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ORIGINAL ARTICLE

$\mathcal{Z}$-Armendariz Rings and Modules

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Abstract

In this paper we introduce and study right $\mathcal{Z}$-Armendariz rings. A ring R is said to be right $\mathcal{Z}$-Armendariz if $f(x)g(x) = 0$ implies that ab is a right singular element of R, where $f(x)$ and $g(x)$ belong to $R[x]$ and a, b are arbitrary coefficients of $f(x)$, $g(x)$. Then we construct some examples of right $\mathcal{Z}$-Armendariz rings by a given one. Finally, we extend this notion for modules.

Keywords Right singular ideal · Armendariz ring · Right $\mathcal{Z}$-Armendariz ring · $\mathcal{Z}$-Armendariz module

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1 Introduction

In this paper, all rings are associative with identity $1 \neq 0$ and all modules are unital. Let R be a ring. The set of nilpotent elements of R is denoted by $\mathrm{Nil}(R)$. A right ideal I of R is essential, if $I \cap I' \neq 0$ for any nonzero right ideal I' of R. An element $x \in R$ is called right singular, if $\mathrm{ann}_r(x) = \{a \in R \mid xa = 0\}$ is an essential right ideal of R. The set of all right singular elements of R is a two-sided ideal and is denoted by $\mathcal{Z}(R_R)$.

In [8], Rege and Chhawachharia introduced the notion of Armendariz rings. A ring R is Armendariz, if whenever $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ are in $R[x]$, the equation $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for every $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. In [3, Lemma 1] the authors proved that every reduced ring is Armendariz and in [6, Lemma 7]

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it is proved that every Armendariz ring is Abelian. Motivated by this definition, we call a ring R right $\mathcal{Z}$-Armendariz, if the above equation implies that $a_ib_j \in \mathcal{Z}(R_R)$. It turns out that this notion is not left-right symmetric. We prove that the property of being right $\mathcal{Z}$-Armendariz is closed under direct products and finite subdirect products but it is not a Morita invariant property. By an example we show that this property is not preserved under homomorphic images. Also we will prove that a ring R is right $\mathcal{Z}$-Armendariz if and only if the polynomial ring $R[x]$ is so. However, if R is right $\mathcal{Z}$-Armendariz, then $R[[x]]$, the ring of formal power series over R, is not necessarily right $\mathcal{Z}$-Armendariz.

A right R-module M_R is called Armendariz ([1, Proposition 12]), if $f(x)g(x) = 0$ implies that $m_ir_j = 0$ for any $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$, where $f(x) = \sum_{i=0}^{m} m_ix^i \in M[x]$ (the corresponding polynomial module over $R[x]$) and $g(x) = \sum_{j=0}^{n} r_jx^j \in R[x]$. Generalizing this notion, an R-module M_R is called $\mathcal{Z}$-Armendariz, if the above equation implies that $m_ir_j \in \mathcal{Z}(M_R)$ for every $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. We show that an R-module M is $\mathcal{Z}$-Armendariz if and only if every (finitely generated) submodule of it is $\mathcal{Z}$-Armendariz, and we prove that every right module over a right duo-ring is $\mathcal{Z}$-Armendariz. It is proved that the class of $\mathcal{Z}$-Armendariz modules is closed under direct sums but it is not closed under infinite direct products. Also it turns out that when R is a right $\mathcal{Z}$-Armendariz ring, flat R-modules and also semisimple R-modules are $\mathcal{Z}$-Armendariz.

2 $\mathcal{Z}$-Armendariz Rings

In this section, we focus on right $\mathcal{Z}$-Armendariz rings and prove some related results. Then we construct some examples of right $\mathcal{Z}$-Armendariz rings.

Definition 1 A ring R is called *right $\mathcal{Z}$-Armendariz*, if for every $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ in $R[x]$, the equation $f(x)g(x) = 0$ implies that $a_ib_j \in \mathcal{Z}(R_R)$ for every $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$.

We define *left $\mathcal{Z}$-Armendariz* rings similarly. If a ring R is both left and right $\mathcal{Z}$-Armendariz, then we say that R is a *$\mathcal{Z}$-Armendariz* ring.

Obviously every Armendariz ring is $\mathcal{Z}$-Armendariz. On the other hand, if R is a right $\mathcal{Z}$-Armendariz ring which is right nonsingular, then clearly it is Armendariz. In the following example we show that every commutative ring is $\mathcal{Z}$-Armendariz and in Example 4, we generalize this result.

Example 1 Every commutative ring R is $\mathcal{Z}$-Armendariz.

Let $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ and $f(x)g(x) = 0$, which implies that

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots + a_mb_nx^{n+m} = 0.$$

So $a_0b_0 = 0 \in \mathcal{Z}(R)$. Multiplying the equation $a_0b_1 + a_1b_0 = 0$ by a_1b_0, we have $(a_1b_0)^2 = 0$. Since all nilpotent elements of a commutative ring are singular, the nilpotent elements a_1b_0 and a_0b_1 belong to $\mathcal{Z}(R)$. Now multiplying the equation $a_0b_2+a_1b_1+a_2b_0 = 0$ by a_2b_0, we have $(a_2b_0)^2 = -a_2b_0a_1b_1 \in \mathrm{Nil}(R)$ so that $a_2b_0 \in \mathrm{Nil}(R) \subseteq \mathcal{Z}(R)$. By continuing this processes we obtain that $a_ib_j \in \mathrm{Nil}(R) \subseteq \mathcal{Z}(R)$, for every $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. So R is $\mathcal{Z}$-Armendariz.

The following example shows that a (commutative) $\mathcal{Z}$-Armendariz ring need not be Armendariz.

Example 2 Let $R = \mathbb{Z}_8(+)\mathbb{Z}_8$ with componentwise addition and multiplication $(a,b)(a',b') = (aa', ab' + ba')$. By [8, Example 3.2], R is not Armendariz and by Example 1, it is $\mathcal{Z}$-Armendariz.

