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ORIGINAL ARTICLE

An Algorithm to Solve Equilibrium Problems and Fixed Points Problems Involving a Finite Family of Multivalued Strictly Pseudo-Contractive Mappings

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Abstract

The aim of this paper is to introduce and study a new iterative algorithm for finding a common element of the set of fixed points of a finite family of multivalued strictly pseudo-contractive mappings and the set of solutions of equilibrium problems in Hilbert spaces. Strong convergence of the proposed method is established under suitable control conditions. Application to optimization problems with constraints is provided to support our main results. Furthermore, numerical example is given to demonstrate the implementability of our algorithm.The algorithm and its convergence results improve and develop previous results in the field.

Keywords Common fixed points · Multivalued strictly pseudo-contractive mappings · Equilibrium problems

Mathematics Subject Classification (2010) 47H05 · 47J05 · 47J25

1 Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the real numbers. The equilibrium problem for F is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \forall y \in C. \tag{1}$$

The set of solutions is denoted by $EP(F)$. Equilibrium problems which were introduced by Fan [9] and Blum and Oettli [10] have had a great impact and influence on the development of several branches of pure and applied sciences. Equilibrium problems include variational inequality problems as well as fixed point problems, complementarity problems, optimization, saddle point problems and Nash equilibrium problems as special cases. Equilibrium problems provide us with a systematic framework to study a wide class of problems

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arising in finance economics, optimization and operation research etc., which motivate the extensive concern. In recent years, equilibrium problems have been deeply and thoroughly researched, see [3, 4, 10, 12, 18, 22, 30] and the references therein. However, there are few iterative algorithms developed for the approximation of solutions of equilibrium problems.

Let (X, d) be a metric space, K be a nonempty subset of X and $T : K \to 2^K$ be a multivalued mapping. An element $x \in K$ is called a fixed point of T if $x \in Tx$. For single valued mapping, this reduces to $Tx = x$. The fixed point set of T is denoted by $\mathrm{Fix}(T) := \{x \in D(T) : x \in Tx\}$.

A point $x \in X$ is called an endpoint (or stationary point) of T if x is a fixed point of T and $T(x) = \{x\}$. We shall denote by $\mathrm{End}(T)$ the set of all endpoints of T. We see that for each mapping T, $\mathrm{End}(T) \subset \mathrm{Fix}(T)$. Thus, the concept of endpoints seems to be more difficult (but more important) than the concept of fixed points. However, both concepts are equivalent when T is a single-valued mapping since, in this case, $\mathrm{End}(T) = \mathrm{Fix}(T)$. Next is an example of a multivalued mapping T with $\mathrm{Fix}(T) \neq \emptyset$, $Tp = \{p\}$ for all $p \in Tp$.

Example 1 Let $X = \mathbb{R}$ (the reals with usual metric). Define $T : [-1, 1] \to 2^{[-1,1]}$ by

$$Tx = \begin{cases} \left[-1, \frac{2}{3}x \sin\frac{1}{x}\right], & x \in (0, 1], \\ \{0\}, & x = 0, \\ \left[\frac{2}{3}x \sin\frac{1}{x}, 1\right], & x \in [-1, 0). \end{cases}$$

Then, clearly $\mathrm{Fix}(T) = \{0\}$.

Many problems arising in different areas of mathematics, such as game theory, control theory, dynamic systems theory, signal and image processing, market economy and in other areas of mathematics, such as in non-smooth differential equations and differential inclusions, optimization theory equations, can be modeled by the equation

$$x \in Tx,$$

where T is a multivalued nonexpansive mapping. The solution set of this equation coincides with the fixed point set of T.

For several years, the study of fixed point theory for *multi-valued nonlinear mappings* has attracted, and continues to attract, the interest of several well known mathematicians (see, for example, Brouwer [7], Kakutani [14], Nash [19, 20]).

Nonsmooth differential equations A large number of problems from mechanics and electrical engineering leads to differential inclusions and differential equations with discontinuous right-hand sides, for example, a dry friction force of some electronic devices. Below are two models.

$$\frac{du}{dt} = f(t, u) \quad \text{a.e. } t \in I := [-a, a],\ u(0) = u_0, \tag{2}$$

a, u_0 fixed in $\mathbb{R}$. These types of differential equations do not have solutions in the classical sense. A generalized notion of solution is what is called a solution in the sense of Fillipov.

Consider the following *multi-valued* initial value problem.

$$\begin{cases} -\frac{d^2u}{dt^2} \in u - \frac{1}{4} - \frac{1}{4}\mathrm{sign}(u-1) \text{ on } \Omega = (0, \pi); \\ u(0) = 0; \\ u(\pi) = 0. \end{cases} \tag{3}$$

Under some conditions, the solutions set of equations (2) and (3) coincides with the fixed point set of some multi-valued mappings.

Let K be a nonempty subset of a normed space E. The set K is called *proximinal* (see, e.g., [21]) if for each $x \in E$, there exists $u \in K$ such that

$$d(x, u) = \inf\{\|x - y\| : y \in K\} = d(x, K),$$

where $d(x, y) = \|x - y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(K)$, $K(K)$ and $P(K)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of K respectively. The *Hausdorff metric* on $CB(K)$ is defined by:

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(b, A)\right\}$$

for all $A, B \in CB(K)$. A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called *L-Lipschitzian* if there exists $L > 0$ such that

$$H(Tx, Ty) \le L\|x - y\| \quad \forall x, y \in D(T).$$

When $L \in (0, 1)$, we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$.

