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Common Fixed Points of Family of Multivalued F-Contraction Mappings on Ordered Metric Spaces

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Abstract
The purpose of this paper is to obtain the existence of common fixed points of family of multivalued mappings satisfying generalized F-contraction conditions in ordered metric spaces. Some examples are presented to support the results proved herein. Our results generalize and extend various comparable results in the existing literature.

Keywords Common fixed point · Multivalued mapping · F-contraction · Upper semi-continuous map

Mathematics Subject Classification (2010) 47H10 · 54H25 · 54E50

1 Introduction and Preliminaries

To study necessary conditions for existence of fixed points of mappings satisfying certain comparison conditions on partially ordered domains equipped with an appropriate distance structure is an active area of research.

The existence of fixed points in partially ordered metric spaces was first considered in 2004 by Ran and Reurings [18], and then by Nieto and Lopez [15]. Later, in 2016, Nieto

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et al. [16] studied random fixed points theorems in partially ordered metric spaces. Further results in this direction under different contractive and comparison conditions were proved in [2, 3, 7, 8].

The theory of multivalued maps has various applications in convex optimization, dynamical systems, commutative algebra, differential equations, and economics. Markin [13] initiated the study of fixed points for multivalued nonexpansive and contractive maps. Later, a rich and interesting fixed point theory for such maps was developed; see, for instance [6, 8, 10]. Recently, Wardowski [21] introduced a new contraction called F-contraction and proved a fixed point result as a generalization of the Banach contraction principle. Very recently, in 2018, Wardowski [22] studied the existence of fixed points of nonlinear F-contraction and sum of this type mapping with a compact operator. Minak et al. [14] proved some fixed point results for Ćirić-type generalized F-contraction. Abbas et al. [4] obtained common fixed point results employing the F-contraction condition. Further, in this direction, Abbas et al. [5] introduced a notion of generalized F-contraction mapping and employed these results to obtain fixed point of generalized nonexpansive mappings on star-shaped subsets of normed linear spaces. Further useful results in this direction were proved in [11, 22].

The aim of this paper is to prove some common fixed point theorems for a family of multivalued generalized F-contraction mappings without using any commutativity condition in the setup of partially ordered metric space. These results extend and unify various comparable results in the existing literature [1, 12, 19, 20].

In the sequel, the letters $\mathbb{N}$, $\mathbb{R}_+$, $\mathbb{R}$ will denote the set of natural numbers, the set of positive real numbers, and the set of real numbers, respectively.

Consistent with [21] and [8], the following definitions will be needed in the sequel.

Let $\digamma$ be the collection of all mappings $F : \mathbb{R}_+ \to \mathbb{R}$ such that the following conditions hold:

(F_1) F is strictly increasing, that is, for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta$ implies that $F(\alpha) < F(\beta)$.

(F_2) For every sequence $\{\alpha_n\}$ of positive real numbers, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} F(\alpha_n) = -\infty$ are equivalent.

(F_3) There exists $h \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^h F(\alpha) = 0$.

Latif and Beg [12] introduced a notion of K-multivalued mapping as an extension of Kannan mapping to multivalued mappings. Rus [19] coined the term R-multivalued mapping as a generalization of a K-multivalued mapping. Abbas and Rhoades [1] gave the notion of a generalized R-multivalued mappings, which in turn generalized R-multivalued mappings, and obtained common fixed point results for such mappings.

Let (X, d) be a metric space. Let $P(X)(P_{cl}(X))$ be the family of all nonempty (nonempty and closed) subsets of X.

A point x in X is a fixed point of a multivalued mapping $T : X \to P(X)$ if and only if $x \in Tx$. The set of all fixed points of multivalued mapping T is denoted by $\mathrm{Fix}(T)$.

Definition 1 Let $(X, \preceq)$ be a partially ordered set. We define

$$\Delta_1 = \{(x, y) \in X \times X : x \preceq y\}$$

and

$$\Delta_2 = \{(x, y) \in X \times X : x \prec y \text{ or } y \prec x\}.$$

That is, Δ_2 is the set of all comparable elements of X.

Definition 2 Let $(X, \preceq)$ be a partially ordered set, A and B two nonempty subsets of $(X, \preceq)$. We say that $A \preceq_1 B$, whenever for every $a \in A$, there exists $b \in B$ such that $a \preceq b$.

