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ORIGINAL ARTICLE

A Strongly Convergent Modified Halpern Subgradient Extragradient Method for Solving the Split Variational Inequality Problem

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Abstract
We propose a method for solving the split variational inequality problem (SVIP) involving Lipschitz continuous and pseudomonotone mappings. The proposed method is inspired by the Halpern subgradient extragradient method for solving the monotone variational inequality problem with a simple step size. A strong convergence theorem for an algorithm for solving such a SVIP is proved without the knowledge of the Lipschitz constants of the mappings. As a consequence, we get a strongly convergent algorithm for finding the solution of the split feasibility problem (SFP), which requires only two projections at each iteration step. A simple numerical example is given to illustrate the proposed algorithm.

Keywords Split variational inequality problem · Split feasibility problem · Halpern subgradient extragradient method · Strong convergence · Pseudomonotone mapping

Mathematics Subject Classification (2010) 49M37 · 90C26 · 65K15

1 Introduction

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. Let C and Q be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Given

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mappings $F_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $F_2 : \mathcal{H}_2 \to \mathcal{H}_2$, the split variational inequality problem (in short, SVIP) introduced first by Censor et al. [8] is to find a solution x^* of the variational inequality problem in the space $\mathcal{H}_1$ so that the image $y^* = A(x^*)$, under a given bounded linear operator A, is a solution of another variational inequality problem in space $\mathcal{H}_2$.

More specifically, the SVIP is to find

$$x^* \in C : \langle F_1(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C$$

such that

$$y^* = A(x^*) \in Q : \langle F_2(y^*), y - y^* \rangle \geq 0 \quad \forall y \in Q.$$

When $F_1 = 0$ and $F_2 = 0$, the SVIP reduces to the split feasibility problem, shortly SFP,

$$\text{Find } x^* \in C \text{ such that } A(x^*) \in Q,$$

which was first introduced by Censor and Elfving [4] in finite-dimensional Hilbert spaces for modeling inverse problems. Recently, it has been found that the SFP can also be used to model the intensity-modulated radiation therapy [3, 5, 10, 23], and other real-world problems.

The SVIP was introduced and investigated by Censor et al. [8] in the case when F_1 is α_1-inverse strongly monotone on $\mathcal{H}_1$ and F_2 is α_2-inverse strongly monotone on $\mathcal{H}_2$. Their algorithm starts from a given point $x^0 \in \mathcal{H}_1$, for all $n \geq 0$, the next iterate is defined as

$$x^{n+1} = P_C^{F_1,\lambda} \left(x^n + \gamma A^* \left(P_Q^{F_2,\lambda} - I \right) (Ax^n) \right),$$

where $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, $0 \leq \lambda \leq 2\min\{\alpha_1, \alpha_2\}$ and $P_C^{F_1,\lambda}$ and $P_Q^{F_2,\lambda}$ stand for $P_C(I - \lambda F_1)$ and $P_Q(I - \lambda F_2)$, respectively. They showed that the sequence $\{x^n\}$ converges weakly to a solution of the split variational inequality problem, provided that the solution set of the SVIP is nonempty.

Since the solution set of the variational inequality problem VIP(C, F), for $F : \mathcal{H} \to \mathcal{H}$, coincides with the set of fixed points of the mapping T from $\mathcal{H}$ to $\mathcal{H}$ by taking $T(x) = P_C(x - \lambda F(x))$ for all $x \in \mathcal{H}$ ($\lambda > 0$ fixed), the SVIP is an instance of the split common fixed point problem, shortly SCFPP, which is introduced in 2009 by Censor and Segal [11]

$$\text{Find } x^* \in \text{Fix}(U) \text{ such that } y^* = A(x^*) \in \text{Fix}(T),$$

where $U : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_2 \to \mathcal{H}_2$ are given mappings. Many authors proposed several methods for solving the SCFPP, see [1, 2, 13, 21, 25] and the references therein.

It is well-known (see e.g. [14, p. 1110]) that the projection method for monotone variational inequality problems (VIPs) may fail to converge. To overcome this difficulty, the extragradient method, first proposed by Korpelevich [19] for saddle problems, can be applied to monotone VIPs ensuring convergence. However, the extragradient method may be costly, since it requires two projections at each step. Motivated by this fact, Censor et al. [6] introduced an algorithm, which is called the subgradient extragradient method, for solving the monotone variational inequality problem

$$\text{VIP}(C, F) \qquad \text{Find } x^* \in C \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C,$$

in which the second projection onto the constrained set C is replaced by the one onto a half-space T_n containing it. Their algorithm is of the form

$$\begin{cases} x^0 \in \mathcal{H}, \\ y^n = P_C(x^n - \lambda F(x^n)), \\ T_n = \{\omega \in \mathcal{H} : \langle x^n - \lambda F(x^n) - y^n, \omega - y^n \rangle \le 0\}, \\ x^{n+1} = P_{T_n}(x^n - \lambda F(y^n)). \end{cases} \tag{1}$$

It was proved that if $F : \mathcal{H} \to \mathcal{H}$ is monotone on C, L-Lipschitz continuous on $\mathcal{H}$ and the stepsize $\lambda \in \left(0, \frac{1}{L}\right)$, then the sequence $\{x^n\}$ generated by (1) converges weakly to a solution x^* of the VIP(C, F). Since the inception of the subgradient extragradient method, they also proposed another modification in Euclidean space (see [9]).

In order to obtain the strong convergence of the subgradient extragradient method, Censor et al. [7] introduced the following hybrid subgradient extragradient method

$$\begin{cases} x^0 \in \mathcal{H}, \\ y^n = P_C(x^n - \lambda F(x^n)), \\ T_n = \{\omega \in \mathcal{H} : \langle x^n - \lambda F(x^n) - y^n, \omega - y^n \rangle \le 0\}, \\ z^n = \alpha_n x^n + (1-\alpha_n) P_{T_n}(x^n - \lambda F(y^n)), \\ C_n = \{z \in \mathcal{H} : \|z^n - z\| \le \|x^n - z\|\}, \\ Q_n = \{z \in \mathcal{H} : \langle x^n - z, x^0 - x^n \rangle \ge 0\}, \\ x^{n+1} = P_{C_n \cap Q_n}(x^0), \end{cases} \tag{2}$$

and they proved, under appropriate conditions, that the sequence $\{x^n\}$ generated by (2) converges strongly to a point $u^* = P_{\mathrm{Sol}(C,F)}(x^0)$.

Inspired by the results in [7], Kraikaew and Saejung [20] introduced the following Halpern subgradient extragradient method for solving VIP(C, F)

$$\begin{cases} x^0 \in \mathcal{H}, \\ y^n = P_C(x^n - \lambda F(x^n)), \\ T_n = \{\omega \in \mathcal{H} : \langle x^n - \lambda F(x^n) - y^n, \omega - y^n \rangle \le 0\}, \\ z^n = P_{T_n}(x^n - \lambda F(y^n)), \\ x^{n+1} = \alpha_n x^0 + (1-\alpha_n) z^n, \end{cases} \tag{3}$$

where $\lambda \in \left(0, \frac{1}{L}\right)$, $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. They proved that the sequence $\{x^n\}$ generated by (3) converges strongly to $P_{\mathrm{Sol}(C,F)}(x^0)$.

In the present paper, inspired by the above mentioned works, we present the modified Halpern subgradient extragradient method for the SVIP when F_1 and F_2 are Lipschitz continuous pseudomonotone mappings but the Lipschitz constants are not required to be known. The strong convergence of the proposed method is established under some suitable conditions.

The paper is organized as follows. In Section 2, we present some preliminaries that will be needed in the sequel. Section 3 deals with the algorithm and its convergence analysis. Finally, in Section 4, we illustrate the proposed method by considering a simple numerical experiment.