Different iterative processes have been developed to approximate fixed points of multi-valued nonexpansive mappings (see, e.g., [1, 15] and the references therein) and their generalizations (see, e.g., [13]).

Recently, viscosity iterative algorithms for finding a common element of the set of fixed points for single-valued nonexpansive mappings and the set of solutions of variational inequality problems have been investigated by many authors; (see, e.g., [21, 31] and the references therein). For example, Moudafi [16] introduced the explicit viscosity approximation method for nonexpansive mappings.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction mapping and T be a single-valued nonexpansive mapping on C. Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \end{cases} \tag{4}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. Then, the sequence $\{x_n\}$ generated by (4) converges strongly to $x^* \in \mathrm{Fix}(T)$, which is a unique solution of the following variational inequality:

$$\langle x^* - f(x^*), x^* - p\rangle \le 0 \quad \forall p \in \mathrm{Fix}(T).$$

In 2007, Takahashi and Takahashi [27] investigate Moudafi's viscosity method (4) for finding a common element of the solutions set of (1) and the fixed points set of a nonexpansive mapping in a Hilbert space, and proved the following strong convergence theorem.

Theorem 1 [27] *Let C be a nonempty, closed and convex subset a real Hilbert space H. Let F be a bifunction from $C \times C \to \mathbb{R}$ satisfying the following assumptions:*

(A1) $F(x, x) = 0$ *for all* $x \in C$;

(A2) *F is monotone, i.e.,* $F(x, y) + F(y, x) \le 0$ *for all* $x, y \in C$;

(A3) *for each* $x, y, z \in C$,

$$\lim_{t\to 0} F(tz + (1 - t)x, y) \le F(x, y);$$

(A4) *for each $x \in C$, $y \to F(x, y)$ is convex and lower semicontinuous.*

Let $f : C \to C$ be a contraction and $T : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \cap EP(F) \neq \emptyset$.

Let $\{x_n\}$ and $\{u_n\}$ be sequences defined iteratively from arbitrary $x_0 \in C$ by:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \end{cases} \tag{5}$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset]0, \infty[$ satisfy:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;
(iii) $\lim_{n\to\infty} \inf r_n > 0$ *and* $\sum_{n=0}^{\infty} |r_{n+1} - r_n| < \infty$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (5) converge strongly to $x^* \in \mathrm{Fix}(T) \cap EP(F)$.

The important class of single-valued k-strictly pseudo-contractive maps on Hilbert spaces was introduced by Browder and Petryshyn [2] as a generalization of the class of nonexpansive mappings.

Definition 1 Let K be a nonempty subset of a real Hilbert space H. A map $T : K \to H$ is called k-strictly pseudo-contractive if there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2 \quad \forall x, y \in K.$$

Motivated by approximating fixed points of multivalued mappings, Chidume et al. [8] introduced the following important class of multivalued strictly pseudo-contractive mappings in real Hilbert spaces which is more general than the class of multivalued nonexpansive mappings.

Definition 2 A multi-valued mapping $T : D(T) \subseteq H \to CB(H)$ is said to be *k-strictly pseudo-contractive*, if there exists $k \in (0, 1)$ such for all $x, y \in D(T)$, we have

$$\big(H(Tx, Ty)\big)^2 \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2 \quad \forall u \in Tx,\ v \in Ty. \tag{6}$$

If $k = 1$ in (6), the map T is said to be pseudo-contractive.

Remark 1 It is easily seen that any multivalued nonexpansive mapping is k-strictly pseudo-contractive for any $k \in (0, 1)$. Moreover, the converse is not true (see, for example, Djitte and Sene [20]).

With this definition at hand, many mathematicians proved some strong convergence theorems for approximating fixed points of multivalued k-strictly pseudo-contractive mappings under some compactness conditions (see, for example, Sene et al. [24], Chidume et al. [8]).

Motivated by Takahashi and Takahashi [27] and the fact that the class of multivalued strictly pseudo-contractive mappings properly includes that of multivalued nonexpansive maps, we construct a new iterative algorithm which is a combination of Krasnoselskii–Mann algorithm and viscosity method for approximating a common element of the set of fixed points of a finite family of multivalued strictly pseudo-contractive mappings and the set of solutions of equilibrium problems which is also the solution of some variational inequality problems. Furthermore, we applied our main results to constrained convex minimization problems. The algorithm and results presented in this paper improve and extend some recents results. Finally, our method of proof is of independent interest.

2 Preliminaries

Let us recall the following definitions and results which will be used in the sequel.

Let H be a real Hilbert space. Let $\{x_n\}$ be a sequence in H and let $x \in H$. Weak convergence of x_n to x is denoted by $x_n \rightharpoonup x$ and strong convergence by $x_n \to x$. Let K be a nonempty, closed convex subset of H. The nearest point projection from H to K, denoted by P_K assigns to each $x \in H$ the unique $P_K x$ with the property

$$\|x - P_K x\| \leq \|y - x\|$$

for all $y \in K$. It is well known that $P_K x$ satisfies

$$\langle x - P_K x, y - P_K x\rangle \leq 0$$

for all $y \in K$.

