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ORIGINAL ARTICLE

Duality for Robust Linear Infinite Programming Problems Revisited

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Abstract

In this paper, we consider the robust linear infinite programming problem (RLIP_c) defined by

$$\begin{array}{ll}(\mathrm{RLIP}_c) & \inf\langle c, x\rangle \\ \text{subject to} & x \in X,\ \langle x^*, x\rangle \leq r,\ \forall (x^*, r) \in \mathcal{U}_t,\ \forall t \in T,\end{array}$$

where X is a locally convex Hausdorff topological vector space, T is an arbitrary index set, $c \in X^*$, and $\mathcal{U}_t \subset X^* \times \mathbb{R}$, $t \in T$ are uncertainty sets. We propose an approach to duality for the robust linear problems with convex constraints (RP_c) and establish corresponding robust strong duality and also, stable robust strong duality, i.e., robust strong duality holds "uniformly" with all $c \in X^*$. With the different choices/ways of setting/arranging data from (RLIP_c), one gets back to the model (RP_c) and the (stable) robust strong duality for (RP_c) applies. By such a way, nine versions of dual problems for (RLIP_c) are proposed. Necessary and sufficient conditions for stable robust strong duality of these pairs of primal-dual problems are given, for which some cover several known results in the literature while the others, due to the best knowledge of the authors, are new. Moreover, as by-products, we obtained from the robust strong duality for variants pairs of primal-dual problems, several robust Farkas-type results for linear infinite systems with uncertainty. Lastly, as extensions/applications, we extend/apply the results obtained to robust linear problems with sub-affine constraints, and to linear infinite problems (i.e., (RLIP_c) with the absence of uncertainty). It is worth noticing even for these cases, we are able to derive new results on (robust/stable robust) duality for the mentioned classes of problems and new robust Farkas-type results for sub-linear systems, and also for linear infinite systems in the absence of uncertainty.

Keywords Linear infinite programming problems · Robust linear infinite problems · Stable robust strong duality for robust linear infinite problems · Robust Farkas-type results for infinite linear systems · Robust Farkas-type results for sub-affine systems

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Dedicated to Professor Marco Antonio López's 70th birthday.

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1 Introduction

In this paper, we are concerned with the *linear infinite programming with uncertainty parameters* of the form

$$\begin{array}{ll} (\mathrm{LIP}_c) & \inf\langle c, x\rangle \\ \text{subject to} & x \in X,\ \langle a_t, x\rangle \leq b_t,\ \forall t \in T, \end{array}$$

where X is a locally convex Hausdorff topological vector space, T is an arbitrary (possible infinite) index set, $c \in X^*$, $a_t \in X^*$ and $b_t \in \mathbb{R}$ for each $t \in T$, and the couple (a_t, b_t) belongs to an uncertainty set $\mathcal{U}_t \subset X^* \times \mathbb{R}$. For such a *linear infinite programming* (LIP_c) with *input-parameter uncertainty*, its robust counterpart is the robust linear infinite programing problem (RLIP_c) defined as follows:

$$\begin{array}{ll} (\mathrm{RLIP}_c) & \inf\langle c, x\rangle \\ \text{subject to} & x \in X,\ \langle x^*, x\rangle \leq r,\ \forall (x^*, r) \in \mathcal{U}_t,\ \forall t \in T. \end{array}$$

The robust linear infinite problems of the model (RLIP_c) together with their duality were considered in several works in the literature such as, [6, 12, 16, 20, 23]. There are variants of duality results for robust convex problems (see [4, 5, 11, 14–16, 18, 22, 24] and the references therein), and also for robust vector optimization/multi-objective problems (see, e.g., [7, 12, 13, 21]). In the mentioned papers, results for robust strong duality are established for classes of problems from linear to convex, non-convex, and vector problems, under various (constraint) qualification conditions.

In this paper we propose a way, which can be considered as a unification approach to duality for the robust linear problems (RLIP_c). Concretely, we propose some model for a bit more general problem, namely, the robust linear problem with convex conical constraints (RP_c) and establish corresponding robust strong duality and also, stable robust strong duality, i.e., robust strong duality holds "uniformly" with all $c \in X^*$. Then, with the different choices/ways of setting, we transfer (RLIP_c) to the models (RP_c), and the (stable) robust strong duality results for (RP_c) apply. By such a way, several forms of dual problems for (RLIP_c) are proposed. Necessary and sufficient conditions for stable robust strong duality of these pairs of primal-dual problems are given, for which some cover results known in the literature while the others, due to the best knowledge of the authors, are new. We point out also that, even in the case with the absence of uncertainty, i.e., in the case where $\mathcal{U}_t$ is singleton for each $t \in T$, the results obtained still lead to new results on duality for robust linear infinite/semi-infinite problems (see Section 6).

The paper is organized as follows: In Section 2, some preliminaries and basic tools are introduced. Concretely, we introduce or quote some robust Farkas lemmas for conical constraint systems under uncertainty, some results on duality of robust linear problems with convex conical constraints. The model of robust linear infinite problem and its seven models of robust dual problems are given in Section 3. The main results: Robust stable strong duality results for (RLIP_c) are given in Section 4 together with two more models of robust dual problems of (RLIP_c). Here, the stable strong duality for the seven pairs of primal-dual problems are established and the ones for two new pairs are mentioned. Some of these results cover or extend some in [11, 20]. In Section 5, from the duality results in Section 4, we derive variants of stable robust Farkas lemmas for linear infinite systems with uncertainty which cover the ones in [12, 16] while the others are new. In Section 6, as an extension/application of the approach, we get robust strong duality results for linear problems with sub-affine constraints. We consider a particular case with the absence of uncertainty (i.e., in the case where $\mathcal{U}_t$ is singleton for each $t \in T$), the results obtained still lead to some new results on duality for robust linear infinite/semi-infinite problems, and, in turn, these results also give rise to several new versions of Farkas lemmas for sub-affine systems under uncertainty and also, some new versions of Farkas-type results for linear infinite/semi-infinite systems.

2 Preliminaries and Basic Tools

Let X and Z be locally convex Hausdorff topological vector spaces with topological dual spaces X^* and Z^*, respectively. The only topology considered on dual spaces is the weak*-topology. Let S be a non-empty closed and convex cone in Z. The positive dual cone S^+ of S is $S^+ := \{z^* \in Z^* : \langle z^*, s\rangle \geq 0\ \forall s \in S\}$. Let further, $\Gamma(X)$ be the set of all proper, convex and lower semi-continuous (briefly, lsc) functions on X. Denote by $\mathcal{L}(X, Z)$ the space of all continuous linear mappings from X to Z and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, $\mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$.

2.1 Notations and Preliminaries

We now give some notations which will be used in the sequel. For $f : X \to \overline{\mathbb{R}}$, the domain and the epigraph of f are defined respectively by

$$\begin{aligned} \operatorname{dom} f &:= \{x \in X :\ f(x) \neq +\infty\}, \\ \operatorname{epi} f &:= \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}, \end{aligned}$$

while its conjugate function $f^* : X \to \overline{\mathbb{R}}$ is

$$f^*\left(x^*\right) := \sup_{x\in X} \left[\langle x^*, x\rangle - f(x)\right] \quad \forall x^* \in X^*.$$

Let $\leq_S$ be the ordering on Z induced by the cone S, i.e.,

$$z_1 \leqq_S z_2 \quad \text{if and only if} \quad z_2 - z_1 \in S.$$

We enlarge Z by attaching a greatest element $+\infty_Z$ and a smallest element $-\infty_Z$ which do not belong to Z by the convention, $-\infty_Z \leqq_S z \leqq_S +\infty_Z$ for all $z \in Z$. Denote $Z^\bullet := Z \cup \{-\infty_Z, +\infty_Z\}$. Let $G : X \to Z^\bullet$. We define

$$\begin{aligned} \operatorname{dom} G &:= \{x \in X : G(x) \neq +\infty_Z\}, \\ \operatorname{epi}_S G &:= \{(x, z) \in X \times Z : z \in G(x) + S\}. \end{aligned}$$

If $-\infty_Z \notin G(X)$ and $\operatorname{dom} G \neq \emptyset$, then we say that G is a *proper mapping*. We say that G is *S-convex* (resp., *S-epi closed*) if $\operatorname{epi}_S G$ is a convex subset (resp., a closed subset) of $X \times Z$. The mapping G is called *positively S-upper semicontinuous*[1] (*positively S-usc*, briefly) if λG is upper semicontinuous (in short, usc) for all $\lambda \in S^+$ (see [1, 2]).

Let T be an index (possibly infinite) set and let $\mathbb{R}^T$ be the product space endowed with the product topology and its dual space, $\mathbb{R}^{(T)}$, the so-called *space of generalized finite sequences* $\lambda = (\lambda_t)_{t\in T}$ such that $\lambda_t \in \mathbb{R}$, for each $t \in T$, and with only finitely many λ_t different from zero. The supporting set of $\lambda \in \mathbb{R}^{(T)}$ is $\operatorname{supp} \lambda := \{t \in T :\ \lambda_t \neq 0\}$. For a pair $(\lambda, v) \in \mathbb{R}^{(T)} \times \mathbb{R}^T$, the dual product is defined by

$$\langle \lambda, v\rangle := \begin{cases} \sum_{t\in \operatorname{supp}\lambda} \lambda_t v_t & \text{if } \lambda \neq 0_T, \\ 0 & \text{otherwise.} \end{cases}$$

[1] In [3] this notion is named as *star S-usc*

The positive cones in $\mathbb{R}^T$ and in $\mathbb{R}^{(T)}$ are denoted by $\mathbb{R}^T_+$ and $\mathbb{R}^{(T)}_+$, respectively.

S^+-Upper Semi-Continuity and Uniform S^+-Convexity. Let $U \neq \emptyset$ be a subset of some topological space. We recall the notions of S^+-upper semi-continuity, S^+-convexity, and uniform S^+-convexity introduced recently in [13].

Definition 1 [13] Let $H : U \to Z \cup \{+\infty_Z\}$. We say that:

– H is *S^+-convex* if for all $(u_i, \lambda_i) \in U \times S^+$ $(i = 1, 2)$ there is $(\bar{u}, \bar{\lambda}) \in U \times S^+$ such that $(\lambda_1 H)(u_1) + (\lambda_2 H)(u_2) \geq (\bar{\lambda} H)(\bar{u})$,

– H is *S^+-upper semi-continuos* (briefly, *S^+-usc*) if for any net $(\lambda_\alpha, u_\alpha, r_\alpha)_{\alpha \in D} \subset S^+ \times U \times \mathbb{R}$ and any $(\bar{\lambda}, \bar{u}, \bar{r}) \in S^+ \times U \times \mathbb{R}$, satisfying

$$\begin{cases} (\lambda_\alpha H)(u_\alpha) \geq r_\alpha \ \forall \alpha \in D, \\ \lambda_\alpha \overset{*}{\rightharpoonup} \bar{\lambda}, \ u_\alpha \to \bar{u}, \ r_\alpha \to \bar{r} \end{cases} \implies (\bar{\lambda} H)(\bar{u}) \geq \bar{r},$$

where the symbol "$\overset{*}{\rightharpoonup}$" means the convergence with respect to weak*-topology.

– H is *S^+-concave* (*S^+-lsc*, resp.) if $-H$ is *S^+-convex* (*S^+-usc*, respectively).

Definition 2 [13] For the collection $(H_j)_{j \in I}$ with $H_j : U \to Z \cup \{+\infty_Z\}$, we say that $(H_j)_{j \in I}$ is *uniformly S^+-convex* if for all $(u_i, \lambda_i) \in U \times S^+$, $i = 1, 2$, there is $(\bar{u}, \bar{\lambda}) \in U \times S^+$ such that $(\lambda_1 H_j)(u_1) + (\lambda_2 H_j)(u_2) \geq (\bar{\lambda} H_j)(\bar{u})$ for all $j \in I$.

The collection $(H_j)_{j \in I}$ is said to be *uniformly S^+-concave* if $(-H_j)_{j \in I}$ is uniformly S^+-convex.

Remark 1 It is worth observing that when $H : U \to Z \cup \{+\infty_Z\}$ is S^+-usc then H is positively S-usc [13]. Moreover, in the case where $Z = \mathbb{R}$ and $S = \mathbb{R}_+$, (and hence, $S^+ = \mathbb{R}_+$), the following assertions hold:

(i) If $H : U \to \mathbb{R}_\infty$ is a convex function then H is $\mathbb{R}_+$-convex.
(ii) If $H_j : U \to \mathbb{R}_\infty$ is convex for all $j \in I$ then $(H_j)_{j \in I}$ is uniformly $\mathbb{R}_+$-convex.
(iii) $H : U \to \mathbb{R}_\infty$ is $\mathbb{R}_+$-usc if and only if it is usc.

