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ORIGINAL ARTICLE

# Optimal Harvesting for Size-Dependent Control

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**Abstract**
We investigate some optimal harvesting problems concerning the age-structured population dynamics. The goal is to maximize the profit with respect to a size-dependent harvesting effort. The harvesting is made either by fishing or by hunting. For each studied problem, we state the necessary optimality conditions and use them to get the structure of the optimal control. For the problem related to hunting, we derive an iterative algorithm to improve at each step the harvesting effort, in order to increase the profit. Numerical tests are performed to show that the proposed methods are effective.

**Keywords** Age-structured population dynamics · Optimal harvesting problem · Size-dependent harvesting effort

**Mathematics Subject Classification (2010)** 92D25 · 49K20 · 35Q92 · 35Q93 · 49M05

## 1 Introduction and Setting of the Problems

In this paper we investigate some optimal harvesting problems related to age-structured population dynamics. There are many studies that have been developed for such problems. For basic results and methods in age-dependent population dynamics see [3, 18, 23, 25]. For particular harvesting problems related to structured population we refer to [1–6, 8–17, 21, 22, 24, 26, 27], while for optimal control related to biological models see [19, 20]. The particularity in this paper is that we study biological populations for which we can not always accurately assess the age of an individual knowing his size. There is, however, a certain relationship between age and size.

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We shall consider optimal control problems for the following age-structured population dynamics:

$$\begin{cases} \partial_t y(a,t) + \partial_a y(a,t) + \mu(a,t)y(a,t) \\ \qquad\qquad + cy(a,t)\int_0^A y(a,t)da = -u(a,t)y(a,t), & (a,t) \in (0,A)\times(0,T), \\ y(0,t) = \int_0^A \beta(a,t)y(a,t)da, & t\in(0,T), \\ y(a,0) = y_0(a), & a \in (0,A). \end{cases} \tag{1}$$

This is McKendrick's well-known model, where $A \in (0,+\infty)$ is the maximal age for the population species, and $T \in (0,+\infty)$ is the final moment of the harvesting. Here $y(a,t)$ is the density of the population of age $a$ at time $t$, and $y_0(a)$ is the initial density of the population of age $a$; $\mu(a,t)$ is the mortality rate and $\beta(a,t)$ is the fertility rate for individuals of age $a$ at the moment $t$. The term $cy(a,t)\int_0^A y(a,t)da$ is a logistic one, with $c > 0$ a constant, where $c\int_0^A y(a,t)da$ is an additional mortality rate due to the competition for resources. The population is subject to a harvesting effort $u = u(a,t)$. The control $u$ can be seen as an additional mortality rate.

Here are the hypotheses that we are going to use in the following:

**(H1)** $\beta \in C([0,A]\times[0,T])$, $\mu \in C([0,A)\times[0,T])$,

$$\beta(a,t) \geq 0, \quad \forall(a,t)\in[0,A]\times[0,T], \quad \mu(a,t)\geq 0, \quad \forall(a,t)\in[0,A)\times[0,T];$$

**(H2)** $y_0 \in C([0,A])$, $y_0(a) > 0$, $a \in (0,A)$.

We denote by $\mathcal{U} = \{w \in L^\infty((0,A)\times(0,T));\ 0 \leq w(a,t) \leq L \text{ a.e. in } (0,A)\times(0,T)\}$, with $L \in (0,+\infty)$ a constant. We say that function $y \in L^\infty((0,A)\times(0,T))$ is the solution to problem (1), if it is absolutely continuous along almost any characteristic line (of equation $a - t =$ const., $(a,t)\in[0,A]\times[0,T]$), and satisfies

$$\begin{cases} Dy(a,t) + \mu(a,t)y(a,t) \\ \qquad\qquad + cy(a,t)\int_0^A y(a,t)da = -u(a,t)y(a,t), & \text{a.e. } (a,t)\in(0,A)\times(0,T), \\ \lim_{\varepsilon\to 0+} y(\varepsilon, t+\varepsilon) = \int_0^A \beta(a,t)y(a,t)da, & \text{a.e. } t\in(0,T) \\ \lim_{\varepsilon\to 0+} y(a+\varepsilon,\varepsilon) = y_0(a), & \text{a.e. } a\in(0,A), \end{cases}$$

where $Dy$ is the directional derivative, $Dy(a,t) = \lim_{\varepsilon\to 0}\frac{y(a+\varepsilon,t+\varepsilon)-y(a,t)}{\varepsilon}$. For the definition of the solutions to age-dependent population models we refer to Chapter 2 in [3]. It follows that, for any $u \in \mathcal{U}$, there exits a unique solution $y^u$ to problem (1) (see [3]).

Our first goal is to investigate the following optimal control problem

$$\underset{u\in\mathcal{U}}{\text{Maximize}}\ \Phi(u), \tag{2}$$

where $\Phi(u) = \int_0^T\int_0^A u(a,t)y(a,t)da\,dt$, and represents the total harvest.