Definition 3 Let H be a real Hilbert space and $T : D(T) \subset H \to 2^H$ be a multivalued mapping. $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $d(x_n, Tx_n)$ converges to zero, then $p \in Tp$.

Lemma 1 (Demiclosedness principle, [6]) *Let H be a real Hilbert space, K be a nonempty closed and convex subset of H. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values. Then $I - T$ is demi-closed at zero.*

Lemma 2 [7] *Let H be a real Hilbert space. Then for any $x, y \in H$, the following inequality hold:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 3 (Xu, [29]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(b) $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 4 [17] *Let K be a nonempty closed convex subset of a real Hilbert space H and $T : K \to K$ be a mapping.*

(i) *If T is a k-strictly pseudo-contractive mapping, then T satisfies the Lipschitzian condition*

$$\|Tx - Ty\| \leq \frac{1+k}{1-k}\|x - y\|.$$

(ii) *If T is a k-strictly pseudo-contractive mapping, then the mapping $I - T$ is demiclosed at* 0.

Lemma 5 (Sene et al. [24]) *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $\lambda_i \in]0, 1[, i = 1, \ldots, n$ such that $\sum_{i=1}^{n} \lambda_i = 1$. Then,*

$$\left\|\sum_{i=1}^{n} \lambda_i u_i\right\|^2 = \sum_{i=1}^{n} \lambda_i \|u_i\|^2 - \sum_{i<j} \lambda_i\lambda_j \|u_i - u_j\|^2 \qquad \forall u_1, u_2, \ldots, u_n \in K.$$

The following lemma appears implicitly in [10].

Lemma 6 [10] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfies (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \forall y \in C.$$

The following lemma was also given in [28].

Lemma 7 [28] *Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows*

$$T_r(x) = \left\{z \in C,\ F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\}$$

for all $x \in H$. Then, the following hold:

1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., $\|T_r(x) - T_r(y)\|^2 \leq \langle T_r x - T_r y, x - y\rangle$ for any $x, y \in H$;*
3. *$\mathrm{Fix}(T_r) = EP(F)$;*
4. *$EP(F)$ is closed and convex.*

3 Main Results

We now prove the following result.

Theorem 2 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4) and $f : C \to C$ be a contraction with coefficient b. Let $m \geq 1$ be a fixed number and for $1 \leq i \leq m$, let $T_i : C \to CB(C)$ be a multivalued k_i-strictly pseudo-contractive mapping such that $G := \bigcap_{i=1}^{m} \mathrm{Fix}(T_i) \cap EP(F) \neq \emptyset$ and $T_i p = \{p\}\ \forall p \in G$.*

Let $\{x_n\}$ and $\{v_n\}$ be sequences defined iteratively from arbitrary $x_0 \in C$ by

$$\begin{cases} F(v_n, y) + \frac{1}{r_n}\langle y - v_n, v_n - x_n\rangle \geq 0\ \ \forall y \in C, \\ y_n = \lambda_0 v_n + \sum_{i=1}^{m} \lambda_i u_n^i, \ \ u_n^i \in T_i v_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \end{cases} \tag{7}$$

where $\lambda_0 \in]\mu, 1[$, $\mu := \max\{k_i, i = 1, \ldots, m\}$ and $\lambda_i \in]0, 1[$ such that $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset]0, \infty[$ satisfy:

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\lambda_0 + \lambda_1 + \cdots + \lambda_m = 1$,
(iv) $\lim_{n\to\infty} \inf r_n > 0$.

Assume that the mappings $I - T_i$ are demiclosed at the origin. Then, the sequences $\{x_n\}$ and $\{y_n\}$ generated by (7) converge strongly to $x^* \in G$, which is the unique solution of the variational inequality:

$$\langle x^* - f(x^*), x^* - p\rangle \leq 0 \quad \forall p \in G. \tag{8}$$

Proof From $(I - f)$ is strongly monotone and G is closed convex, then the variational inequality (8) has a unique solution in G. Below, we use x^* to denote the unique solution of (8).

Let $p \in G$. Then from $v_n = T_{r_n} x_n$, we have

$$\|v_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \le \|x_n - p\| \quad \forall n \ge 0.$$

We prove that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Using (7) and Lemma 5, we have

$$\begin{aligned}\|y_n - x^*\|^2 &= \left\|\lambda_0(v_n - x^*) + \sum_{i=1}^{m} \lambda_i (u_n^i - x^*)\right\|^2 \\ &= \lambda_0 \|v_n - x^*\|^2 + \sum_{i=1}^{m} \lambda_i \|u_n^i - x^*\|^2 - \sum_{i=1}^{m} \lambda_0 \lambda_i \|u_n^i - v_n\|^2 \\ &\quad - \sum_{1 \le i < j}^{m} \lambda_i \lambda_j \|u_n^i - u_n^j\|^2.\end{aligned}$$

Using that, for $i = 1, \ldots, m$, $T_i x^* = \{x^*\}$, we get

$$\begin{aligned}\|y_n - x^*\|^2 \le\ & \lambda_0 \|v_n - x^*\|^2 + \sum_{i=1}^{m} \lambda_i \left(H(T_i v_n, T_i x^*)\right)^2 - \sum_{i=1}^{m} \lambda_0 \lambda_i \|u_n^i - v_n\|^2 \\ & - \sum_{1 \le i < j}^{m} \lambda_i \lambda_j \|u_n^i - u_n^j\|^2.\end{aligned}$$