Example 3 For any ring R and $n \geq 2$, $M_n(R)$, the ring of all $n \times n$ matrices and also the ring of all $n \times n$ upper (lower) triangular matrices over R are not right $\mathcal{Z}$-Armendariz.

Let $S = M_n(R)$ and $E_{ij} \in S$ be the matrix unit with 1 in the (i,j)th entry and 0 elsewhere. Let $f(x) = E_{12}+E_{11}x$ and $g(x) = E_{12}-E_{22}x \in S[x]$. We have $f(x)g(x) = 0$, but $E_{11}E_{12} = E_{12} \notin \mathcal{Z}(S_S)$, since $\mathrm{ann}_r(E_{12}) \cap E_{22}S = 0$. A similar proof can be used for the ring of $n \times n$ upper (lower) triangular matrices over R.

Proposition 1 *Let $\{R_i\}_{i\in I}$ be a family of rings and $R = \prod_{i\in I} R_i$. Then R is right $\mathcal{Z}$-Armendariz if and only if each R_i is so.*

Proof The proof follows from the fact that $\mathcal{Z}(R_R) = \prod_{i\in I} \mathcal{Z}(R_{i\,R_i})$. □

To show that the class of right $\mathcal{Z}$-Armendariz rings is closed under finite subdirect products, we need the following lemma.

Lemma 1 *Let $I_1, \dots, I_t$ be ideals of a ring R such that $\cap_{k=1}^{t} I_k = 0$. If $x + I_k$ is a right singular element of the ring $\frac{R}{I_k}$ for each $k = 1, \dots, t$, then $x \in \mathcal{Z}(R_R)$.*

Proof Let $0 \neq y \in R$. Since $\cap_{k=1}^{t} I_k = 0$, we can assume that $y \notin I_1$. So there exists $r_1 \in R$ such that $yr_1 \notin I_1$ and $xyr_1 \in I_1$. If $yr_1 \in I_k$ for $i = 2, \dots, t$, then $xyr_1 \in \cap_{k=1}^{t} I_k = 0$. If $yr_1 \notin I_2$, then there exists $r_2 \in R$ such that $yr_1r_2 \notin I_2$ and $xyr_1r_2 \in I_1 \cap I_2$. By continuing this process, we can find $r \in R$ with $yr \neq 0$ and $xyr = 0$. Thus, $x \in \mathcal{Z}(R_R)$. □

Theorem 1 *A finite subdirect product of right $\mathcal{Z}$-Armendariz rings is right $\mathcal{Z}$-Armendariz.*

Proof Suppose that $I_1, \cdots, I_t$ are ideals of a ring R such that $\cap_{k=1}^{t} I_k = 0$ and for each $k = 1, \dots, t$, the ring $\frac{R}{I_k}$ is right $\mathcal{Z}$-Armendariz. Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ and $f(x)g(x) = 0$. Then $a_i b_j + I_k$ is a right singular element of the ring $\frac{R}{I_k}$, for all $k = 1, \dots, t$. So by Lemma 1, $a_i b_j \in \mathcal{Z}(R_R)$ for $i = 0, 1, \dots, m$ and $j = 0, 1, \dots, n$. Therefore, R is right $\mathcal{Z}$-Armendariz. □

Suppose that $I_1, \dots, I_n$ are ideals of a ring R such that $\frac{R}{I_1}, \dots, \frac{R}{I_n}$ are right $\mathcal{Z}$-Armendariz rings. Then $\frac{R}{\cap_{k=1}^{n} I_k}$, as a subdirect product of $\frac{R}{I_1}, \dots, \frac{R}{I_n}$ is right $\mathcal{Z}$-Armendariz.

Remark 1 In general, a subdirect product of right $\mathcal{Z}$-Armendariz rings is not necessarily right $\mathcal{Z}$-Armendariz. For example, let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$. By Example 3, R is not right $\mathcal{Z}$-Armendariz. For any $n \geq 1$, suppose that $I_n = \begin{bmatrix} 0 & n\mathbb{Z} \\ 0 & 0 \end{bmatrix}$. Then $\cap_{n=1}^{\infty} I_n = 0$, which implies that R is a subdirect product of $\left\{\frac{R}{I_n}\right\}_{n=1}^{\infty}$. If $R_n := \frac{R}{I_n} = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_n \\ 0 & \mathbb{Z} \end{bmatrix}$, then $\mathcal{Z}(R_{n\,R_n}) =$

$\begin{bmatrix} 0 & \mathbb{Z}_n \\ 0 & 0 \end{bmatrix}$. So $\frac{R_n}{\mathcal{Z}(R_{n\,R_n})}$ is reduced and by [3, Lemma 1] it is Armendariz. As we shall see in Proposition 4, each $\frac{R}{I_n}$ is right $\mathcal{Z}$-Armendariz for any $n \geq 1$.

In the sequel, we use the following observation. Let R be a ring and $S = R[X]$, where X is a set of commuting indeterminates over R. Then $\mathcal{Z}(S_S) = \mathcal{Z}(R_R)[X]$, see [7, Exercise 7.35].

Proposition 2 *Let R be a ring. Then R is a right $\mathcal{Z}$-Armendariz ring if and only if $R[x]$ is so.*

Proof For the "only if part" let R be a right $\mathcal{Z}$-Armendariz ring and $f(t) = f_0 + f_1 t + \cdots + f_n t^n$, $g(t) = g_0 + g_1 t + \cdots + g_m t^m \in R[x][t]$ and $f(t)g(t) = 0$, where $f_i, g_j \in R[x]$ for each $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$. We show that $f_i g_j \in \mathcal{Z}(R[x]_{R[x]})$. Let $k = \deg f_0 + \cdots + \deg f_n + \deg g_0 + \cdots + \deg g_m$. Then $f(x^k) = f_0 + f_1 x^k + \cdots + f_n x^{kn}$, $g(x^k) = g_0 + g_1 x^k + \cdots + g_m x^{km} \in R[x]$ and $f(x^k)g(x^k) = 0$. So the product of each coefficient of f_i with every coefficient of g_j belongs to $\mathcal{Z}(R_R)$. Thus, $f_i g_j \in \mathcal{Z}(R[x]_{R[x]})$.