Using the ideas in [1], it can be proved the following result:

**Theorem 1** *Problem* (2) *admits at least one optimal control* $u^*$.

In the following, we shall state the necessary optimality conditions and use them to get the structure of the optimal effort. Let us denote by $p = p(a,t)$, the solution to the following problem:

$$\begin{cases} \partial_t p(a,t) + \partial_a p(a,t) - \mu(a,t)p(a,t) = -\beta(a,t)p(0,t) \\ \qquad + c\, p(a,t)\int_0^A y^{u^*}(a,t)da + c\int_0^A y^{u^*}(a,t)p(a,t)da \\ \qquad + u^*(a,t)(1+p(a,t)), & (a,t) \in (0,A)\times(0,T), \\ p(A,t) = 0, & t \in (0,T), \\ p(a,T) = 0, & a \in (0,A), \end{cases} \tag{3}$$

where $u^*$ is an optimal control for (2), and $y^{u^*}$ is the solution to (1) corresponding to the harvesting effort $u^*$. As above, we say that a function $p \in L^\infty((0,A)\times(0,T))$ is the solution to adjoint state system (3), if it is absolutely continuous along almost any characteristic line, and satisfies

$$\begin{cases} Dp(a,t) - \mu(a,t)p(a,t) = -\beta(a,t)p(0,t) \\ \quad + c\, p(a,t)\int_0^A y^{u^*}(a,t)da + c\int_0^A y^{u^*}(a,t)p(a,t)da \\ \quad + u^*(a,t)(1+p(a,t)), & \text{a.e. } (a,t) \in (0,A)\times(0,T), \\ \lim_{\varepsilon\to 0+} p(A-\varepsilon, t-\varepsilon) = 0, & \text{a.e. } t \in (0,T), \\ \lim_{\varepsilon\to 0+} p(a-\varepsilon, T-\varepsilon) = 0, & \text{a.e. } a \in (0,A). \end{cases}$$

By $p(0,t)$ we mean $\lim_{\varepsilon\to 0+} p(\varepsilon, t+\varepsilon)$ a.e. $t \in (0,T)$. The existence and uniqueness of the solution to (3) follow a fixed point result (for details, see [3]).

It is possible to prove that the following necessary optimality conditions hold (see Section 2)

**Theorem 2** *If* $u^*$ *is an optimal effort for problem* (2), *and* $p$ *is the solution to* (3), *then*

$$u^*(a,t) = \begin{cases} 0 & \text{if } 1 + p(a,t) < 0, \\ L & \text{if } 1 + p(a,t) > 0. \end{cases} \tag{4}$$

When we intend to develop a harvesting strategy for a population species, we are interested in determining the age of individuals. In order to do that, we try to estimate their size. We know that the size of individuals is a nondecreasing and continuous function of age. We denote by $s = s(a)$, $a \in [0,A]$ the size of an individual over its lifetime. An individual grows beginning from his birth until he reaches a certain age $\bar{a}$, $\bar{a} \in (0,A)$, when the growth stops and the size as well. Actually, the function $s$ is a strictly increasing on $[0,\bar{a}]$ and constant on $[\bar{a},A]$ (see Fig. 1). Thus, $s : [0,A] \longrightarrow [0,S]$ is a nondecreasing and continuous function, with $S = s(\bar{a})$.

We can precisely specify the age of an individual, if his size is less than $s(\bar{a})$. Otherwise, if his size is equal to $s(\bar{a}) = S$, we can only say that its age is in the interval $[\bar{a},A]$.

It is obvious now that a special attention is due to the problems where the harvesting effort is dependent on the size of individuals. We shall deal with two such problems: the first is specific to fishing and the second, to hunting.

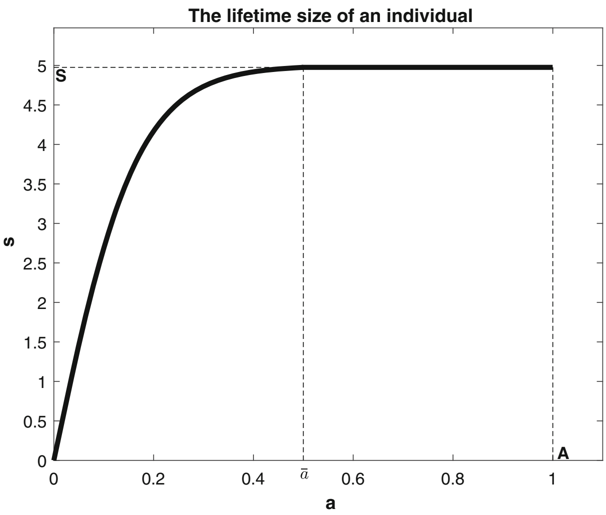



**Fig. 1** The function $s = s(a)$, $a \in [0, A]$

For the first kind of problem, let assume that we are interested in industrial fishing. Such problems were studied in [6]. Because we cannot distinguish between individuals of age in the interval $[\bar{a}, A]$, the harvesting effort will be the same for them. In this case, we can vary in time the intensity of harvesting effort. We denote by $v(t)$ the intensity of the harvesting at the moment $t$. The individuals with a size greater than a certain value, which corresponds to ages greater than another certain value $\gamma(t) \in [0, \bar{a}]$, are harvested at the same rate. The two functions, $v : [0, T] \longrightarrow [0, L]$, and $\gamma : [0, T] \longrightarrow [0, \bar{a}]$ has to be determined. In order to maximize the harvest, we must obtain the optimal intensity of the harvesting, denoted by $v^*$, and the optimal minimal age of the harvested population $\gamma^*$, which indicates the optimal region where the control acts (the hashed area in Fig. 2). Thus, we can treat this problem as a shape optimization one.

The optimal control problem (2) becomes

$$\underset{(v,\gamma)\in\mathcal{V}\times\mathcal{G}_1}{\text{Maximize}}\ J_1(v, \gamma), \tag{5}$$

where $J_1(v, \gamma) = \int_0^T \int_0^A v(t)H(a - \gamma(t))y(a, t)da\,dt$, $\mathcal{V} = \{r \in L^\infty(0, T);\ 0 \le r(t) \le L$ a.e. in $(0, T)\}$, $\mathcal{G}_1 = \{\zeta \in L^\infty(0, T);\ 0 \le \zeta(t) \le \bar{a}$ a.e. in $(0, T)\}$, and $y$ is the solution to (1) corresponding to the harvesting effort $u(a, t) = v(t)H(a - \gamma(t))$.