2 Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. The strong convergence of $\{x^n\}$ to x is written as $x^n \to x$, while the weak convergence of $\{x^n\}$ to x is denoted by

$x^n \rightharpoonup x$. Recall that the metric projection from $\mathcal{H}$ onto C, denoted P_C, is defined in such a way that, for each $x \in \mathcal{H}$, $P_C(x)$ is the unique element in C with the property

$$\|x - P_C(x)\| = \min\{\|x - y\| : y \in C\}.$$

Some important properties of the projection operator P_C are gathered in the following lemma.

Lemma 1 ([15])

(i) *For given* $x \in \mathcal{H}$ *and* $y \in C$, $y = P_C(x)$ *if and only if*

$$\langle x - y, z - y\rangle \leq 0 \quad \forall z \in C.$$

(ii) P_C *is nonexpansive, that is,*

$$\|P_C(x) - P_C(y)\| \leq \|x - y\| \quad \forall x, y \in \mathcal{H}.$$

(iii) *For all* $x \in \mathcal{H}$ *and* $y \in C$, *we have*

$$\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|P_C(x) - x\|^2.$$

For more information on the projection operator P_C, see [16, Section 3] and [18].

Definition 1 Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. An operator $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ with the property

$$\langle A(x), y\rangle = \langle x, A^*(y)\rangle$$

for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, is called the adjoint operator of A.

The adjoint operator of a bounded linear operator A between Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ always exists and is uniquely determined. Furthermore, A^* is a bounded linear operator and $\|A^*\| = \|A\|$.

Definition 2 ([12, 17]) A mapping $F : \mathcal{H} \to \mathcal{H}$ is said to be

(i) L-Lipschitz continuous on $\mathcal{H}$ if

$$\|F(x) - F(y)\| \leq L\|x - y\| \quad \forall x, y \in \mathcal{H};$$

(ii) monotone on C if

$$\langle F(x) - F(y), x - y\rangle \geq 0 \quad \forall x, y \in C;$$

(iii) pseudomonotone on C if

$$\langle F(y), x - y\rangle \geq 0 \quad \Longrightarrow \quad \langle F(x), x - y\rangle \geq 0 \quad \forall x, y \in C.$$

The next lemmas will be used for proving the convergence of the algorithm proposed in the next section.

Lemma 2 *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $F : \mathcal{H} \to \mathcal{H}$ be pseudomonotone on C and L-Lipschitz continuous on $\mathcal{H}$ such that the solution set* $\mathrm{Sol}(C, F)$ *of the* $\mathrm{VIP}(C, F)$ *is nonempty. Let $x \in \mathcal{H}$, $\mu \in (0, 1)$, $\lambda > 0$ and define*

$$\begin{aligned} y &= P_C(x - \lambda F(x)), \\ z &= P_T(x - \lambda F(y)), \\ T &= \{\omega \in \mathcal{H} : \langle x - \lambda F(x) - y, \omega - y\rangle \le 0\}, \\ \gamma &= \begin{cases} \min\left\{\dfrac{\mu\|x - y\|}{\|F(x) - F(y)\|}, \lambda\right\} & \text{if } F(x) \ne F(y), \\ \lambda & \text{if } F(x) = F(y). \end{cases} \end{aligned}$$

Then for all $x^* \in \mathrm{Sol}(C, F)$

$$\|z - x^*\|^2 \le \|x - x^*\|^2 - \left(1 - \mu\frac{\lambda}{\gamma}\right)\|x - y\|^2 - \left(1 - \mu\frac{\lambda}{\gamma}\right)\|y - z\|^2.$$

Proof By the definition of y and Lemma 1, it follows that

$$\langle x - \lambda F(x) - y, z - y\rangle \le 0 \quad \forall z \in C.$$

Combining this inequality and the definition of T, we get $C \subset T$.

Since $x^* \in \mathrm{Sol}(C, F)$ and $y \in C$, we have, in particular, $\langle F(x^*), y - x^*\rangle \ge 0$. Using the pseudomonotonicity on C of F, we get

$$\langle F(y), y - x^*\rangle \ge 0. \tag{4}$$

From $z = P_T(x - \lambda F(y))$, we have $z \in T$. This together with the definition of T implies

$$\langle x - \lambda F(x) - y, z - y\rangle \le 0. \tag{5}$$

Since $x^* \in C$ and $C \subset T$, we get $x^* \in T$. Thus, using Lemma 1, (4) and (5), we obtain

$$\begin{aligned} \|z - x^*\|^2 &= \|P_T(x - \lambda F(y)) - x^*\|^2 \\ &\le \|x - \lambda F(y) - x^*\|^2 - \|x - \lambda F(y) - z\|^2 \\ &= \|x - x^*\|^2 - \|x - z\|^2 + 2\lambda\langle x^* - z, F(y)\rangle \\ &= \|x - x^*\|^2 - \|x - z\|^2 - 2\lambda\langle F(y), y - x^*\rangle + 2\lambda\langle y - z, F(y)\rangle \\ &\le \|x - x^*\|^2 - \|x - z\|^2 + 2\lambda\langle y - z, F(y)\rangle \\ &= \|x - x^*\|^2 + 2\lambda\langle y - z, F(y)\rangle - \|x - y\|^2 - \|y - z\|^2 - 2\langle y - z, x - y\rangle \\ &= \|x - x^*\|^2 - \|x - y\|^2 - \|y - z\|^2 + 2\langle y - z, \lambda F(y) - x + y\rangle \\ &= \|x - x^*\|^2 - \|x - y\|^2 - \|y - z\|^2 + 2\langle x - \lambda F(x) - y, z - y\rangle \\ &\quad + 2\lambda\langle F(x) - F(y), z - y\rangle \\ &\le \|x - x^*\|^2 - \|x - y\|^2 - \|y - z\|^2 + 2\lambda\langle F(x) - F(y), z - y\rangle. \end{aligned} \tag{6}$$

If $F(x) \ne F(y)$ then from the definition of γ, we have

$$\|F(x) - F(y)\| \le \frac{\mu}{\gamma}\|x - y\|. \tag{7}$$

Using the Cauchy–Schwarz inequality, (7) and the inequality of arithmetic and geometric means, we obtain

$$\begin{aligned} 2\langle F(x) - F(y), z - y\rangle &\leq 2\|F(x) - F(y)\|\|z - y\| \\ &\leq 2\frac{\mu}{\gamma}\|x - y\|\|z - y\| \\ &\leq \frac{\mu}{\gamma}\left(\|x - y\|^2 + \|y - z\|^2\right). \end{aligned} \tag{8}$$

Substituting (8) into (6), we get

$$\begin{aligned} \|z - x^*\|^2 &\leq \|x - x^*\|^2 - \|x - y\|^2 - \|y - z\|^2 + \lambda\frac{\mu}{\gamma}\left(\|x - y\|^2 + \|y - z\|^2\right) \\ &= \|x - x^*\|^2 - \left(1 - \mu\frac{\lambda}{\gamma}\right)\|x - y\|^2 - \left(1 - \mu\frac{\lambda}{\gamma}\right)\|y - z\|^2. \end{aligned}$$

If $F(x) = F(y)$ then $\gamma = \lambda$. From (6), we have

$$\begin{aligned} \|z - x^*\|^2 &\leq \|x - x^*\|^2 - \|x - y\|^2 - \|y - z\|^2 \\ &\leq \|x - x^*\|^2 - \left(1 - \mu\frac{\lambda}{\gamma}\right)\|x - y\|^2 - \left(1 - \mu\frac{\lambda}{\gamma}\right)\|y - z\|^2. \end{aligned}$$

This completes the proof of Lemma 2. □

Lemma 3 *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $F : \mathcal{H} \to \mathcal{H}$ be monotone and L-Lipschitz continuous on $\mathcal{H}$. Assume that $\lambda_n \geq a > 0$ for all n, $\{x^n\}$ is a sequence in $\mathcal{H}$ satisfying $x^n \rightharpoonup \overline{x}$ and $\lim_{n\to\infty}\|x^n - y^n\| = 0$, where $y^n = P_C(x^n - \lambda_n F(x^n))$ for all n. Then $\overline{x} \in \mathrm{Sol}(C, F)$.*

Proof It follows from $x^n \rightharpoonup \overline{x}$ and $\lim_{n\to\infty}\|x^n - y^n\| = 0$ that $\{x^n\}$ is bounded and $y^n \rightharpoonup \overline{x}$. Then $\{y^n\}$, $\{F(x^n)\}$ are also bounded thanks to $y^n \rightharpoonup \overline{x}$ and the Lipschitz continuity of F. Since $\{y^n\} \subset C$, $y^n \rightharpoonup \overline{x}$ and C is closed and convex, it is also weakly closed, and thus $\overline{x} \in C$.

