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On the Euler–Lagrange Equation in Calculus of Variations

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Abstract In 1985, Clarke and Vinter proved that, in the classical Bolza problem of the calculus of variations, if the Lagrangian is coercive and autonomous, all minimizers are Lipschitz and satisfy the Euler–Lagrange equation. I give a short and direct proof of this result.

Keywords Euler–Lagrange equation · Lavrentiev phenomenon

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1 The Theorem

We consider the classical Bolza problem in the one-dimensional calculus of variations

$$\begin{aligned} &\inf \int_0^T L(t, x, \dot{x})dt \\ x(0) = &\ \xi_0, \quad x(T) = \xi_1 \\ \dot{x} \in &\ L^1(0, T; \mathbb{R}^d),\ x(t) = \int \dot{x}(t)dt, \end{aligned}$$

the Lagrangian $L : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is assumed to be C^1 and coercive, meaning that there exists a continuous and increasing function g such that

$$\begin{aligned} \frac{g(t)}{t} &\to \infty \text{ when } t \to \infty, \\ L(t, x, y) &\geq g(\|y\|) \quad \forall (t, x, y). \end{aligned}$$

This paper is dedicated to Michel Théra in honour of his 70th birthday.

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If in addition f is convex with respect to y, the Bolza problem will have a (possibly non-unique) solution (see [3]). However, we will not be assuming convexity, and we will not be studying the existence of solution(s).

In this paper, we are concerned with another question: if the solution exists, does it satisfy the Euler–Lagrange equation

$$\frac{d}{dt}\frac{\partial}{\partial y}L(t,x,\dot{x}) = \frac{\partial L}{\partial x}.$$

It is by now well-known that the answer is no: this is the so-called Lavrentiev phenomenon. After the initial work of Lavrentiev in 1926, Ball and Mizel [2] gave an example where the minimizer exists and does not satisfy the E-L equation (1984). Their example was simplified by Loewen [7], and further by Willem [8]. To understand the problem, assume that $x_0(t)$ is a solution, and consider another admissible trajectory, namely $x_\varepsilon(y) = x_0(t) + \varepsilon h(t)$, with $h(0) = h(T) = 0$. Comparing the values of the criterion, we are led to the relation

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^T \left[L\left(t, x_0(t)+\varepsilon h(t), \dot{x}_0(t)+\varepsilon\dot{h}(t)\right) - L(t,x,\dot{x})\right]dt \geq 0.$$

If we could interchange the limit and the integral, we would be done, since we then get

$$\int_0^T \left[\frac{\partial L}{\partial x}(t,x_0,\dot{x}_0)h + \frac{\partial L}{\partial y}(t,x_0,\dot{x}_0)\dot{h}\right]dt \geq 0$$

and we get the E-L equation by integrating by parts. However, we must be able to interchange the limit and the integral, so a further condition is needed. Here is one such condition:

Lemma 1 *If the solution x_0 is Lipschitz, it satisfies the E-L equation.*

Proof Assume x_0 is Lipschitz, so that there exists some constant k such that

$$\|x_0(t_1) - x_0(t_2)\| \leq k\|t_2 - t_1\|.$$

Since x_0 is Lipschitz, its derivative is bounded. Take h to be Lipschitz. Then, there is a constant M_1 such that $\|\dot{x}_0(t) + \varepsilon\dot{h}(t)\| \leq M_1$ for $0 \leq t \leq T$ and $0 \leq \varepsilon \leq 1$. Since x_0 and h are continuous, there is some constant M_2 such that $\|x_0(t) + \varepsilon h(t)\| \leq M_2$ for $0 \leq t \leq T$ and $0 \leq \varepsilon \leq 1$. Since the Lagrangian L is C^1, its partial derivatives are continuous, and bounded on every compact subset. It follows that L must be Lipschitz on the set $[0,T] \times B(M_1) \times B(M_2)$, where $B(M)$ denotes the ball of radius M. Denote by R the Lipschitz norm for h and K the Lipschitz constant for L. We have

$$\begin{aligned}\frac{1}{\varepsilon}\left[L\left(t, x_0(t)+\varepsilon h(t), \dot{x}_0(t)+\varepsilon\dot{h}(t)\right) - L(t,x_0(t),\dot{x}_0(t))\right] &\leq K\left(|h(t)| + |\dot{h}(t)|\right)\\ &\leq KR.\end{aligned}$$