If we have to pay a cost in order to harvest in the hashed region (see Fig. 2), the optimal harvesting problem becomes

$$\underset{(v,\gamma)\in\mathcal{V}\times\mathcal{G}_2}{\text{Maximize}}\ J_2(v, \gamma), \tag{6}$$

where $J_2(v, \gamma) = \int_0^T \int_0^A v(t)H(a-\gamma(t))y(a, t)da\,dt - k\int_0^T (1+\gamma'(t)^2)^{\frac{1}{2}}dt$, assuming that $\gamma$ is smooth, $k \in (0, +\infty)$ is a constant, and $\mathcal{G}_2 = \{\zeta \in C^1([0, T]); 0 \le \zeta(t) \le \bar{a}\ \ \forall t \in [0, T]\}$. Here, the cost is proportional to the length of $\{(t, \gamma(t));\ t \in [0, T]\}$.

Let assume now that we are interested in hunting. In this case, the harvesting effort is no longer the same for the individuals with a size greater than a certain value. Because the size of an individual is a strictly increasing function on $[0, \bar{a}]$ and constant on $[\bar{a}, A]$, we may

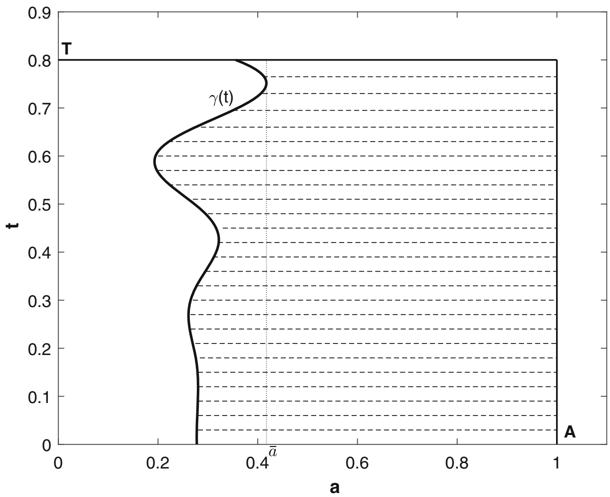



**Fig. 2** The region where the control acts

consider that the harvesting effort is constant on $[\bar{a}, A]$, with respect to age. This means that the harvesting effort satisfies $u(s(a), t) = u(S, t)$ for any $a \in [\bar{a}, A]$ and $t \in [0, T]$.

We denote by $\mathcal{W} = \{w : [0, S] \times [0, T] \longrightarrow \mathbb{R};\ w$ is measurable, $\exists h \in L^{\infty}(0, T)$ such as $w(s(a), t) = w(S, t) = h(t)$ a.e. in $(\bar{a}, A) \times (0, T)\}$, and by $\bar{\mathcal{U}} = \{w \in \mathcal{W};\ 0 \le w(s(a), t) \le L$ a.e. in $(0, A) \times (0, T)\}$. We shall investigate the following optimal harvesting problem related to hunting:

$$\underset{u \in \bar{\mathcal{U}}}{\text{Maximize}}\ J_3(u), \tag{7}$$

where $J_3(u) = \int_0^T \int_0^A u(s(a), t) y(a, t) da\, dt$, $u = u(s(a), t)$ is the harvesting effort, and $y$ is the solution to the following problem:

$$\begin{cases} \partial_t y(a, t) + \partial_a y(a, t) + \mu(a, t) y(a, t) \\ \qquad + c y(a, t) \displaystyle\int_0^A y(a, t) da = -u(s(a), t) y(a, t), & (a, t) \in (0, A) \times (0, T), \\ y(0, t) = \displaystyle\int_0^A \beta(a, t) y(a, t) da, & t \in (0, T), \\ y(a, 0) = y_0(a), & a \in (0, A). \end{cases} \tag{8}$$

The cost of the harvesting effort (if is not negligible) can be taken into account for this third problem as well. One possibility is to modify the cost functional by

$$\tilde{J}_3(u) = \int_0^T \int_0^A u(s(a), t) y(a, t) da\, dt - \eta \int_0^T \int_0^A u(s(a), t) da\, dt,$$

where $\eta > 0$ is a constant. The approach in this case is similar.

The paper is organized as follows. The proof of Theorem 2 is given in the second section. In Section 3 we remind the results obtained in [6] for the two optimal problems related to fishing. The optimal harvesting problem proposed for the case when the harvesting is made by hunting is treated in Section 4. In Section 5 is derived an approximation for the optimal harvesting effort obtained for the problem (7).

## 2 Proof of Theorem 2

This proof follows the lines in [3]. Let us consider $w \in L^\infty((0, A) \times (0, T))$, arbitrary but fixed, such that $u^* + \varepsilon w \in \mathcal{U}$ for sufficiently small $\varepsilon > 0$. It can be proved that

$$y^{u^*+\varepsilon w} \longrightarrow y^{u^*}, \quad \frac{y^{u^*+\varepsilon w} - y^{u^*}}{\varepsilon} \longrightarrow z \quad \text{in} \quad L^\infty((0, A) \times (0, T)),$$

as $\varepsilon \longrightarrow 0+$, where $z = z(a, t)$ is the solution to

$$\begin{cases} \partial_t z(a,t) + \partial_a z(a,t) + \mu(a,t) z(a,t) \\ +c\, z(a,t) \int_0^A y^{u^*}(a,t)da + c\, y^{u^*}(a,t) \int_0^A z(a,t)da \\ = -[u^*(a,t)z(a,t) + w(a,t)y^{u^*}(a,t)], & (a,t) \in (0,A) \times (0,T), \\ z(0,t) = \int_0^A \beta(a,t) z(a,t) da, & t \in (0,T), \\ z(a,0) = 0, & a \in (0,A). \end{cases} \tag{9}$$