Now, we give the following definition:

Definition 3 Let $\{T_i\}_{i=1}^m$ be a family of mappings such that $T_i : X \to P_{cl}(X)$ for each $i \in \{1, 2, \ldots, m\}$ and $T_{m+1} = T_1$. The set $\{T_i\}_{i=1}^m$ is said to be

1. F_1-contraction family, whenever for any $x, y \in X$ with $(x, y) \in \triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in \triangle_2$ such that the following condition holds
$$\tau(U(x, y; u_x, u_y)) + F\left(d(u_x, u_y)\right) \leq F(U(x, y; u_x, u_y)),$$
where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a mapping with $\liminf_{s \to t^+} \tau(s) \geq 0$ for all $t \geq 0$ and
$$U(x, y; u_x, u_y) = \max\left\{d(x, y), d(x, u_x), d(y, u_y), \frac{d(x, u_y) + d(y, u_x)}{2}\right\}.$$
2. F_2-contraction family, whenever for any $x, y \in X$ with $(x, y) \in \triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in \triangle_2$ such that
$$\tau(U(x, y; u_x, u_y)) + F\left(d(u_x, u_y)\right) \leq F(U(x, y; u_x, u_y))$$
holds where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\liminf_{s \to t^+} \tau(s) \geq 0$ for all $t \geq 0$ and
$$U_2(x, y; u_x, u_y) = \alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y) + \delta_1 d(x, u_y) + \delta_2 d(y, u_x)$$
for $\alpha, \beta, \gamma, \delta_1, \delta_2 \geq 0$, $\delta_1 \leq \delta_2$ with $\alpha + \beta + \gamma + \delta_1 + \delta_2 \leq 1$.

Note that for different choices of mappings F, one can obtains different contractive conditions.

Recall that, a map $T : X \to P_{cl}(X)$ is said to be upper semi-continuous, if for $x_n \in X$ and $y_n \in Tx_n$ with $x_n \to x_0$ and $y_n \to y_0$, then we have $y_0 \in Tx_0$.

2 Common Fixed Point Theorems

In this section, we obtain several common fixed point results for family of multivalued mappings in the framework of partially ordered metric space. We begin with the following result.

Theorem 1 *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $\{T_i\}_{i=1}^m$ an F_1-contraction family of multivalued maps. Then, the following hold*

(i) $\mathrm{Fix}(T_i) \neq \emptyset$ *for any* $i \in \{1, 2, \ldots, m\}$ *if and only if* $\mathrm{Fix}(T_1) = \mathrm{Fix}(T_2) = \cdots = \mathrm{Fix}(T_m) \neq \emptyset$.

(ii) $\mathrm{Fix}(T_1) = \mathrm{Fix}(T_2) = \cdots = \mathrm{Fix}(T_m) \neq \emptyset$ *provided that there exists some* $x_0 \in X$ *such that* $\{x_0\} \preceq_1 T_k(x_0)$ *for any* $k \in \{1, 2, \ldots, m\}$ *and any one of* T_i *is upper semi-continuous for* $i \in \{1, 2, \ldots, m\}$.

(iii) $\cap_{i=1}^m \mathrm{Fix}(T_i)$ *is well ordered if and only if* $\cap_{i=1}^m \mathrm{Fix}(T_i)$ *is a singleton set.*

Proof To prove (i): Let $x^* \in T_k(x^*)$ for any $k \in \{1, 2, \ldots, m\}$. If $x^* \notin T_{k+1}(x^*)$, then there exists an $x \in T_{k+1}(x^*)$ with $(x^*, x) \in \triangle_2$ such that

$$\tau(U(x^*, x^*; x^*, x)) + F(d(x^*, x)) \leq F(U(x^*, x^*; x^*, x)),$$

holds, where

$$\begin{aligned} U(x^*, x^*; x^*, x) &= \max\left\{d(x^*, x^*), d(x^*, x^*), d(x, x^*), \frac{d(x^*, x) + d(x^*, x^*)}{2}\right\} \\ &= d(x, x^*). \end{aligned}$$

Thus, we have

$$\tau(d(x^*, x)) + F(d(x^*, x)) \leq F(d(x^*, x)),$$

a contradiction as $\tau(d(x^*, x)) > 0$. Thus $x^* = x$. Hence, $x^* \in T_{k+1}(x^*)$ and $\mathrm{Fix}(T_k) \subseteq \mathrm{Fix}(T_{k+1})$. Similarly, we obtain that $\mathrm{Fix}(T_{k+1}) \subseteq \mathrm{Fix}(T_{k+2})$. Continuing this way, we get $\mathrm{Fix}(T_1) = \mathrm{Fix}(T_2) = \cdots = \mathrm{Fix}(T_k)$. The converse is straightforward.

To prove (ii): Suppose that x_0 is an arbitrary point of X. If $x_0 \in T_{k_0}(x_0)$ for any $k_0 \in \{1, 2, \ldots, m\}$, then by using (i), the proof is finished.