For all $x \in C$, from $y^n = P_C(x^n - \lambda_n F(x^n))$, we have

$$\langle x^n - \lambda_n F(x^n) - y^n, x - y^n\rangle \leq 0 \quad \forall n.$$

This together with the monotonicity of F and the Cauchy–Schwarz inequality would imply that

$$\begin{aligned} \langle F(x), x^n - x\rangle &\leq \langle F(x^n), x^n - x\rangle \\ &= \langle F(x^n), x^n - y^n\rangle + \tfrac{1}{\lambda_n}\langle x^n - y^n, y^n - x\rangle + \tfrac{1}{\lambda_n}\langle x^n - \lambda_n F(x^n) - y^n, x - y^n\rangle \\ &\leq \langle F(x^n), x^n - y^n\rangle + \tfrac{1}{\lambda_n}\langle x^n - y^n, y^n - x\rangle \\ &\leq \|F(x^n)\|\|x^n - y^n\| + \tfrac{1}{\lambda_n}\|x^n - y^n\|\|y^n - x\| \\ &\leq \|F(x^n)\|\|x^n - y^n\| + \tfrac{1}{a}\|x^n - y^n\|\|y^n - x\|. \end{aligned} \tag{9}$$

Taking the limit in (9) as $n \to \infty$, using the boundedness of $\{F(x^n)\}$, $\{y^n\}$, and recalling that $\lim_{n\to\infty} \|x^n - y^n\| \to 0$, $x^n \rightharpoonup \overline{x}$, we obtain $\langle F(x), \overline{x} - x\rangle \leq 0$ and hence,

$$\langle F(x), x - \overline{x}\rangle \geq 0 \quad \forall x \in C. \tag{10}$$

Let $x_t = (1-t)\overline{x} + tx \in C$ for $t \in [0, 1]$. From (10), we have

$$0 \leq \langle F(x_t), x_t - \overline{x}\rangle = t\langle F(x_t), x - \overline{x}\rangle.$$

Then, for all $0 < t \leq 1$

$$\begin{aligned} 0 \leq \langle F(x_t), x - \overline{x}\rangle &= \langle F(x_t) - F(\overline{x}), x - \overline{x}\rangle + \langle F(\overline{x}), x - \overline{x}\rangle \\ &\leq L\|x_t - \overline{x}\|\|x - \overline{x}\| + \langle F(\overline{x}), x - \overline{x}\rangle \\ &= Lt\|x - \overline{x}\|^2 + \langle F(\overline{x}), x - \overline{x}\rangle. \end{aligned}$$

Taking the limit as $t \to 0^+$, we have $\langle F(\overline{x}), x - \overline{x}\rangle \geq 0$, i.e., $\overline{x} \in \mathrm{Sol}(C, F)$. □

Lemma 4 ([22, Remark 4.4]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that for any integer m, there exists an integer p such that $p \geq m$ and $a_p \leq a_{p+1}$. Let n_0 be an integer such that $a_{n_0} \leq a_{n_0+1}$ and define, for all integer $n \geq n_0$, by*

$$\tau(n) = \max\{k \in \mathbb{N} : n_0 \leq k \leq n, a_k \leq a_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence satisfying $\lim_{n\to\infty} \tau(n) = \infty$ and the following inequalities hold true:

$$a_{\tau(n)} \leq a_{\tau(n)+1}, \quad a_n \leq a_{\tau(n)+1} \qquad \forall n \geq n_0.$$

3 The Algorithm and Convergence Analysis

In this section, we propose a strong convergence algorithm for solving SVIP by using the modified Halpern subgradient extragradient method. We impose the following assumptions on the mappings F_1 and F_2 associated with the SVIP.

- (A_1) $F_1 : \mathcal{H}_1 \to \mathcal{H}_1$ is pseudomonotone on C and L_1-Lipschitz continuous on $\mathcal{H}_1$.
- (A_2) $\limsup_{n\to\infty}\langle F_1(x^n), y - y^n\rangle \leq \langle F_1(\overline{x}), y - \overline{y}\rangle$ for every sequence $\{x^n\}$, $\{y^n\}$ in $\mathcal{H}_1$ converging weakly to $\overline{x}$ and $\overline{y}$, respectively.
- (A_3) $F_2 : \mathcal{H}_2 \to \mathcal{H}_2$ is pseudomonotone on Q and L_2-Lipschitz continuous on $\mathcal{H}_2$.
- (A_4) $\limsup_{n\to\infty}\langle F_2(u^n), v - v^n\rangle \leq \langle F_2(\overline{u}), v - \overline{v}\rangle$ for every sequence $\{u^n\}$, $\{v^n\}$ in $\mathcal{H}_2$ converging weakly to $\overline{u}$ and $\overline{v}$, respectively.

Remark 1 (i) In finite dimensional spaces conditions (A_2) and (A_4) automatically follow from the Lipschitz continuity of F_1, F_2.

(ii) If F_1 and F_2 satisfy the assumptions (A_1)–(A_4), then the solution sets $\mathrm{Sol}(C, F_1)$ and $\mathrm{Sol}(Q, F_2)$ of $\mathrm{VIP}(C, F_1)$ and $\mathrm{VIP}(Q, F_2)$ are closed and convex (see e.g. [24]). Therefore, the solution set $\Omega = \{x^* \in \mathrm{Sol}(C, F_1) : Ax^* \in \mathrm{Sol}(Q, F_2)\}$ of the SVIP is also closed and convex.

The algorithm can be expressed as follows:

Algorithm 1

Step 0. Choose $\mu_0 > 0$, $\lambda_0 > 0$, $\mu \in (0, 1)$, $\lambda \in (0, 1)$, $\{\delta_n\} \subset [\underline{\delta}, \overline{\delta}] \subset \left(0, \frac{1}{\|A\|^2+1}\right)$, $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Step 1. Let $x^0 \in \mathcal{H}_1$. Set $n := 0$.
Step 2. Compute

$$\begin{aligned} u^n &= A(x^n), \\ v^n &= P_Q(u^n - \mu_n F_2(u^n)), \\ w^n &= P_{Q_n}(u^n - \mu_n F_2(v^n)), \end{aligned}$$

where

$$Q_n = \{\omega_2 \in \mathcal{H}_2 : \langle u^n - \mu_n F_2(u^n) - v^n, \omega_2 - v^n \rangle \leq 0\}$$

and

$$\mu_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|u^n - v^n\|}{\|F_2(u^n) - F_2(v^n)\|}, \mu_n\right\} & \text{if } F_2(u^n) \neq F_2(v^n), \\ \mu_n & \text{if } F_2(u^n) = F_2(v^n). \end{cases}$$

Step 3. Compute

$$\begin{aligned} y^n &= x^n + \delta_n A^*(w^n - u^n), \\ z^n &= P_C(y^n - \lambda_n F_1(y^n)), \\ t^n &= P_{C_n}(y^n - \lambda_n F_1(z^n)), \end{aligned}$$

where

$$C_n = \{\omega_1 \in \mathcal{H}_1 : \langle y^n - \lambda_n F_1(y^n) - z^n, \omega_1 - z^n \rangle \leq 0\}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\lambda\|y^n - z^n\|}{\|F_1(y^n) - F_1(z^n)\|}, \lambda_n\right\} & \text{if } F_1(y^n) \neq F_1(z^n), \\ \lambda_n & \text{if } F_1(y^n) = F_1(z^n). \end{cases}$$

Step 4. Compute

$$x^{n+1} = \alpha_n x^0 + (1 - \alpha_n) t^n.$$

Step 5. Set $n := n + 1$, and go to Step 2.