A function $z \in L^\infty((0, A) \times (0, T))$ is a solution to (9) if it is absolutely continuous along almost any characteristic line and satisfies

$$\begin{cases} Dz(a,t) + \mu(a,t) z(a,t) \\ +c\, z(a,t) \int_0^A y^{u^*}(a,t)da + c\, y^{u^*}(a,t) \int_0^A z(a,t)da \\ = -[u^*(a,t)z(a,t) + w(a,t)y^{u^*}(a,t)], & \text{a.e. } (a,t) \in (0,A) \times (0,T), \\ \lim_{\varepsilon \to 0+} z(\varepsilon, t+\varepsilon) = \int_0^A \beta(a,t) z(a,t) da, & \text{a.e. } t \in (0,T), \\ \lim_{\varepsilon \to 0+} z(a+\varepsilon, \varepsilon) = 0, & \text{a.e. } a \in (0,A) \end{cases}$$

(for the existence and uniqueness of the solution to (9), see [3, Chapter 2]).

From the optimality of $u^*$, we know that $\Phi(u^*) \geq \Phi(u^* + \varepsilon w)$, and than

$$\int_0^T \int_0^A u^*(a,t) \frac{y^{u^*+\varepsilon w}(a,t) - y^{u^*}(a,t)}{\varepsilon} da\, dt + \int_0^T \int_0^A w(a,t) y^{u^*+\varepsilon w}(a,t) da\, dt \leq 0.$$

Passing to the limit for $\varepsilon \longrightarrow 0+$, we get that

$$\int_0^T \int_0^A u^*(a,t) z(a,t) dadt + \int_0^T \int_0^A w(a,t) y^{u^*}(a,t) dadt \leq 0. \tag{10}$$

Multiplying the first equation in (3) by $z$ and integrating over $(0, T) \times (0, A)$, we obtain

$$\begin{aligned} &\int_0^T \int_0^A z(a,t)[Dp(a,t) - \mu(a,t)p(a,t)] dadt \\ &= \int_0^T \int_0^A z(a,t) \Big[ -\beta(a,t)p(0,t)dadt + cp(a,t) \int_0^A y^{u^*}(a,t)da \\ &\quad + c \int_0^A y^{u^*}(a,t)p(a,t)da + u^*(1 + p(a,t)) \Big] dadt. \end{aligned}$$

Integrating by parts with respect to $t$ and to $a$, and using also (3) and (9), we get that

$$-\int_0^T z(0,t)p(0,t)dt - \int_0^T\int_0^A p(a,t)[Dz(a,t)+\mu(a,t)z(a,t)]dadt$$
$$= -\int_0^T z(0,t)p(0,t)dt + \int_0^T\int_0^A z(a,t)\left[cp(a,t)\int_0^A y^{u^*}(a,t)da\right.$$
$$\left. +c\int_0^A y^{u^*}(a,t)p(a,t)da + u^*(a,t)(1+p(a,t))\right] da\, dt,$$

and

$$\int_0^T\int_0^A p(a,t)\left[cz(a,t)\int_0^A y^{u^*}(a,t)da + c\, y^{u^*}(a,t)\int_0^A z(a,t)da\right.$$
$$\left. +u^*(a,t)z(a,t) + w(a,t)y^{u^*}(a,t)\right] da\, dt$$
$$= \int_0^T\int_0^A z(a,t)\left[cp(a,t)\int_0^A y^{u^*}(a,t)da\right.$$
$$\left. +c\int_0^A y^{u^*}(a,t)p(a,t)da + u^*(a,t)(1+p(a,t))\right] dadt.$$

We may infer now that

$$\int_0^T\int_0^A u^*(a,t)z(a,t)da\, dt = \int_0^T\int_0^A w(a,t)y^{u^*}(a,t)p(a,t)da\, dt.$$

Using (10) we get

$$\int_0^T\int_0^A w(a,t)\left[y^{u^*}(a,t) + y^{u^*}(a,t)p(a,t)\right] da\, dt \le 0$$

for any $w \in L^\infty((0,A)\times(0,T))$, such that $u^* + \varepsilon w \in \mathcal{U}$, for sufficiently small $\varepsilon > 0$. Since $y^{u^*}(a,t) > 0$ a.e. $(a,t) \in (0,A)\times(0,T)$, we obtain (4).

*Remark 1* This implies that $p$ is the solution to

$$\begin{cases} \partial_t p(a,t) + \partial_a p(a,t) - \mu(a,t)p(a,t) = -\beta(a,t)p(0,t) & \\ +c\, p(a,t)\int_0^A y^{u^*}(a,t)da + c\int_0^A y^{u^*}(a,t)p(a,t)da & \\ +L(1+p(a,t))^+, & (a,t) \in (0,A)\times(0,T), \\ p(A,t) = 0, & t \in (0,T), \\ p(a,T) = 0, & a \in (0,A). \end{cases}$$

If $p(a,t) \neq -1$ a.e. in $(0,A)\times(0,T)$, then $u^*(a,t) = LH(1+p(a,t))$, where $H$ is the Heaviside function. It can be found sufficient conditions on the mortality and fertility rates such that $p(a,t) \neq -1$ a.e. in $(0,A)\times(0,T)$.

If, for example, for almost any $t \in (0,T)$ and for any real number $\eta$, $\mu(\cdot,t) + \eta\beta(\cdot,t)$ is not a constant on a subset of positive measure of $(0,A)$, then the set

$$\{(a,t) \in (0,A)\times(0,T);\; p(a,t) = -1\}$$

is negligible, and we get the conclusion. This also implies that the optimal control is of bang-bang type.