Since, for $i = 1, \ldots, m$, T_i is k_i-strictly pseudo-contractive, we have

$$\begin{aligned}\|y_n - x^*\|^2 \le\ & \lambda_0 \|v_n - x^*\|^2 + \sum_{i=1}^{m} \lambda_i \left(\|v_n - x^*\|^2 + k_i \|u_n^i - v_n\|^2\right) \\ & - \sum_{i=1}^{m} \lambda_0 \lambda_i \|u_n^i - v_n\|^2 - \sum_{1 \le i < j}^{m} \lambda_i \lambda_j \|u_n^i - u_n^j\|^2.\end{aligned}$$

Hence,

$$\|y_n - x^*\|^2 \le \|v_n - x^*\|^2 - \sum_{i=1}^{m} \lambda_i (\lambda_0 - k_i) \|u_n^i - v_n\|^2. \tag{9}$$

Since $\lambda_0 \in]\mu, 1[$, we obtain

$$\|y_n - x^*\| \le \|v_n - x^*\| \le \|x_n - x^*\|. \tag{10}$$

From (7) and (10), we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n f(x_n) + (1 - \alpha_n) y_n - x^*\| \\ &\le \alpha_n \|f(x_n) - f(x^*)\| + (1 - \alpha_n)\|y_n - x^*\| + \alpha_n \|f(x^*) - x^*\| \\ &\le (1 - \alpha_n(1 - b))\|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| \\ &\le \max\left\{\|x_n - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - b}\right\}.\end{aligned}$$

By induction, it is easy to see that

$$\|x_n - x^*\| \le \max\left\{\|x_0 - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - b}\right\}, \quad n \ge 1.$$

Hence, $\{x_n\}$ is bounded and also are $\{f(x_n)\}$, and $\{y_n\}$.

Consequently, by inequality (9) and property of μ we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n f(x_n)+(1-\alpha_n)y_n-x^*\|^2\\ &\le \|\alpha_n(f(x_n)-x^*)+(1-\alpha_n)(y_n-x^*)\|^2\\ &\le \alpha_n^2\|f(x_n)-x^*\|^2+(1-\alpha_n)^2\|y_n-x^*\|^2\\ &\quad +2\alpha_n(1-\alpha_n)\|f(x_n)-x^*\|\|y_n-x^*\|\\ &\le \alpha_n^2\|f(x_n)-x^*\|^2+(1-\alpha_n)^2\|v_n-x^*\|^2\\ &\quad -(1-\alpha_n)^2\sum_{i=1}^m\lambda_i(\lambda_0-k_i)\|u_n^i-v_n\|^2\\ &\quad +2\alpha_n(1-\alpha_n)\|f(x_n)-x^*\|\|x_n-x^*\|.\end{aligned}$$

Thus, for every $i,\ 1\le i\le m$, we get

$$\begin{aligned}(1-\alpha_n)^2\sum_{i=1}^m\lambda_i(\lambda_0-k_i)\|u_n^i-v_n\|^2 &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n^2\|f(x_n)-x^*\|^2\\ &\quad +2\alpha_n(1-\alpha_n)\|f(x_n)-x^*\|\|x_n-x^*\|.\end{aligned}$$

Since $\{x_n\}$ and $\{f(x_n)\}$ are bounded, there exists a constant $B>0$ such that for every $i,\ 1\le i\le m$,

$$(1-\alpha_n)^2\sum_{i=1}^m\lambda_i(\lambda_0-k_i)\|u_n^i-v_n\|^2\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_nB. \tag{11}$$

Now we prove that $\{x_n\}$ converges strongly to x^*. We divide the rest of the proof into two cases.

Case 1 Assume that there is $n_0\in N$ such that $\{\|x_n-p\|\}$ is decreasing for all $n\ge n_0$. Since $\{\|x_n-x^*\|\}$ is monotonic and bounded, $\{\|x_n-x^*\|\}$ is convergent. Clearly, we have

$$\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\to 0.$$

This implies from (11) that

$$\lim_{n\to\infty}\sum_{i=1}^m\lambda_i(\lambda_0-k_i)\|u_n^i-v_n\|^2=0\quad\forall i=1,\dots,m.$$

Since $\lambda_0\in]\mu,1[$, we have

$$\lim_{n\to\infty}\|v_n-u_n^i\|^2=0.$$

Since $u_n^i\in T_iv_n$ for each n, it follows that

$$\lim_{n\to\infty}d(v_n,T_iv_n)=0\quad\forall i=1,\dots,m. \tag{12}$$

Let $p\in G$, then for each n, we have

$$\begin{aligned}\|v_n-p\|^2 &= \|T_{r_n}x_n-T_{r_n}p\|^2\\ &\le \langle T_{r_n}x_n-T_{r_n}p,x_n-p\rangle\\ &\le \langle v_n-p,x_n-p\rangle\\ &= \frac{1}{2}(\|v_n-p\|^2+\|x_n-p\|^2-\|x_n-v_n\|^2)\end{aligned}$$

