{-4}|u|^2 + |x|^{-2}|\nabla u|^2\right) dx. \end{aligned} \tag{12}$$

Assuming $\gamma \geq \gamma_1 := \max\{\gamma_0, 2C_0^{2/3}C_2^{1/3}, 8C_1^2C_2\}$, the first term on the right-hand side of (12) can be absorbed by its left-hand side. Thus, we deduce that

$$2\int |x|^{-n} e^{2\gamma\varphi}|v|^2 dx \leq 4C_2C^2 \int_E |x|^{-n+4} e^{2\gamma\varphi}\left(|x|^{-4}|u|^2 + |x|^{-2}|\nabla u|^2\right) dx.$$

Using Lemma 4 to bound the gradient terms on the right-hand side, we get

$$\begin{aligned} 2\int_{\mathcal{A}(2r, \frac{R_0}{2})} |x|^{-n} e^{2\gamma\varphi}|u|^2 dx \leq\ & C_4 e^{2\gamma\varphi(r)} \int_{\mathcal{A}(r,2r)} |x|^{-n}|u|^2 dx \\ &+ C_4 e^{2\gamma\varphi\left(\frac{R_0}{2}\right)} \int_{\mathcal{A}\left(\frac{R_0}{2}, R_0\right)} |x|^{-n}|u|^2 dx, \end{aligned} \tag{13}$$

where $C_4 = 32C_2C^2(C_3 + 1)$.

We now fix

$$\gamma = \max\left\{\gamma_1, \frac{\log\left(C_4 \int_{\mathcal{A}\left(\frac{R_0}{2}, R_0\right)} |x|^{-n}|u|^2 / \int_{\mathcal{A}\left(\frac{R_0}{4}, \frac{R_0}{3}\right)} |x|^{-n}|u|^2\right)}{2\varphi\left(\frac{R_0}{3}\right) - 2\varphi\left(\frac{R_0}{2}\right)}\right\}.$$

For this choice of γ,

$$C_4 e^{2\gamma\varphi\left(\frac{R_0}{2}\right)} \int_{\mathcal{A}\left(\frac{R_0}{2}, R_0\right)} |x|^{-n}|u|^2 \leq \int_{\mathcal{A}\left(\frac{R_0}{4}, \frac{R_0}{3}\right)} |x|^{-n} e^{2\gamma\varphi}|u|^2,$$

hence the last term on the right-hand side of (13) can be absorbed by the left-hand side, giving

$$\int_{\mathcal{A}\left(2r,\frac{R_0}{2}\right)} |x|^{-n} e^{2\gamma\varphi}|u|^2 \le C_4 e^{2\gamma\varphi(r)} \int_{\mathcal{A}(r,2r)} |x|^{-n}|u|^2. \tag{14}$$

Note that this would give a lower bound that is worse than polynomial. We will use Lemma 1 to improve upon (14) to reach the conclusion. Let $R_1 \in (0, R_0/8]$ satisfy

$$|\log R_1| \ge \max\left\{\left(2\sqrt{C_2}C_\delta(5\gamma + \log C_4)\right)^{\frac{1}{\delta}}, (8C_0C_2C_\delta)^{\frac{1}{1+\delta}}\right\}$$

and η be a smooth cut-off function such that $\chi_{\mathcal{A}(7r/4,5R_1/8)} \le \eta \le \chi_{\mathcal{A}(5r/4,7R_1/8)}$ and $|\partial^\alpha\eta(x)| \le 10|x|^{-|\alpha|}$, $\forall x \in \mathbb{R}^n$ and $|\alpha| \le 2$.