For the "if part" suppose that the polynomial ring $R[x]$ is right $\mathcal{Z}$-Armendariz and $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ such that $f(x)g(x) = 0$. Consider $F(t) = \sum_{i=0}^{m} f_i t^i$ and $G(t) = \sum_{j=0}^{n} g_j t^j \in R[x][t]$, where $f_i = a_i x^i$ and $g_j = b_j x^j$. We have $F(t)G(t) = 0$, so that $f_i g_j \in \mathcal{Z}(R[x]_{R[x]})$ which implies that $a_i b_j \in \mathcal{Z}(R_R)$, for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. Thus, R is right $\mathcal{Z}$-Armendariz. □

Corollary 1 *A ring R is right $\mathcal{Z}$-Armendariz if and only if the polynomial ring $S = R[\{x_\alpha\}_{\alpha \in A}]$ is right $\mathcal{Z}$-Armendariz.*

Proof Let R be a right $\mathcal{Z}$-Armendariz ring and $f, g \in R[\{x_\alpha\}_{\alpha \in A}][t]$ with $fg = 0$. Then $f, g \in T[t] = R[x_{\alpha_1}, \ldots, x_{\alpha_n}][t]$ for some finite subset $\{\alpha_1, \ldots, \alpha_n\} \subseteq A$. By Proposition 2, the ring $R[x_{\alpha_1}, \ldots, x_{\alpha_n}]$ is right $\mathcal{Z}$-Armendariz, so that $ab \in \mathcal{Z}(T_T) \subseteq \mathcal{Z}(S_S)$ for each coefficient a of f and b of g. Therefore, S is right $\mathcal{Z}$-Armendariz. The converse is trivial. □

Remark 2 If R is a right $\mathcal{Z}$-Armendariz ring, then $S = R[[x]]$, the formal power series ring over R, is not necessarily right $\mathcal{Z}$-Armendariz. For example, let K be a field and $R = \frac{K\langle a,b\rangle}{\langle b^2\rangle}$. In [2, Example 1], it is shown that R is an Armendariz ring but $R[[x]]$ is not. We show that S is not right $\mathcal{Z}$-Armendariz. Let $u = (1 - ax) \in S$. Clearly u is a unit in S with $u^{-1} = (1 + ax + a^2x^2 + a^3x^3 + \cdots) \in S$ and $f = ubu^{-1}$ is such that $f^2 = 0$. In the polynomial ring $S[y]$, $(b+bfy)(b-fby) = 0$ but $bfb \notin \mathcal{Z}(S_S)$, since $\mathrm{ann}_r(bfb) \cap aS = 0$. Hence, $S = R[[x]]$ is not right $\mathcal{Z}$-Armendariz. Also S is an example of an Abelian ring which is not right $\mathcal{Z}$-Armendariz.

Proposition 3 *Let R be a ring and G be a group. If the group ring RG or $R[[x]]$ is right $\mathcal{Z}$-Armendariz, then so is R.*

Proof Let S be one of the rings RG or $R[[x]]$. We can show that $\mathcal{Z}(S_S) \cap R \subseteq \mathcal{Z}(R_R)$. Now the rest of the proof follows easily. □

Proposition 4 *Let I be an ideal of a ring R such that the factor ring $\bar{R} = \frac{R}{I}$ is Armendariz. Then for $f_1, f_2, \dots, f_n \in R[x]$ the equation $f_1 f_2 \dots f_n \in I[x]$ implies that $a_1 a_2 \dots a_n \in I$, where a_i is an arbitrary coefficient of f_i for $i = 1, 2, \dots, n$. In particular, if $I \subseteq \mathcal{Z}(R_R)$, then R is right $\mathcal{Z}$-Armendariz.*

Proof Suppose that $f_1, f_2, \dots, f_n \in R[x]$ such that $f_1 f_2 \dots f_n \in I[x]$. Then in $\bar{R}[x]$, we have $\bar{f}_1 \bar{f}_2 \dots \bar{f}_n = 0$. By [1, Proposition 1], $a_1 a_2 \dots a_n \in I$ where a_i is an arbitrary coefficient of f_i for $i = 1, 2, \dots, n$. □

Corollary 2 *Let R be a ring. If $Nil(R)$ is an ideal of R contained in $\mathcal{Z}(R_R)$, then R is right $\mathcal{Z}$-Armendariz.*

Proof The factor ring $\frac{R}{\mathrm{Nil}(R)}$ is reduced and by [3, Lemma 1], it is Armendariz. So by Proposition 4, R is right $\mathcal{Z}$-Armendariz. □

Recall that a ring R is *right duo*, if all right ideals are two-sided, also a ring R is called *reversible*, if $ab = 0$ implies that $ba = 0$ for all $a, b \in R$.

Example 4 Right duo rings and reversible rings are examples of right $\mathcal{Z}$-Armendariz rings. By an easy calculation, we can show that $\frac{R}{\mathcal{Z}(R_R)}$ is reduced, whenever R is a right duo or a reversible ring. So by [3, Lemma 1], it is Armendariz. Now, applying Proposition 4, we get that R is right $\mathcal{Z}$-Armendariz.

The next example shows that for a ring R, being $\mathcal{Z}$-Armendariz is not left-right symmetric and also it is not preserved under homomorphic images.

Example 5 Let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_4 \end{bmatrix}$. Since $\mathcal{Z}(R_R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 2\mathbb{Z}_4 \end{bmatrix} = \mathrm{Nil}(R)$, Corollary 2 implies that R is right $\mathcal{Z}$-Armendariz. However, it is not left $\mathcal{Z}$-Armendariz, because for $f(x) = E_{12} + E_{11}x$ and $g(x) = E_{12} - E_{22}x \in R[x]$, where E_{ij}'s are those introduced in Example 3, we have $f(x)g(x) = 0$, but $E_{11}E_{12} = E_{12} \notin \mathcal{Z}({}_RR)$, since $\mathrm{ann}_l(E_{12}) \cap RE_{11} = 0$. Note that R is an example of a noncommutative right $\mathcal{Z}$-Armendariz ring which is not Armendariz. Moreover, let $I = \begin{bmatrix} 0 & 0 \\ 0 & 2\mathbb{Z}_4 \end{bmatrix}$. Then $\frac{R}{I}$ is isomorphic to $\begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ which is not right $\mathcal{Z}$-Armendariz by Example 3. Therefore, a homomorphic image of a right $\mathcal{Z}$-Armendariz ring need not be right $\mathcal{Z}$-Armendariz.