So, we assume that $x_0 \notin T_{k_0}(x_0)$ for any $k_0 \in \{1, 2, \ldots, m\}$. For $i \in \{1, 2, \ldots, m\}$, $x_1 \in T_i(x_0)$, there exists $x_2 \in T_{i+1}(x_1)$ with $(x_1, x_2) \in \triangle_2$ such that

$$\tau(U(x_0, x_1; x_1, x_2)) + F\,(d(x_1, x_2)) \leq F(U(x_0, x_1; x_1, x_2)),$$

holds where

$$\begin{aligned} U(x_0, x_1; x_1, x_2) &= \max\left\{d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2) + d(x_1, x_1)}{2}\right\} \\ &= \max\left\{d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2)}{2}\right\} \\ &= \max\{d(x_0, x_1), d(x_1, x_2)\}. \end{aligned}$$

If $U(x_0, x_1; x_1, x_2) = d(x_1, x_2)$, then

$$\tau(d(x_1, x_2)) + F(d(x_1, x_2)) \leq F(d(x_1, x_2))$$

gives a contradiction as $\tau(d(x_1, x_2)) > 0$. Therefore, $U(x_0, x_1; x_1, x_2) = d(x_0, x_1)$ and we have

$$\tau\,(d(x_0, x_1)) + F\,(d(x_1, x_2)) \leq F\,(d(x_0, x_1))\,.$$

Similarly, for the point x_2 in $T_{i+1}(x_1)$, there exists $x_3 \in T_{i+2}(x_2)$ with $(x_2, x_3) \in \triangle_2$ such that

$$\tau(U(x_1, x_2; x_2, x_3)) + F\,(d(x_2, x_3)) \leq F(U(x_1, x_2; x_2, x_3)),$$

holds where

$$\begin{aligned} U(x_1, x_2; x_2, x_3) &= \max\left\{d(x_1, x_2), d(x_1, x_2), d(x_2, x_3), \frac{d(x_1, x_3) + d(x_2, x_2)}{2}\right\} \\ &= \max\{d(x_1, x_2), d(x_2, x_3)\}. \end{aligned}$$

In case $U(x_1, x_2; x_2, x_3) = d(x_2, x_3)$, we have

$$\tau(d(x_2, x_3)) + F\,(d(x_2, x_3)) \leq F(d(x_2, x_3)),$$

a contradiction as $\tau(d(x_2, x_3)) > 0$. Therefore, $U(x_1, x_2; x_2, x_3) = d(x_1, x_2)$ and we have

$$\tau\,(d(x_1, x_2)) + F\,(d(x_2, x_3)) \leq F\,(d(x_1, x_2))\,.$$

Continuing this way, for $x_{2n} \in T_i(x_{2n-1})$, there exists $x_{2n+1} \in T_{i+1}(x_{2n})$ with $(x_{2n}, x_{2n+1}) \in \triangle_2$ such that

$$\tau\left(U(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1})\right) + F\left(d(x_{2n}, x_{2n+1})\right) \leq F\left(U(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1})\right),$$

holds that is,

$$\tau\left(d(x_{2n-1}, x_{2n})\right) + F\left(d(x_{2n}, x_{2n+1})\right) \leq F\left(d(x_{2n-1}, x_{2n})\right).$$

Similarly, for $x_{2n+1} \in T_{i+1}(x_{2n})$, there exist $x_{2n+2} \in T_{i+2}(x_{2n+1})$ with $(x_{2n+1}, x_{2n+2}) \in \triangle_2$ such that

$$\tau\left(d(x_{2n}, x_{2n+1})\right) + F\left(d(x_{2n+1}, x_{2n+2})\right) \leq F\left(d(x_{2n}, x_{2n+1})\right)$$

holds. Hence, we obtain a sequence $\{x_n\}$ in X such that $x_n \in T_i(x_{n-1})$ and $x_{n+1} \in T_{i+1}(x_n)$ with $(x_n, x_{n+1}) \in \triangle_2$ and it satisfies

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau(d(x_{n-1}, x_n)) \\ &< F(d(x_{n-1}, x_n)). \end{aligned}$$

Thus, $\{d(x_n, x_{n+1})\}$ is decreasing and hence convergent. We now show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. By property of mapping τ, there exists $c > 0$ with $n_0 \in \mathbb{N}$ such that $\tau(d(x_n, x_{n+1})) > c$ for all $n \geq n_0$. Note that