and hence,

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - v_n\|^2. \tag{13}$$

Therefore, from (7) and inequality (13), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)y_n - x^*\|^2 \\
&\le (1-\alpha_n)^2\|y_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)^2\|v_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)^2\|v_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle \\
&\quad +2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\le (1-\alpha_n)^2(\|x_n - x^*\|^2 - \|x_n - v_n\|^2) + 2\alpha_n b\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad +2\alpha_n\|f(x^*) - x^*\|\|x_{n+1} - x^*\| \\
&= (1 - 2\alpha_n + \alpha_n^2)\|x_n - x^*\|^2 - (1-\alpha_n)^2\|x_n - v_n\|^2 \\
&\quad +2\alpha_n b\|x_n - x^*\|\|x_{n+1} - x^*\| + 2\alpha_n\|f(x^*) - x^*\|\|x_{n+1} - x^*\| \\
&\le \|x_n - x^*\|^2 + \alpha_n\|x_n - x^*\|^2 - (1-\alpha_n)^2\|x_n - v_n\|^2 \\
&\quad +2\alpha_n b\|x_n - x^*\|\|x_{n+1} - x^*\| + 2\alpha_n\|f(x^*) - x^*\|\|x_{n+1} - x^*\|,
\end{aligned}$$

and hence

$$\begin{aligned}
(1-\alpha_n)^2\|x_n - v_n\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\|x_n - x^*\|^2 \\
&\quad +2\alpha_n b\|x_n - x^*\|\|x_{n+1} - x^*\| + 2\alpha_n\|f(x^*) - x^*\|\|x_{n+1} - x^*\|.
\end{aligned}$$

So, we have

$$\lim_{n\to\infty}\|x_n - v_n\| = 0.$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^* - f(x^*), x^* - x_n\rangle \le 0$. Since H is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to a in C and

$$\limsup_{n\to+\infty}\langle x^* - f(x^*), x^* - x_n\rangle = \lim_{j\to+\infty}\langle x^* - f(x^*), x^* - x_{n_j}\rangle.$$

From (12) and the fact that the operators $I - T_i$ are demiclosed, we obtain $a \in \bigcap_{i=1}^m \mathrm{Fix}(T_i)$. Without loss of generality, we can assume that $v_{n_k} \rightharpoonup a$. Let us show $a \in EP(F)$. It follows by Lemma 7 and (A2) that

$$\frac{1}{r_n}\langle y - v_n, v_n - x_n\rangle \ge F(y, v_n)$$

and hence

$$\left\langle y - v_{n_k}, \frac{v_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle \ge F(y, v_{n_k}).$$

Since $\frac{v_{n_k} - x_{n_k}}{r_{n_k}} \to 0$ and $v_{n_k} \rightharpoonup a$, it follows from (A4) that $F(y, a) \le 0$ for all $y \in C$. For t with $0 < t < 1$ and $y \in C$, let $y_t = ty + (1-t)a$. Since $y \in C$ and $a \in C$, we have $y_t \in C$ and hence $F(y_t, a) \le 0$. So, from (A1) and (A4) we have

$$0 = F(y_t, y_t) \le tF(y_t, y) + (1-t)F(y_t, a) \le tF(y_t, y)$$

and hence $0 \le F(y_t, y)$. From (A3), we have $F(a, y) \ge 0$ for all $y \in C$ and hence $a \in EP(F)$. Therofore, $a \in G$.

Hence,

$$\begin{aligned}
\limsup_{n\to+\infty}\langle x^* - f(x^*), x^* - x_n\rangle &= \lim_{k\to+\infty}\langle x^* - f(x^*), x^* - x_{n_k}\rangle \\
&= \langle x^* - f(x^*), x^* - a)\rangle \le 0.
\end{aligned}$$

Finally, we show that $x_n \to x^*$. From (7) and Lemma 2, we get that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)y_n - x^*\|^2\\ &\leq \|\alpha_n(f(x_n)-f(x^*)) + (1-\alpha_n)(y_n - x^*)\|^2\\ &\quad +2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle\\ &\leq \big(\alpha_n\|f(x_n)-f(x^*)\| + \|(1-\alpha_n)(y_n-x^*)\|\big)^2\\ &\quad +2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle\\ &\leq \big(\alpha_n b\|x_n - x^*\| + (1-\alpha_n)\|y_n - x^*\|\big)^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle\\ &\leq \big((1-\alpha_n(1-b))\|x_n - x^*\|\big)^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle\\ &\leq (1-\alpha_n(1-b))\|x_n - x^*\|^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle.\end{aligned}$$

From Lemma 3, its follows that $x_n \to x^*$.

Case 2 Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing. Set $B_n = \|x_n - x^*\|^2$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ B_k \leq B_{k+1}\}$.