Every Armendariz ring is Abelian [6, Lemma 7]. But a $\mathcal{Z}$-Armendariz ring is not necessarily Abelian. For example, let $R = \begin{bmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{bmatrix}$. We have $\mathcal{Z}(R_R) = \mathcal{Z}({}_RR) = \begin{bmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & 2\mathbb{Z}_4 \end{bmatrix}$. So $\frac{R}{\mathcal{Z}(R_R)} = \frac{R}{\mathcal{Z}({}_RR)}$ is reduced and so it is Armendariz. Therefore, according to Proposition 4, R is $\mathcal{Z}$-Armendariz. However, R is not Abelian.

Now, we need the following lemma whose proof is the same as the proof of [5, Lemma 7].

Lemma 2 *If a, b, c are elements in a right $\mathcal{Z}$-Armendariz ring R such that $ab = 0$ and $ac^n b = 0$ for some $n \in \mathbb{N}$, then $acb \in \mathcal{Z}(R_R)$.*

Proof We have $f(x)g(x) = 0$, where $f(x) = a(1 - cx)$ and $g(x) = (1 + cx + \cdots + c^{n-1}x^{n-1})b$. Thus, $acb \in \mathcal{Z}(R_R)$. □

Proposition 5 *If R is a right $\mathcal{Z}$-Armendariz ring and idempotents lift modulo $\mathcal{Z}(R_R)$, then the ring $\bar{R} = \frac{R}{\mathcal{Z}(R_R)}$ is Abelian.*

Proof Let $\bar{e} \in \mathrm{Id}(\bar{R})$. By the hypothesis, we can assume that $e \in \mathrm{Id}(R)$. Thus, it is sufficient to show that for any $r \in R$, $er - re \in \mathcal{Z}(R_R)$. Let $a = e$, $b = (1-e)$ and $c = er(1-e)$. Clearly $ab = 0$ and $c^2 = 0$. By Lemma 2, $er - ere = acb \in \mathcal{Z}(R_R)$. Similarly, we have $re - ere \in \mathcal{Z}(R_R)$. So $er - re \in \mathcal{Z}(R_R)$. □

Proposition 6 *Every right $\mathcal{Z}$-Armendariz ring is Dedekind-finite.*

Proof Suppose that R is a right $\mathcal{Z}$-Armendariz ring and $uv = 1$ for some $u, v \in R$. The element $c = v(1 - vu)$ is nilpotent of nilpotency index two. If we put $a = vu$ and $b = 1 - vu$, then by Lemma 2, $v(1 - vu) = acb \in \mathcal{Z}(R_R)$. Thus, $uv(1 - vu) = (1 - vu) \in \mathrm{Id}(R) \cap \mathcal{Z}(R_R) = 0$. □

Remark 3 Let R be a ring and Γ be an infinite set. Then the ring of column (respectively, row) finite $\Gamma \times \Gamma$ matrices over R is not Dedekind-finite and so is neither left nor right $\mathcal{Z}$-Armendariz.

Note that the converse of Proposition 6 is not true in general. For example, the matrix ring $M_n(F)$, where F is any field and $n \geq 2$ is Dedekind-finite but is neither left nor right $\mathcal{Z}$-Armendariz (Example 3). Therefore, the class of (right) $\mathcal{Z}$-Armendariz rings lies strictly between the classes of Armendariz and Dedekind-finite rings.

Recall that a ring R is subdirectly irreducible, if every representation of R as a subdirect product of other rings is trivial, equivalently the intersection of all nonzero ideals of R is nonzero.

Example 6 A subdirectly irreducible ring is not necessarily right $\mathcal{Z}$-Armendariz. Let R be the ring of $\mathbb{N} \times \mathbb{N}$ column finite matrices over a field F. Then R has exactly one nonzero proper ideal and so it is subdirectly irreducible. However, R is not right $\mathcal{Z}$-Armendariz.

For the rest of this section we construct some right $\mathcal{Z}$-Armendariz rings by a given one.

Proposition 7 *Let R be a ring and $e \in \mathrm{Id}(R)$ such that $eR(1-e) = 0$. If R is a right $\mathcal{Z}$-Armendariz ring, then so is $S = eRe$.*

Proof First, we show that $\mathcal{Z}(R_R) \cap S \subseteq \mathcal{Z}(S_S)$. Let $a \in \mathcal{Z}(R_R) \cap S$ and $0 \neq s \in S$. There exists $r \in R$ such that $sr \neq 0$ and $asr = 0$. Thus, $as(ere) = 0$ and $s(ere) \neq 0$, since $eR(1-e) = 0$. This implies that $a \in \mathcal{Z}(S_S)$. Now, suppose that $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in S[x]$ such that $f(x)g(x) = 0$. So $a_i b_j \in \mathcal{Z}(R_R) \cap S \subseteq \mathcal{Z}(S_S)$ for every $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. □

Proposition 8 *Let R be a ring and M be an ideal of R containing an element r such that $ann_l(r) = 0$. Then the ring $S = \left\{ \begin{bmatrix} a & m \\ 0 & a \end{bmatrix} \mid a \in R \text{ and } m \in M \right\}$ is right $\mathcal{Z}$-Armendariz if and only if R is a right $\mathcal{Z}$-Armendariz ring.*

Proof By some calculations we can show that

$$\mathcal{Z}(S_S) = \left\{ \begin{bmatrix} a & m \\ 0 & a \end{bmatrix} \in S \mid a \in \mathcal{Z}(R_R) \right\}.$$