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau(d(x_{n-1}, x_n)) \\ &\leq F(d(x_{n-2}, x_{n-1})) - \tau(d(x_{n-2}, x_{n-1})) - \tau(d(x_{n-1}, x_n)) \\ &\leq \cdots \\ &\leq F(d(x_0, x_1)) - \tau(d(x_{n-1}, x_n)) + \tau(d(x_{n-2}, x_{n-1})) \\ &\quad + \cdots + \tau(d(x_0, x_1)) \\ &\leq F(d(x_0, x_1)) - n_0, \end{aligned} \tag{1}$$

gives $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$ which together with (F_2) implies that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. By (F_3), there exists $h \in (0, 1)$ such that

$$\lim_{n\to\infty} [d(x_n, x_{n+1})]^h F(d(x_n, x_{n+1})) = 0.$$

From (1), we have

$$\begin{aligned} &[d(x_n, x_{n+1})]^h F\left(d(x_n, x_{n+1})\right) - [d(x_n, x_{n+1})]^h F\left(d(x_0, x_1)\right) \\ &\leq [d(x_n, x_{n+1})]^h (F(d(x_0, x_1) - n_0)) - [d(x_n, x_{n+1})]^h F\left(d(x_0, x_1)\right) \\ &\leq -n_0[d(x_n, x_{n+1})]^h \leq 0. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain that $\lim_{n\to\infty} n[d(x_n, x_{n+1})]^h = 0$ and $\lim_{n\to\infty} n^{\frac{1}{h}} d(x_n, x_{n+1}) = 0$. There exists $n_1 \in \mathbb{N}$ such that $n^{\frac{1}{h}} d(x_n, x_{n+1}) \leq 1$ for all $n \geq n_1$ and hence $d(x_n, x_{n+1}) \leq \frac{1}{n^{1/h}}$ for all $n \geq n_1$. So, for all $m, n \in \mathbb{N}$ with $m > n \geq n_1$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/h}}. \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/h}}$, we obtain that $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Now, if T_i is upper semi-continuous for any of $i \in \{1, 2, \ldots, m\}$, then $x_{2n} \in X$, $x_{2n+1} \in T_i(x_{2n})$ with $x_{2n} \to x^*$ and $x_{2n+1} \to x^*$ as $n \to \infty$ imply that $x^* \in T_i(x^*)$. Using (i), we get $x^* \in T_1(x^*) = T_2(x^*) = \cdots = T_m(x^*)$.

Finally, to prove (iii): Suppose the set $\cap_{i=1}^{m} \mathrm{Fix}(T_i)$ is well ordered. Assume that there exist u and v such that $u, v \in \cap_{i=1}^{m} \mathrm{Fix}(T_i)$ but $u \neq v$. As $(u, v) \in \triangle_2$, we have

$$\begin{aligned}\tau(U(u, v; u, v)) + F(d(u, v)) &\leq F(U(u, v; u, v)) \\ &= F\left(\max\left\{d(u, v), d(u, u), d(v, v), \frac{d(u, v) + d(v, u)}{2}\right\}\right) \\ &= F(d(u, v)),\end{aligned}$$

that is, $\tau(d(u, v)) + F(d(u, v)) \leq F(d(u, v))$, a contradiction as $\tau(d(u, v)) > 0$. Hence, $u = v$. The converse is obvious. □

Corollary 1 *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $T_1, T_2 : X \to P_{cl}(X)$. Suppose that for every $(x, y) \in \triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ with $i \neq j$ with $(u_x, u_y) \in \triangle_2$ such that*

$$\tau(U(x, y; u_x, u_y)) + F(d(u_x, u_y)) \leq F(U(x, y; u_x, u_y)$$

holds, where $i, j \in \{1, 2\}$, $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\liminf_{s \to t^+} \tau(s) \geq 0$ *for all $t \geq 0$ and*

$$U(x, y; u_x, u_y) = \max\left\{d(x, y), d(x, u_x), d(y, u_y), \frac{d(x, u_y) + d(y, u_x)}{2}\right\}.$$

Then, the following statements hold:

(I) $\mathrm{Fix}(T_i) \neq \emptyset$ *for any $i \in \{1, 2\}$ if and only if* $\mathrm{Fix}(T_1) = \mathrm{Fix}(T_2) \neq \emptyset$.
(II) $\mathrm{Fix}(T_1) = \mathrm{Fix}(T_2) \neq \emptyset$ *provided that either T_1 or T_2 is upper semi-continuous.*
(III) $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$ *is well ordered if and only if* $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$ *is singleton set.*

Example 1 Let $X = [0, 10]$ be endowed with usual order $\leq$. Define the mappings $T_1, T_2 : X \to P_{cl}(X)$ by

$$T_1(x) = \left[0, \frac{x}{10}\right] \quad \text{and} \quad T_2(x) = \left[0, \frac{x}{12}\right] \quad \text{for all } x \in X.$$