For details, see [13].

2.2 Conical Constrained Systems with Uncertainty

Let $\mathcal{U}$ be an uncertainty parameter set, $(G_u)_{u \in \mathcal{U}}$ with $G_u : X \to Z \cup \{+\infty_Z\}$ be a proper S-convex and S-epi closed mapping for each $u \in \mathcal{U}$. We are concerned with the robust cone constraint system:

$$G_u(x) \in -S \quad \forall u \in \mathcal{U}. \tag{1}$$

Denote

$$\mathcal{F}_u := \{x \in X : G_u(x) \in -S\}, \quad u \in \mathcal{U}, \tag{2}$$

and $\mathcal{F}$ the *solution set* of (1), i.e.,

$$\mathcal{F} := \{x \in X : G_u(x) \in -S \ \forall u \in \mathcal{U}\}. \tag{3}$$

It is clear that $\mathcal{F} = \bigcap_{u \in \mathcal{U}} \mathcal{F}_u$. Assume that $\mathcal{F} \neq \emptyset$.

Corresponding to system (1), let us consider the set (also called: *robust moment cone* corresponding to system (1))

$$\mathcal{M}_0 := \bigcup_{(u,\lambda)\in\mathcal{U}\times S^+} \operatorname{epi}(\lambda G_u)^*. \tag{4}$$

It is easy to check that $\mathcal{M}_0$ (generalizing the one in [22, Proposition 2.2]) is a cone in $X^* \times \mathbb{R}$. Moreover, $\mathcal{M}_0$ (and also $\mathcal{M}_1$ in (9)) leads to the cone M in [20].

We now introduce a version of robust Farkas-type result and some consequences involving system (1), which will be useful in the sequel.

Proposition 1 (Farkas-type result involving robust system (1)) *For all* $(x^*, r) \in X^* \times \mathbb{R}$, *the next statements are equivalent:*

(i) $G_u(x) \in -S \;\; \forall u \in \mathcal{U} \implies \langle x^*, x\rangle \geq r$.
(ii) $(x^*, r) \in -\overline{\mathrm{co}}\, \mathcal{M}_0$.

Proof It is easy to see that (i) is equivalent to $-r \geq -\langle x^*, x\rangle$ for all $x \in \mathcal{F}$, which also means $(x^*, r) \in -\operatorname{epi} \delta^*_{\mathcal{F}}$. So, to prove the equivalence (i) $\Longleftrightarrow$ (ii), it suffices to show that $\operatorname{epi} \delta^*_{\mathcal{F}} = \overline{\mathrm{co}}\, \mathcal{M}_0$.

Now, for each $u \in \mathcal{U}$, $\mathcal{F}_u$ is closed and convex subsets of X, and hence, $\delta_{\mathcal{F}_u} \in \Gamma(X)$ and so $\delta_{\mathcal{F}} = \sup_{u\in\mathcal{U}} \delta_{\mathcal{F}_u} \in \Gamma(X)$. By [25, Lemma 2.2], one gets $\operatorname{epi} \delta^*_{\mathcal{F}} = \overline{\mathrm{co}} \bigcup_{u\in\mathcal{U}} \operatorname{epi} \delta^*_{\mathcal{F}_u}$. On the other hand, for each $u \in \mathcal{U}$, one has $\operatorname{epi} \delta^*_{\mathcal{F}_u} = \overline{\bigcup_{\lambda\in S^+} \operatorname{epi}(\lambda G_u)^*}$ (see [17]), and so, $\operatorname{epi} \delta^*_{\mathcal{F}} = \overline{\mathrm{co}}\, \mathcal{M}_0$ and we are done. □

Remark 2 Proposition 1 generalized [23, Theorem 3.1], [12, Theorem 4.2(iii)], [16, Theorem 5.5], and in some sense, it extends the robust semi-infinite Farkas' lemmas in [20], [19, Corollary 3.1.2].

Let $\emptyset \neq B \subset X^*$ and $\beta \in \mathbb{R}$. The function $\sigma_B(\cdot) - \beta$, where $\sigma_B(x) := \sup\{\langle x^*, x\rangle : x^* \in B\}$, is known as a sub-affine function [15]. We next give a version of robust Farkas lemma for a system involving sub-affine functions.

Corollary 1 *Let* $(\mathcal{A}_t)_{t\in T}$ *be a family of nonempty, w^*-closed convex subsets of* X^* *and* $(b_t)_{t\in T} \subset \mathbb{R}$. *Then, for each* $(x^*, r) \in X^* \times \mathbb{R}$, *the next statements are equivalent:*

(i) $\sigma_{\mathcal{A}_t}(x) \leq b_t \;\forall t \in T \implies \langle x^*, x\rangle \geq r$.
(ii) $(x^*, r) \in -\overline{\mathrm{co}}\,\mathrm{cone}\left[\bigcup_{t\in T} (\mathcal{A}_t \times \{b_t\}) \cup \{(0_{X^*}, 1)\}\right]$.

Proof Take $Z = \mathbb{R}$, $S = \mathbb{R}_+$ (and hence, $Z^* = \mathbb{R}$ and $S^+ = \mathbb{R}^+$), $\mathcal{U} = T$, and $G_t := \sigma_{\mathcal{A}_t} - b_t$ for each $t \in T$. Then, for any $(t, \lambda) \in T \times \mathbb{R}_+$, one has

$$\begin{aligned} \operatorname{epi}(\lambda G_t)^* &= \lambda \operatorname{epi}(G_t)^* = \lambda \operatorname{epi}(\sigma_{\mathcal{A}_t} - b_t)^* = \lambda(\mathcal{A}_t \times \{b_t\}) + \mathbb{R}_+(0_{X^*}, 1), \\ \mathcal{M}_0 &= \bigcup_{t\in T} \mathrm{co}\,\mathrm{cone}\left[(\mathcal{A}_t \times \{b_t\}) \cup \{(0_{X^*}, 1)\}\right], \end{aligned}$$

and so, $\overline{\mathrm{co}}\, \mathcal{M}_0 = \overline{\mathrm{co}}\,\mathrm{cone}\left[\bigcup_{t\in T} (\mathcal{A}_t \times \{b_t\}) \cup \{(0_{X^*}, 1)\}\right]$. The conclusion now follows from Proposition 1. □

2.3 Duality of Robust Linear Problems with Convex Conical Constraints

Let $c \in X^*$. We consider the pair of primal-dual robust problems:

$$
\begin{array}{ll}
(\mathrm{RP}_c) & \inf \langle c, x\rangle \\
 & \text{subject to } x \in X,\ G_u(x) \in -S\ \forall u \in \mathcal{U}, \\
(\mathrm{RD}_c) & \sup\limits_{(u,\lambda)\in\mathcal{U}\times S^+} \inf\limits_{x\in X} (\langle c, x\rangle + \lambda G_u(x)).
\end{array}
$$

Let $\mathcal{F}_u$ and $\mathcal{F}$ be as in (2) and (3). Let further $\bar{x} \in \mathcal{F}$ and $(\bar{u}, \bar{\lambda}) \in \mathcal{U} \times S^+$. As $\bar{x} \in \mathcal{F}$, $G_u(\bar{x}) \in -S$ for all $u \in \mathcal{U}$, and in particular, $G_{\bar{u}}(\bar{x}) \in -S$. Moreover, as $\bar{\lambda} \in S^+$, one has $\bar{\lambda} G_{\bar{u}}(\bar{x}) \leq 0$. Therefore, $\langle c, \bar{x}\rangle + (\bar{\lambda} G_{\bar{u}})(x) \leq \langle c, \bar{x}\rangle$, and so,

$$\inf_{x\in X} \left[\langle c, x\rangle + (\bar{\lambda} G_{\bar{u}})(x)\right] \leq \langle c, \bar{x}\rangle + (\bar{\lambda} G_{\bar{u}})(x) \leq \langle c, \bar{x}\rangle,$$

leading to

$$\inf_{x\in X} \left[\langle c, x\rangle + (\bar{\lambda} G_{\bar{u}})(x)\right] \leq \inf_{\bar{x}\in A} \langle c, \bar{x}\rangle.$$

Consequently,

$$\sup_{(\bar{u},\bar{\lambda})\in\mathcal{U}\times S^+} \inf_{x\in X} \left[\langle c, x\rangle + (\bar{\lambda} G_{\bar{u}})(x)\right] \leq \inf_{\bar{x}\in A} \langle c, \bar{x}\rangle, \tag{5}$$

which means that the *weak duality* holds for the pair (RP_c)–(RD_c).

Definition 3 We say that

- *the robust strong duality holds for the pair* (RP_c)–(RD_c) if $\inf(\mathrm{RP}_c) = \max(\mathrm{RD}_c)$,
- *the stable robust strong duality holds for the pair* (RP_c)–(RD_c) if $\inf(\mathrm{RP}_c) = \max(\mathrm{RD}_c)$ for all $c \in X^*$.

The next theorem, Theorem 1, can be derived from [16, Theorem 6.3]. However, for the sake of completeness we will give here a short and direct proof.

Theorem 1 (Characterization of stable robust strong duality for (RP_c)) *Assume that* $r_0 := \inf(\mathrm{RP}_c) > -\infty$. *Then the following statements are equivalent:*

(a) $\mathcal{M}_0$ *is a closed and convex subset of* $X^* \times \mathbb{R}$.
(b) *The stable robust strong duality holds for the pair* (RP_c)–(RD_c)*, i.e.,*

$$\inf(\mathrm{RP}_c) = \max(\mathrm{RD}_c) \quad \forall c \in X^*.$$

Proof Take arbitrarily $c \in X^*$. Observe firstly that

$$\begin{aligned}
\sup(\mathrm{RD}_c) &= \sup_{(u,\lambda)\in\mathcal{U}\times S^+} \inf_{x\in X} \{\langle c, x\rangle + (\lambda G_u)(x)\} \\
&= \sup_{(u,\lambda)\in\mathcal{U}\times S^+} \left[-\sup_{x\in X} \{\langle -c, x\rangle - (\lambda G_u)(x)\} \right] = \sup_{(u,\lambda)\in\mathcal{U}\times S^+} [-(\lambda G_u)^*(-c)] \\
&= \sup \left\{ r : (c,r) \in -\bigcup_{(u,\lambda)\in\mathcal{U}\times S^+} \mathrm{gph}(\lambda G_u)^* \right\} \\
&= \sup \left\{ r : (c,r) \in -\bigcup_{(u,\lambda)\in\mathcal{U}\times S^+} \mathrm{gph}(\lambda G_u)^* - \mathbb{R}_+(0_{X^*}, 1) \right\} \\
&= \sup \left\{ r : (c,r) \in -\bigcup_{(u,\lambda)\in\mathcal{U}\times S^+} \left[\mathrm{gph}(\lambda G_u)^* + \mathbb{R}_+(0_{X^*}, 1)\right] \right\} \\
&= \sup \left\{ r : (c,r) \in -\bigcup_{(u,\lambda)\in\mathcal{U}\times S^+} \mathrm{epi}(\lambda G_u)^* \right\} = \sup\{r : (c,r) \in -\mathcal{M}_0\}. \quad (6)
\end{aligned}$$

Observe also that $r_0 < +\infty$ as (RP_c) is feasible (i.e., its feasible set $\mathcal{F}$ is non-empty) and so, we can assume that $r_0 \in \mathbb{R}$.

• [(a) $\Longrightarrow$ (b)] Assume that (a) holds. As $r_0 = \inf(\mathrm{RP}_c)$, one has

$$G_u \in -S, \ \forall u \in \mathcal{U} \quad \Longrightarrow \quad \langle c, x\rangle \geq r_0. \tag{7}$$

As (a) holds, it follows from Proposition 1 that

$$(c, r_0) \in -\overline{\mathrm{co}}\,\mathcal{M}_0 = -\mathcal{M}_0 = -\bigcup_{(u,\lambda)\in\mathcal{U}\times S^+} \mathrm{epi}(\lambda G_u)^*,$$

and so, by (6) and the weak duality (5), we get

$$r_0 \leq \sup\{r : (c,r) \in -\mathcal{M}_0\} = \sup(\mathrm{RD}_c) \leq r_0 = \inf(\mathrm{RP}_c),$$

yielding $r_0 = \sup\{r : (c,r) \in -\mathcal{M}_0\} = \sup(\mathrm{RD}_c) = \inf(\mathrm{RP}_c)$. As $r_0 \in \{r : (c,r) \in -\mathcal{M}_0\}$ there exist $(\bar{u}, \bar{\lambda}) \in \mathcal{U} \times S^+$ satisfying (by (4))

$$r_0 = -(\bar{\lambda} G_{\bar{u}})^*(-c) = \max(\mathrm{RD}_c) = \inf(\mathrm{RP}_c),$$

which means that (b) holds.