The following theorem shows the convergence of the algorithm.

Theorem 1 *Suppose that the assumptions (A_1)–(A_4) hold. Then the sequence $\{x^n\}$ generated by Algorithm 1 converges strongly to an element $x^* \in \Omega$, where $x^* = P_\Omega(x^0)$, provided the solution set Ω of the SVIP is nonempty.*

Proof The proof of the theorem is divided into several steps.

Step 1 The sequences $\{x^n\}$, $\{y^n\}$, $\{z^n\}$, $\{t^n\}$ and $\{v^n\}$ are bounded.

Since $x^* \in \Omega$, we have $x^* \in \text{Sol}(C, F_1)$ and $A(x^*) \in \text{Sol}(Q, F_2)$. From Lemma 2, we have, for all $n \geq 0$

$$\begin{aligned} \|w^n - A(x^*)\|^2 \leq\; & \|u^n - A(x^*)\|^2 - \left(1 - \mu\frac{\mu_n}{\mu_{n+1}}\right)\|u^n - v^n\|^2 \\ & - \left(1 - \mu\frac{\mu_n}{\mu_{n+1}}\right)\|v^n - w^n\|^2, \end{aligned} \tag{11}$$

$$\|t^n - x^*\|^2 \le \|y^n - x^*\|^2 - \left(1 - \lambda\frac{\lambda_n}{\lambda_{n+1}}\right)\|y^n - z^n\|^2 - \left(1 - \lambda\frac{\lambda_n}{\lambda_{n+1}}\right)\|z^n - t^n\|^2. \quad (12)$$

Since F_2 is L_2-Lipschitz continuous on $\mathcal{H}_2$, we get $\|F_2(u^n) - F_2(v^n)\| \le L_2\|u^n - v^n\|$. Thus, by induction, for every $n \ge 0$, we have

$$\mu_n \ge \min\left(\frac{\mu}{L_2}, \mu_0\right) > 0. \quad (13)$$

By the definition of μ_{n+1}, we have $\mu_{n+1} \le \mu_n$ for all $n \ge 0$. This together with (13) implies that the limit of $\{\mu_n\}$ exists. We denote $\lim_{n\to\infty}\mu_n = \mu^*$. It is clear that $\mu^* \ge \min\left(\frac{\mu}{L_2}, \mu_0\right) > 0$.

Using the same argument as above, we have

$$\lambda_n \ge \min\left(\frac{\lambda}{L_1}, \lambda_0\right) > 0 \quad \forall n \ge 0 \quad \text{and} \quad \lim_{n\to\infty}\lambda_n = \lambda^* \ge \min\left(\frac{\lambda}{L_1}, \lambda_0\right) > 0.$$

From $\lim_{n\to\infty}\mu_n = \mu^* > 0$ and $\lim_{n\to\infty}\lambda_n = \lambda^* > 0$, we get $\lim_{n\to\infty}\left(1 - \mu\frac{\mu_n}{\mu_{n+1}}\right) = 1 - \mu > 0$, $\lim_{n\to\infty}\left(1 - \lambda\frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \lambda > 0$. This implies that there exists $n_0 \in \mathbb{N}$ such that $1 - \mu\frac{\mu_n}{\mu_{n+1}} > 0$ and $1 - \lambda\frac{\lambda_n}{\lambda_{n+1}} > 0$ for all $n \ge n_0$. By (11) and (12), we get

$$\|w^n - A(x^*)\| \le \|u^n - A(x^*)\| \quad \forall n \ge n_0, \quad (14)$$

$$\|t^n - x^*\| \le \|y^n - x^*\| \quad \forall n \ge n_0. \quad (15)$$

From (14), since $u^n = A(x^n)$, we obtain, for all $n \ge n_0$

$$\begin{aligned}\langle A(x^n - x^*), w^n - u^n\rangle &= \langle w^n - A(x^*), w^n - u^n\rangle - \|w^n - u^n\|^2 \\ &= \frac{1}{2}\left[(\|w^n - A(x^*)\|^2 - \|u^n - A(x^*)\|^2) - \|w^n - u^n\|^2\right] \\ &\le -\frac{1}{2}\|w^n - u^n\|^2.\end{aligned}$$

Hence

$$2\delta_n\langle A(x^n - x^*), w^n - u^n\rangle \le -\delta_n\|w^n - u^n\|^2 \quad \forall n \ge n_0. \quad (16)$$

On the other hand

$$\begin{aligned}\|y^n - x^*\|^2 &= \|(x^n - x^*) + \delta_n A^*(w^n - u^n)\|^2 \\ &= \|x^n - x^*\|^2 + \|\delta_n A^*(w^n - u^n)\|^2 + 2\delta_n\langle x^n - x^*, A^*(w^n - u^n)\rangle \\ &\le \|x^n - x^*\|^2 + \delta_n^2\|A^*\|^2\|w^n - u^n\|^2 + 2\delta_n\langle A(x^n - x^*), w^n - u^n\rangle \\ &= \|x^n - x^*\|^2 + \delta_n^2\|A\|^2\|w^n - u^n\|^2 + 2\delta_n\langle A(x^n - x^*), w^n - u^n\rangle. \quad (17)\end{aligned}$$

Combining (16) and (17), we obtain

$$\|y^n - x^*\|^2 \le \|x^n - x^*\|^2 - \delta_n(1 - \delta_n\|A\|^2)\|w^n - u^n\|^2 \quad \forall n \ge n_0. \quad (18)$$

From (15), (18) and $\{\delta_n\} \subset [\underline{\delta}, \overline{\delta}] \subset \left(0, \frac{1}{\|A\|^2+1}\right)$, we get

$$\|t^n - x^*\| \le \|y^n - x^*\| \le \|x^n - x^*\| \quad \forall n \ge n_0. \quad (19)$$

Since $\lambda_n \leq \lambda_0$, $\mu_n \leq \mu_0$ for all $n \geq 0$, F_1 is L_1-Lipschitz continuous on $\mathcal{H}_1$, F_2 is L_2-Lipschitz continuous on $\mathcal{H}_2$, we have

$$\begin{aligned}
\|z^n - x^*\| &= \|P_C(y^n - \lambda_n F_1(y^n)) - P_C(x^*)\| \\
&\leq \|y^n - x^* - \lambda_n F_1(y^n)\| \\
&= \|y^n - x^* - \lambda_n (F_1(y^n) - F_1(x^*)) - \lambda_n F_1(x^*)\| \\
&\leq \|y^n - x^*\| + \lambda_n \|F_1(y^n) - F_1(x^*)\| + \lambda_n \|F_1(x^*)\| \\
&\leq \|y^n - x^*\| + \lambda_n L_1 \|y^n - x^*\| + \lambda_n \|F_1(x^*)\| \\
&\leq (1 + \lambda_0 L_1)\|y^n - x^*\| + \lambda_0 \|F_1(x^*)\|, \qquad (20)
\end{aligned}$$

$$\begin{aligned}
\|v^n - A(x^*)\| &= \|P_Q(u^n - \mu_n F_2(u^n)) - P_Q(A(x^*))\| \\
&\leq \|u^n - A(x^*) - \mu_n F_2(u^n)\| \\
&= \|u^n - A(x^*) - \mu_n [F_2(u^n) - F_2(A(x^*))] - \mu_n F_2(A(x^*))\| \\
&\leq \|u^n - A(x^*)\| + \mu_n \|F_2(u^n) - F_2(A(x^*))\| + \mu_n \|F_2(A(x^*))\| \\
&\leq \|u^n - A(x^*)\| + \mu_n L_2 \|u^n - A(x^*)\| + \mu_n \|F_2(A(x^*))\| \\
&\leq (1 + \mu_0 L_2)\|u^n - A(x^*)\| + \mu_0 \|F_2(A(x^*))\| \\
&= (1 + \mu_0 L_2)\|A(x^n - x^*)\| + \mu_0 \|F_2(A(x^*))\| \\
&\leq (1 + \mu_0 L_2)\|A\| \|x^n - x^*\| + \mu_0 \|F_2(A(x^*))\|. \qquad (21)
\end{aligned}$$