Take $F(\gamma) = \ln \gamma + \gamma$ for all $\gamma > 0$. The mapping $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as follows:

$$\tau(t) = \begin{cases} \frac{t}{20} & \text{if } t \in (0, 10], \\ \frac{1}{2} & \text{if } t > 10. \end{cases}$$

We consider the following cases:

1. When $x, y \in (0, 10]$ with $(x, y) \in \triangle_1$, then for $u_x \in T_1(x)$, there exists $u_y = 0 \in T_2(y)$ with $(u_x, u_y) \in \triangle_2$ such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-U(x,y;u_x,u_y)+\tau(U(x,y;u_x,u_y))} &= u_x e^{u_x - U(x,y;u_x,u_y)+\frac{U(x,y;u_x,u_y)}{20}} \\ &\le \frac{x}{10}e^{\frac{x}{10}-\frac{19}{20}U(x,y;u_x,u_y)} \\ &\le \frac{x}{10}e^{\frac{x}{10}-\frac{19}{20}\left(\frac{d(x,u_y)+d(y,u_x)}{2}\right)} \\ &\le \frac{9}{10}xe^{\frac{-131x-190y}{400}} \\ &= d(x, u_x)e^0 \le U(x, y; u_x, u_y). \end{aligned}$$

2. If $x = 0$ and $y \in (0, 10]$ with $(x, y) \in \triangle_1$, then for $u_x = 0 \in T_1(x)$, there exists $0 \neq u_y \in T_2(y)$ with $(u_x, u_y) \in \triangle_2$ such that

$$\begin{aligned} & d(u_x, u_y)e^{d(u_x,u_y)-U(x,y;u_x,u_y)+\tau(U(x,y;u_x,u_y))} \\ &= u_y e^{u_y - U(x,y;u_x,u_y)+\frac{U(x,y;u_x,u_y)}{20}} \\ &\le \frac{y}{12}e^{\frac{y}{12}-U(x,y;u_x,u_y)+\frac{U(x,y;u_x,u_y)}{20}} \\ &= \frac{y}{12}e^{\frac{y}{12}-\frac{19}{20}U(x,y;u_x,u_y)} \le \frac{y}{12}e^{\frac{y}{12}-\frac{19}{20}d(y,u_y)} \\ &\le ye^{\frac{y}{12}-\frac{19}{20}(\frac{11y}{12})} \le d(x, y)e^0 \le U(x, y; u_x, u_y). \end{aligned}$$

3. In case $x \in (0, 10]$ and $y = 0$ with $(x, y) \in \triangle_1$, we have for $u_x \in T_1(x)$, there exists $u_y = 0 \in T_2(y)$, such that

$$\begin{aligned} & d(u_x, u_y)e^{d(u_x,u_y)-U(x,y;u_x,u_y)+\tau(U(x,y;u_x,u_y))} \\ &\le \frac{x}{10}e^{\frac{x}{10}-\frac{19}{20}U(x,y;u_x,u_y)} \\ &\le \frac{x}{10}e^{\frac{x}{10}-\frac{19}{20}d(x,u_x)} \\ &\le \frac{x}{10}e^{\frac{x}{10}-\frac{19}{20}(x-\frac{x}{10})} = \frac{x}{10}e^{\frac{x}{10}-\frac{19}{20}(\frac{9x}{10})} \\ &\le xe^0 \le d(x, y) \le U(x, y; u_x, u_y). \end{aligned}$$

4. When $x = 0$ and $y \in (0, 10]$ with $(x, y) \in \triangle_1$, we have for $u_x = 0 \in T_2(x)$, there exists $0 \neq u_y \in T_1(y)$ with $(u_x, u_y) \in \triangle_2$ such that

$$\begin{aligned} & d(u_x, u_y)e^{d(u_x,u_y)-U(x,y;u_x,u_y)+\tau(U(x,y;u_x,u_y))} \\ &= u_y e^{u_y - U(x,y;u_x,u_y)+\frac{U(x,y;u_x,u_y)}{20}} \\ &\le \frac{y}{12}e^{\frac{y}{10}-U(x,y;u_x,u_y)+\frac{U(x,y;u_x,u_y)}{20}} \\ &= \frac{y}{12}e^{\frac{y}{10}-\frac{19}{20}U(x,y;u_x,u_y)} \\ &\le \frac{y}{12}e^{\frac{y}{10}-\frac{19}{20}d(y,u_y)} \\ &\le ye^{\frac{y}{10}-\frac{19}{20}(\frac{11y}{12})} \le d(x, y)e^0 \le U(x, y; u_x, u_y). \end{aligned}$$