*Remark 2* Multiplying the first equation in (3) by $y^{u^*}$ and integrating over $(0, T) \times (0, A)$, we get the maximal harvest

$$\Phi(u^*) = -\int_0^A y_0(a)p(a,0)da - c\int_0^T \int_0^A \int_0^A y^{u^*}(a',t)p(a',t)y^{u^*}(a,t)da'dadt.$$

## 3 Optimal Harvesting by Fishing

In [6] we have investigated both optimal harvesting problems related to the fishing issue, namely the problems (5) and (6). For these problems we have stated the necessary optimality conditions and we have used them to get the structure of the optimal effort.

Firstly, let us remind the results that we have obtained in [6] for the optimal harvesting problem (5).

**Theorem 3** *Problem* (5) *admits at least one optimal control* $(v^*, \gamma^*)$.

We denote by $p = p(a,t)$ the adjoint state, meaning the solution to:

$$\begin{cases} \partial_t p(a,t) + \partial_a p(a,t) - \mu(a,t)p(a,t) = -\beta(a,t)p(0,t) \\ +c\, p(a,t)\int_0^A y^{v^*,\gamma^*}(a,t)da + c\int_0^A y^{v^*,\gamma^*}(a,t)p(a,t)da \\ +v^*(t)H(a-\gamma^*(t))(1+p(a,t)), & (a,t) \in (0,A)\times(0,T), \\ p(A,t) = 0, & t \in (0,T), \\ p(a,T) = 0, & a \in (0,A), \end{cases} \tag{11}$$

where $(v^*, \gamma^*)$ is an optimal control for (5), and $y^{v^*,\gamma^*}$ is the solution to (1) corresponding to the harvesting effort $u(a,t) = v^*(t)H(a-\gamma^*(t))$. The following result has been proved in [6].

**Theorem 4** *If* $(v^*, \gamma^*)$ *is an optimal control for problem* (5)*, and* $p$ *is solution to* (11)*, then*

$$v^*(t) = \begin{cases} 0 & \text{if } \int_{\gamma^*(t)}^A y^{v^*,\gamma^*}(a,t)(1+p(a,t))da < 0, \\ L & \text{if } \int_{\gamma^*(t)}^A y^{v^*,\gamma^*}(a,t)(1+p(a,t))da > 0 \end{cases}$$

*and*

$$\gamma^*(t) = \begin{cases} 0 & \text{if } v^*(t) > 0 \text{ and } y^{v^*,\gamma^*}(\gamma^*(t),t)(1+p(\gamma^*(t),t)) > 0, \\ \bar{a} & \text{if } v^*(t) > 0 \text{ and } y^{v^*,\gamma^*}(\gamma^*(t),t)(1+p(\gamma^*(t),t)) < 0. \end{cases} \tag{12}$$

*Remark 3* If $y^{v^*,\gamma^*}(a,t) > 0$ a.e. $(a,t) \in (0,A)\times(0,T)$, then we may rewrite (12) as

$$\gamma^*(t) = \begin{cases} 0 & \text{if } v^*(t) > 0 \text{ and } p(\gamma^*(t),t) > -1, \\ \bar{a} & \text{if } v^*(t) > 0 \text{ and } p(\gamma^*(t),t) < -1. \end{cases}$$

Based on these theoretical results, we have developed in [6] an iterative algorithm to increase the profit at each iteration, by varying the intensity of the harvesting effort and changing the region where the control is acting. Some numerical tests are given in order to illustrate the effectiveness of the theoretical results.

A particular problem was also studied in [6], which was obtained for $\gamma \equiv 0$. In this situation, the harvesting effort is $v(t)$, the intensity of the fishing. Remark that, when $v(t) = 0$ we have prohibition periods for fishing, and when $v(t) > 0$, the intensity of harvest is the same for all ages. The unique solution to (1) is separable. Based on necessary optimality conditions, it was obtained the structure of the optimal control (which is in this case $u(a,t) = v(t)$).

Regarding the optimal harvesting problem (6) we obtaind in [6] the following result:

**Theorem 5** *Problem* (6) *admits at least one optimal control* $(v^*, \gamma^*)$.

If $(v^*, \gamma^*)$ is an optimal effort for problem (6), and $p$ is solution to (11), then

$$v^*(t) = \begin{cases} 0 & \text{if } \int_{\gamma^*(t)}^{A} y^{v^*,\gamma^*}(a,t)(1+p(a,t))da < 0, \\ L & \text{if } \int_{\gamma^*(t)}^{A} y^{v^*,\gamma^*}(a,t)(1+p(a,t))da > 0 \end{cases}$$

and the gradient ascend with respect to $\gamma$ is

$$\begin{cases} \partial_\theta \gamma(t,\theta) = k\dfrac{\partial_{tt}\gamma(t,\theta)}{(1+\partial_t\gamma(t,\theta)^2)^{\frac{3}{2}}} \\ +v^*(t)y^{v^*,\gamma^*}(\gamma(t,\theta),t)(1+p(\gamma(t,\theta),t)) - \psi(\gamma(t,\theta)), \quad t \in (0,T), \ \theta > 0, \\ \partial_t\gamma(T,\theta) + \psi(\gamma(T,\theta)) = \partial_t\gamma(0,\theta) - \psi(\gamma(0,\theta)) = 0, \quad \theta > 0, \end{cases}$$

where $\psi = \partial\Psi$ is the subdifferential of $\Psi$,

$$\Psi(r) = \begin{cases} 0 & \text{if } r \in [0,\bar{a}], \\ +\infty & \text{if } r \in \mathbb{R} \setminus [0,\bar{a}]. \end{cases}$$

As in [5], we may derive an iterative algorithm to improve, at each step, $w$ and $\gamma$, in order to obtain a bigger value for $J_2$.