• [(b) $\Longrightarrow$ (a)] Assume that (b) holds. To prove (a), it suffices to show that $\overline{\mathrm{co}}\,\mathcal{M}_0 \subset \mathcal{M}_0$. Take $(c, r) \in -\overline{\mathrm{co}}\,\mathcal{M}$. It follows from Proposition 1 that (7) holds with $r_0 = r$, which, taking (b) and (6) into account, entails

$$r \leq r_0 = \inf(\mathrm{RP}_c) = \max(\mathrm{RD}_c) = \max_{(u,\lambda)\in\mathcal{U}\times S^+} [-(\lambda G_u)^*(-c)].$$

This means that there exists $(\bar{u}, \bar{\lambda}) \in \mathcal{U} \times S^+$ such that $(-c, -r_0) \in \mathrm{epi}(\bar{\lambda} G_{\bar{u}})^*$. Now, as $r \leq r_0$, one has $(-c, -r) \in \mathrm{epi}(\bar{\lambda} G_{\bar{u}})^*$, and hence, $(c, r) \in -\mathcal{M}_0$. We have proved that $\overline{\mathrm{co}}\,\mathcal{M}_0 \subset \mathcal{M}_0$ and the proof is complete. □

We now provide some sufficient conditions for the convexity and closedness of the robust moment cone $\mathcal{M}_0$. Assume from now to end this section that $\mathcal{U}$ is a subset of some topological vector space. The next result is a consequence of [13, Propositions 5.1, 5.2].

Proposition 2 *Assume that $\mathcal{U}$ is a subset of some topological vector space and* $\operatorname{int} S \neq \emptyset$. *Then*

(i) *If the collection $(u \mapsto G_u(x))_{x \in X}$ is uniformly S^+-concave, then $\mathcal{M}_0$ is convex.*
(ii) *If $\mathcal{U}$ is a compact set, Z is a normed space, $u \mapsto G_u(x)$ is S^+-usc for all $x \in X$, and the following Slater-type condition holds:*

$$(C_0) \qquad \forall u \in \mathcal{U},\ \exists x_u \in X :\ G_u(x_u) \in -\operatorname{int} S,$$

then $\mathcal{M}_0$ is closed.

Remark 3 If $\mathcal{U}$ is a singleton then it is easy to see that the assumption of Proposition 2(i) automatically holds, and consequently, $\mathcal{M}_0$ is convex. Moreover, if the Slater condition (C_0) holds then $\mathcal{M}_0$ is closed.

Remark 4 It is worth noticing that the Proposition 2 and the next Corollary 2 constitute generalizations of Proposition 2 and Corollary 1 in [20], respectively. Propositions 6–7 on the sufficient conditions for the convexity and closedness of moment cones are of the same line of generalization which show the role played by the Slater constraint qualification condition.

Corollary 2 (Sufficient condition for stable robust strong duality of (RP_c)) *Assume that the following conditions hold:*

(i) *$\mathcal{U}$ is a compact set, Z is a normed space,*
(ii) *$(u \mapsto G_u(x))_{x \in X}$ is uniformly S^+-concave,*
(iii) *$u \mapsto G_u(x)$ is S^+-usc for all $x \in X$,*
(iv) *The Slater-condition (C_0) holds.*

Then, the stable robust strong duality holds for the pair (RP_c)–(RD_c).

Proof The conclusion follows from Theorem 1 and Proposition 2. □

Example 1 Let X, Z, and S be as in this section. Let further $\mathcal{U}$ be an uncertainty set, $(A_u)_{u \in \mathcal{U}} \subset \mathcal{L}(X, Z)$, $(\omega_u)_{u \in \mathcal{U}} \subset Z$.

Let $c \in X^*$ and consider the problem (RLP_c)[2]:

$$(\mathrm{RLP}_c) \qquad \inf \langle c, x \rangle \qquad \text{subject to } A_u(x) \in \omega_u - S, \quad \forall u \in \mathcal{U}, x \in X. \tag{8}$$

It is clear that (RLP_c) a special case of (RP_c) when setting $G_u(x) := A_u(x) - \omega_u$, $u \in \mathcal{U}$. Denote λA_u an element of X^* defined by $(\lambda A_u)(x) = \langle \lambda, A_u(x) \rangle$, for all $x \in X$.

Then the set $\mathcal{M}_0$ defined in (4) becomes

$$\mathcal{M}_1 := \{(\lambda A_u, \langle \lambda, \omega_u \rangle),\ (u, \lambda) \in \mathcal{U} \times S^+\} + \mathbb{R}_+(0_{X^*}, 1). \tag{9}$$

The dual problem of (RLP_c) (specialized from (RD_c)), turns to be

$$(\mathrm{RLD}_c) \qquad \sup -\langle \lambda, \omega_u \rangle \qquad \text{subject to } (u, \lambda) \in \mathcal{U} \times S^+,\ c = -\lambda A_u.$$

[2]The model of Problem (RLP_c) was considered in [12] where some characterizations of its solutions were proposed.

We get from Theorem 1 characterization of stable robust strong duality for (RLP$_c$) as follows:

The following statements are equivalent:

(a) $\mathcal{M}_1$ is a closed and convex subset of $X^* \times \mathbb{R}$,

(b) The stable robust strong duality holds for the pair (RLP$_c$)–(RLD$_c$), i.e.,

$$\inf(\mathrm{RLP}_c) = \max(\mathrm{RLD}_c) \quad \forall c \in X^*.$$

3 Robust Linear Infinite Problem and its Robust Duals

We retain the notations in Section 2 and let $c \in X^*$.

3.1 Statement of Robust Linear Infinite Problems and their Robust Duals

Consider the *linear infinite programming* with *uncertain input-parameters* of the form:

$$(\mathrm{ULIP}_c) \qquad \inf\langle c, x\rangle$$
$$\text{subject to } \langle a_t, x\rangle \le b_t \quad \forall t \in T, x \in X,$$

where (a_t, b_t) belongs to an uncertainty set $\mathcal{U}_t$ with $\emptyset \neq \mathcal{U}_t \subset X^* \times \mathbb{R}$ for all $t \in T$.

The *robust counterpart* of (ULIP$_c$) is

$$(\mathrm{RLIP}_c) \qquad \inf\langle c, x\rangle$$
$$\text{subject to } \langle x^*, x\rangle \le r, \quad \forall (x^*, r) \in \mathcal{U}_t \quad \forall t \in T, x \in X.$$

Assume that the problem (RLIP$_c$) is feasible for each $c \in X^*$, i.e.,

$$\mathcal{F} := \{x \in X : \langle x^*, x\rangle \le r\ \forall (x^*, r) \in \mathcal{U}_t,\ \forall t \in T\} \neq \emptyset \quad \forall c \in X^*$$

and set

$$\mathscr{U} := \prod_{t\in T} \mathcal{U}_t \quad \text{and} \quad \mathscr{V} := \bigcup_{t\in T} \mathcal{U}_t. \tag{10}$$

By convention, we write $v = (v^1, v^2) \in X^* \times \mathbb{R}$ and $u = (u_t)_{t\in T} \in \mathscr{U}$, with $u_t = (u_t^1, u_t^2) \in \mathcal{U}_t$. For brevity, we also write: $u = (u_t^1, u_t^2)_{t\in T} \in \mathscr{U}$ instead of $u = ((u_t^1, u_t^2))_{t\in T} \in \mathscr{U}$.

The robust problem of the model (RLIP$_c$) was considered in several earlier works such as [12, 20] (where $X = \mathbb{R}^n$, i.e., a robust semi-infinite linear problem), [24] where X is a Banach space, T is finite, objective function is a convex function, and for each $t \in T$, $\mathcal{U}_t$ has a special form (problem (SP), p. 2335), and in [11] with a bit more general on constraint linear inequalities, concretely, for all $t \in T$, (x^*, r) is a function defined on $\mathcal{U}_t$ instead of $(x^*, r) \in \mathcal{U}_t$.

We now propose variants of robust dual problems for (RLIP$_c$):

$$(\mathrm{RLID}_c^1) \qquad \sup[-\lambda v^2]$$
$$\text{s.t. } v \in \mathscr{V},\ \lambda \ge 0,\ c = -\lambda v^1,$$

$$(\mathrm{RLID}_c^2) \qquad \sup\left[-\sum_{u\in\operatorname{supp}\lambda} \lambda_u u_t^2\right]$$
$$\text{s.t. } t \in T,\ \lambda \in \mathbb{R}_+^{(\mathscr{U})},\ c = -\sum_{u\in\operatorname{supp}\lambda} \lambda_u u_t^1,$$

$$(\mathrm{RLID}_c^3)\qquad \sup\left[-\sum_{t\in\operatorname{supp}\lambda}\lambda_t u_t^2\right]$$
$$\text{s.t. } u\in\mathscr{U},\ \lambda\in\mathbb{R}_+^{(T)},\ c=-\sum_{t\in\operatorname{supp}\lambda}\lambda_t u_t^1,$$

$$(\mathrm{RLID}_c^4)\qquad \sup_{\lambda\geq 0,\ t\in T}\ \inf_{x\in X}\ \sup_{v\in\mathcal{U}_t}\left[\langle c+\lambda v^1, x\rangle-\lambda v^2\right],$$

$$(\mathrm{RLID}_c^5)\qquad \sup_{\lambda\geq 0,\ u\in\mathscr{U}}\ \inf_{x\in X}\ \sup_{t\in T}\left[\langle c+\lambda u_t^1, x\rangle-\lambda u_t^2\right],$$

$$(\mathrm{RLID}_c^6)\qquad \sup\left[-\sum_{v\in\operatorname{supp}\lambda}\lambda_v v^2\right]$$
$$\text{s.t. } \lambda\in\mathbb{R}_+^{(\mathscr{V})},\ c=-\sum_{v\in\operatorname{supp}\lambda}\lambda_v v^1,$$

$$(\mathrm{RLID}_c^7)\qquad \sup_{\lambda\geq 0}\ \inf_{x\in X}\ \sup_{v\in\mathscr{V}}\left[\langle c+\lambda v^1, x\rangle-\lambda v^2\right].$$

It is worth observing firstly that (RLID_c^3) and (RLID_c^6) are (ODP) and (DRSP) in [20], respectively. These two classes are also special case of (OLD) and (RLD) in [22] (where the constraint functions are affine) and of (RLD^O) and (RLD^C) in [11], respectively.

The *"robust strong duality (and also, stable robust strong duality) holds for the pair* (RLIP_c)–(RLID_c^i)*"*, $i=1,2,\ldots,7$, is understood as in Definition 3. Note that robust strong duality holds for (RLIP_c)–(RLID_c^3) is known as "primal worst equals dual best problem" with the attainment of dual problem [11, 20].

3.2 Relationship Between the Values of Dual Problems and Weak Duality

In this subsection we will establish some relations between the values of the dual problems (RLID_c^i) to each other and the weak duality to each of the dual pairs (RLIP_c)–(RLID_c^i), $i=1,2,\ldots,7$.

Proposition 3 *One has*

$$\sup(\mathrm{RLID}_c^1)\leq\begin{matrix}\sup(\mathrm{RLID}_c^2)\\ \sup(\mathrm{RLID}_c^3)\end{matrix}\leq\sup(\mathrm{RLID}_c^6). \tag{11}$$

Proof Observe that, for $k=1,2,3,6$, it holds $\sup(\mathrm{RLID}_c^k)=\sup E_k$ with

$$E_1 := \{\alpha : v\in\mathscr{V},\ \lambda\geq 0,\ (c,\alpha)=-\lambda v\},$$
$$E_2 := \left\{\alpha : t\in T,\ \lambda\in\mathbb{R}_+^{(\mathscr{U})},\ (c,\alpha)=-\sum_{u\in\operatorname{supp}\lambda}\lambda_u u_t\right\},$$
$$E_3 := \left\{\alpha : u\in\mathscr{U},\ \lambda\in\mathbb{R}_+^{(T)},\ (c,\alpha)=-\sum_{t\in\operatorname{supp}\lambda}\lambda_t u_t\right\},$$
$$E_6 := \left\{\alpha : \lambda\in\mathbb{R}_+^{(\mathscr{V})},\ (c,\alpha)=-\sum_{v\in\operatorname{supp}\lambda}\lambda_v v\right\}.$$

So, to prove (11), it suffices to verify that $E_i \subset E_j$ for $(i, j) \in \{(1,2), (1,3), (2,6), (3,6)\}$.