5. Finally, if $x \in (0, 10]$ and $y = 0$ with $(y, x) \in \triangle_1$, then for $0 \neq u_x \in T_2(x)$, there exists $u_y = 0 \in T_1(y)$ with $(u_y, u_x) \in \triangle_2$ such that

$$\begin{aligned} & d(u_x, u_y)e^{d(u_x,u_y)-U(x,y;u_x,u_y)+\tau\left(U(x,y;u_x,u_y)\right)} \\ & \leq \frac{x}{12}e^{\frac{x}{12}-U(x,y;u_x,u_y)-\frac{U(x,y;u_x,u_y)}{20}} \\ & = \frac{x}{12}e^{\frac{x}{12}-\frac{19}{20}d(x,u_x)} \\ & \leq \frac{x}{12}e^{\frac{x}{12}-\frac{19}{20}(\frac{11x}{12})} \\ & \leq \frac{11x}{12}e^{0} \leq d(x, u_x) \leq U(x, y; u_x, u_y). \end{aligned}$$

Thus, all the conditions of Corollary 1 are satisfied. Moreover, $\mathrm{Fix}(T_1) = \mathrm{Fix}(T_2) = \{0\}$.

The following results generalizes [19, Theorem 3.4].

Theorem 2 *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $\{T_i\}_{i=1}^m$ be F_2-contraction family of multivalued maps. Then, the following hold*

(i) $\mathrm{Fix}(T_i) \neq \emptyset$ *for any $i \in \{1, 2, \ldots, m\}$ if and only if* $\mathrm{Fix}(T_1) = \mathrm{Fix}(T_2) = \cdots = \mathrm{Fix}(T_m) \neq \emptyset$.

(ii) $\mathrm{Fix}(T_1) = \mathrm{Fix}(T_2) = \cdots = \mathrm{Fix}(T_m) \neq \emptyset$ *provided that there exists some $x_0 \in X$ such that $\{x_0\} \preceq_1 T_k(x_0)$ for any $k \in \{1, 2, \ldots, m\}$ and any one of T_i is upper semi-continuous for $i \in \{1, 2, \ldots, m\}$.*

(iii) $\cap_{i=1}^m \mathrm{Fix}(T_i)$ *is well ordered if and only if* $\cap_{i=1}^m \mathrm{Fix}(T_i)$ *is singleton set.*

Proof To prove (i): Let $x^* \in T_k(x^*)$ for any $k \in \{1, 2, \ldots, m\}$. If $x^* \notin T_{k+1}(x^*)$, then there exists an $x \in T_{k+1}(x^*)$ with $(x^*, x) \in \triangle_2$ such that

$$\tau(U_2(x^*, x^*; x^*, x)) + F\left(d(x^*, x)\right) \leq F(U_2(x^*, x^*; x^*, x)),$$

where

$$\begin{aligned} U_2(x^*, x^*; x^*, x) &= \alpha d(x^*, x^*) + \beta d(x^*, x^*) + \gamma d(x, x^*) + \delta_1 d(x^*, x) + \delta_2 d(x^*, x^*) \\ &= (\gamma + \delta_1)d(x, x^*). \end{aligned}$$

Thus, we have

$$\tau((\gamma + \delta_1)d(x^*, x)) + F\left(d(x^*, x)\right) \leq F((\gamma + \delta_1)d(x^*, x)) < F(d(x^*, x)),$$

a contradiction as $\tau((\gamma + \delta_1)d(x^*, x)) > 0$. Thus, $x^* = x$ and hence $x^* \in T_{k+1}(x^*)$ and $\mathrm{Fix}(T_k) \subseteq \mathrm{Fix}(T_{k+1})$. Similarly, we obtain that $\mathrm{Fix}(T_{k+1}) \subseteq \mathrm{Fix}(T_{k+2})$. Continuing this way, we get $\mathrm{Fix}(T_1) = \mathrm{Fix}(T_2) = \cdots = \mathrm{Fix}(T_k)$. The converse is straightforward.

To prove (ii): Suppose that x_0 is an arbitrary point of X. If $x_0 \in T_{k_0}(x_0)$ for any $k_0 \in \{1, 2, \ldots, m\}$ then by using (i) the proof is finished. So, we assume that $x_0 \notin T_{k_0}(x_0)$ for any $k_0 \in \{1, 2, \ldots, m\}$. For $i \in \{1, 2, \ldots, m\}$, $x_1 \in T_i(x_0)$, there exists $x_2 \in T_{i+1}(x_1)$ with $(x_1, x_2) \in \triangle_2$ such that

$$\tau(U_2(x_0, x_1; x_1, x_2)) + F\left(d(x_1, x_2)\right) \leq F(U_2(x_0, x_1; x_1, x_2)),$$

where

$$\begin{aligned} U_2(x_0, x_1; x_1, x_2) &= \alpha d(x_0, x_1) + \beta d(x_0, x_1) + \gamma d(x_1, x_2) + \delta_1 d(x_0, x_2) + \delta_2 d(x_1, x_1) \\ &\leq (\alpha + \beta + \delta_1)d(x_0, x_1) + (\gamma + \delta_1)d(x_1, x_2). \end{aligned}$$