## 4 Optimal Harvesting by Hunting

The goal of this section is to investigate the optimal harvesting problem (7). If $a > \bar{a}$, the harvesting effort is constant with respect to age, thus $u = u(s(a),t) = u(s(\bar{a}),t) = u(S,t) = \tilde{u}(t)$, for $a > \bar{a}$. Following the ideas in [1] it can be proved that

**Theorem 6** *Problem* (7) *admits at least one optimal control* $u^*$.

Let us denote by $p = p(a,t)$ the the solution to the problem

$$\begin{cases} \partial_t p(a,t) + \partial_a p(a,t) - \mu(a,t)p(a,t) = -\beta(a,t)p(0,t) \\ \quad +c\, p(a,t)\displaystyle\int_0^A y^{u^*}(a,t)da + c\int_0^A y^{u^*}(a,t)p(a,t)da \\ \quad +u^*(s(a),t)(1+p(a,t)), & (a,t) \in (0,A)\times(0,T), \\ p(A,t) = 0, & t \in (0,T), \\ p(a,T) = 0, & a \in (0,A), \end{cases} \tag{13}$$

where $u^*$ is an optimal control for (7), and $y^{u^*}$ is the solution to (8) corresponding to the harvesting effort $u^*$. For the existence and uniqueness of the solution to (13) see [3].

**Theorem 7** *If* $u^*$ *is an optimal effort for problem* (7)*, and* $p$ *is solution to* (13)*, then*

$$u^*(s(a),t) = \begin{cases} 0 & \textit{if } 1+p(a,t) < 0, \\ L & \textit{if } 1+p(a,t) > 0 \ \textit{for } a \in [0,\bar{a}), \end{cases} \tag{14}$$

*and*

$$\tilde{u}^*(t) = \begin{cases} 0 & \text{if } \int_{\bar{a}}^{A} y^{u^*}(a,t)(1+p(a,t))da < 0, \\ L & \text{if } \int_{\bar{a}}^{A} y^{u^*}(a,t)(1+p(a,t))da > 0 \text{ for } a > \bar{a}. \end{cases} \tag{15}$$

*Proof* We take $w \in \mathcal{W}$, arbitrary but fixed, such that $u^* + \varepsilon w \in \bar{\mathcal{U}}$ for sufficiently small $\varepsilon > 0$. It is possible to prove that

$$y^{u^*+\varepsilon w} \longrightarrow y^{u^*}, \quad \frac{y^{u^*+\varepsilon w} - y^{u^*}}{\varepsilon} \longrightarrow z \quad \text{in} \quad L^\infty((0,A)\times(0,T)),$$

as $\varepsilon \longrightarrow 0+$, where $z = z(a,t)$ is the solution to

$$\begin{cases} \partial_t z(a,t) + \partial_a z(a,t) + \mu(a,t)z(a,t) \\ +c\, z(a,t)\int_0^A y^{u^*}(a,t)da + c\, y^{u^*}(a,t)\int_0^A z(a,t)da \\ \qquad = -[u^*(s(a),t)z(a,t) + w(s(a),t)y^{u^*}(a,t)], & (a,t) \in (0,A)\times(0,T), \\ z(0,t) = \int_0^A \beta(a,t)z(a,t)da, & t \in (0,T), \\ z(a,0) = 0, & a \in (0,A). \end{cases} \tag{16}$$

From the optimality of $u^*$ we have that

$$\int_0^T\int_0^A u^*(s(a),t)y^{u^*}(a,t)dadt \geq \int_0^T\int_0^A (u^*(s(a),t) + \varepsilon w(s(a),t))y^{u^*+\varepsilon w}(a,t)dadt,$$

and than

$$0 \geq \int_0^T\int_0^A u^*(s(a),t)\frac{y^{u^*+\varepsilon w}(a,t) - y^{u^*}(a,t)}{\varepsilon}dadt + \int_0^T\int_0^A w(s(a),t)y^{u^*+\varepsilon w}(a,t)dadt,$$

for any sufficiently small $\varepsilon > 0$ and $w \in \mathcal{W}$ such that $u^* + \varepsilon w \in \bar{\mathcal{U}}$. We pass now to the limit for $\varepsilon \longrightarrow 0+$, and we obtain that

$$0 \geq \int_0^T\int_0^A (u^*(s(a),t)z(a,t) + w(s(a),t)y^{u^*}(a,t))dadt, \tag{17}$$

for any $w \in \mathcal{W}$, such that $u^* + \varepsilon w \in \bar{\mathcal{U}}$, for sufficiently small $\varepsilon > 0$. Integrating the first equation in (13) over $(0,T)\times(0,A)$, after multiplying it by $z$, we get that

$$\begin{aligned} &\int_0^T\int_0^A z(a,t)[Dp(a,t) - \mu(a,t)p(a,t)]dadt \\ &= \int_0^T\int_0^A z(a,t)\left[-\beta(a,t)p(0,t)dadt + cp(a,t)\int_0^A y^{u^*}(a,t)da \right. \\ &\quad \left. + c\int_0^A y^{u^*}(a,t)p(a,t)da + u^*(s(a),t)(1+p(a,t))\right]dadt. \end{aligned}$$

Integrating by parts with respect to $t$ and to $a$, and using also (13) and (16), we obtain that

$$\int_0^T\int_0^A u^*(s(a),t)z(a,t)dadt = \int_0^T\int_0^A w(s(a),t)y^{u^*}(a,t)p(a,t)dadt.$$

Using (17) we get

$$0 \geq \int_0^T\int_0^A w(s(a),t)y^{u^*}(a,t)(1+p(a,t))dadt$$

for any $w \in \mathcal{W}$, such that $u^* + \varepsilon w \in \bar{\mathcal{U}}$, for sufficiently small $\varepsilon > 0$. Equivalently, we may write

$$0 \geq \int_0^T \int_0^{\bar{a}} w(s(a), t) y^{u^*}(a, t)(1 + p(a, t)) da\, dt + \int_0^T \int_{\bar{a}}^A w(s(\bar{a}), t) y^{u^*}(a, t)(1 + p(a, t)) da\, dt$$

and

$$0 \geq \int_0^T \int_0^{\bar{a}} w(s(a), t) y^{u^*}(a, t)(1 + p(a, t)) da\, dt + \int_0^T w(s(\bar{a}), t) \int_{\bar{a}}^A y^{u^*}(a, t)(1 + p(a, t)) da\, dt, \tag{18}$$

for any $w \in \mathcal{W}$, such that $u^* + \varepsilon w \in \bar{\mathcal{U}}$, for sufficiently small $\varepsilon > 0$.