• $[E_1 \subset E_2]$ Take $\bar{\alpha} \in E_1$. Then, there are $\bar{v} \in \mathscr{V}$ and $\bar{\lambda} \geq 0$ such that $(c, \bar{v}) = -\bar{\lambda}\bar{v}$. Now, take $\bar{t} \in T$ and $\bar{u} \in \mathscr{U}$ such that $\bar{u}_{\bar{t}} = \bar{v}$. Define $\bar{\lambda} \in \mathbb{R}_+^{(\mathscr{U})}$ by $\bar{\lambda}_{\bar{u}} = \bar{\lambda}$ and $\bar{\lambda}_u = 0$ whenever $u \neq \bar{u}$. Then, it is easy to see that

$$-\sum_{u \in \operatorname{supp} \bar{\lambda}} \bar{\lambda}_u u_{\bar{t}} = -\bar{\lambda}_{\bar{u}} \bar{u}_{\bar{t}} = -\bar{\lambda}\bar{v} = (c, \bar{\alpha}),$$

yielding $\bar{\alpha} \in E_2$.

• $[E_1 \subset E_3]$ Can be done by using the same argument as in the proof of $E_1 \subset E_2$, just replace $\bar{\lambda} \in \mathbb{R}_+^{(\mathscr{U})}$ by $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that $\bar{\lambda}_{\bar{t}} = \bar{\lambda}$ and $\bar{\lambda}_t = 0$ for all $t \neq \bar{t}$.

• $[E_2 \subset E_6]$ Take $\bar{\alpha} \in E_2$. Then, there exists $(\bar{t}, \bar{\lambda}) \in T \times \mathbb{R}_+^{(\mathscr{U})}$ satisfying

$$-\sum_{u \in \operatorname{supp} \bar{\lambda}} \bar{\lambda}_u u_{\bar{t}} = (c, \bar{\alpha}).$$

Consider the set-valued mapping $\mathcal{K} \colon \mathscr{V} \rightrightarrows \mathscr{U}$ defined by

$$\mathcal{K}(v) := \left\{u \in \operatorname{supp} \bar{\lambda} : u_{\bar{t}} = v\right\}.$$

It is easy to see that the decomposition $\operatorname{supp} \bar{\lambda} = \bigcup_{v \in \mathscr{V}} \mathcal{K}(v)$ holds. Moreover, as $\operatorname{supp} \bar{\lambda}$ is finite, $\operatorname{dom} \mathcal{K}$ is also finite (where $\operatorname{dom} \mathcal{K} := \{v \in \mathscr{V} : \mathcal{K}(v) \neq \emptyset\}$). Take $\hat{\lambda} \in \mathbb{R}_+^{(\mathscr{V})}$ such that $\hat{\lambda}_v = \sum_{u \in \mathcal{K}(v)} \bar{\lambda}_u$ if $v \in \operatorname{dom} \mathcal{K}$ and $\hat{\lambda}_v = 0$ if $v \notin \operatorname{dom} \mathcal{K}$. Then, one has

$$-\sum_{v \in \operatorname{supp} \hat{\lambda}} \hat{\lambda}_v v = -\sum_{v \in \operatorname{dom} \mathcal{K}} \sum_{u \in \mathcal{K}(v)} \bar{\lambda}_u u_{\bar{t}} = -\sum_{u \in \operatorname{supp} \bar{\lambda}} \bar{\lambda}_u u_{\bar{t}} = (c, \bar{\alpha}),$$

yielding $\bar{\alpha} \in E_6$.

• $[E_3 \subset E_6]$ Similar to the proof of $[E_2 \subset E_6]$. □

Proposition 4 *One has*

$$\sup(\mathrm{RLID}_c^1) \leq \begin{matrix} \sup(\mathrm{RLID}_c^4) \\ \sup(\mathrm{RLID}_c^5) \end{matrix} \leq \sup(\mathrm{RLID}_c^7). \tag{12}$$

Proof It is worth noting firstly that, for any non-empty sets Y_1 and Y_2, any function $f : Y_1 \times Y_2 \to \mathbb{R}$, it always holds

$$\sup_{y_1 \in Y_1} \inf_{y_2 \in Y_2} f(y_1, y_2) \leq \inf_{y_2 \in Y_2} \sup_{y_1 \in Y_1} f(y_1, y_2). \tag{13}$$

By a simple calculation, one easily gets

$$\begin{aligned} \sup(\mathrm{RLID}_c^1) &= \sup_{\lambda \geq 0,\ v \in \mathscr{V}} \inf_{x \in X} (\langle c + \lambda v^1, x\rangle - \lambda v^2) \\ &= \sup_{\lambda \geq 0,\ t \in T} \sup_{w \in \mathcal{U}_t} \inf_{x \in X} (\langle c + \lambda w^1, x\rangle - \lambda w^2) \\ &= \sup_{\lambda \geq 0,\ u \in \mathscr{U}} \sup_{t \in T} \inf_{x \in X} [\langle c + \lambda u_t^1, x\rangle - \lambda u_t^2] \end{aligned}$$

(as $\mathscr{V} = \bigcup_{t \in T} \mathcal{U}_t = \{u_t : u \in \mathscr{U},\ t \in T\}$). So, according to (13),

$$\begin{aligned} \sup(\mathrm{RLID}_c^1) &\leq \sup_{\lambda \geq 0,\ t \in T} \inf_{x \in X} \sup_{w \in \mathcal{U}_t} [\langle c + \lambda w^1, x\rangle - \lambda w^2] = \sup(\mathrm{RLID}_c^4), \\ \sup(\mathrm{RLID}_c^1) &\leq \sup_{\lambda \geq 0,\ u \in \mathscr{U}} \inf_{x \in X} \sup_{t \in T} [\langle c + \lambda u_t^1, x\rangle - \lambda u_t^2] = \sup(\mathrm{RLID}_c^5). \end{aligned}$$

The other desired inequalities in (14) follow from (13) in a similar way as above. □

The weak duality for the primal-dual pairs of problems (RLIP_c^i)–(RLID_c^i), $i = 1, 2, \ldots, 7$, will be given in the next proposition.

Proposition 5 (Weak duality) *One has*

$$\begin{matrix} \sup(\text{RLID}_c^6) \\ \sup(\text{RLID}_c^7) \end{matrix} \leq \inf(\text{RLIP}_c). \tag{14}$$

Consequently, $\sup(\text{RLID}_c^i) \leq \inf(\text{RLIP}_c)$ *for all* $i = 1, 2, \ldots, 7$.

Proof • Proof of $\sup(\text{RLID}_c^6) \leq \inf(\text{RLIP}_c)$: Take $\bar{\lambda} \in \mathbb{R}_+^{(\mathscr{V})}$, and $\bar{x} \in X$ such that $c = -\sum_{v \in \operatorname{supp} \lambda} \bar{\lambda}_v v^1$ and

$$\langle v^1, \bar{x} \rangle - v^2 \leq 0, \quad \forall v \mathscr{V}. \tag{15}$$

Then it is easy to see that $-\sum_{v \in \operatorname{supp} \bar{\lambda}} v^2 \leq -\sum_{v \in \operatorname{supp} \bar{\lambda}} \langle v^1, \bar{x} \rangle = \langle c, \bar{x} \rangle$. So, by the definitions of (RLID_c^6) one has $\sup(\text{RLID}_c^6) \leq \langle c, \bar{x} \rangle$ for any $\bar{x} \in X$ satisfying (15), which yields $\sup(\text{RLID}_c^6) \leq \inf(\text{RLIP}_c)$.

- Proof of $\sup(\text{RLID}_c^7) \leq \inf(\text{RLIP}_c)$: Take $\bar{\lambda} \geq 0$ and $\bar{x} \in X$ such that (15) holds. For all $v \in \mathscr{V}$, as (15) holds, one has $\langle c + \bar{\lambda} v^1, \bar{x} \rangle - \bar{\lambda} v^2 \leq \langle c, \bar{x} \rangle$. This yields that $\sup_{v \in \mathscr{V}} [\langle c + \bar{\lambda} v^1, \bar{x} \rangle - \bar{\lambda} v^2] \leq \langle c, \bar{x} \rangle$ which, in turn, amounts for

$$\inf_{x \in X} \sup_{v \in \mathscr{V}} [\langle c + \bar{\lambda} v^1, x \rangle - \bar{\lambda} v^2] \leq \langle c, \bar{x} \rangle.$$

The conclusion follows. □

4 Robust Stable Strong Duality for (RLIP_c)

In this section, we will establish variants of stable robust strong duality results for (RLIP_c). Some of them cover the ones in [20, 22] and the others are new.

Let us introduce variants of *robust moment cones* of (RLIP_c):

$$\mathscr{N}_1 := \operatorname{cone} \mathscr{V} + \mathbb{R}_+(0_{X^*}, 1), \qquad \mathscr{N}_2 := \bigcup_{t \in T} \operatorname{co} \operatorname{cone}[\mathscr{U}_t \cup \{(0_{X^*}, 1)\}],$$

$$\mathscr{N}_3 := \bigcup_{u \in \mathscr{U}} \operatorname{co} \operatorname{cone}[u(T) \cup \{(0_{X^*}, 1)\}], \qquad \mathscr{N}_4 := \bigcup_{t \in T} \operatorname{cone} \overline{\operatorname{co}}[\mathscr{U}_t + \mathbb{R}_+(0_{X^*}, 1)],$$

$$\mathscr{N}_5 := \bigcup_{u \in \mathscr{U}} \operatorname{cone} \overline{\operatorname{co}}[u(T) + \mathbb{R}_+(0_{X^*}, 1)], \qquad \mathscr{N}_6 := \operatorname{co} \operatorname{cone}\left[\mathscr{V} \cup \{(0_{X^*}, 1)\}\right],$$

$$\mathscr{N}_7 := \operatorname{cone} \overline{\operatorname{co}}\left[\mathscr{V} + \mathbb{R}_+(0_{X^*}, 1)\right],$$

where $u(T) := \{u_t : t \in T\}$ if $u \in \mathscr{U}$.

Observe that $\mathscr{N}_3$ is $M_{\ell f}$ in [12], and $\mathscr{N}_3$ and $\mathscr{N}_6$ were introduced in [20] and known as "robust moment cone" and "characteristic cone", respectively.

Theorem 2 (1st characterization of stable robust strong duality for (RLIP_c)) *For* $i \in \{1, 2, \ldots, 5\}$, *consider the following statements:*

(c_i) $\mathscr{N}_i$ *is a closed and convex subset of* $X^* \times \mathbb{R}$.

(d_i) *The stable robust strong duality holds for the pair* (RLIP_c)–(RLID_c^i).

Then, one has $[(c_i) \Leftrightarrow (d_i)]$ *for all* $i \in \{1, 2, \ldots, 5\}$.

Proof $\bullet[(c_1) \Leftrightarrow (d_1)]$ Set $Z = \mathbb{R}$, $S = \mathbb{R}_+$, $\mathcal{U} = \mathscr{V}$, $A_v = v^1$ and $\omega_v = v^2$ for all $v = (v^1, v^2) \in \mathscr{V}$. Then, (RLIP$_c$) has the form of (RLP$_c$) in (8). In such a setting, the robust moment cone $\mathcal{M}_1$ in (9) reduces to

$$\begin{aligned}\mathcal{M}_1 &= \{(\lambda A_v, \langle \lambda, \omega_v \rangle) : v \in \mathscr{V},\ \lambda \geq 0\} + \mathbb{R}_+(0_{X^*}, 1)\\ &= \{\lambda v : v \in \mathscr{V},\ \lambda \geq 0\} + \mathbb{R}_+(0_{X^*}, 1)\\ &= \operatorname{cone} \mathscr{V} + \mathbb{R}_+(0_{X^*}, 1) = \mathcal{N}_1.\end{aligned}$$

It is easy to see that the robust dual problem (RLD$_c$) of the resulting robust problem (RLP$_c$) now turns be exactly (RLID$_c^1$), and so, the equivalence $[(c_1) \Leftrightarrow (d_1)]$ follows directly from Theorem 1 (see also Example 1).