We have τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \leq B_{\tau(n)+1}$ for $n \geq n_0$. Let $i \in \mathbb{N}^*$, from (11), we have

$$\big(1-\alpha_{\tau(n)}\big)^2 \sum_{i=1} \lambda_i(\lambda_0 - k_i)\left\|v_{\tau(n)} - u^i_{\tau(n)}\right\|^2 \leq \alpha_{\tau(n)} B.$$

Furthermore, we have

$$\lim_{n\to+\infty} \sum_{i=1}^{m} \lambda_i(\lambda_0 - k_i)\left\|v_{\tau(n)} - u^i_{\tau(n)}\right\|^2 = 0.$$

Since $\lambda_0 \in]\mu, 1[$, we can deduce

$$\lim_{n\to\infty} \left\|v_{\tau(n)} - u^i_{\tau(n)}\right\|^2 = 0.$$

Since $u^i_{\tau(n)} \in T_i v_{\tau(n)}$, it follows that

$$\lim_{n\to\infty} d\left(v_{\tau(n)}, T_i v_{\tau(n)}\right) = 0 \quad \forall i = 1, \ldots, m.$$

By a similar argument as in Case 1, we can show that $x_{\tau(n)}$ and $y_{\tau(n)}$ are bounded in C and $\limsup_{\tau(n)\to+\infty}\langle x^* - f(x^*), x^* - x_{\tau(n)})\rangle \leq 0$. We have for all $n \geq n_0$,

$$\begin{aligned}0 &\leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2\\ &\leq \alpha_{\tau(n)}\Big[-(1-b)\|x_{\tau(n)} - x^*\|^2 + 2\langle x^* - f(x^*), x^* - x_{\tau(n)+1}\rangle\Big],\end{aligned}$$

which implies that

$$\|x_{\tau(n)} - x^*\|^2 \leq \frac{2}{1-b}\langle x^* - f(x^*), x^* - x_{\tau(n)+1}\rangle.$$

Then, we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - x^*\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Furthermore, for all $n \geq n_0$, we have $B_{\tau(n)} \leq B_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $n > \tau(n)$); because $B_j > B_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As consequence, we have for all $n \geq n_0$,

$$0 \leq B_n \leq \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

We now apply Theorem 2 when multivalued mappings are nonexpansive mappings with convex-values. In this case demiclosedness assumption is not necessary.

Theorem 3 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4) and $f : C \to C$ be a contraction with coefficient b. Let $m \geq 1$ be a fixed number and $1 \leq i \leq m$, let $T_i : C \to CB(C)$ be a multivalued nonexpansive mapping and convex-values such that $G := \bigcap_{i=1}^{m} \mathrm{Fix}(T_i) \cap EP(F) \neq \emptyset$ and $T_i p = \{p\}\ \forall p \in G$.*

Let $\{x_n\}$ and $\{v_n\}$ be sequences defined iteratively from arbitrary $x_0 \in C$ by:

$$\begin{cases} F(v_n, y) + \frac{1}{r_n}\langle y - v_n, v_n - x_n\rangle \geq 0 \ \ \forall y \in C, \\ y_n = \lambda_0 v_n + \sum_{i=1}^{m} \lambda_i u_n^i, \ \ u_n^i \in T_i v_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \end{cases} \tag{14}$$

where $\lambda_i \in]0, 1[$, $i = 0, \ldots, m$ such that $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset]0, \infty[$ satisfy:

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\lambda_0 + \lambda_1 + \cdots + \lambda_m = 1$.
(iv) $\lim_{n\to\infty} \inf r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{v_n\}$ generated by (14) converge strongly to $x^* \in G$, which is a unique solution of the following variational inequality (8).

Proof Since every multivalued nonexpansive mapping is multivalued strictly pseudo-contractive mapping, then, the proof follows from Lemma 1 and Theorem 2. □

Since every single-valued mapping can be viewed as a multivalued mapping, we obtain from Lemma 4 the following corollary.

Corollary 1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $f : C \to C$ be a contraction with coefficient b. Let $m \geq 1$ be a fixed number and $1 \leq i \leq m$, let $T_i : C \to C$ be a k_i-strictly pseudo-contractive mapping such that $\bigcap_{i=1}^{m} \mathrm{Fix}(T_i) \neq \emptyset$. Let $\{x_n\}$ and $\{v_n\}$ be sequences defined iteratively from arbitrary $x_0 \in C$ by:*

$$\begin{cases} y_n = \lambda_0 v_n + \sum_{i=1}^{m} \lambda_i T_i x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \end{cases} \tag{15}$$

where $\lambda_0 \in]\mu, 1[$, $\mu := \max\{k_i, i = 1, \ldots, m\}$, $\lambda_i \in]0, 1[$, $i = 1, \ldots, m$ and $\{\alpha_n\}$ is a real sequence in $(0, 1)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\lambda_0 + \lambda_1 + \cdots + \lambda_m = 1$.

Then, the sequences $\{x_n\}$ *and* $\{v_n\}$ *generated by* (15) *converge strongly to* $x^* \in \bigcap_{i=1}^m \text{Fix}(T_i)$, *which is the unique solution of the variational inequality*

$$\langle x^* - f(x^*), x^* - p\rangle \leq 0 \quad \forall p \in \bigcap_{i=1}^{m} \text{Fix}(T_i).$$

Proof Put $F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$, we get $u_n = x_n$ in Theorem 2. The proof follows from Theorem 2 and Lemma 4. □

Let K be a nonempty, closed and convex subset of a real Hilbert space, let $T : K \to P(K)$ be a multivalued map and $P_T : K \to CB(K)$ be defined by

$$P_T(x) := \{y \in Tx : \|y - x\| = d(x, Tx)\}.$$

We will need the following result.