Suppose that R is a right $\mathcal{Z}$-Armendariz ring and $F(x)G(x) = 0$, where $F(x) = \sum_{i=0}^{m} \begin{bmatrix} a_i & m_i \\ 0 & a_i \end{bmatrix} x^i$ and $G(x) = \sum_{j=0}^{n} \begin{bmatrix} b_j & m'_j \\ 0 & b_j \end{bmatrix} x^j$. So we have $f(x)g(x) = 0$, where $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$. Since R is right $\mathcal{Z}$-Armendariz, $a_i b_j \in \mathcal{Z}(R_R)$, which implies that $\begin{bmatrix} a_i & m_i \\ 0 & a_i \end{bmatrix} \begin{bmatrix} b_j & m'_j \\ 0 & b_j \end{bmatrix} \in \mathcal{Z}(S_S)$ for every $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. Therefore, S is a right $\mathcal{Z}$-Armendariz ring. Now, suppose that S is a right $\mathcal{Z}$-Armendariz ring and $f(x)g(x) = 0$, where $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$. So $F(x)G(x) = 0$ where $F(x) = \sum_{i=0}^{m} \begin{bmatrix} a_i & 0 \\ 0 & a_i \end{bmatrix} x^i$ and $G(x) = \sum_{j=0}^{n} \begin{bmatrix} b_j & 0 \\ 0 & b_j \end{bmatrix} x^j \in S[x]$. So $\begin{bmatrix} a_i & 0 \\ 0 & a_i \end{bmatrix} \begin{bmatrix} b_j & 0 \\ 0 & b_j \end{bmatrix} \in \mathcal{Z}(S_S)$ for every $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$, which implies that $a_i b_j \in \mathcal{Z}(R_R)$. Thus, R is right $\mathcal{Z}$-Armendariz. □

Corollary 3 *Let R be a ring. Then R is right $\mathcal{Z}$-Armendariz if and only if the ring $\frac{R[x]}{\langle x^2 \rangle}$ is so.*

Proof The ring $\frac{R[x]}{\langle x^2 \rangle}$ is isomorphic to the ring $S = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in R \right\}$. Now, apply Proposition 8. □

Proposition 9 *For a ring R, the following are equivalent*

(1) *R is right $\mathcal{Z}$-Armendariz;*
(2) *$\frac{R[x]}{\langle x \rangle^n}$ is right $\mathcal{Z}$-Armendariz for every $n \in \mathbb{N}$;*
(3) *$\frac{R[x]}{\langle x \rangle^n}$ is right $\mathcal{Z}$-Armendariz for some $n \in \mathbb{N}$.*

Proof The proof follows from the fact that the ring $\frac{R[x]}{\langle x \rangle^n}$ is isomorphic to the ring

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ 0 & a_1 & \ldots & a_{n-1} \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & a_1 \end{bmatrix} \mid a_i \in R, i = 1, \ldots, n \right\}$$

and $\mathcal{Z}(S_S) = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_n \\ 0 & a_1 & \ldots & a_{n-1} \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & a_1 \end{bmatrix} \in S \mid a_1 \in \mathcal{Z}(R_R) \right\}$. □

Proposition 10 *Let R be a ring and M be an ideal of R such that $M \subseteq \mathcal{Z}(R_R)$. Then R is right $\mathcal{Z}$-Armendariz if and only if the ring $S = \begin{bmatrix} R & M \\ 0 & R \end{bmatrix}$ is so.*

Proof It is not difficult to show that $\mathcal{Z}(S_S) = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in S \mid a, b \in \mathcal{Z}(R_R) \right\}$. The rest of the proof is similar to the proof of Proposition 8. □

Proposition 11 *Let R and S be rings, ${}_RM_S$ be an (R, S)-bimodule and $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$. If R is Armendariz, S is right $\mathcal{Z}$-Armendariz and $\mathcal{Z}(M_S) = M$, then T is right $\mathcal{Z}$-Armendariz.*

Proof First note that $\begin{bmatrix} 0 & M \\ 0 & \mathcal{Z}(S_S) \end{bmatrix} \subseteq \mathcal{Z}(T_T)$. Now suppose that $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in T[x]$ and $f(x)g(x) = 0$, where $a_i = \begin{bmatrix} r_i & m_i \\ 0 & s_i \end{bmatrix}$ and $b_j = \begin{bmatrix} r'_j & m'_j \\ 0 & s'_j \end{bmatrix}$. Thus, $(\sum_{i=0}^{m} r_i x^i)(\sum_{j=0}^{n} r'_j x^j) = 0$ in $R[x]$ and $(\sum_{i=0}^{m} s_i x^i)(\sum_{j=0}^{n} s'_j x^j) = 0$ in $S[x]$. Since R is Armendariz and S is right $\mathcal{Z}$-Armendariz, for any $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$ we have $r_i r'_j = 0$ and $s_i s'_j \in \mathcal{Z}(S_S)$ and hence $a_i b_j \in \mathcal{Z}(T_T)$. □

Remark 4 Let R and S be rings, ${}_RM_S$ be an (R, S)-bimodule and $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$. If M_S is not a singular S-module, then T is not a right $\mathcal{Z}$-Armendariz ring. For if $m \in M - \mathcal{Z}(M_S)$, then

$$\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x\right)\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x\right) = 0.$$

But $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \notin \mathcal{Z}(T_T)$.

Proposition 12 *Let R be a ring and S be a multiplicatively closed set of central regular elements of R. Then R is right $\mathcal{Z}$-Armendariz if and only if the ring $T = RS^{-1}$ is so.*

Proof It is easy to see that $\frac{a}{s} \in \mathcal{Z}(T_T)$ if and only if $a \in \mathcal{Z}(R_R)$. Now the rest of the proof is straightforward. □

Corollary 4 *A ring R is right $\mathcal{Z}$-Armendariz if and only if the ring*

$$R\left[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\right]$$

is right $\mathcal{Z}$-Armendariz.