$\bullet[(c_2) \Leftrightarrow (d_2)]$ Set $Z = \mathbb{R}^{\mathscr{U}}$, $S = \mathbb{R}_+^{\mathscr{U}}$ (and consequently, $Z^* = \mathbb{R}^{(\mathscr{U})}$ and $S^+ = \mathbb{R}_+^{(\mathscr{U})}$), $\mathcal{U} = T$, $A_t = (u_t^1)_{u \in \mathscr{U}}$ and $\omega_t = (u_t^2)_{u \in \mathscr{U}}$ for all $t \in T$. Then the problem (RLIP$_c$) possesses the form (RLP$_c$). In this setting, the set $\mathcal{M}_1$ in (9) becomes

$$\begin{aligned}\mathcal{M}_1 &= \left\{(\lambda A_t, \langle \lambda, \omega_t \rangle) : t \in T, \lambda \in \mathbb{R}_+^{(\mathscr{U})}\right\} + \mathbb{R}_+(0_{X^*}, 1)\\ &= \left\{\left(\sum_{u \in \operatorname{supp} \lambda} \lambda_u u_t^1, \sum_{u \in \operatorname{supp} \lambda} \lambda_u u_t^1\right) : t \in T, \lambda \in \mathbb{R}_+^{(\mathscr{U})}\right\} + \mathbb{R}_+(0_{X^*}, 1)\\ &= \left\{\sum_{u \in \operatorname{supp} \lambda} \lambda_u u_t : t \in T, \lambda \in \mathbb{R}_+^{(\mathscr{U})}\right\} + \mathbb{R}_+(0_{X^*}, 1)\\ &= \left[\bigcup_{t \in T} \left\{\sum_{u \in \operatorname{supp} \lambda} \lambda_u u_t : \lambda \in \mathbb{R}_+^{(\mathscr{U})}\right\}\right] + \mathbb{R}_+(0_{X^*}, 1)\\ &= \left[\bigcup_{t \in T} \operatorname{co} \operatorname{cone} \mathcal{U}_t\right] + \mathbb{R}_+(0_{X^*}, 1) \quad (\text{note that } \{u_t : u \in \mathscr{U}\} = \mathcal{U}_t)\\ &= \bigcup_{t \in T} \left[\operatorname{co} \operatorname{cone} \mathcal{U}_t + \mathbb{R}_+(0_{X^*}, 1)\right] = \bigcup_{t \in T} \operatorname{co} \operatorname{cone} \left[\mathcal{U}_t \cup \{(0_{X^*}, 1)\}\right] = \mathcal{N}_2,\end{aligned}$$

and the dual problem of (RLD$_c$) (in the new format) has the form (RLID$_c^2$). The equivalence $[(c_2) \Leftrightarrow (d_2)]$ then follows from Theorem 1.

$\bullet[(c_3) \Leftrightarrow (d_3)]$ We transform (RLIP$_c$) to (RLP$_c$) by setting: $Z = \mathbb{R}^T$, $S = \mathbb{R}_+^T$ (hence, $Z^* = \mathbb{R}^{(T)}$ and $S^+ = \mathbb{R}_+^{(T)}$), $\mathcal{U} = \mathscr{U}$, $A_u = (u_t^1)_{t \in T}$ and $\omega_u = (u_t^2)_{t \in T}$ for all $u \in \mathscr{U}$. Then, one has

$$\begin{aligned}\mathcal{M}_1 &= \left\{(\lambda A_u, \langle \lambda, \omega_u \rangle) : u \in \mathscr{U},\ \lambda \in \mathbb{R}_+^{(T)}\right\} + \mathbb{R}_+(0_{X^*}, 1)\\ &= \left\{\sum_{t \in \operatorname{supp} \lambda} \lambda_t u_t : u \in \mathscr{U}, \lambda \in \mathbb{R}_+^{(T)}\right\} + \mathbb{R}_+(0_{X^*}, 1)\\ &= \left[\bigcup_{u \in \mathscr{U}} \operatorname{co} \operatorname{cone} u(T)\right] + \mathbb{R}_+(0_{X^*}, 1) \quad (\text{note that } \{u_t : t \in T\} = u(T))\\ &= \bigcup_{u \in \mathscr{U}} \operatorname{co} \operatorname{cone} [u(T) \cup \{(0_{X^*}, 1)\}] = \mathcal{N}_3\end{aligned}$$

and the dual problem (RLD_c) of the resulting problem (RLP_c) is exactly (RLID_3). The desired equivalence follows from Theorem 1.

•[(c_4) ⇔ (d_4)] We now consider another way of transforming (RLIP_c) to the form (RP_c) by letting $Z = \mathbb{R}$, $S = \mathbb{R}_+$, $\mathcal{U} = T$, and $G_t : X \to \overline{\mathbb{R}}$ such that $G_t(x) = \sup_{v\in\mathcal{U}_t}[\langle v^1, x\rangle - v^2]$ for all $t \in T$. Then, one has (see (4))

$$\begin{aligned}\mathcal{M}_0 &= \bigcup_{t\in T,\lambda\geq 0} \mathrm{epi}(\lambda G_t)^* = \bigcup_{t\in T,\ \lambda\geq 0} \lambda \mathrm{epi}(G_t)^* \\ &= \bigcup_{t\in T} \mathrm{cone}\,\mathrm{epi}(G_t)^* = \bigcup_{t\in T} \mathrm{cone}\,\mathrm{epi}\left[\sup_{v\in\mathcal{U}_t}(\langle v^1,\cdot\rangle - v^2)\right]^* \\ &= \bigcup_{t\in T} \mathrm{cone}\,\overline{\mathrm{co}} \bigcup_{v\in\mathcal{U}_t} \mathrm{epi}\left(\langle v^1,\cdot\rangle - v^2\right)^*\end{aligned}$$

(the last equalities follows from [25, Lemma 2.2]). On the other hand, for each $t \in T$ and $v \in \mathcal{U}_t$, by simple calculation one gets $\mathrm{epi}(\langle v^1,\cdot\rangle - v^2)^* = v + \mathbb{R}_+(0_{X^*}, 1)$. So,

$$\mathcal{M}_0 = \bigcup_{t\in T} \mathrm{cone}\,\overline{\mathrm{co}} \bigcup_{v\in\mathcal{U}_t} [v + \mathbb{R}_+(0_{X^*}, 1)] = \bigcup_{t\in T} \mathrm{cone}\,\overline{\mathrm{co}}[\mathcal{U}_t + \mathbb{R}_+(0_{X^*}, 1)] = \mathcal{N}_4.$$

It is easy to see that the dual problem (RD_c) of the resulting problem (RP_c) is nothing else but (RLID_c^4), and the equivalence [(c_4) ⇔ (d_4)] is a consequence of Theorem 1.

•[(c_5) ⇔ (d_5)] Again, we transform (RLIP_c) to (RP_c) but by another setting: $Z = \mathbb{R}$, $S = \mathbb{R}_+$, $\mathcal{U} = \mathscr{U}$, and $G_u : X \to \overline{\mathbb{R}}$ such that $G_u(x) = \sup_{t\in T}[\langle u_t^1, x\rangle - u_t^2)]$ for all $u \in \mathscr{U}$. Then, one has (see (4))

$$\begin{aligned}\mathcal{M}_0 &= \bigcup_{u\in\mathscr{U},\ \lambda\geq 0} \mathrm{epi}(\lambda G_u)^* = \bigcup_{u\in\mathscr{U}} \mathrm{cone}\,\mathrm{epi}(G_u)^* \\ &= \bigcup_{u\in\mathscr{U}} \mathrm{cone}\,\mathrm{epi}\left[\sup_{t\in T}(\langle u_t^1,\cdot\rangle - u_t^2)\right]^* = \bigcup_{u\in\mathscr{U}} \mathrm{cone}\,\overline{\mathrm{co}} \bigcup_{t\in T} \mathrm{epi}(\langle u_t^1,\cdot\rangle - u_t^2)^* \\ &= \bigcup_{u\in\mathscr{U}} \mathrm{cone}\,\overline{\mathrm{co}} \bigcup_{t\in T} [u_t + \mathbb{R}_+(0_{X^*}, 1)] = \bigcup_{u\in\mathscr{U}} \mathrm{cone}\,\overline{\mathrm{co}}[u(T) + \mathbb{R}_+(0_{X^*}, 1)] = \mathcal{N}_5,\end{aligned}$$

and the robust dual problem (RD_c) of the new problem (RP_c) is exactly (RLID_c^5). The desired equivalence again follows from Theorem 1. □

Remark 5 Theorem 2 with $i = 3$ is [20, Theorem 2] while $i = 6$ ($i = 3$, resp.) is similar to [11, Proposition 5.2(ii)] with $i = C$ ($i = O$, resp.).

Theorem 3 (2nd characterization for stable robust strong duality for (RLIP_c)) *For $i = 6, 7$, consider the next statements:*

(c_i) *$\mathcal{N}_i$ is a closed subset of $X^* \times \mathbb{R}$.*
(d_i) *The stable robust strong duality holds for the pair* (RLIP_c)–(RLID_c^i).

Then [(c_i) ⇔ (d_i)] *for $i = 6, 7$.*

Proof •[(c_6) ⇔ (d_6)] The robust problem (RLIP_c) turns to be (RLP_c) if we set $Z = \mathbb{R}^{\mathscr{V}}$, $S = \mathbb{R}_+^{\mathscr{V}}$, $\mathcal{U}$ to be a singleton, $A = (v^1)_{v\in\mathscr{V}}$ and $\omega = (v^2)_{v\in\mathscr{V}}$. In such a setting, one gets

$$\begin{aligned}\mathcal{M}_1 &= \left\{(\lambda A, \langle\lambda,\omega\rangle) : \lambda \in \mathbb{R}_+^{(\mathscr{V})}\right\} + \mathbb{R}_+(0_{X^*}, 1)\\ &= \left\{\sum_{v\in \operatorname{supp}\lambda} \lambda_v v : \lambda \in \mathbb{R}_+^{(\mathscr{V})}\right\} + \mathbb{R}_+(0_{X^*}, 1)\\ &= \operatorname{co}\operatorname{cone}\mathscr{V} + \mathbb{R}_+(0_{X^*}, 1) = \operatorname{co}\operatorname{cone}\left[\mathscr{V}\cup\{(0_{X^*}, 1)\}\right] = \mathcal{N}_6,\end{aligned}$$

while the robust dual problem of the new problem (RLP_c) (i.e., (RLD_c)) is non other than (RLID_c^6). The equivalence [(c_6) ⇔ (d_6)] now follows from Theorem 1 and the fact that the robust moment cone is always convex whenever $\mathcal{U}$ is a singleton (see Proposition 2 and Remark 3).

•[(c_7) ⇔ (d_7)] Set $Z = \mathbb{R}$, $S = \mathbb{R}$, $\mathcal{U}$ to be a singleton, and $G = \sup_{v\in\mathscr{V}}(\langle v^1, \cdot\rangle - v^2)$. The problem ($\text{RLIP}_c$) now becomes ($\text{RP}_c$). On the other hand, one has (see (4))

$$\begin{aligned}\mathcal{M}_0 &= \bigcup_{\lambda\geq 0}\operatorname{epi}(\lambda G)^* = \operatorname{cone}\operatorname{epi}(\lambda G)^*\\ &= \operatorname{cone}\operatorname{epi}\left[\sup_{v\in\mathscr{V}}(\langle v^1,\cdot\rangle - v^2)\right]^* = \operatorname{cone}\overline{\operatorname{co}}\bigcup_{v\in\mathscr{V}}\operatorname{epi}(\langle v^1,\cdot\rangle - v^2)^*\\ &= \operatorname{cone}\overline{\operatorname{co}}\bigcup_{v\in\mathscr{V}}[v + \mathbb{R}_+(0_{X^*}, 1)] = \operatorname{cone}\overline{\operatorname{co}}\left[\mathscr{V} + \mathbb{R}_+(0_{X^*}, 1)\right] = \mathcal{N}_7,\end{aligned}$$

while the dual problem of (RD_c) of the new problem (RP_c) is (RLID_c^7). The equivalence [(c_7) ⇔ (d_7)] is a consequence of Theorem 1, Proposition 2 (see also Remark 3). □

Remark 6 There may have sone more ways of transforming (RLIP_c) to the form of (RP_c) which give rise to some more robust dual problems for (RLIP_c), for instance,

(α) Set $Z = \mathbb{R}^T$, $S = \mathbb{R}_+^T$, $\mathcal{U}$ to be a singleton, and $G = \left(\sup_{v\in\mathcal{U}_t}[\langle v^1,\cdot\rangle - v^2]\right)_{t\in T}$. Then ($\text{RLIP}_c$) reduces to the form of (RP_c) with no uncertainty as now $\mathcal{U}$ is a singleton. In this setting, the moment cone $\mathcal{M}_0$ becomes

$$\mathcal{M}_0 = \bigcup_{\lambda\in\mathbb{R}_+^{(T)}}\operatorname{epi}\left[\sum_{t\in T}\lambda_t\sup_{v\in\mathcal{U}_t}(\langle v^1,\cdot\rangle - v^2)\right]^* =: \mathcal{N}_8,$$

and the robust dual problems now collapses to

$$(\text{RLID}_c^8) \qquad \sup_{\lambda\in\mathbb{R}_+^{(T)}}\inf_{x\in X}\left[\langle c, x\rangle + \sum_{t\in\operatorname{supp}\lambda}\lambda_t\sup_{v\in\mathcal{U}_t}\left(\langle v^1, x\rangle - v^2\right)\right].$$