Lemma 8 (Song and Cho [25]) *Let* K *be a nonempty subset of a real Banach space and* $T : K \to P(K)$ *be a multi-valued map. Then the following are equivalent:*

(i) $x^* \in \text{Fix}(T)$;
(ii) $P_T(x^*) = \{x^*\}$;
(iii) $x^* \in \text{Fix}(P_T)$. *Moreover,* $\text{Fix}(T) = \text{Fix}(P_T)$.

Now, using the similar arguments as in the proof of Theorem 2 and Lemma 8, we obtain the following result by replacing T by P_T and removing the rigid restriction on $\text{Fix}(T)$ $(Tp = \{p\}\ \forall p \in F(T))$.

Theorem 4 *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C \to \mathbb{R}$ *satisfying (A1)–(A4) and* $f : C \to C$ *be a contraction with coefficient b. Let* $T : C \to CB(C)$ *be a multivalued mapping such that* $G := \text{Fix}(T) \cap EP(F) \neq \emptyset$. *Assume that* P_T *is k-strictly pseudo-contractive.*

Let $\{x_n\}$ *and* $\{v_n\}$ *be sequences defined iteratively from arbitrary* $x_0 \in C$ *by:*

$$\begin{cases} F(v_n, y) + \frac{1}{r_n}\langle y - v_n, v_n - x_n\rangle \geq 0 \ \ \forall y \in C, \\ y_n = \lambda_0 v_n + (1 - \lambda_0)u_n, \quad u_n \in P_T v_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n, \end{cases} \tag{16}$$

where $\lambda_0 \in]k, 1[$ *and* $\{\alpha_n\} \subset (0, 1)$ *and* $\{r_n\} \subset]0, \infty[$ *satisfy:*

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\lim_{n\to\infty} \inf r_n > 0$.

Assume that the mappings $I - P_T$ *is demiclosed at the origin. Then, the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by* (16) *converge strongly to* $x^* \in G$, *which is the unique solution of the variational inequality:*

$$\langle x^* - f(x^*), x^* - p\rangle \leq 0 \quad \forall p \in G.$$

4 Application to Constrained Optimization Problems

Convex optimization theory is a powerful tool for solving many practical problems in operational research. In particular, it has been widely used to solve practical minimization problems over complicated constraints [5, 11], e.g., convex optimization problems with a fixed point constraint and with a variational inequality constraint. Consider the following constrained optimization problem: Let C be a nonempty, closed and convex subset a real Hilbert space H. Given a convex objective function $g : C \to \mathbb{R}$, the problem can be expressed as

$$\text{Minimize } g(x) \quad \text{subject to } x \in C.$$

The set of solutions of (17) is denoted by $\mathrm{Sol}(g)$.

Proposition 1 [26] *Let H be a real Hilbert space. Let $A : H \to H$ be a monotone mapping such that $K := D(A)$ is closed and convex. Assume that A is bounded on bounded subsets and hemi-continuous on K. Then, the bifunction $F(x, y) := \langle Ax, y-x\rangle$ satisfies conditions (A1)–(A4).*

The following basic results are well known.

Lemma 9 *Let H be a real Hilbert space and K be a nonempty closed and convex subset of H. Let $g : H \to \mathbb{R}$ be a real valued differentiable convex function. Let $\nabla g : K \to H$ denotes the differential map associated to g. Then the following hold. If g is bounded, then g is locally Lipschitzian, i.e., for every $x_0 \in K$ and $r > 0$, there exists $\gamma > 0$ such that g is γ-Lipschitzian on $B(x_0, r)$, i.e.,*

$$|g(x) - g(y)| \leq \gamma \|x - y\| \quad \forall x, y \in B(x_0, r).$$

Lemma 10 *Let K be a nonempty, closed convex subset of H and let $g : K \to \mathbb{R}$ a real valued differentiable convex function. Then x^* is a minimizer of g over K if and only if x^* solves the following variational inequality $\langle \nabla g(x^*), x - x^*\rangle \geq 0$ for all $x \in K$.*

Remark 2 Let K be a nonempty, closed convex subset of H. Let $g : K \to \mathbb{R}$ a real valued differentiable convex function. It is well known that the differential map associated to g is monotone.

Lemma 11 *Let K be a nonempty, closed and convex subset of a real Hilbert space H and $g : K \to \mathbb{R}$ be a real valued differentiable convex function. Assume that g is bounded. Then the differentiable map, $\nabla g : K \to H$ is bounded.*

Proof For $x_0 \in K$ and $r > 0$, let $B := B(x_0, r)$. We show that $\nabla g(B)$ is bounded. From Lemma 9, there exists $\gamma > 0$ such that

$$|g(x) - g(y)| \leq \gamma \|x - y\| \quad \forall x, y \in B. \tag{17}$$

Let $z^* \in \nabla g(B)$ and $x^* \in B$ such that $z^* = \nabla g(x^*)$. For $u \in H$, since B is open, there exists $t > 0$ such that $x^* + tu \in B$. Using the fact that $z^* = \nabla g(x^*)$, the convexity of g and the inequality (17), it follows

$$\langle z^*, tu\rangle \leq g(x^* + tu) - g(x^*) \leq t\gamma \|u\|.$$

So that, $\langle z^*, u\rangle \leq \gamma \|u\| \; \forall u \in H$. Therefore, $\|z^*\| \leq \gamma$. Hence, $\nabla g(B)$ is bounded. □

Theorem 5 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $g : C \to \mathbb{R}$ a real valued continuously differentiable convex and bounded function and $f : C \to C$ be a contraction with coefficient b. Let $m \geq 1$ be a fixed number and*

$1 \le i \le m$, let $T_i : C \to CB(C)$ be a multivalued k_i-strictly pseudo-contractive mapping such that $G := \bigcap_{i=1}^m \text{Fix}(T_i) \cap \text{Sol}(g) \neq \emptyset$ and $T_i p = \{p\}\ \forall p \in G$. Assume that $I - T_i$ are demiclosed at the origin.