To complete the proof, let suppose by contradiction that exists a set $\mathcal{B}_0 \subset [0, \bar{a}) \times (0, T)$, with $m(\mathcal{B}_0) > 0$, such that $1 + p(a, t) < 0$ and $u^*(s(a), t) \in (0, L]$ on $\mathcal{B}_0$, $a \in [0, \bar{a})$. In order that $0 \leq u^*(s(a), t) + \varepsilon w(s(a), t) \leq L$, we take $w \equiv -1$ on $\mathcal{B}_0$, and $w \equiv 0$, otherwise. We get that

$$0 \geq \iint_{\mathcal{B}_0} (-1) y^{u^*}(a, t)(1 + p(a, t)) da\, dt > 0,$$

which is a contradiction. We obtain that $u^*(s(a), t) = 0$, for $1 + p(a, t) < 0$ and $a \in [0, \bar{a})$. In the same manner we can also prove that $u^*(s(a), t) = L$, for $1 + p(a, t) > 0$ and $a \in [0, \bar{a})$. Let now suppose by contradiction that exists a set $\mathcal{B}_1 \subset (0, T)$, with $m(\mathcal{B}_1) > 0$, such that $\int_{\bar{a}}^A y^{u^*}(a, t)(1 + p(a, t)) da < 0$ and $\tilde{u}^*(t) = u^*(s(\bar{a}), t) \in (0, L]$ on $\mathcal{B}_1$, $a > \bar{a}$. Because $0 \leq u^*(s(\bar{a}), t) + \varepsilon w(s(\bar{a}), t) \leq L$, we consider $w(s(\bar{a}), t) = -1$ on $\mathcal{B}_1$, and 0, otherwise. From (18), we get that

$$0 \geq \int_{\mathcal{B}_1} (-1) \int_{\bar{a}}^A y^{u^*}(a, t)(1 + p(a, t)) da\, dt > 0,$$

which is absurd. Thus, $\tilde{u}^*(t) = 0$, for $\int_{\bar{a}}^A y^{u^*}(a, t)(1 + p(a, t)) da < 0$, $a > \bar{a}$. In the same manner we can prove also that $\tilde{u}^*(t) = L$, for $\int_{\bar{a}}^A y^{u^*}(a, t)(1 + p(a, t)) da > 0$, $a > \bar{a}$. Thus, we obtain the conclusion of the theorem. □

## 5 An Iterative Algorithm for Problem (7)

Using Theorem 7, we can develop an iterative algorithm to improve, at each iteration, the harvesting effort $u$, in order to obtain a higher value for the cost functional $J_3$.

**Step 0:** `Set` $k := 0$ `and` $J_3^{(0)}$ `to a small value;`
`Initialize` $u^{(0)}(s(a), t)$`;`
**Step 1:** `Compute` $y^{(k+1)}$ `the solution of (8) corresponding to` $u^{(k)}$`;`
`Compute the integral`

$$J_3^{(k+1)} = \int_0^T \int_0^A u^{(k)}(s(a), t) y^{(k+1)}(a, t) da\, dt.$$

**Step 2:** `If` $\left|J_3^{(k+1)} - J_3^{(k)}\right| < \varepsilon_1$ `or` $J_3^{(k+1)} \le J_3^{(k)}$ `then` **`STOP;`**
`Else go to` **`Step 3.`**
**Step 3:** `Compute` $p^{(k+1)}$ `the solution of (13) corresponding to` $u^{(k)}$`;`
`Compute` $\bar{u}^{(k+1)}$ `according to the formulae (14) and (15).`
`Define` $\hat{u}_\lambda^{(k+1)} = \lambda u^{(k)} + (1-\lambda)\bar{u}^{(k+1)}$`, for` $\lambda \in [0, 1]$`.`
`Compute` $\lambda^* \in [0, 1]$ `the solution of the maximization problem`

$$\text{Maximize } J_3\left(\hat{u}_\lambda^{(k+1)}\right) \text{ subject to } \lambda \in [0, 1];$$

`Set` $u^{(k+1)} = \hat{u}_{\lambda^*}^{(k+1)}$`.`
**Step 4:** `If` $\|u^{(k+1)} - u^{(k)}\| < \varepsilon_2$ `then` **`STOP.`**
`Else` $k := k + 1$`;`
`Go to` **`Step 1.`**

The constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ (in Step 2 and Step 4) are given prescribed convergence parameters, and $\|\cdot\|$ is the $L^2$-norm. The algorithm is a gradient type one. For details about the gradient methods see [7, Section 2.3].

The domain $[0, A] \times [0, T]$ is discretized by a grid of $M \times N$ equidistant nodes $(a_i, t_j)$, with $a_i = (i-1)h$, $i = 1, 2, \ldots, M$, $M = 1 + A/h$, and $t_j = (j-1)h$, $j = 1, 2, \ldots, N$, $N = 1 + T/h$, where $h > 0$ is the grid step.