(β) Set $Z = \mathbb{R}^{\mathscr{U}}$, $S = \mathbb{R}_+^{\mathscr{U}}$, $\mathcal{U}$ to be a singleton, and $G = (\sup_{t\in T}[\langle u_t^1,\cdot\rangle - u_t^2])_{u\in\mathscr{U}}$. Then, the problem ($\text{RLIP}_c$) turns to be of the model (RP_c), and one has

$$\mathcal{M}_0 = \operatorname{co}\operatorname{cone}\bigcup_{u\in\mathscr{U}}\overline{\operatorname{co}}\left[u(T) + \mathbb{R}_+(0_{X^*}, 1)\right] =: \mathcal{N}_9.$$

The corresponding dual problem is

$$(\text{RLID}_c^9) \qquad \sup_{\lambda\in\mathbb{R}_+^{(\mathscr{U})}}\inf_{x\in X}\left[\langle c, x\rangle + \sum_{u\in\operatorname{supp}\lambda}\lambda_u\sup_{t\in T}\left(\langle u_t^1, x\rangle - u_t^2\right)\right].$$

For the mentioned cases, we get also the relation between the values of these two dual problems:

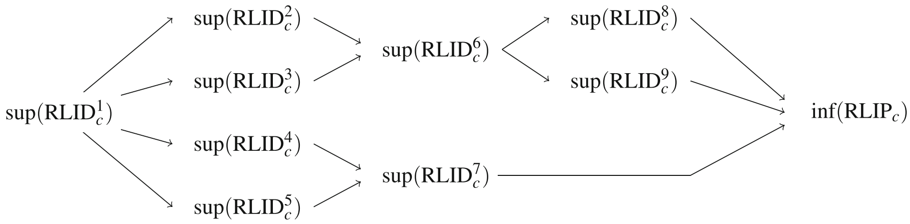
$$\sup(\mathrm{RLID}_c^6) \le \sup(\mathrm{RLID}_c^8) \quad \text{and} \quad \sup(\mathrm{RLID}_c^6) \le \sup(\mathrm{RLID}_c^9),$$

and weak duality hold as well:

$$\begin{matrix} \sup(\mathrm{RLID}_c^8) \\ \sup(\mathrm{RLID}_c^9) \end{matrix} \le \inf(\mathrm{RLIP}_c).$$

Moreover, under some suitable conditions, robust strong duality holds, similar to the ones in [11, Proposition 5.2(ii)].

Remark 7 From Propositions 3–5 and Remark 6, we get an overview of the relationship between the values of robust dual problems and weak duality of each pair of primal-dual problems which can be described as in the next figure:



where by $a \longrightarrow b$ we mean $a \le b$.

As we have seen from the previous theorems and from the previous section, the closedness and convexity of robust moment cones play crucial roles in closing the dual gaps for the primal-dual pairs of robust problems. In the left of this section, we will give some sufficient conditions for the mentioned properties of these cones whose proofs are rather long and will be put in the last section: Appendices.

Proposition 6 (Convexity of moment cones) *The next assertions hold:*

(i) *If $\mathcal{V}$ is convex then $\mathcal{N}_1$ is convex.*
(ii) *If $\{x^* \in X^* : (x^*, r) \in \mathcal{U}_t\}$ is convex for all $t \in T$ then $\mathcal{N}_3$ is convex.*
(iii) *Assume that T is a convex subset of some vector space, and that, for all $t \in T$, $\mathcal{U}_t = \mathcal{U}_t^1 \times \mathcal{U}_t^2$ with $\mathcal{U}_t^1 \subset X^*$ and $\mathcal{U}_t^2 \subset \mathbb{R}$. Assume further that, for each $t \in T$ and $x \in X$, the function $t \mapsto \sup_{x^* \in \mathcal{U}_t^1} \langle x^*, x\rangle$ is affine and the function $t \mapsto \inf \mathcal{U}_t^2$ is convex. Then, $\mathcal{N}_4$ is convex.*
(iv) *The sets $\mathcal{N}_6$, $\mathcal{N}_7$ are convex*[3].

Proof See Appendix A. □

Proposition 7 (Closedness of moment cones) *The following assertions are true.*

(i) *If $\mathcal{V}$ is compact and*

$$\forall v \in \mathcal{V},\ \exists \bar{x} \in X : \quad \langle v^1, \bar{x}\rangle < v^2, \tag{16}$$

[3] $\mathcal{N}_8, \mathcal{N}_9$ are also convex.

then $\mathcal{N}_1$ *is closed.*

(ii) *If* T *is compact,* $t \mapsto \sup_{v\in\mathcal{U}_t}[\langle v^1, x\rangle - v^2]$ *is usc for all* $x \in X$*, and*

$$\forall t \in T,\ \exists x_t \in X: \quad \sup_{v\in\mathcal{U}_t}[\langle v^1, x_t\rangle - v^2] < 0, \tag{17}$$

then $\mathcal{N}_4$ *is closed.*

(iii) *If* $\mathcal{U}_t$ *is compact for all* $t \in T$*,* $u \mapsto \sup_{t\in T}[\langle u_t^1, x\rangle - u_t^2]$ *is usc for all* $x \in X$*, and*

$$\forall u \in \mathscr{U},\ \exists x_u \in X: \quad \sup_{t\in T}[\langle u_t^1, x_u\rangle - u_t^2] < 0,$$

then $\mathcal{N}_5$ *is closed.*

(iv) *If the following condition holds*

$$\exists x \in X: \quad \sup_{v\in\mathscr{V}}[\langle v^1, x\rangle - v^2] < 0,$$

then $\mathcal{N}_7$ *is closed.*

Proof See Appendix B. □

5 Farkas-Type Results for Infinite Linear Systems with Uncertainty

We retain the notations used in Sections 2, 3, and 4.

Let $c \in X^*$, T be an index set (possibly infinite), and let $\mathcal{U}_t$ be uncertainty set for each $t \in T$. Consider the robust linear system of the form

$$\langle x^*, x\rangle \le r, \quad \forall (x^*, r) \in \mathcal{U}_t,\ \forall t \in T,$$

which is the constraint system of the problem (RLIP$_c$) considered in Section 4.

Based on the stable strong robust duality results established in Section 4, we can derive the next robust Farkas-type results for the linear systems with uncertainty parameters (for a short survey on Farkas-type results, see, e.g., [10]).

Corollary 3 (Robust Farkas lemma for linear system I) *Let* $\mathscr{V}$ *be the set defined by* (10). *The following statements are equivalent:*

(i) *For all* $(c, s) \in^* \times\mathbb{R}$ *such that* $\inf(\text{RLIP}_c) > -\infty$*, the next assertions are equivalent:*

   (α) $\langle x^*, x\rangle \le r\ \ \forall (x^*, r) \in \mathscr{V} \implies \langle c, x\rangle \ge s.$,

   (β) $\exists (\bar{x}^*, \bar{r}) \in \mathscr{V},\ \exists \bar{\lambda} \ge 0: \begin{cases} \bar{\lambda}\bar{x}^* = -c, \\ \bar{\lambda}\bar{r} \le -s. \end{cases}$

(ii) $\operatorname{cone}\mathscr{V} + \mathbb{R}_+(0_{X^*}, 1)$ *is convex and closed.*

Proof Take $(c, s) \in X^* \times \mathbb{R}$. Set $\Lambda := \{(x^*, r, \lambda) : (x^*, r) \in \mathscr{V}, \lambda \in \mathbb{R}_+, \lambda x^* = -c\}$ and $\Phi(x^*, r, \lambda) = -\lambda r$ for all $(x^*, r, \lambda) \in \Lambda$. So, $\sup(\text{RLID}_c^1) = \sup_{(x^*,r,\lambda)\in\Lambda} \Phi(x^*, r, \lambda)$. From the statements of the problems (RLIP$_c$) and (RLID$_c^1$), one has

$$(\alpha) \iff \inf(\text{RLIP}_c) \ge s, \tag{18}$$

$$(\beta) \iff \left(\exists (\bar{x}^*, \bar{r}, \bar{\lambda}) \in \Lambda: \ \sup(\text{RLID}_c^1) \ge \Phi(\bar{x}^*, \bar{r}, \bar{\lambda}) = -\bar{\lambda}\bar{r} \ge s\right). \tag{19}$$

• [(ii) $\Longrightarrow$ (i)] Assume that (ii) holds. Then it follows from Theorem 2 (with $i = 1$),

$$\begin{aligned}(\text{ii}) \iff & \left(\text{the stable robust strong duality holds for the pair } (\text{RLIP}_c)\text{–}(\text{RLID}_c^1)\right)\\ \iff & \left(\forall c \in X^*,\ \ \inf(\text{RLIP}_c) = \max(\text{RLID}_c^1)\right). \qquad (20)\end{aligned}$$

So, for $c \in X^*$, if (α) holds then $\inf(\text{RLIP}_c) \geq s$ and hence, we get from (20),

$$\inf(\text{RLIP}_c) = \max(\text{RLID}_c) = \Phi(\bar{x}^*, \bar{r}, \bar{\lambda}) = -\bar{\lambda}\bar{r} \geq s,$$

for some $(\bar{x}^*, \bar{r}, \bar{\lambda}) \in \Lambda$, which means that (β) holds, and so $[(\alpha) \Longrightarrow (\beta)]$.

Conversely, if (β) holds, then from (19) and the weak duality of the primal-dual pair (RLIP_c)–(RLID_c^1), one gets the existence of $(\bar{x}^*, \bar{r}, \bar{\lambda}) \in \Lambda$ such that

$$\inf(\text{RLIP}_c) \geq \sup(\text{RLID}_c^1) \geq \Phi(\bar{x}^*, \bar{r}, \bar{\lambda}) = -\bar{\lambda}\bar{r} \geq s,$$

yielding (α). So, $[(\beta) \Longrightarrow (\alpha)]$ and consequently, we have proved that [(ii) $\Longrightarrow$ (i)].

•[(i) $\Longrightarrow$ (ii)] Assume that (i) holds. Take $s = \inf(\text{RLIP}_c) \in \mathbb{R}$ and $c \in X^*$. Then (α) holds and as (i) holds, (β) holds as well. This, together with the weak duality, yields, for some $(\bar{x}^*, \bar{r}, \bar{\lambda}) \in \Lambda$ (see (19)),

$$\inf(\text{RLIP}_c) \geq \sup(\text{RLID}_c^1) = \Phi(\bar{x}^*, \bar{r}, \bar{\lambda}) = -\bar{\lambda}\bar{r} \geq s = \inf(\text{RLIP}_c),$$

meaning that the robust dual problem (RLID_c^1) attains and $\inf(\text{RLIP}_c) = \max(\text{RLID}_c^1)$. Since $c \in X^*$ is arbitrary, the stable robust strong duality holds for the pair (RLIP_c)–(RLID_c^1). The fulfillment of (ii) now follows from Theorem 2 (with $i = 1$). □

Remark 8 Assume that $\mathscr{V}$ is a convex and compact subset of $X^* \times \mathbb{R}$ and that the Slater-type condition (16) holds. According to Propositions 6 and 7, one has $\mathcal{N}_1 := \operatorname{cone} \mathscr{V} + \mathbb{R}_+(0_{X^*}, 1)$ is closed and convex. So, it follows from Corollary 3, (α) and (β) in Corollary 3 are equivalent. This observation may apply to some of the next corollaries.

The next versions of robust Farkas lemmas follows from the same way as Corollary 3, using Theorem 2 with $i = 2, 3$, and $i = 4$.

Corollary 4 (Robust Farkas lemma for linear system II) *The following statements are equivalent:*

(i) *For all* $(c, s) \in X^* \times \mathbb{R}$ *such that* $\inf(\text{RLIP}_c) > -\infty$, *the next assertions are equivalent:*

(α) $\langle x^*, x\rangle \leq r \ \ \forall (x^*, r) \in \mathscr{V} \implies \langle c, x\rangle \geq s,$

(γ) $\exists \bar{t} \in T,\ \exists \bar{\lambda} \in \mathbb{R}_+^{(\mathscr{U})} : \begin{cases} \sum_{u \in \operatorname{supp}\lambda} \bar{\lambda}_u u_{\bar{t}}^1 = -c, \\ \sum_{u \in \operatorname{supp}\lambda} \bar{\lambda}_u u_{\bar{t}}^2 \leq -s. \end{cases}$

(ii) $\bigcup_{t \in T} \operatorname{co}\operatorname{cone}[\mathcal{U}_t \cup \{(0_{X^*}, 1)\}]$ *is convex and closed.*

Corollary 5 (Robust Farkas lemma for linear system III), [16, Theorem 5.6], [20, Corollary 3], [12, Theorem 6.1(i)]) *The following statements are equivalent:*

(i) *For all* $(c, s) \in X^* \times \mathbb{R}$, *such that* $\inf(\text{RLIP}_c) > -\infty$, *the next assertions are equivalent:*

(α) $\langle x^*, x\rangle \leq r \ \ \forall (x^*, r) \in \mathscr{V} \implies \langle c, x\rangle \geq s,$

(δ) $\exists \bar{u} \in \mathscr{U},\ \exists \bar{\lambda} \in \mathbb{R}_+^{(T)} : \begin{cases} \sum_{t \in \operatorname{supp}\lambda} \bar{\lambda}_t \bar{u}_t^1 = -c, \\ \sum_{t \in \operatorname{supp}\lambda} \bar{\lambda}_t \bar{u}_t^2 \leq -s. \end{cases}$

(ii) $\bigcup_{u\in\mathscr{U}} \operatorname{co}\operatorname{cone}[u(T)\cup\{(0_{X^*},1)\}]$ *is convex and closed, where* $u(T):=\{u_t : t\in T\}$ *for all* $u\in\mathscr{U}$.