Let $\{x_n\}$ and $\{v_n\}$ be sequences generated iteratively from arbitrary $x_0 \in C$ by:

$$\begin{cases} \langle \nabla g(v_n), y - v_n \rangle + \frac{1}{r_n} \langle y - v_n, v_n - x_n \rangle \ge 0 \ \ \forall y \in C, \\ y_n = \lambda_0 v_n + \sum_{i=1}^m \lambda_i u_n^i, \ \ u_n^i \in T_i v_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \end{cases}$$

where $\lambda_0 \in]\mu, 1[$, $\mu := \max\{k_i,\ i = 1, \dots, m\}$ and $\lambda_i \in]0, 1[$ such that $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset]0, \infty[$ satisfy:

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) $\lambda_0 + \lambda_1 + \cdots + \lambda_m = 1$,
(iv) $\lim_{n\to\infty} \inf r_n > 0$.

Then, the sequence $\{x_n\}$ converges strongly to x^* solution of (17).

Proof Let $F(x, y) := \langle \nabla g(x), y - x \rangle$ for all $x, y \in C$. From the properties of g, Proposition 1, Remark 2 and Lemma 11, it follows that ∇g is monotone, continuous and bounded on bounded subset on C. So, F satisfies (A1)–(A4). Using the assumption that (17) has a solution and Lemma 10, we have x^* is solution of (17) if and only if $x^* \in EP(F)$. Then, the proof follows from Theorem 2. □

5 Numerical Example

In this section, we present a numerical example to illustrate the convergence behavior of our iteration scheme (16).

Let $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the inner product defined by

$$\langle x, y \rangle = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$$

and let $\| \cdot \| : \mathbb{R}^3 \to \mathbb{R}$ be the usual norm defined by $\|x\| = \sqrt{x_1{}^2 + x_2{}^2 + x_2{}^2}$ for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. For all $x \in \mathbb{R}^3$, let $T : \mathbb{R}^3 \to CB(\mathbb{R}^3)$ defined by

$$Tx = \begin{cases} \left[0, \frac{x}{2}\right], & x \in (0, \infty)^3, \\ \left[\frac{x}{2}, 0\right], & x \in (-\infty, 0]^3. \end{cases}$$

Then P_T is strictly pseudo-contractive. In fact, $P_T(x) = \{\frac{x}{2}\}$ for all $x \in \mathbb{R}^3$. It is easy to see that $\text{Fix}(T) = \{0\}$. Let $F(x, y) := y^2 + yx - 2x^2$, $f(x) = \frac{1}{3}x$ and $T_r(x) = \{z \in \mathbb{R}^3, f(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \ge 0\ \forall y \in \mathbb{R}^3\}$. We can observe that $T_r(x) = \frac{1}{1+3r}x$ and $0 \in \text{Fix}(T) \cap EP(F)$. Choose $r = 1$, $\alpha_n = \frac{1}{n+1}$ and $\lambda_0 = \frac{1}{2}$. Then, the scheme (16) can be simplified as

$$\begin{cases} v_n = \frac{1}{4} x_n, \\ y_n = \frac{3}{16} x_n, \\ x_{n+1} = \frac{1}{3n+3} x_n + \frac{3n}{16n+16} x_n, \ \ n \ge 1. \end{cases}$$

Taking the initial point $x_1 = (1, 2, 3)$, the result of the numerical example obtained by using MATLAB is given in Fig. 1 where it is shown that the sequence of iterates $\{x_n\}$ strongly converges to 0.

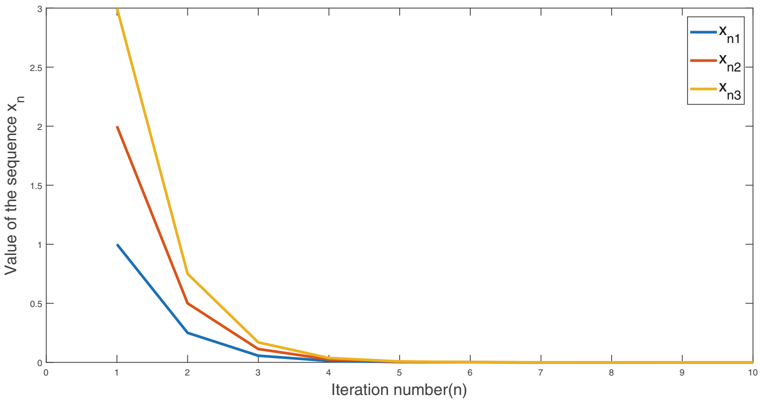

Fig. 1 Two dimensions

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