On the other hand

$$\begin{aligned}
\|x^{n+1} - x^*\| &= \|(1 - \alpha_n)(t^n - x^*) + \alpha_n (x^0 - x^*)\| \\
&\leq (1 - \alpha_n)\|t^n - x^*\| + \alpha_n \|x^0 - x^*\|. \qquad (22)
\end{aligned}$$

Using (19) and (22), we have

$$\|x^{n+1} - x^*\| \leq (1 - \alpha_n)\|x^n - x^*\| + \alpha_n \|x^0 - x^*\| \quad \forall n \geq n_0.$$

This implies that

$$\|x^{n+1} - x^*\| \leq \max\{\|x^n - x^*\|, \|x^0 - x^*\|\} \quad \forall n \geq n_0.$$

So, by induction, we obtain, for every $n \geq n_0$ that

$$\|x^n - x^*\| \leq \max\{\|x^{n_0} - x^*\|, \|x^0 - x^*\|\}.$$

Hence, the sequence $\{x^n\}$ is bounded and so are the sequences $\{y^n\}$, $\{z^n\}$, $\{t^n\}$ and $\{v^n\}$ thanks to (19), (20) and (21).

Step 2 We prove that $\{x^n\}$ converges strongly to x^*.

We have

$$\begin{aligned}
\|x^{n+1} - x^*\|^2 &= \|\alpha_n x^0 + (1 - \alpha_n) t^n - x^*\|^2 \\
&= \|t^n - x^* + \alpha_n (x^0 - t^n)\|^2 \\
&= \|t^n - x^*\|^2 + 2\alpha_n \langle x^0 - t^n, t^n - x^* \rangle + \alpha_n^2 \|t^n - x^0\|^2, \qquad (23)
\end{aligned}$$

which together with (19) implies, for all $n \geq n_0$

$$\begin{aligned}
0 &\leq \|y^n - x^*\|^2 - \|t^n - x^*\|^2 \\
&\leq \|x^n - x^*\|^2 - \|t^n - x^*\|^2 \\
&= (\|x^n - x^*\|^2 - \|x^{n+1} - x^*\|^2) + 2\alpha_n \langle x^0 - t^n, t^n - x^* \rangle + \alpha_n^2 \|t^n - x^0\|^2. \qquad (24)
\end{aligned}$$

Let us consider two cases.

Case 1. There exists n_1 such that $\{\|x^n - x^*\|\}$ is decreasing for $n \geq n_1$. In this case the limit of $\{\|x^n - x^*\|\}$ exists and we denote $\lim_{n\to\infty} \|x^n - x^*\|^2 = \xi \geq 0$. It follows from (24), $\lim_{n\to\infty} \alpha_n = 0$ and the boundedness of $\{t^n\}$ that

$$\lim_{n\to\infty} (\|y^n - x^*\|^2 - \|t^n - x^*\|^2) = 0, \quad \lim_{n\to\infty} (\|x^n - x^*\|^2 - \|t^n - x^*\|^2) = 0. \tag{25}$$

It follows from (25) that

$$\lim_{n\to\infty} (\|x^n - x^*\|^2 - \|y^n - x^*\|^2) = 0. \tag{26}$$

Combining (12), (25) and $\lim_{n\to\infty} \left(1 - \lambda \frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \lambda > 0$, we obtain

$$\lim_{n\to\infty} \|y^n - z^n\| = 0, \quad \lim_{n\to\infty} \|z^n - t^n\| = 0. \tag{27}$$

From (27) and the triangle inequality, we get

$$\lim_{n\to\infty} \|y^n - t^n\| = 0. \tag{28}$$

Using (18) and $\{\delta_n\} \subset [\underline{\delta}, \overline{\delta}] \subset \left(0, \frac{1}{\|A\|^2+1}\right)$, we have

$$\underline{\delta}(1 - \overline{\delta}\|A\|^2)\|w^n - u^n\|^2 \leq \|x^n - x^*\|^2 - \|y^n - x^*\|^2 \quad \forall n \geq n_0. \tag{29}$$

Combining (26) and (29), we get

$$\lim_{n\to\infty} \|w^n - u^n\| = 0. \tag{30}$$

Note that, for all n,

$$\|y^n - x^n\| = \|\delta_n A^*(w^n - u^n)\| \leq \delta_n \|A^*\| \|w^n - u^n\| \leq \overline{\delta}\|A\|\|w^n - u^n\|.$$

It follows from the above inequality and (30) that

$$\lim_{n\to\infty} \|y^n - x^n\| = 0. \tag{31}$$

From (28) and (31), we have

$$\lim_{n\to\infty} \|x^n - t^n\| = 0. \tag{32}$$

We now prove that

$$\limsup_{n\longrightarrow\infty} \langle x^0 - x^*, t^n - x^*\rangle \leq 0. \tag{33}$$

Choose a subsequence $\{t^{n_k}\}$ of $\{t^n\}$ such that

$$\limsup_{n\longrightarrow\infty} \langle x^0 - x^*, t^n - x^*\rangle = \lim_{k\to\infty} \langle x^0 - x^*, t^{n_k} - x^*\rangle.$$

Since $\{t^{n_k}\}$ is bounded, we may assume that $\{t^{n_k}\}$ converges weakly to some $\overline{t} \in \mathcal{H}_1$.

Therefore

$$\limsup_{n\longrightarrow\infty} \langle x^0 - x^*, t^n - x^*\rangle = \langle x^0 - x^*, \overline{t} - x^*\rangle. \tag{34}$$

From (32), (28), (27) and $t^{n_k} \rightharpoonup \overline{t}$, we conclude that x^{n_k}, y^{n_k} and z^{n_k} converge weakly to $\overline{t}$. Since $\{z^{n_k}\} \subset C$ and C is weakly closed then $\overline{t} \in C$.

We prove $\overline{t} \in \mathrm{Sol}(C, F_1)$.

Indeed, let $x \in C$. From the definition of z^{n_k} and Lemma 1, we have

$$\langle y^{n_k} - \lambda_{n_k} F_1(y^{n_k}) - z^{n_k}, x - z^{n_k}\rangle \leq 0 \quad \forall k.$$

Since $\lambda_{n_k} > 0$ for every k, it follows from the above inequality that

$$\langle F_1(y^{n_k}), x - z^{n_k}\rangle \geq \frac{\langle y^{n_k} - z^{n_k}, x - z^{n_k}\rangle}{\lambda_{n_k}}. \tag{35}$$

From $\lim_{k\to\infty}\|y^{n_k}-z^{n_k}\|=0$, $\lim_{k\to\infty}\lambda_{n_k}=\lambda^*>0$ and the boundedness of $\{z^{n_k}\}$, we get

$$\lim_{k\to\infty}\frac{\langle y^{n_k}-z^{n_k},x-z^{n_k}\rangle}{\lambda_{n_k}}=0.$$

Using (35), condition (A$_2$) and the weak convergence of two sequences $\{y^{n_k}\}$, $\{z^{n_k}\}$ to $\bar{t}$, we have

$$0\le\limsup_{k\longrightarrow\infty}\langle F_1(y^{n_k}),x-z^{n_k}\rangle\le\langle F_1(\bar{t}),x-\bar{t}\rangle,$$

i.e., $\bar{t}\in \mathrm{Sol}(C,F_1)$.