Corollary 6 (Robust Farkas lemma for linear system IV) *The following statements are equivalent:*

(i) *For all* $(c,s)\in X^*\times\mathbb{R}$*, such that* $\inf(\mathrm{RLIP}_c)>-\infty$*,* $x\in X$*, the next assertions are equivalent:*

  (α) $\langle x^*,x\rangle\le r,\ \forall(x^*,r)\in\mathscr{V}\implies\langle c,x\rangle\ge s,$
  (ϵ) $\exists\bar t\in T,\ \exists\bar\lambda\ge 0$ *such that* $\forall x\in X,\ \forall\varepsilon>0,\ \exists(x_0^*,r_0)\in\mathcal{U}_{\bar t}$ *satisfying*

$$\langle c+\bar\lambda x_0^*,x\rangle-\bar\lambda r_0\ge s-\varepsilon,$$

(ii) $\bigcup_{t\in T}\operatorname{cone}\overline{\operatorname{co}}[\mathcal{U}_t+\mathbb{R}_+(0_{X^*},1)]$ *is convex and closed.*

Remark 9 It worth noting that robust Farkas-type results can be established in the same way as in the previous corollaries, corresponding to the stable robust strong duality for pairs (RLIP_c)–(RLID_c^j) with $j=5,\dots,9$. The results corresponding to $i=6$ can be considered as a version of [20, Corollary 4] with $\mathscr{V}$ replacing gph $\mathscr{U}$.

6 Linear Infinite Problems with Sub-affine Constraints

The results in previous sections for robust linear infinite problems (RLIP_c) $(c\in X^*)$ can be extended to a rather broader class of robust problems by a similar approaching. Here we consider a concrete class of problems: The robust linear problems with sub-affine constraints.

Denote by $\mathscr{P}_0(X^*)$ the set of all the nonempty, w^*-closed convex subsets of X^*. Let T be a possibly infinite index set, $(\mathcal{U}_t)_{t\in T}\subset\mathscr{P}_0(X^*)\times\mathbb{R}$ be a collection of nonempty uncertainty sets. We introduce the sets

$$\mathfrak{V}:=\bigcup_{t\in T}\mathcal{U}_t \quad\text{and}\quad \mathfrak{U}=\prod_{t\in T}\mathcal{U}_t.$$

By convention, for each $V\in\mathscr{P}_0(X^*)\times\mathbb{R}$, we write $V=(V^1,V^2)$ with $V^1\subset X^*$ and $V^2\in\mathbb{R}$. In some case, we also considered $V=(V^1,V^2)\in\mathscr{P}_0(X^*)\times\mathbb{R}$ as a subset of the set $X^*\times\mathbb{R}$ by identifying V with $V^1\times\{V_2\}$. In the same way, for $U\in\mathfrak{U}$, we write $U=(U_t)_{t\in T}$ with $U_t=(U_t^1,U_t^2)\in\mathcal{U}_t$ for each $t\in T$.

For each $c\in X^*$, consider the robust linear problem with sub-affine constraints:

$$(\mathrm{RSAP}_c)\qquad \begin{aligned}&\inf\langle c,x\rangle\\ &\text{subject to } \sigma_{\mathcal{A}_t}(x)\le b_t,\ \forall(\mathcal{A}_t,b_t)\in\mathcal{U}_t,\ \forall t\in T, x\in X.\end{aligned}$$

Here $\sigma_{\mathcal{A}_t}$ denotes the support function of the set $\mathcal{A}_t\subset X^*$, i.e., $\sigma_{\mathcal{A}_t}(x):=\sup_{x^*\in\mathcal{A}_t}\langle x^*,x\rangle$.

We now introduce two robust dual problems for (RSAP_c):

$$(\mathrm{RSAD}_c^1)\qquad \begin{aligned}&\inf -\lambda v^2\\ &\text{subject to } V\in\mathfrak{V},\ v=(v^1,v^2)\in V,\lambda\ge 0,\ c=-\lambda v^1.\end{aligned}$$

$$(\text{RSAD}_c^2) \qquad \inf - \sum_{U \in \operatorname{supp}\lambda} \lambda_U v_U^2$$
$$\text{subject to } (v_U)_{U\in\mathfrak{U}} \in (U_t)_{U\in\mathfrak{U}},\ c = -\sum_{U \in \operatorname{supp}\lambda} \lambda_U v_U^1,\ t \in T,\ \lambda \in \mathbb{R}_+^{(\mathfrak{U})}.$$

We can state stable robust strong duality for the pairs (RSAP_c)–(RSAD_c^1) and (RSAP_c)–(RSAD_c^2) which are consequences of Theorem 1.

Corollary 7 (Stable robust strong duality for (RSAP_c)) *Consider the following statements:*

(g_1) $\mathcal{R}_1 := \operatorname{cone}\mathfrak{U} + \mathbb{R}_+(0_{X^*}, 1)$ *is a closed and convex subset of* $X^* \times \mathbb{R}$.
(g_2) $\mathcal{R}_2 := \bigcup_{t\in T} \operatorname{co}\operatorname{cone}[\mathrm{U}_t \cup \{(0_{X^*}, 1)\}]$ *is a closed and convex subset of* $X^* \times \mathbb{R}$.
(h_1) *The stable robust strong duality holds for the pair* (RSAP_c)–(RSAD_c^1).
(h_2) *The stable robust strong duality holds for the pair* (RSAP_c)–(RSAD_c^2).

Then, it holds $[(g_1) \Leftrightarrow (h_1)]$. *If for any* $V = (V^1, V^2) \in \mathfrak{V}$, V^1 *is a* w^**-compact subset of* X^* *then* $[(g_2) \Leftrightarrow (h_2)]$.

Proof We transform (RSAP_c) to the model (RP_c) in two different ways, for which the dual problems in each of such a way is (RSAD_c^1) or (RSAD_c^2), and then Theorem 1 applies.

• *Proof of* $[(g_1) \Leftrightarrow (h_1)]$*:* We transform (RSAP_c) to (RP_c) by the setting $Z = \mathbb{R}$, $S = \mathbb{R}^+, \mathcal{U} = \mathfrak{V}$ and $G_V(\cdot) = \sigma_{V^1}(\cdot) - V^2$ for all $V = (V^1, V^2) \in \mathfrak{V}$. Then $\mathcal{M}_0$ becomes

$$\mathcal{M}_0 = \bigcup_{(V,\lambda)\in\mathfrak{V}\times\mathbb{R}_+} \operatorname{epi}(\lambda G_V)^* = \operatorname{cone}\mathfrak{U} + \mathbb{R}_+(0_{X^*}, 1) = \mathcal{R}_1$$

and the dual problem (RD_c) is nothing else but (RSAD_c^1). The equivalence $[(g_1) \Leftrightarrow (h_1)]$ follows from Theorem 1.

• *Proof of* $[(g_2) \Leftrightarrow (h_2)]$*:* Take $Z = \mathbb{R}^{\mathfrak{U}}$, $S = \mathbb{R}_+^{\mathfrak{U}}, \mathcal{U} = T$, $G_t = (\sigma_{U_t^1}(\cdot) - U_t^2)_{U\in\mathfrak{U}}$ for all $t \in T$. Then the problem (RSAP_c) turns to the model (RP_c) and in this setting, the moment cone $\mathcal{M}_0$ becomes:

$$\mathcal{M}_0 = \bigcup_{(t,\lambda)\in T\times\mathbb{R}_+^{(\mathfrak{U})}} \operatorname{epi}(\lambda G_t)^* = \bigcup_{(t,\lambda)\in T\times\mathbb{R}_+^{(\mathfrak{U})}} \operatorname{epi}\left(\sum_{U\in\operatorname{supp}\lambda} \lambda_U \sigma_{U_t^1}(\cdot) - U_t^2\right)^* \tag{21}$$

while the dual problem (RD_c) now becomes exactly (RSAD_c^2).

Assume that V^1 is a w^*-compact subset of X^* for all $V = (V^1, V^2) \in \mathfrak{V}$. Then, σ_{V^1} is continuous on X for all $V = (V^1, V^2) \in \mathfrak{V}$. This, together with (21), yields

$$\mathcal{M}_0 = \bigcup_{(t,\lambda)\in T\times\mathbb{R}_+^{(\mathfrak{U})}} \sum_{U\in\operatorname{supp}\lambda} \lambda_U \operatorname{epi}\left(\sigma_{U_t^1}(\cdot) - U_t^2\right)^* = \bigcup_{t\in T} \operatorname{co}\operatorname{cone}\left[\mathcal{U}_t \cup \{(0_{X^*}, 1)\}\right] = \mathcal{R}_2,$$

and the equivalence $[(g_2) \Leftrightarrow (h_2)]$ follows from Theorem 1. □

Using the same argument as the one in Section 5 to get some versions of (stable) robust Farkas lemma for systems involved sub-affine functions with uncertain parameters. For instance, from the equivalence $[(g_1) \Leftrightarrow (h_1)]$ in Corollary 7 we get

Corollary 8 *The following statements are equivalent:*

(i) *For all* $(c, s) \in X^* \times \mathbb{R}$, *next assertions are equivalent:*

(α'') $\sigma_{\mathcal{A}_t}(x) \leq b_t, \ \forall(\mathcal{A}_t, b_t) \in \mathcal{U}_t, \ \forall t \in T \implies \langle c, x\rangle \geq s.$

(β'') $\exists \bar{V} \in \mathfrak{V}, \ \exists(\bar{x}^*, \bar{r}) \in \bar{V}, \ \exists \bar{\lambda} \geq 0 : \begin{cases} \bar{\lambda}\bar{x}^* = -c, \\ \bar{\lambda}\bar{r} \leq -s. \end{cases}$

(ii) $\operatorname{cone} \mathfrak{U} + \mathbb{R}_+(0_{X^*}, 1)$ *is a closed and convex subset of* $X^* \times \mathbb{R}$.

To conclude this section, we consider an application of the results for (RLIP$_c$)) (the same, for (RSAP$_c$)) to the linear infinite programming problems. It turns out that even in this simple case, we are able to derive new results on duality for this class of problems and new Farkas-type results for systems associated to the problems in the absence of uncertainty.

Example 2 (Linear infinite programming problems) Consider a *linear infinite programming problem* of the model

$$(\mathrm{LIP}_c) \qquad \inf\langle c, x\rangle \quad \text{s.t. } x \in X, \ \langle a_t, x\rangle \leq b_t, \ \forall t \in T,$$

where T is an arbitrary (possible infinite) index set, $c \in X^*$, $a_t \in X^*$, and $b_t \in \mathbb{R}$ for all $t \in T$. It is clear that the problem is a special case of (RLIP$_c$) and also (RSAP$_c$). In the case where $X = \mathbb{R}^n$ this problem is often known as linear semi-infinite problem (see [19] and also, [8, 9] for applications of this model in finance).

We consider (LIP$_c$) in a new look: a special case of (RLIP$_c$) where all uncertainty sets $\mathcal{U}_t$, $t \in T$, are singletons for all $t \in T$, say, $\mathcal{U}_t = \{(a_t, b_t)\}$, and then $\mathscr{U} = \prod_{t\in T} \mathcal{U}_t$ is also a singleton, say $\mathscr{U} = \{((a_t, b_t))_{t\in T}\}$, while $\mathscr{V} = \{(a_t, b_t) : t \in T\}$. Then

• All the three "robust" dual problems (RLID$_c^1$), (RLID$_c^2$), (RLID$_c^4$) of the problem (LIP$_c$) (considered as (RLIP$_c$)) collapse to

$$(\mathrm{LID}_c^1) \qquad \sup[-\lambda b_t] \quad \text{subject to } t \in T, \ \lambda \geq 0, \ c = -\lambda a_t,$$

and in this situation, the three moments cones $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_4$ reduce to

$$\mathcal{E}_1 := \bigcup_{t\in T} \operatorname{co}\operatorname{cone}\{(a_t, b_t), (0_{X^*}, 1)\}.$$

• The dual problems (RLID$_c^3$), (RLID$_c^6$), (RLID$_c^8$) of the new-formulated problem (RLIP$_c$) collapse to

$$(\mathrm{LID}_c^2) \qquad \sup\left[-\sum_{t\in \operatorname{supp}\lambda} \lambda_t b_t\right] \quad \text{subject to } \lambda \in \mathbb{R}_+^{(T)}, \ c = -\sum_{t\in \operatorname{supp}\lambda} \lambda_t a_t,$$

and moment cones $\mathcal{N}_3$, $\mathcal{N}_6$, and $\mathcal{N}_8$ reduce to:

$$\mathcal{E}_2 := \operatorname{co}\operatorname{cone}\{(a_t, b_t), t \in T; (0_{X^*}, 1)\}.$$

The dual problem (LID$_c^2$) introduced in [19] when $X = \mathbb{R}^n$, an in such a setting, when the uncertainty sets are all singletons, the dual problems (ODP) and (RDSP) in [20] also collapse to (LID$_c^2$).