If $d(x_0, x_1) \leq d(x_1, x_2)$, then

$$\begin{aligned}&\tau((\alpha+\beta+\gamma+2\delta_1)d(x_{1,}x_2))+F\left(d(x_1,x_2)\right)\\&\leq F((\alpha+\beta+\gamma+2\delta_1)d(x_1,x_2))\\&\leq F(d(x_1,x_2)),\end{aligned}$$

gives a contradiction as $\tau((\alpha+\beta+\gamma+2\delta_1)d(x_{1,}x_2)) > 0$. Thus, we have

$$\tau(d(x_{0,}x_1)) + F\left(d(x_1, x_2)\right) \leq F\left(d(x_0, x_1)\right).$$

Continuing this way, for $x_{2n} \in T_i(x_{2n-1})$, there exist $x_{2n+1} \in T_{i+1}(x_{2n})$ with $(x_{2n}, x_{2n+1}) \in \triangle_2$ such that

$$\tau(U_2(x_{2n-1,}x_{2n;}x_{2n,}x_{2n+1})) + F\left(d(x_{2n}, x_{2n+1})\right) \leq F\left(U_2(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1})\right)$$

holds, where

$$\begin{aligned}U_2(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1}) &= \alpha d(x_{2n-1}, x_{2n}) + \beta d(x_{2n-1}, x_{2n}) + \gamma d(x_{2n}, x_{2n+1})\\&\quad+\delta_1 d(x_{2n-1}, x_{2n+1}) + \delta_2 d(x_{2n}, x_{2n})\\&\leq (\alpha+\beta+\delta_1)d(x_{2n-1}, x_{2n}) + (\gamma+\delta_1)d(x_{2n}, x_{2n+1}).\end{aligned}$$

If $d(x_{2n-1}, x_{2n}) \leq d(x_{2n}, x_{2n+1})$, then

$$\begin{aligned}&\tau((\alpha+\beta+\gamma+2\delta_1)d(x_{2n,}x_{2n+1}) + F\left(d(x_{2n}, x_{2n+1})\right)\\&\leq F\left((\alpha+\beta+\gamma+2\delta_1)d(x_{2n}, x_{2n+1})\right)\\&\leq F\left(d(x_{2n}, x_{2n+1})\right),\end{aligned}$$

gives a contradiction as $\tau((\alpha+\beta+\gamma+2\delta_1)d(x_{2n,}x_{2n+1})) > 0$. Therefore,

$$\tau(d(x_{2n-1,}x_{2n})) + F\left(d(x_{2n}, x_{2n+1})\right) \leq F\left(d(x_{2n-1}, x_{2n})\right).$$

Similarly, for $x_{2n+1} \in T_{i+1}(x_{2n})$, there exist $x_{2n+2} \in T_{i+2}(x_{2n+1})$ with $(x_{2n+1}, x_{2n+2}) \in \triangle_2$ such that

$$\tau(d(x_{2n,}x_{2n+1})) + F\left(d(x_{2n+1}, x_{2n+2})\right) \leq F\left(d(x_{2n}, x_{2n+1})\right)$$

holds. Hence, we obtain a sequence $\{x_n\}$ in X such that $x_n \in T_i(x_{n-1})$ and $x_{n+1} \in T_{i+1}(x_n)$ with $(x_n, x_{n+1}) \in \triangle_2$ and it satisfies

$$\begin{aligned}F(d(x_n, x_{n+1})) &\leq F\left(d(x_{n-1}, x_n)\right) - \tau\left(d(x_{n-1}, x_n)\right)\\&< F\left(d(x_{n-1}, x_n)\right).\end{aligned}$$

Thus, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and hence convergent. We show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. By the property of mapping τ, there exists $c > 0$ with $n_0 \in \mathbb{N}$ such that $\tau(d(x_n, x_{n+1})) > c$ for all $n \geq n_0$. Note that

$$\begin{aligned}F\left(d(x_n, x_{n+1})\right) &\leq F(d(x_{n-1}, x_n)) - \tau(d(x_{n-1}, x_n))\\&\leq F(d(x_{n-2}, x_{n-1})) - \tau(d(x_{n-2}, x_{n-1})) - \tau\left(d(x_{n-1}, x_n)\right)\\&\leq \cdots\\&\leq F(d(x_0, x_1)) - (\tau\left(d(x_{n-1}, x_n)\right) + \tau(d(x_{n-2}, x_{n-1}))\\&\quad+\cdots+\tau(d((x_0, x_1))\\&\leq F(d(x_0, x_1)) - n_0.\end{aligned}$$