On the other hand

$$\begin{aligned}\|w^n-A(x^*)\|^2&=\|u^n-A(x^*)-(u^n-w^n)\|^2\\&=\|u^n-A(x^*)\|^2-2\langle u^n-A(x^*),u^n-w^n\rangle+\|u^n-w^n\|^2\\&=\|u^n-A(x^*)\|^2-2\langle A(x^n-x^*),u^n-w^n\rangle+\|u^n-w^n\|^2\\&\ge\|u^n-A(x^*)\|^2-2\|A(x^n-x^*)\|\|u^n-w^n\|+\|u^n-w^n\|^2\\&\ge\|u^n-A(x^*)\|^2-2\|A\|\|x^n-x^*\|\|u^n-w^n\|+\|u^n-w^n\|^2.\end{aligned}\tag{36}$$

Combining (11) and (36) yields

$$\left(1-\mu\frac{\mu_n}{\mu_{n+1}}\right)\|u^n-v^n\|^2+\left(1-\mu\frac{\mu_n}{\mu_{n+1}}\right)\|v^n-w^n\|^2$$
$$\le 2\|A\|\|x^n-x^*\|\|u^n-w^n\|-\|u^n-w^n\|^2.$$

Using the above inequality, $\lim_{n\to\infty}\|u^n-w^n\|=0$, $\lim_{n\to\infty}\left(1-\mu\frac{\mu_n}{\mu_{n+1}}\right)=1-\mu>0$ and the fact that $\{x^n\}$ is bounded, we obtain

$$\lim_{n\to\infty}\|u^n-v^n\|=0.$$

From $x^{n_k}\rightharpoonup\bar{t}$, we get $u^{n_k}=A(x^{n_k})\rightharpoonup A(\bar{t})$. This together with $\lim_{n\to\infty}\|u^n-v^n\|=0$ implies $v^{n_k}\rightharpoonup A(\bar{t})$. Since $\{v^{n_k}\}\subset Q$ and Q is closed and convex, it is also weakly closed, and thus $A(\bar{t})\in Q$.

We prove $A(\bar{t})\in\mathrm{Sol}(Q,F_2)$.

Indeed, let $y\in Q$. From the definition of v^{n_k} and Lemma 1, we get

$$\langle u^{n_k}-\mu_{n_k}F_2(u^{n_k})-v^{n_k},y-v^{n_k}\rangle\le 0\quad\forall k.\tag{37}$$

Since $\mu_{n_k}>0$ for every k, it follows from (37) that

$$\langle F_2(u^{n_k}),y-v^{n_k}\rangle\ge\frac{\langle u^{n_k}-v^{n_k},y-v^{n_k}\rangle}{\mu_{n_k}}.\tag{38}$$

Since $\lim_{k\to\infty}\|u^{n_k}-v^{n_k}\|=0$, $\lim_{k\to\infty}\mu_{n_k}=\mu^*>0$ and the sequence $\{v^{n_k}\}$ is bounded, we get

$$\lim_{k\to\infty}\frac{\langle u^{n_k}-v^{n_k},y-v^{n_k}\rangle}{\mu_{n_k}}=0.$$

Using (38), condition (A$_4$) and the weak convergence of $\{u^{n_k}\}$, $\{v^{n_k}\}$ to $A(\bar{t})$, we obtain

$$0\le\limsup_{k\longrightarrow\infty}\langle F_2(u^{n_k}),y-v^{n_k}\rangle\le\langle F_2(A(\bar{t})),y-A(\bar{t})\rangle,$$

i.e., $A(\bar{t})\in\mathrm{Sol}(Q,F_2)$.

It follows from $\bar{t}\in\mathrm{Sol}(C,F_1)$ and $A(\bar{t})\in\mathrm{Sol}(Q,F_2)$ that $\bar{t}\in\Omega$. Which together with $x^*=P_\Omega(x^0)$ implies that $\langle x^0-x^*,\bar{t}-x^*\rangle\le 0$. So, from (34), we have $\limsup_{n\to\infty}\langle x^0-x^*,t^n-x^*\rangle\le 0$.

From $\lim_{n\to\infty}\|x^n - x^*\|^2 = \xi$ and (25), we have

$$\lim_{n\to\infty}\|t^n - x^*\|^2 = \xi. \tag{39}$$

From $\lim_{n\to\infty}\alpha_n = 0$, the boundedness of $\{t^n\}$, (33) and (39), we obtain

$$\begin{aligned}\limsup_{n\longrightarrow\infty}(2\langle x^0 - t^n, t^n - x^*\rangle + \alpha_n\|t^n - x^0\|^2) &= 2\limsup_{n\longrightarrow\infty}\langle x^0 - t^n, t^n - x^*\rangle \\ &= 2\limsup_{n\longrightarrow\infty}[\langle x^0 - x^*, t^n - x^*\rangle - \|t^n - x^*\|^2] \\ &\le -2\xi. \end{aligned} \tag{40}$$

Assume, to get a contradiction, that $\xi > 0$, and choose $\varepsilon = \xi > 0$. It follows from (40) that there exists $n_2 \ge 0$ such that

$$2\langle x^0 - t^n, t^n - x^*\rangle + \alpha_n\|t^n - x^0\|^2 \le -2\xi + \xi = -\xi \quad \forall n \ge n_2. \tag{41}$$

Then, from (19) and (23), we get

$$\|x^{n+1} - x^*\|^2 \le \|x^n - x^*\|^2 + \alpha_n\left[2\langle x^0 - t^n, t^n - x^*\rangle + \alpha_n\|t^n - x^0\|^2\right] \quad \forall n \ge n_0,$$

which together with (41) implies

$$\|x^{n+1} - x^*\|^2 - \|x^n - x^*\|^2 \le -\alpha_n\xi \quad \forall n \ge n_3 = \max(n_0, n_2).$$

Thus, after a summation, we obtain

$$\|x^{n+1} - x^*\|^2 - \|x^{n_3} - x^*\|^2 \le -\xi\left(\sum_{j=n_3}^{n}\alpha_j\right) \quad \forall n \ge n_3.$$

Therefore, we arrive at a contradiction

$$\xi\left(\sum_{j=n_3}^{n}\alpha_j\right) \le \|x^{n_3} - x^*\|^2 \quad \forall n \ge n_3$$

because $\sum_{n=0}^{\infty}\alpha_n = \infty$. Hence $\xi = 0$, which implies $x^n \to x^*$.