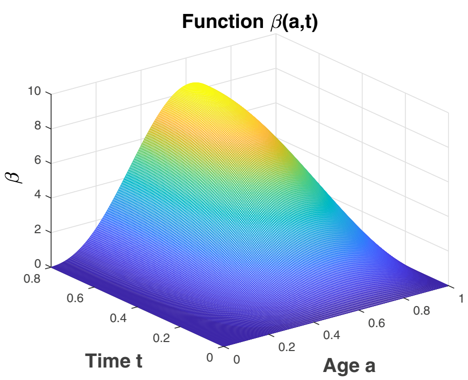
System (8) (Step 1) is approximated by an Euler type scheme, ascending with respect to time levels. We approximate also system (13) (Step 4), descending with respect to time levels.

For the numerical test, we take the fertility rate $\beta(a, t) = 30a^2(A-a)(1 + \sin(\pi/Aa))(\sin(2t))^+$, and the mortality rate $\mu(a) = \exp(-a)/(A-a)$, with $a \in [0, A]$ and $t \in [0, T]$. These rates are represented in Fig. 3. We also consider $p_0(a) = c_1(A-a)$, with $c_1 > 0$ and $a \in [0, A]$, $A = 1$, $T = 0.8$, $h = 0.005$, $L = 10$, $\bar{a} = 0.3$, $c = 1$, $c_1 = 1$, and $\varepsilon_1 = \varepsilon_2 = 0.000001$.

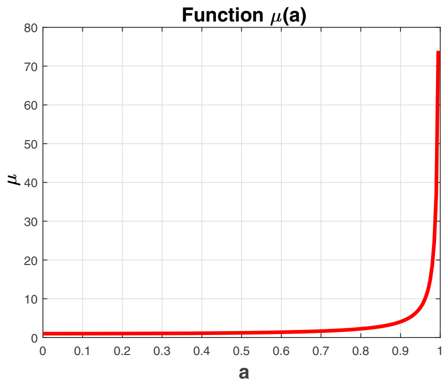
The algorithm stops in 20 iterations, when the first condition in Step 2 is fulfilled. As we can see in Table 1 below, the algorithm gives a higher value for $J_3$ at each iteration.

The control $u$ corresponding to the last iteration is given in Fig. 4a. The control $u(s(a), t)$ for the time level $t = 0.1$ and $t = 0.6$ can be seen in Fig. 4b,c.

*Remark 4* We see that the suboptimal control is of bang-bang type. Figure 4 shows that there are periods when the best strategy is to hunt only individuals older than a certain age, periods when the best strategy is to hunt only young individuals (and to allow the matures to produce offsprings) and, as it was expected, when we are close to the end of hunting, a period when the best strategy is to hunt at the maximum individuals of any ages.




(a) The fertility rate $\beta(a,t)$

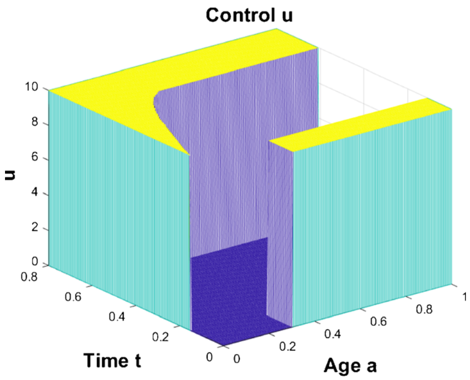



(b) The mortality rate $\mu(a)$

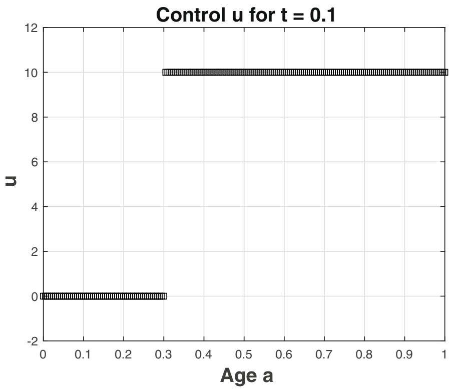
**Fig. 3** The fertility and mortality rates

**Table 1** $J_3$ for each iteration

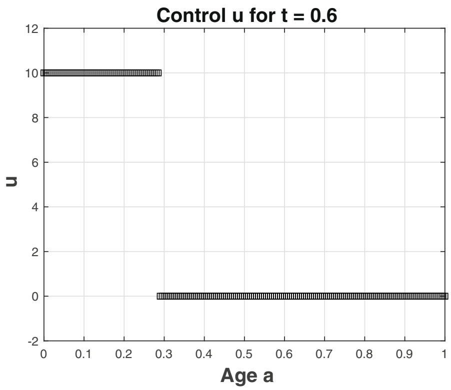
| Iteration | J3 |
|---|---|
| 1 | 0 |
| 2 | 0.4757085681364940 |
| 3 | 0.533321665809279 |
| 4 | 0.554101830342712 |
| 5 | 0.568331164047589 |
| 6 | 0.576608831379407 |
| 7 | 0.581035285632760 |
| 8 | 0.583332908850150 |
| 9 | 0.584560826649836 |
| 10 | 0.585227134622686 |
| 11 | 0.585562528362632 |
| 12 | 0.585730721638206 |
| 13 | 0.585814967570220 |
| 14 | 0.585857127896071 |
| 15 | 0.585878217403755 |
| 16 | 0.585888764494402 |
| 17 | 0.585894038624005 |
| 18 | 0.585896675834886 |
| 19 | 0.585897994476848 |
| 20(STOP) | 0.585898653806959 |




(a) $u(s(a), t)$ from the last iteration




(b) $u(s(a), t)$ for $t = 0.1$




(c) $u(s(a), t)$ for $t = 0.6$

**Fig. 4** The harvesting effort

Additional numerical tests and the analysis of the convergence of the algorithms will be performed in a future paper.

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