Proof Consider the multiplicatively closed set

$$S = \left\{x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \mid i_1, i_2, \ldots, i_n \geq 0\right\}$$

in $R[x_1, x_2, \ldots, x_n]$. Now, apply Proposition 12 and Corollary 1. □

Corollary 5 *Let R be a ring. Then $R[[x]]$ is a right $\mathcal{Z}$-Armendariz ring if and only if $R((x))$, the Laurent series ring over R, is so.*

Proof Use Proposition 12 when $S = \{1, x, x^2, \ldots\} \subseteq R[[x]]$. □

Proposition 13 *Let R be a ring and consider the ring*

$$S = \{(a, b) \in R \times R \mid a - b \in \mathcal{Z}(R_R)\}$$

with component-wise addition and multiplication. Then R is right $\mathcal{Z}$-Armendariz if and only if S is so.

Proof Let R be right $\mathcal{Z}$-Armendariz. If

$$F(x)=\sum_{i=0}^{m}(a_i,b_i)x^i,\quad G(x)=\sum_{j=0}^{n}(a'_j,b'_j)x^j\in S[x]$$

and $F(x)G(x)=0$, then $f(x)g(x)=0$, where $f(x)=\sum_{i=0}^{m}a_ix^i$ and $g(x)=\sum_{j=0}^{n}a'_jx^j\in R[x]$. So $a_ia'_j\in\mathcal{Z}(R_R)$ for $i=0,1,\dots,m$ and $j=0,1,\dots,n$. Similarly, we can show that $b_ib'_j\in\mathcal{Z}(R_R)$. Thus, $(a_i,b_i)(a'_j,b'_j)\in\mathcal{Z}(R_R)\times\mathcal{Z}(R_R)\subseteq\mathcal{Z}(S_S)$. So S is right $\mathcal{Z}$-Armendariz.

Now, suppose that S is right $\mathcal{Z}$-Armendariz. If $f(x)=\sum_{i=0}^{m}a_ix^i$, $g(x)=\sum_{j=0}^{n}b_jx^j\in R[x]$ and $f(x)g(x)=0$, then $(\sum_{i=0}^{m}(a_i,a_i)x^i)(\sum_{j=0}^{n}(b_j,b_j)x^j)=0$ in $S[x]$. So for any i,j we have $(a_i,a_i)(b_j,b_j)\in\mathcal{Z}(S_S)$. Thus, $a_ib_j\in\mathcal{Z}(R_R)$. Therefore, R is right $\mathcal{Z}$-Armendariz. □

3 $\mathcal{Z}$-Armendariz Modules

Recall that a right R-module M is Armendariz if $f(x)g(x)=0$ implies that $mr=0$, where $f(x)\in M[x]$, $g(x)\in R[x]$, m is an arbitrary coefficient of $f(x)$ and r is an arbitrary coefficient of $g(x)$ [1]. In [4], it is shown that the class of Armendariz modules is closed under direct products and submodules, and also every flat module over an Armendariz ring is Armendariz. In general a homomorphic image of an Armendariz module need not be Armendariz [4, Example 2.12]. However, as we shall see below, for an Armendariz module M_R, the factor module $\frac{M}{\mathcal{Z}(M_R)}$ is Armendariz too. But first we need a lemma.

Lemma 3 *Let M_R be a right R-module. Then $\mathcal{Z}(M[x]_{R[x]})=\mathcal{Z}(M_R)[x]$.*

Proof The proof is similar to [7, Exercise 7.35], for the right singular ideal of a polynomial ring. □

Proposition 14 *If M_R is an Armendariz R-module, then so is $\bar{M}=\frac{M}{\mathcal{Z}(M_R)}$.*

Proof Assume that $f(x)=\sum_{i=0}^{m}m_ix^i\in M[x]$ and $g(x)=\sum_{j=0}^{n}r_jx^j\in R[x]$ such that $f(x)g(x)\in\mathcal{Z}(M_R)[x]=\mathcal{Z}(M[x]_{R[x]})$. We will show that for every $i=0,1,\dots,m$ and $j=0,1,\dots,n$, $m_ir_j\in\mathcal{Z}(M_R)$. All coefficients of $f(x)g(x)$ are in $\mathcal{Z}(M_R)$, so that for every nonzero $c\in R$ there exists $r\in R$ such that $cr\neq 0$ and $f(x)g(x)cr=0$. Since M_R is Armendariz, $m_ir_jcr=0$ for every $i=0,1,\dots,m$ and $j=0,1,\dots,n$ and therefore, $m_ir_j\in\mathcal{Z}(M_R)$. □

A similar technique can be used to show that for any Armendariz ring R, the factor rings $\frac{R}{\mathcal{Z}(R_R)}$ and $\frac{R}{\mathcal{Z}({}_RR)}$ are Armendariz. Note that the converse of this statement is not true, for example, let R be a commutative ring. Then $\frac{R}{\mathcal{Z}(R)}$ is reduced and so is Armendariz. However, commutative rings are not necessarily Armendariz.

In the rest of this section, we study $\mathcal{Z}$-Armendariz modules as a generalization of Armendariz modules.

Definition 2 A right R-module M_R is called *$\mathcal{Z}$-Armendariz*, if the equation $f(x)g(x)=0$ implies that $m_ir_j\in\mathcal{Z}(M_R)$ for every $i=0,1,\dots,m$ and $j=0,1,\dots,n$, where $f(x)=\sum_{i=0}^{m}m_ix^i\in M[x]$ and $g(x)=\sum_{j=0}^{n}r_jx^j\in R[x]$.

Clearly every Armendariz module (for example, every vector space over a division ring) is $\mathcal{Z}$-Armendariz. Also every singular right R-module is $\mathcal{Z}$-Armendariz and if M_R is a non-singular $\mathcal{Z}$-Armendariz module, then M_R is Armendariz. A ring R is right $\mathcal{Z}$-Armendariz, if R_R is a $\mathcal{Z}$-Armendariz module.