Case 2. Suppose that for any integer m, there exists an integer n such that $n \ge m$ and $\|x^n - x^*\| \le \|x^{n+1} - x^*\|$. According to Lemma 4, there exists a nondecreasing sequence $\{\tau(n)\}_{n\ge N}$ of $\mathbb{N}$ such that $\lim_{n\to\infty}\tau(n) = \infty$ and the following inequalities hold

$$\|x^{\tau(n)} - x^*\| \le \|x^{\tau(n)+1} - x^*\|, \quad \|x^n - x^*\| \le \|x^{\tau(n)+1} - x^*\| \quad \forall n \ge N. \tag{42}$$

Choose $n_4 \ge N$ such that $\tau(n) \ge n_0$ for all $n \ge n_4$. From (42) and (22), we get

$$\begin{aligned}\|x^{\tau(n)} - x^*\| &\le \|x^{\tau(n)+1} - x^*\| \\ &\le (1 - \alpha_{\tau(n)})\|t^{\tau(n)} - x^*\| + \alpha_{\tau(n)}\|x^0 - x^*\| \quad \forall n \ge n_4.\end{aligned} \tag{43}$$

From (43), we have

$$\|x^{\tau(n)} - x^*\| - \|t^{\tau(n)} - x^*\| \le \alpha_{\tau(n)}\|x^0 - x^*\| - \alpha_{\tau(n)}\|t^{\tau(n)} - x^*\| \quad \forall n \ge n_4,$$

which together with (19) implies, for all $n \ge n_4$, that

$$\begin{aligned}\alpha_{\tau(n)}\|x^0 - x^*\| - \alpha_{\tau(n)}\|t^{\tau(n)} - x^*\| &\ge \|x^{\tau(n)} - x^*\| - \|t^{\tau(n)} - x^*\| \\ &\ge \|x^{\tau(n)} - x^*\| - \|y^{\tau(n)} - x^*\| \\ &\ge 0.\end{aligned}$$

Then, it follows from the above inequality, the boundedness of $\{t^n\}$ and $\lim_{n\to\infty}\alpha_n = 0$ that

$$\lim_{n\to\infty}(\|x^{\tau(n)} - x^*\| - \|t^{\tau(n)} - x^*\|) = 0, \quad \lim_{n\to\infty}(\|x^{\tau(n)} - x^*\| - \|y^{\tau(n)} - x^*\|) = 0. \quad (44)$$

From (44) and the boundedness of $\{x^n\}$, $\{y^n\}$ and $\{t^n\}$, we obtain

$$\lim_{n\to\infty}(\|x^{\tau(n)} - x^*\|^2 - \|t^{\tau(n)} - x^*\|^2) = 0,$$
$$\lim_{n\to\infty}(\|x^{\tau(n)} - x^*\|^2 - \|y^{\tau(n)} - x^*\|^2) = 0.$$

Arguing similarly as in the first case, we can conclude that

$$\limsup_{n\longrightarrow\infty}\langle x^0 - x^*, t^{\tau(n)} - x^*\rangle \le 0.$$

Then, the boundedness of $\{t^n\}$ and $\lim_{n\to\infty}\alpha_n = 0$ yield

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\langle x^0 - x^*, x^{\tau(n)+1} - x^*\rangle &= \limsup_{n\longrightarrow\infty}\langle x^0 - x^*, t^{\tau(n)} - x^* + \alpha_{\tau(n)}(x^0 - t^{\tau(n)})\rangle\\ &= \limsup_{n\longrightarrow\infty}\langle x^0 - x^*, t^{\tau(n)} - x^*\rangle \le 0. \quad (45)\end{aligned}$$

Using the inequality

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle \quad \forall x, y \in \mathcal{H}_1,$$

as well as (19) and (42), we obtain, for all $n \ge n_4$

$$\begin{aligned}\|x^{\tau(n)+1} - x^*\|^2 &= \|(1-\alpha_{\tau(n)})(t^{\tau(n)} - x^*) + \alpha_{\tau(n)}(x^0 - x^*)\|^2\\ &\le \|(1-\alpha_{\tau(n)})(t^{\tau(n)} - x^*)\|^2 + 2\langle\alpha_{\tau(n)}(x^0 - x^*), x^{\tau(n)+1} - x^*\rangle\\ &= (1-\alpha_{\tau(n)})^2\|t^{\tau(n)} - x^*\|^2 + 2\alpha_{\tau(n)}\langle x^0 - x^*, x^{\tau(n)+1} - x^*\rangle\\ &\le (1-\alpha_{\tau(n)})\|t^{\tau(n)} - x^*\|^2 + 2\alpha_{\tau(n)}\langle x^0 - x^*, x^{\tau(n)+1} - x^*\rangle\\ &\le (1-\alpha_{\tau(n)})\|x^{\tau(n)} - x^*\|^2 + 2\alpha_{\tau(n)}\langle x^0 - x^*, x^{\tau(n)+1} - x^*\rangle\\ &\le (1-\alpha_{\tau(n)})\|x^{\tau(n)+1} - x^*\|^2 + 2\alpha_{\tau(n)}\langle x^0 - x^*, x^{\tau(n)+1} - x^*\rangle.\end{aligned}$$

In particular, since $\alpha_{\tau(n)} > 0$

$$\|x^{\tau(n)+1} - x^*\|^2 \le 2\langle x^0 - x^*, x^{\tau(n)+1} - x^*\rangle \quad \forall n \ge n_4.$$

Combining the above inequality with (42), we get

$$\|x^n - x^*\|^2 \le 2\langle x^0 - x^*, x^{\tau(n)+1} - x^*\rangle \quad \forall n \ge n_4. \quad (46)$$

Taking the limit in (46) as $n \to \infty$, and using (45), we obtain

$$\limsup_{n\longrightarrow\infty}\|x^n - x^*\|^2 \le 0,$$

which implies $x^n \to x^*$. This complete the proof of Theorem 1. □

Remark 2 Theorem 1 is still true if the assumptions (A_1)–(A_4) are replaced by the following assumptions:

(A) $F_1 : \mathcal{H}_1 \to \mathcal{H}_1$ is monotone on $\mathcal{H}_1$ and L_1-Lipschitz continuous on $\mathcal{H}_1$.
(B) $F_2 : \mathcal{H}_2 \to \mathcal{H}_2$ is monotone on $\mathcal{H}_2$ and L_2-Lipschitz continuous on $\mathcal{H}_2$.

Proof Note that in the proof of Theorem 1, the assumptions (A_2) and (A_4) are used to prove $\bar{t} \in \mathrm{Sol}(C, F_1)$ and $A(\bar{t}) \in \mathrm{Sol}(Q, F_2)$, respectively. Now we will prove $\bar{t} \in \mathrm{Sol}(C, F_1)$ and $A(\bar{t}) \in \mathrm{Sol}(Q, F_2)$ by using assumptions (A), (B) and Lemma 3.

Indeed, from assumption (A), $z^n = P_C(y^n - \lambda_n F_1(y^n))$, $\lim_{n\to\infty} \|y^n - z^n\| = 0$, $\lambda_n \geq \min\left(\frac{\lambda}{L_1}, \lambda_0\right) > 0$, $y^{n_k} \rightharpoonup \bar{t}$ and Lemma 3, we imply $\bar{t} \in \mathrm{Sol}(C, F_1)$.

Using the same argument, from (B), $v^n = P_Q(u^n - \mu_n F_2(u^n))$, $\lim_{n\to\infty} \|u^n - v^n\| = 0$, $\mu_n \geq \min\left(\frac{\mu}{L_2}, \mu_0\right) > 0$, $u^{n_k} \rightharpoonup A(\bar{t})$ and Lemma 3, we have $A(\bar{t}) \in \mathrm{Sol}(Q, F_2)$. □

When $F_1 = F_2 = 0$, we have the following corollary from Algorithm 1 and Theorem 1.

Corollary 1 *Let C and Q be two nonempty closed convex subset of two real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Suppose that positive sequences $\{\alpha_n\}$, $\{\delta_n\}$ satisfy the following conditions*

$$\begin{cases} \{\delta_n\} \subset [\underline{\delta}, \overline{\delta}] \subset \left(0, \dfrac{1}{\|A\|^2 + 1}\right), \\ \{\alpha_n\} \subset (0, 1), \ \ \lim_{n\to\infty} \alpha_n = 0, \ \ \sum_{n=0}^{\infty} \alpha_n = \infty. \end{cases}$$

Let $\{x^n\}$ be the sequence generated by $x^0 \in \mathcal{H}_1$ and

$$x^{n+1} = \alpha_n x^0 + (1 - \alpha_n) P_C(x^n + \delta_n A^*(P_Q(Ax^n) - Ax^n)) \quad \forall n \geq 0.$$

Then the sequence $\{x^n\}$ converges strongly to an element $x^* \in \Gamma$, where $x^* = P_\Gamma(x^0)$, provided the solution set $\Gamma = \{x^* \in C : Ax^* \in Q\}$ of the SFP is nonempty.