Thus, $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$ which together with (F_2) gives $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Following the arguments similar to those in the proof of Theorem 1, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an element

$x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Now, if T_i is upper semi-continuous for any $i \in \{1, 2, \ldots, m\}$, then as $x_{2n} \in X$, $x_{2n+1} \in T_i(x_{2n})$ with $x_{2n} \to x^*$ and $x_{2n+1} \to x^*$ as $n \to \infty$, so we have $x^* \in T_i(x^*)$. Using (i), we get $x^* \in T_1(x^*) = T_2(x^*) = \cdots = T_m(x^*)$.

To prove (iii): Suppose the set $\cap_{i=1}^m \text{Fix}(T_i)$ is well ordered. Assume that there exist u and v such that $u, v \in \cap_{i=1}^m \text{Fix}(T_i)$ but $u \neq v$. As $(u, v) \in \triangle_2$, we have

$$\tau(U_2(u, v; u, v)) + F(d(u, v)) \leq F(U_2(u, v; u, v)),$$

where

$$\begin{aligned} U_2(u, v; u, v) &= \alpha d(u, v) + \beta d(u, u) + \gamma d(v, v) + \delta_1 d(u, v) + \delta_2 d(v, u) \\ &= (\alpha + \delta_1 + \delta_2) d(u, v), \end{aligned}$$

that is,

$$\tau(d(u, v)) + F(d(u, v)) = F((\alpha + \delta_1 + \delta_2) d(u, v)) \leq F(d(u, v)),$$

a contradiction as $\tau(d(u, v)) > 0$. Hence, $u = v$. The converse is obvious. □

Corollary 2 *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $\{T_i\}_{i=1}^m : X \to P_{cl}(X)$ with $T_{m+1} = T_1$. Suppose that for any $x, y \in X$ with $(x, y) \in \triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in \triangle_2$ such that*

$$\begin{aligned} &\tau(\alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y)) + F(d(u_x, u_y)) \\ &\leq F(\alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y)) \end{aligned}$$

holds, where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\liminf_{s \to t^+} \tau(s) \geq 0$ for all $t \geq 0$ and $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma \leq 1$. Then, the conclusions obtained in Theorem 2 remain true.

Corollary 3 *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $\{T_i\}_{i=1}^m : X \to P_{cl}(X)$ with $T_{m+1} = T_1$. Suppose that for any $x, y \in X$ with $(x, y) \in \triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in \triangle_2$ such that*

$$\tau(h[d(x, u_x) + d(y, u_y)]) + F(d(u_x, u_y)) \leq F(h[d(x, u_x) + d(y, u_y)])$$

holds, where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\liminf_{s \to t^+} \tau(s) \geq 0$ for all $t \geq 0$ and $h \in [0, \frac{1}{2}]$. Then the conclusions obtained in Theorem 2 remain true.

Corollary 4 *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $\{T_i\}_{i=1}^m : X \to P_{cl}(X)$ with $T_{m+1} = T_1$. Suppose that for any $x, y \in X$ with $(x, y) \in \triangle_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ with $(u_x, u_y) \in \triangle_2$ such that*

$$\tau(d(x, y)) + F(d(u_x, u_y)) \leq F(d(x, y)),$$

holds, where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\liminf_{s \to t^+} \tau(s) \geq 0$ for all $t \geq 0$. Then, the conclusions obtained in Theorem 2 remain true.

Remark 1

1. Theorem 1 extends, improves and generalizes (i) Theorem 1.9 in [1], (ii) Theorem 4.1 in [12], (iii) Theorem 3.4 of [19], (iv) Theorem 2.1 of [17], and (v) Theorem 3.1 of [20].
2. Corollary 1 improves and generalizes (i) Theorem 1.9 in [1], (ii) Theorem 4.1 in [12], (iii) Theorem 3.4 of [19], and (iv) Theorem 3.1 of [20].
3. Theorem 2 improves and extends (i) Theorem 3.4 and Theorem 4.1 in [9], (ii) Theorem 3.4 in [19], and (iii) Theorem 3.4 in [20].

4. Corollary 2 extends and generalizes (i) Theorem 3.4 in [19] and (ii) Theorem 4.1 of [12].
5. Corollary 3 improves and generalizes Theorem 4.1 in [12].
6. If we take $T_1 = T_2 = \cdots = T_m$ in F_1 and F_2-contraction family of multivalued maps, then we obtain the fixed point results for F_1-contraction and F_2-contraction of a multivalued map, respectively.

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