Proposition 15 *The class of $\mathcal{Z}$-Armendariz modules over a ring R, is closed under submodules and arbitrary direct sums.*

Proof The proof follows from the fact that if $N_R \leq M_R$, then $\mathcal{Z}(N_R) = \mathcal{Z}(M_R) \cap N$ and for a family of right R-modules $\{M_i\}_{i\in I}$, $\mathcal{Z}(\oplus_{i\in I} M_i) = \oplus_{i\in I}\mathcal{Z}(M_i)$. □

Corollary 6 *A ring R is right $\mathcal{Z}$-Armendariz if and only if every submodule of a free right R-module is $\mathcal{Z}$-Armendariz.*

Corollary 7 *Every semisimple right module over a right $\mathcal{Z}$-Armendariz ring is $\mathcal{Z}$-Armendariz.*

Proof By Proposition 15, it is sufficient to prove the corollary for simple modules. Suppose that R is a right $\mathcal{Z}$-Armendariz ring and M_R is a simple module. By [7, Exercise 7.12A], every simple module over an arbitrary ring is either singular or projective. According to Corollary 6, M_R is a $\mathcal{Z}$-Armendariz module. □

In the next example we see that an infinite direct product of $\mathcal{Z}$-Armendariz modules is not necessarily $\mathcal{Z}$-Armendariz.

Example 7 Let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$. For every $n \geq 2$, $M_n = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_n \\ 0 & \mathbb{Z} \end{bmatrix}$ is a right R-module with $\mathcal{Z}(M_n) = \begin{bmatrix} 0 & \mathbb{Z}_n \\ 0 & 0 \end{bmatrix}$. Since $\mathbb{Z}$ is an Armendariz ring, one can show that M_n is a $\mathcal{Z}$-Armendariz R-module. Now consider $M = \prod_{n\geq 2} M_n$. Put $a = \left(\begin{bmatrix} 0 & \bar{1} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \bar{1} \\ 0 & 0 \end{bmatrix}, \ldots\right)$, $b = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ldots\right) \in M$ and $f(x) = a - bx \in M[x]$ and $g(x) = E_{12} + E_{22}x \in R[x]$, where E_{ij}'s are those introduced in Example 3. We have $f(x)g(x) = 0$. But $aE_{22} = a$ is not contained in $\mathcal{Z}(M_R)$, since $\text{ann}_r(a) = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$, which is not an essential right ideal.

Proposition 16 *A module M_R is $\mathcal{Z}$-Armendariz if and only if every finitely generated submodule of M is $\mathcal{Z}$-Armendariz.*

Proof The only if part follows from Proposition 15. For the if part, note that for any $f \in M[x]$ there exists a finitely generated submodule N of M such that $f \in N[x]$. □

Corollary 8 *Let R be a ring such that every finitely generated right R-module can be embedded in a free module (for example, let R be a quasi-Frobenious ring). Then the following are equivalent:*

(1) *R is a right $\mathcal{Z}$-Armendariz ring;*
(2) *Every right R-module is $\mathcal{Z}$-Armendariz.*
(3) *Every cyclic right R-module is $\mathcal{Z}$-Armendariz.*

Proof (1) ⇒ (2) Let M_R be an R-module and K_R be a finitely generated submodule of M_R. Since K_R can be embedded in a free R-module, by Proposition 15, it is $\mathcal{Z}$-Armendariz. Now Proposition 16 implies that M_R is $\mathcal{Z}$-Armendariz. The proofs of (2) ⇒ (3) and (3) ⇒ (1) are clear. □

Proposition 17 *Let I be a right ideal of a ring R such that I is not contained in $\mathcal{Z}(R_R)$ and $\mathcal{Z}(R_R)$ is a prime ideal of R. If I_R is a $\mathcal{Z}$-Armendariz module, then R is a right $\mathcal{Z}$-Armendariz ring.*

Proof Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ and $f(x)g(x) = 0$. For every $a \in I$ and $r \in R$, we have $af(x)g(x)r = 0$. Obviously, $af(x) \in I[x]$, so that $aa_ib_jr \in \mathcal{Z}(I_R) \subseteq \mathcal{Z}(R_R)$. Thus, $Ia_ib_jR \subseteq \mathcal{Z}(R_R)$ for any $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. As $\mathcal{Z}(R_R)$ is a prime ideal and $I \not\subseteq \mathcal{Z}(R_R)$, we have $a_ib_j \in \mathcal{Z}(R_R)$. Therefore, R is right $\mathcal{Z}$-Armendariz. □

Proposition 18 *Every flat right R-module over a right $\mathcal{Z}$-Armendariz ring R is $\mathcal{Z}$-Armendariz.*

Proof In view of the fact that for any modules F_R and M_R and any R-homomorphism $\varphi : F \to M$, $\varphi(\mathcal{Z}(F_R)) \subseteq \mathcal{Z}(M_R)$, the proof is similar to the proof of [4, Theorem 2.15]. □

The proof of the following lemma is similar to the proof of [1, Proposition 1].

Lemma 4 *Let M_R be an Armendariz module, $f \in M[x]$ and $g_1, g_2, \ldots, g_n \in R[x]$. If $fg_1g_2 \cdots g_n = 0$, then $mb_1b_2 \cdots b_n = 0$, where m is an arbitrary coefficient of f and b_i is an arbitrary coefficient of g_i for $i = 1, 2, \ldots, n$.*

Proposition 19 *Let M_R be a right R-module and $\frac{M}{K}$ be an Amendariz module for some submodule K of $\mathcal{Z}(M_R)$. For any $f \in M[x]$ and $g_1, g_2, \ldots, g_n \in R[x]$, if $fg_1g_2 \cdots g_n \in K[x]$, then $mb_1b_2 \cdots b_n \in K$, where m is any coefficient of f and b_i is any coefficient of g_i for $i = 1, 2, \ldots, n$. In particular, M_R is a $\mathcal{Z}$-Armendariz module.*

Proof Using Lemma 4, the proof is clear. □

Similar to the case for the Armendariz modules (Proposition 14), we have the following result.