• The dual problems (RLID_c^5), (RLID_c^7), (RLID_c^9) of the resulting problem (RLIP_c) reduce to:

$$(\mathrm{LID}_c^3) \qquad \sup_{\lambda\geq 0}\inf_{x\in X}\sup_{t\in T}\left[\langle c,x\rangle+\langle\lambda a_t,x\rangle-\lambda b_t\right],$$

while "robust" moment cones $\mathcal{N}_5$, $\mathcal{N}_7$, and $\mathcal{N}_9$ reduce to:

$$\mathcal{E}_3 := \operatorname{cone}\overline{\operatorname{co}}\{(a_t,b_t),\ t\in T;\ (0_{X^*},1)\}.$$

Moreover, for all $c\in X^*$, one has (see Remark 7),

$$\sup(\mathrm{LID}_c^1)\leq \begin{matrix}\sup(\mathrm{LID}_c^2)\\ \sup(\mathrm{LID}_c^3)\end{matrix}\leq \inf(\mathrm{LIP}_c).$$

Now, from Theorems 2 and 3, we get principles for stable robust strong duality for (LIP_c) which state as follows:

(i) *The next two statements are equivalent:*

(e$_1$) *$\mathcal{E}_1$ is a closed and convex subset of* $X^*\times\mathbb{R}$.
(f$_1$) *The stable robust strong duality holds for the pair* (LIP_c)–(LID_c^1).

(ii) *For each* $i=2,3$, *the following statements are equivalent:*

(e$_i$) *$\mathcal{E}_i$ is a closed subset of* $X^*\times\mathbb{R}$.
(f$_i$) *The stable robust strong duality holds for the pair* (LIP_c)–(LID_c^i).

Similar to what is done in the Section 5, the duality results of the primal-dual pairs of problems (LIP_c)–(LID_c^j), $j=1,2,3$ will give rise to some new variants of generalized Farkas lemmas for linear infinite systems. Realize this process for $j=2$ we will get a version of Farkas lemma which covers [19, Corollary 3.1.2] (where $X=\mathbb{R}^n$) while with $j=1$ and $j=3$, the resulting versions of Farkas lemmas for linear infinite systems obtained, due to the best knowledge of the authors, are new, which state as follows:

Farkas lemma for linear infinite systems: Consider the statements:

(i) For all $(c,s)\in X^*\times\mathbb{R}$, next assertions are equivalent:

(α') $\langle a_t,x\rangle\leq b_t,\ \forall t\in T \implies \langle c,x\rangle\geq s.$
(β') $\exists\bar{t}\in T,\ \exists\bar{\lambda}\geq 0:\bar{\lambda}a_{\bar{t}}=-c$ and $\bar{\lambda}b_{\bar{t}}\leq -s.$

(ii) For all $(c,s)\in X^*\times\mathbb{R}$, next assertions are equivalent:

(α') $\langle a_t,x\rangle\leq b_t,\ \forall t\in T \implies \langle c,x\rangle\geq S.$
(δ') $\exists\bar{\lambda}\geq 0:[\forall x\in X,\ \forall\varepsilon>0,\ \exists t_0\in T:\langle c+\bar{\lambda}a_{t_0},x\rangle-\bar{\lambda}b_{t_0}+\varepsilon\geq s].$

(iii) $\bigcup_{t\in T}\operatorname{co}\operatorname{cone}\{(a_t,b_t),(0_{X^*},1)\}$ is a closed and convex subset of $X^*\times\mathbb{R}$.
(iv) $\operatorname{cone}\overline{\operatorname{co}}\{(a_t,b_t),t\in T;(0_{X^*},1)\}$ is a closed subset of $X^*\times\mathbb{R}$.

Then [(i) $\Leftrightarrow$ (iii)] and [(ii) $\Leftrightarrow$ (iv)].

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Appendix A: Proof of Proposition 6

(i) From the proof of Theorem 2 for the case $i = 1$, we can see that the problem (RLIP$_c$) can be transformed to (RP$_c$) with $Z = \mathbb{R}$, $S = \mathbb{R}_+, \mathcal{U} = \mathscr{V}$, $G_v(\cdot) = v^1(\cdot) - v^2$ for all $v \in \mathscr{V}$, and in such a case, $\mathcal{M}_0 = \mathcal{N}_1$. Observe that the functions $v \mapsto \langle v^1, x\rangle - v_2$, $x \in X$, are concave (actually, they are affine). Together with the fact that $\mathscr{V}$ is convex and $Z = \mathbb{R}$, the collection $(v \mapsto G_v(x))_{x\in X}$ is uniformly $\mathbb{R}_+$-concave. So, in the light of Proposition 2(i), $\mathcal{M}_0$ is convex, and so is $\mathcal{N}_1$.

(ii) By the same argument as above, to prove $\mathcal{N}_3$ is convex, it is sufficient to show that the collection $(u \mapsto G_u(x))_{x\in X}$ is uniformly $\mathbb{R}_+^{(T)}$-concave with $\mathcal{U} = \mathscr{U}$, $Z = \mathbb{R}^T$ and $G_u(\cdot) = (\langle u_t^1, \cdot\rangle - u_t^2)_{t\in T}$ for all $u \in \mathscr{U}$ (the setting in the proof of Theorem 2 for the case $i = 3$). Now, take arbitrarily $\lambda, \mu \in \mathbb{R}_+^{(T)}$ and $u, w \in \mathscr{U}$. Let $\bar\lambda \in \mathbb{R}_+^{(T)}$ and $\bar u \in \mathscr{U}$ such that $\bar\lambda_t = \lambda_t + \mu_t$, $\bar u_t^2 = \min\{u_t^2, w_t^2\}$ and

$$\bar u_t^1 = \begin{cases} \frac{1}{\bar\lambda_t}(\lambda_t u_t^1 + \mu_t w_t^1) & \text{if } \lambda_t + \mu_t \neq 0, \\ u_t^1 & \text{otherwise} \end{cases}$$

($\bar u \in \mathscr{U}$ as $\{x^* \in X^* : (x^*, r) \in \mathcal{U}_t\}$ is convex for all $t \in T$). Then, it is easy to check that

$$\lambda_t(\langle u_t^1, x\rangle - u_t^2) + \mu_t(\langle w_t^1, x\rangle - w_t^2) \le \bar\lambda_t(\langle \bar u_t^1, x\rangle - \bar u_t^2) \quad \forall t \in T,\ \forall x \in X,$$

and consequently,

$$\sum_{t\in T}\lambda_t^1(\langle u_t^1, x\rangle - u_t^2) + \sum_{t\in T}\lambda_t^2(\langle w_t^1, x\rangle - w_t^2) \le \sum_{t\in T}\bar\lambda_t(\langle \bar u_t^1, x\rangle - \bar u_t^1) \quad \forall x \in X,$$

which means $\lambda G_u(x) + \mu G_w(x) \le \bar\lambda G_{\bar u}(x)$ for all $x \in X$, yielding the uniform $\mathbb{R}_+^{(T)}$-concavity of the collection $(u \mapsto G_u(x))_{x\in X}$. The conclusion now follows from Proposition 2(i).

(iii) Recall that $\mathcal{N}_4$ is a specific form of $\mathcal{M}_0$ with $Z = \mathbb{R}$, $S = \mathbb{R}_+, \mathcal{U} = T$, and $G_t(\cdot) = \sup_{v\in\mathcal{U}_t}[\langle v^1, \cdot\rangle - v^2]$ for all $t \in T$ (the setting in the proof of Theorem 2 for the case $i = 4$). Now, for each $t \in T$ and $x \in X$, as $\mathcal{U}_t = \mathcal{U}_t^1 \times \mathcal{U}_t^2$ (with $\mathcal{U}_t^1 \subset X^*$ and $\mathcal{U}_t^2 \subset \mathbb{R}$), it holds

$$G_t(x) = \sup_{x^*\in\mathcal{U}_t^1}\langle x^*, x\rangle - \inf_{r\in\mathcal{U}_t^2} r = \sup_{x^*\in\mathcal{U}_t^1}\langle x^*, x\rangle - \inf\mathcal{U}_t^2.$$

So, for all $x \in X$, because T is convex, $t \mapsto \sup_{x^*\in\mathcal{U}_t^1}\langle x^*, x\rangle$ is affine, and $t \mapsto \inf\mathcal{U}_t^2$ is convex, the function $t \mapsto G_t(x)$ is concave. This accounts for the uniform $\mathbb{R}_+^{(T)}$-concavity of the collection $(t \mapsto G_t(x))_{x\in X}$. The conclusion again follows from Proposition 2(i).

(iv) Consider the ways of transforming (RLIP$_c$) to (RP$_c$) in the proofs of Theorem 3 for the case $i = 6, 7$. Note that, in these ways, the uncertain set $\mathcal{U}$ is always a singleton. So, the corresponding qualifying sets (i.e, $\mathcal{N}_6$ and $\mathcal{N}_7$) are always convex (see Remark 3).

Appendix B: Proof of Proposition 7

Recall that $\mathcal{N}_i$, $i = 1, 2, \ldots, 7$, are specific forms of $\mathcal{M}_0$ following the corresponding ways transforming of (RLIP$_c$) to (RP$_c$) considered in the proofs of Theorems 2 and 3. So, to prove that $\mathcal{N}_i$ is closed, we make use of Proposition 2(ii), which provides some sufficient condition for the closedness of the robust moment cone $\mathcal{M}_0$.

(i) For $i = 1$, let us consider the way of transforming (RLIP$_c$) to (RP$_c$) by setting $Z = \mathbb{R}$, $S = \mathbb{R}_+, \mathcal{U} = \mathscr{V}$, and $G_v(\cdot) = \langle v^1, \cdot \rangle - v^2$ for all $v \in \mathscr{V}$. For all $x \in X$, it is easy to see that the function $v \mapsto G_v(x) = \langle v^1, x \rangle - v^2$ is continuous, and hence, it is $\mathbb{R}^+$-usc (see Remark 1(iii)). Moreover, gph $\mathscr{U}$ is compact, $\mathbb{R}$ is normed space, and (16) ensures the fulfilling of condition (C_0) in Proposition 2. The closedness of $\mathcal{N}_1$ follows from Proposition 2(ii).

(ii) For $i = 4$, consider the way of transforming with the setting $Z = \mathbb{R}$, $S = \mathbb{R}_+, \mathcal{U} = T$, and $G_t(\cdot) = \sup_{v \in \mathcal{U}_t} [\langle v^1, \cdot \rangle - v^2]$ for all $t \in T$. One has that $\mathcal{U} = T$ is a compact set, that $t \mapsto G_t(x) = \sup_{v \in \mathcal{U}_t} [\langle v^1, x \rangle - v^2]$ is usc and hence, it is $\mathbb{R}^+$-usc, and that Slater-type condition (C_0) holds (as (17) holds). The conclusion now follows from Proposition 2(ii).

(iii) Consider the way of transforming which corresponds to $i = 5$, i.e., we consider $Z = \mathbb{R}$, $S = \mathbb{R}_+, \mathcal{U} = \mathscr{U}$, and $G_u(\cdot) = \sup_{t \in T} [\langle u_t^1, \cdot \rangle - u_t^2]$ for all $u \in \mathscr{U}$. As $\mathscr{U} = \prod_{t \in T} \mathcal{U}_t$, the assumption that $\mathcal{U}_t$ is compact for all $t \in T$ which entails the compactness of $\mathscr{U}$. The other assumptions ensure the fulfillment of conditions in Proposition 2(ii) and the conclusion follows from this very proposition.

(iv) For $i = 7$, using the same argument as above in transforming (RLIP$_c$) to (RP$_c$) in the proof of Theorem 3. As by this way, the uncertainty set is a singleton, and hence, $\mathcal{N}_7$ is convex (see Remark 3). Now from Proposition 2(ii), Slater-type condition ensures the closedness of the robust moment cone $\mathcal{N}_7$, as desired.

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