Setting $\varepsilon = 1/n$ and letting $n \to \infty$, we conclude by applying Lebesgue's dominated convergence theorem. □

The purpose of this paper is to give a short and direct proof of the following result:

Theorem 1 *Suppose the Lagrangian $L(x,y)$ is coercive and does not depend on t. Then, all solutions of the Bolza problem are Lipschitzian.*

It follows that all minimizers of the Bolza problem satisfy the E-L equation. Clarke and Vinter [4, 5] were the first to prove this theorem, and since their seminal work, many researchers have extended it to other situations, including time-dependent Lagrangians (see [1, 6, 8, 9]). No such result is known for the multidimensional case, where $t = (t_1, \dots, t_D)$ and the interval is replaced by a bounded domain of R^D.

2 The Proof

Let x_0 be a solution of the Bolza problem, i.e., a minimizer of the integral under the boundary constraints. Assume that it is not Lipschitzian. Then, for any $M > 0$, the set

$$\ell^M = \{t \mid \|\dot{x}_0(t)\| > M\}$$

has a positive measure. We will derive a contradiction.

Comparing x_0 with the trajectory $t \to \xi_0 + \frac{1}{T}(\xi_1 - \xi_0)t$, which also satisfies the constraints, we get

$$\int_0^T L(x_0, \dot{x}_0)dt \le \int_0^T L\left(\xi_0 + (\xi_1 - \xi_0)\frac{t}{T}, \frac{\xi_1 - \xi_0}{T}\right)dt.$$

Since L is coercive and x_0 is a minimizer, we have

$$\int_0^T g(|\dot{x}_0|)dt \le \int_0^T L(x_0, \dot{x}_0)dt \le \int_0^T L\left(\xi_0 + (\xi_1 - \xi_0)\frac{t}{T}, \frac{\xi_1 - \xi_0}{T}\right)dt. \tag{1}$$

With every $m > 0$, we associate the subset ℓ_m defined by

$$\ell_m = \{t \mid \|\dot{x}_0(t)\| \le m\}.$$

Denoting by C, the right-hand side of equation (1), we find that for every m, we have

$$(T - \text{meas}(\ell_m))g(m) + \text{meas}(\ell_m) \inf g \le C.$$

Hence,

$$\text{meas}(\ell_m) \ge \frac{Tg(m) - C}{g(m) - \inf g}.$$

When $m \to \infty$, $\text{meas}(\ell_m) \to T$. Choose m so large that $\text{meas}(\ell_m) \ge T/2$. With every $M > m$, we associate a change of variables $s = \sigma(t)$ defined as follows. Set

$$\begin{aligned} \ell_m^M &= \{t \mid m < \|\dot{x}_0(t)\| \le M\}, \\ \ell^M &= \{t \mid \|\dot{x}_0(t)\| > M\}. \end{aligned}$$

We set $\sigma(0) = 0$ and

$$\frac{d\sigma}{dt} = \begin{cases} r_M & \text{for } t \in \ell_m, \\ 1 & \text{for } t \in \ell_m^M, \\ \|\dot{x}_0(t)\| & \text{for } t \in \ell^M. \end{cases}$$

We have to adjust r_M so that $\sigma(T) = T$ and σ sends $[0, T]$ into itself. By assumption, ℓ^M has positive measure, so $r_M < 1$. This yields

$$\int_0^T \frac{d\sigma}{dt}dt = T = r_M \text{meas}(\ell_m) + [T - \text{meas}(\ell_m) - \text{meas}(\ell^M)] + \int_{\ell^M} \|\dot{x}_0\|dt,$$

$$(1 - r_M)\text{meas}(\ell_m) = \int_{\ell^M} (\|\dot{x}_0\| - 1)dt. \tag{2}$$

Since $\dot{x}_0$ is integrable, there is some $M_0 > 1$ such that the right-hand side is less than $T/4$ for all $M > M_0$. So, for all $M > M_0$, we get

$$r_M > \frac{1}{2}.$$

We now compare the solution x_0 to the trajectory $x_1 = x_0 \circ \sigma^{-1}$. It satisfies the boundary conditions and we have

$$\begin{aligned} \frac{dx_1}{ds}(s) &= \frac{dx_1}{dt}(t)\frac{dt}{ds} \quad \text{with } s = \sigma(t), \\ \frac{dx_1}{ds}(s) &= \frac{dx_1}{dt}\left(\sigma^{-1}(s)\right)\left[\frac{d\sigma}{dt}\left(\sigma^{-1}(s)\right)\right]^{-1}. \end{aligned}$$