Proposition 20 *Let R be a ring. If M_R is a $\mathcal{Z}$-Armendariz module, then so is the factor module $\bar{M} = \frac{M}{\mathcal{Z}(M_R)}$.*

Proof Suppose that $\bar{f}(x) = \sum_{i=0}^{m} \bar{a}_i x^i \in \bar{M}[x]$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ such that $\bar{f}(x)g(x) = \bar{0}$ in $\frac{M}{\mathcal{Z}(M_R)}[x]$. We show that $\bar{a}_ib_j \in \mathcal{Z}\left(\frac{M}{\mathcal{Z}(M_R)}\right)$ for any $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. We have $f(x)g(x) \in \mathcal{Z}(M_R)[x]$, where $f(x) = \sum_{i=0}^{m} a_i x^i$. Since every coefficient of $f(x)g(x)$ is a singular element of M, for every nonzero element $c \in R$, there exists $r \in R$ such that $cr \neq 0$ and $f(x)g(x)cr = 0$. Now $a_ib_jcr \in \mathcal{Z}(M_R)$ for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$, since M_R is $\mathcal{Z}$-Armendariz. Hence, $\bar{a}_ib_jcr = \bar{0}$ in $\frac{M}{\mathcal{Z}(M_R)}$. Thus, $\bar{a}_ib_j \in \mathcal{Z}\left(\frac{M}{\mathcal{Z}(M_R)}\right)$. □

Corollary 9 *Let R be a right nonsingular ring. Then for every $\mathcal{Z}$-Armendariz right R-module M, the factor module $\frac{M}{\mathcal{Z}(M_R)}$ is Armendariz.*

Proof By Proposition 20, $\frac{M}{\mathcal{Z}(M_R)}$ is $\mathcal{Z}$-Armendariz and by [7, Theorem 7.21], $\mathcal{Z}\left(\frac{M}{\mathcal{Z}(M_R)}\right) = 0$. Therefore, $\frac{M}{\mathcal{Z}(M_R)}$ is an Armendariz module. □

The proof of the next result is similar to the proof of Proposition 2.

Proposition 21 *Let M_R be an R-module. Then M_R is $\mathcal{Z}$-Armendariz if and only if $M[x]_{R[x]}$ is so.*

Note that if $\theta : R \to S$ is a ring homomorphism and M is an S-module, then M is an R-module via $mr = m\theta(r)$.

Proposition 22 *Let $\theta : R \to S$ be a ring epimorphism. If M_S is a $\mathcal{Z}$-Armendariz S-module, then M_R is $\mathcal{Z}$-Armendariz as an R-module.*

Proof Observe that $\mathcal{Z}(M_S) \subseteq \mathcal{Z}(M_R)$, now the rest of the proof is clear. □

In the next theorem, we show that over a right duo-ring, every right module is $\mathcal{Z}$-Armendariz. But first we state the following lemma.

Lemma 5 *Let R be a right duo-ring and M_R be a right R-module. If $mr^2 \in \mathcal{Z}(M_R)$ for some $m \in M$ and $r \in R$, then $mr \in \mathcal{Z}(M_R)$.*

Proof Suppose that $mr^2 \in \mathcal{Z}(M_R)$ and $mr \notin \mathcal{Z}(M_R)$. So there exists $a \in R - \{0\}$ such that $\text{ann}_r(mr) \cap aR = 0$. On the other hand, $mr^2ab = 0$ for some $b \in R$ such that $ab \neq 0$. Thus, $mr(rab) = 0$, which implies that $rab \in \text{ann}_r(mr) \cap aR = 0$. Hence, $ab \in \text{ann}_r(mr) \cap aR = 0$, which is a contradiction. □

Theorem 2 *For a right duo-ring R, every right R-module is $\mathcal{Z}$-Armendariz.*

Proof Let $f(x) = \sum_{i=0}^{m} m_i x^i \in M[x]$ and $g(x) = \sum_{j=0}^{n} r_j x^j \in R[x]$ such that $f(x)g(x) = 0$. We will show that $m_i r_j \in \mathcal{Z}(M_R)$ for every $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. We prove by induction on $i + j$. Clearly $m_0 r_0 = 0 \in \mathcal{Z}(M_R)$. Suppose that the statement is true when $i + j < k$. If $i + j = k$, we multiply the equation

$$m_0 r_k + m_1 r_{k-1} + \cdots + m_k r_0 = 0 \tag{1}$$

by r_0. Since R is a right duo-ring, for each $i = 0, 1, \ldots, (k-1)$, we have $m_i r_{k-i} r_0 = m_i r_0 r_i'$ for some $r_i' \in R$. By the induction hypotheses, $m_i r_0 \in \mathcal{Z}(M_R)$ for $i < k$. Thus, $m_k r_0^2 \in \mathcal{Z}(M_R)$. By Lemma 5, $m_k r_0 \in \mathcal{Z}(M_R)$. Now multiplying (1), by r_1, we deduce that $m_{k-1} r_1^2 \in \mathcal{Z}(M_R)$ and again by Lemma 5, $m_{k-1} r_1 \in \mathcal{Z}(M_R)$. By continuing this proses, we have $m_i r_{k-i} \in \mathcal{Z}(M_R)$ for every $i = 0, 1, \ldots, k$. □

Remark 5 (1) We show that the converse of Theorem 2 is not true. Recall that a ring is right distributive if its lattice of right ideals is distributive. By [10, Corollary 7], over a right distributive ring, any right module is Armendariz (and hence $\mathcal{Z}$-Armendariz). But there is a right distributive ring which is not right duo ([9, Example 7.1.6]).

(2) Recall that an R-module M_R is Dedekind-finite if $M \cong M \oplus N$ (for some R-module N_R) implies that $N = 0$. We show that a $\mathcal{Z}$-Armendariz module is not necessarily Dedekind-finite. For example, let R be a commutative ring and $M = R^{(\mathbb{N})}$. By Theorem 2, M is a $\mathcal{Z}$-Armendariz module but clearly it is not Dedekind-finite.

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