From $|\nabla A(x)| \le C_\delta|x|^{-1}|\log|x||^{-2-\delta}$ and $A(0) = \mathrm{Id}$, we see that

$$|A(x) - \mathrm{Id}| \le C_\delta|\log|x||^{-1-\delta} \le C_\delta|\log R_1|^{-1-\delta} \qquad \forall x \in B_{R_1}.$$

Appplying Lemma 1 to $w = \eta u$, we obtain

$$\begin{aligned}\sum_{j=0}^{2}\int \tau^{2-2j}|x|^{-2\tau+2j-n}|\nabla^j w|^2 &\le C_2\int |x|^{-2\tau+4-n}|\Delta w|^2\\ &\le 2C_2\int |x|^{-2\tau+4-n}|Pw|^2\\ &\quad +2C_2C_\delta^2|\log R_1|^{-2-2\delta}\int |x|^{-2\tau+4-n}|\nabla^2 w|^2.\end{aligned}$$

Choosing $\tau = \lfloor\frac{1}{2\sqrt{C_2}C_\delta}|\log R_1|^{1+\delta}\rfloor$, the last term can be absorbed by the left-hand side, hence we obtain

$$\tau^2\int |x|^{-2\tau-n}|w|^2 + \int |x|^{-2\tau+2-n}|\nabla w|^2 \le 2C_2\int |x|^{-2\tau+4-n}|Pw|^2. \tag{15}$$

Note that by our choice of C^*, R_1, and τ, we have $\tau^2 \ge 16C_0^2C_2$ and $1 \ge 16C_1^2C_2$. Hence, using the same arguments that lead to (13) from Lemma 2, we obtain from (15) that

$$2\int_{\mathcal{A}(2r,R_1)} |x|^{-2\tau-n}|u|^2 \le C_4 r^{-2\tau}\int_{\mathcal{A}(r,2r)} |x|^{-n}|u|^2 + C_4R_1^{-2\tau}\int_{\mathcal{A}(R_1,2R_1)} |x|^{-n}|u|^2. \tag{16}$$

From our choice of R_1 and τ, we have

$$\tau \ge (5\gamma + \log C_4)|\log R_1|,$$

which implies that for $R_2 = \frac{1}{2}R_1^2$,

$$(2R_2)^{-2\tau}e^{-2\gamma\varphi(R_2)+2\gamma\varphi(2R_1)} \ge C_4^2R_1^{-2\tau}.$$

Hence, for $0 < r < R_2/2$, using (14) we have

$$\begin{aligned}\int_{\mathcal{A}(2r,R_1)} |x|^{-2\tau-n}|u|^2 &\geq (2R_2)^{-2\tau}\int_{\mathcal{A}(R_2,2R_2)} |x|^{-n}|u|^2 \\ &\geq C_4^{-1}(2R_2)^{-2\tau}e^{-2\gamma\varphi(R_2)}\int_{\mathcal{A}\left(2R_2,\frac{R_0}{2}\right)} |x|^{-n}e^{2\gamma\varphi}u^2 \\ &\geq C_4^{-1}(2R_2)^{-2\tau}e^{-2\gamma\varphi(R_2)+2\gamma\varphi(2R_1)}\int_{\mathcal{A}(R_1,2R_1)} |x|^{-n}|u|^2 \\ &\geq C_4R_1^{-2\tau}\int_{\mathcal{A}(R_1,2R_1)} |x|^{-n}|u|^2.\end{aligned}$$

Thus, the last term of (16) can be absorbed by its left-hand side. Hence, for $r < R_2/2$,

$$\int_{\mathcal{A}(2r,R_1)} |x|^{-2\tau-n}|u|^2 \leq C_4r^{-2\tau}\int_{\mathcal{A}(r,2r)} |x|^{-n}|u|^2. \tag{17}$$

From this, (6) follows with $k = 2\tau + n$ and

$$M_1 = C_4^{-1}\int_{\mathcal{A}(R_1/2,R_1)} |x|^{-2\tau-n}|u|^2.$$

Moreover, we can deduce from (17) that

$$\int_{\mathcal{A}(2r,4r)} |u|^2 \leq 4^{2\tau+n}C_4\int_{\mathcal{A}(r,2r)} |u|^2.$$

Adding $\int_{B_{2r}} |u|^2$ to both sides, it follows that

$$\int_{B_{2r}} |u|^2 \geq \frac{1}{4^{2\tau+n}C_4+1}\int_{B_{4r}} |u|^2.$$

Thus, (7) follows with $M_2 = 4^{2\tau+n}C_4 + 1$

To finish the proof, note that for $r \in (R_2/2, R_0/8]$, (6) and (7), possibly with different M_1 and M_2, follow from (14).

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