Since x_0 is a minimizer, we must have

$$\int_0^T L(x_1(s), \dot{x}_1(s))ds \geq \int_0^T L(x_0(s), \dot{x}_0(s))ds.$$

Decomposing each integral into three, one on ℓ_m, one on ℓ_m^M, and one on ℓ^M, we find that the integrals on ℓ_m^M cancel each other, and we are left with

$$\int_{\ell_m} \left[r_M L\left(x_0, \frac{1}{r_M}\dot{x}_0\right) - L(x_0, \dot{x}_0)\right]dt + \int_{\ell^M}\left[L\left(x_0, \frac{\dot{x}_0}{\|\dot{x}_0\|}\right)\|\dot{x}_0\| - L(x_0, \dot{x}_0)\right]dt \geq 0.$$

Using the coercivity of L, this becomes

$$\int_{\ell_m} \left[r_M L\left(x_0, \frac{1}{r_M}\dot{x}_0\right) - L(x_0, \dot{x}_0)\right]dt + \int_{\ell^M}\left[L\left(x_0, \frac{\dot{x}_0}{\|\dot{x}_0\|}\right)\|\dot{x}_0\| - g(\|\dot{x}_0\|)\right]dt \geq 0. \tag{3}$$

Set:

$$A = \max\{|L(x_0(t), y)| \mid 0 \leq t \leq T,\ \|y\| \leq 1\}.$$

This number does not depend on m or M, and we have

$$L\left(x_0, \frac{\dot{x}_0}{\|\dot{x}_0\|}\right)\|\dot{x}_0\| \leq A\|\dot{x}_0\|.$$

Similarly, set

$$\begin{aligned} B &= \max\{|L(x_0(t), y)| \mid 0 \leq t \leq T,\ \|y\| \leq 2m\}, \\ K &= \max\left\{\left|\frac{\partial L}{\partial y}(x_0(t), y)\right| \mid 0 \leq t \leq T,\ \|y\| \leq 2m\right\}. \end{aligned}$$

We have, by the mean value theorem

$$r_M L\left(x_0, \frac{1}{r_M}\dot{x}_0\right) - L(x_0, \dot{x}_0) = L\left(x_0, \frac{1}{r_M}\dot{x}_0\right) - L(x_0, \dot{x}_0) + (r_M - 1)L\left(x_0, \frac{1}{r_M}\dot{x}_0\right),$$

$$\left|r_M L\left(x_0, \frac{1}{r_M}\dot{x}_0\right) - L(x_0, \dot{x}_0)\right| \leq K\left(\frac{1}{r_M} - 1\right)m + (1 - r_M)B \leq (1 - r_M)(2Km + B).$$

We have used the fact that, for each t in ℓ^m, one has $\|\dot{x}_0\| r_M^{-1} \leq 2\|\dot{x}_0\| \leq 2m$. Writing all this into inequality (3), we get

$$(1 - r_M)(2Km + B)\text{meas}(\ell_m) + \int_{\ell^M}[A\|\dot{x}_0\| - g(\|\dot{x}_0\|)]dt \geq 0.$$

We now remember (2). Writing it into the preceding one, we get

$$\int_{\ell^M} [(2Km + B)(\|\dot{x}_0\| - 1) + A\|\dot{x}_0\| - g(\|\dot{x}_0\|)]dt \geq 0. \tag{4}$$

This inequality holds for all $M > M_0$. But $\|\dot{x}_0\| \geq M$ on ℓ^M. Since $g(t)/t \to \infty$ when $t \to \infty$, there is some $M_1 > M_0$ such that

$$\|y\| > M_1 \Longrightarrow [(2Km + B)(\|y\| - 1) + A\|y\| - g(y)] < 0.$$

In other words, for $M > M_1$, the integrand of (4) is negative. This is the desired contradiction.

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