4 Numerical Results

Let $\mathcal{H}_1 = \mathbb{R}^4$ with the norm $\|x\| = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{1}{2}}$ for $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ and $\mathcal{H}_2 = \mathbb{R}^2$ with the standard norm $\|y\| = (y_1^2 + y_2^2)^{\frac{1}{2}}$. Let $A(x) = (x_1 + x_3 + x_4, x_2 + x_3 - x_4)^T$ for all $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ then A is a bounded linear operator from $\mathbb{R}^4$ into $\mathbb{R}^2$ with $\|A\| = \sqrt{3}$. For $y = (y_1, y_2)^T \in \mathbb{R}^2$, let $B(y) = (y_1, y_2, y_1 + y_2, y_1 - y_2)^T$, then B is a bounded linear operator from $\mathbb{R}^2$ into $\mathbb{R}^4$ with $\|B\| = \sqrt{3}$. Moreover, for any $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ and $y = (y_1, y_2)^T \in \mathbb{R}^2$, $\langle A(x), y\rangle = \langle x, B(y)\rangle$, so $B = A^*$ is an adjoint operator of A.

Let

$$C = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : x_1 - x_2 - x_3 + 2x_4 \geq -1\}$$

and define a mapping $F_1 : \mathbb{R}^4 \to \mathbb{R}^4$ by $F_1(x) = (\sin\|x\| + 2)a^0$ for all $x \in \mathbb{R}^4$, where $a^0 = (1, -1, -1, 2)^T \in \mathbb{R}^4$. It is easy to verify that F_1 is pseudomonotone on $\mathbb{R}^4$.

Furthermore, for all $x, y \in \mathbb{R}^4$, we have

$$\begin{aligned} \|F_1(x) - F_1(y)\| &= \|a^0\| |\sin\|x\| - \sin\|y\|| \\ &= \sqrt{7} |\sin\|x\| - \sin\|y\|| \\ &\leq \sqrt{7} |\|x\| - \|y\|| \\ &\leq \sqrt{7} \|x - y\|. \end{aligned}$$

So F_1 is $\sqrt{7}$-Lipschitz continuous on $\mathbb{R}^4$.

It is easy to see that the solution set $\mathrm{Sol}(C, F_1)$ of $\mathrm{VIP}(C, F_1)$ is given by

$$\mathrm{Sol}(C, F_1) = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : x_1 - x_2 - x_3 + 2x_4 = -1\}.$$

Now let $Q = \{(u_1, u_2)^T \in \mathbb{R}^2 : 2u_1 - 3u_2 \geq -4\}$ and define another mapping $F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ by $F_2(u) = (\sin\|u\| + 3)b^0$ for all $u \in \mathbb{R}^2$, where $b^0 = (2, -3)^T \in \mathbb{R}^2$. Similarly,

F_2 is pseudomonotone on $\mathbb{R}^2$, $\sqrt{13}$-Lipschitz continuous on $\mathbb{R}^2$ and that the solution set $\mathrm{Sol}(Q, F_2)$ of $\mathrm{VIP}(Q, F_2)$ is given by

$$\mathrm{Sol}(Q, F_2) = \{(u_1, u_2)^T \in \mathbb{R}^2 : 2u_1 - 3u_2 = -4\}.$$

The solution set Ω of the SVIP is given by

$$\begin{aligned}\Omega &= \{(x_1, x_2, x_3, x_4)^T \in \mathrm{Sol}(C, F_1) : A(x_1, x_2, x_3, x_4) \in \mathrm{Sol}(Q, F_2)\}\\ &= \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : x_1 - x_2 - x_3 + 2x_4 = -1,\\ &\quad 2(x_1 + x_3 + x_4) - 3(x_2 + x_3 - x_4) = -4\}\\ &= \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : x_1 - x_2 - x_3 + 2x_4 = -1, 2x_1 - 3x_2 - x_3 + 5x_4 = -4\}\\ &= \{(2a - b + 1, a + b + 2, a, b)^T : a, b \in \mathbb{R}\}.\end{aligned}$$

Select a random starting point $x^0 = (-1, 1, 2, -3)^T$ for the Algorithm 1. We choose $\mu = 0.7$, $\mu_0 = 1$, $\lambda = 0.4$, $\lambda_0 = 2$, $\alpha_n = \frac{1}{n+2}$, $\delta_n = \frac{n+1}{6n+8}$. An elementary computation shows that $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\{\delta_n\} \subset \left[\frac{1}{8}, \frac{1}{6}\right] \subset \left(0, \frac{1}{4}\right) = \left(0, \frac{1}{\|A\|^2+1}\right)$.

Suppose $x = (2a - b + 1, a + b + 2, a, b)^T \in \Omega$ then

$$\begin{aligned}\|x - x^0\| &= \sqrt{(2a - b + 2)^2 + (a + b + 1)^2 + (a - 2)^2 + (b + 3)^2}\\ &= \sqrt{6a^2 + 3b^2 - 2ab + 6a + 4b + 18}\\ &= \sqrt{\frac{1}{3}(3b - a + 2)^2 + \frac{1}{51}(17a + 11)^2 + \frac{243}{17}}\\ &\geq \sqrt{\frac{243}{17}}.\end{aligned}$$

The above equality holds if and only if $3b - a + 2 = 0$ and $a = -\frac{11}{17}$. So, we obtain $a = -\frac{11}{17}$, $b = -\frac{15}{17}$.

Therefore

$$x^* = P_\Omega(x^0) = \left(\frac{10}{17}, \frac{8}{17}, -\frac{11}{17}, -\frac{15}{17}\right)^T.$$

With $\varepsilon = 10^{-9}$, the approximate solution obtained after 225081 iterations (with elapsed time 118.6242 seconds) is

$$x^{225081} = (0.5882, 0.4705, -0.6468, -0.8823)^T,$$

which is a good approximation to $x^* = \left(\frac{10}{17}, \frac{8}{17}, -\frac{11}{17}, -\frac{15}{17}\right)^T$.

Table 1 presents the numerical result of Algorithm 1 with different tolerances. From the preliminary numerical results reported in the table, we observe that the running time of Algorithm 1 depends very much on the tolerance.

Table 1 Algorithm 1 for the above example with different tolerances

Tolerance	Iter(n)	Elapsed Time(s)	x^n
$\epsilon = 10^{-5}$	2575	0.7088	$(0.5864, 0.4599, -0.6404, -0.8801)^T$
$\epsilon = 10^{-6}$	8142	3.3164	$(0.5871, 0.4690, -0.6423, -0.8809)^T$
$\epsilon = 10^{-7}$	25746	12.7714	$(0.5879, 0.4701, -0.6449, -0.8817)^T$
$\epsilon = 10^{-8}$	81415	47.4449	$(0.5881, 0.4704, -0.6466, -0.8822)^T$

We perform the iterative schemes in MATLAB R2018a running on a laptop with Intel(R) Core(TM) i5-3230M CPU @ 2.60GHz, 4 GB RAM.

5 Conclusion

In this paper, we have proposed an iterative algorithm for solving the split variational inequality problem involving Lipschitz continuous pseudomonotone mappings. The proof of convergence of the algorithm is performed without the prior knowledge of the Lipschitz constants of cost operators. The strong convergence of the iterative sequence generated by the proposed iterative algorithm to the solution of the SVIP is obtained. When applied to the well-known SFP, our method is reduced to a strongly convergent algorithm, which requires only two projections at each iteration step.

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