Vietnam J. Math. (2018) 46:531–555
DOI 10.1007/s10013-017-0254-y

Some Generalizations of Fixed Point Theorems in Partially Ordered Metric Spaces and Applications to Partial Differential Equations with Uncertainty

Hoang Viet Long[1] · **Nguyen Thi Kim Son**[2] · **Rosana Rodríguez-López**[3]

Received: 4 January 2017 / Accepted: 19 June 2017 / Published online: 9 August 2017

Abstract Some generalized contractions using altering distances in partially ordered metric spaces are investigated and their applications to fuzzy partial differential equations are considered. Starting from the Banach contraction principle, our theorems presented here generalize, extend, and improve different results existing in the literature on the existence of coincidence points for a pair of mappings. In terms of their applicability, this might constitute the first paper dealing with the solvability of fuzzy partial differential equations from the point of view of considering the structure of the fuzzy number space as a partially ordered space. Under the generalized contractive-like property over comparable items, which is weaker than the Lipschitz condition, we show that the existence of just a lower or an upper solution is enough to prove the existence and uniqueness of two types of fuzzy solutions in the sense of gH-differentiability.

Keywords Contractive-like mapping principle · Well-posed boundary value problems · Fuzzy partial hyperbolic differential equations · Generalized Hukuhara derivatives

Mathematics Subject Classification (2010) 47H10 · 47H04 · 03E72 · 46S40

✉ Hoang Viet Long
longhv08@gmail.com

1 Faculty of Information Technology, People's Police University of Technology and Logistics, Bac Ninh, Vietnam

2 Department of Mathematics, Hanoi University of Education, Hanoi, Vietnam

3 Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782, Santiago de Compostela, Spain

1 Introduction

Fixed point theory is one of the most powerful and fruitful tools in nonlinear analysis. The Banach contraction principle is widely considered as the source of fixed point theory. It is a very popular tool to deal with the existence problems in many branches of mathematical analysis. There has been a large number of generalizations of the Banach contraction principle. In particular, an interesting aspect is to deduce the existence and uniqueness of fixed point for self-maps on a metric space by altering distances between the points with the use of a certain control function. These control functions were introduced by Khan et al. in [16] and then applied in many works as, for instance, [3, 9, 14, 27, 34], where some fixed point theorems were investigated with the help of such altering distance functions.

Recently, a new technique was proposed in order to weaken the requirements on the contraction property by considering metric spaces endowed with a partial ordering. This approach was initiated by Ran and Reurings in [33] with some applications to matrix equations. It was later refined and extended in [28] by Nieto and Rodríguez-López and applied to periodic boundary value problems for ordinary differential equations (ODEs). Following this direction, in this paper, we generalize some fixed point theorems in partially ordered sets of Amini-Harandi and Emami [3] by using altering distances. With the help of the weak contractivity coefficient function $\beta \in \mathcal{S} := \mathcal{S}_0 \cup \{1_{[0,\infty)}\}$, where $\mathcal{S}_0$ is the class of functions $\beta : [0, \infty) \to [0, 1)$ that satisfy the condition

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0,$$

and $1_{[0,\infty)}$ is the indicator function on $[0, +\infty)$, i.e., $1_{[0,\infty)}(t) = 1$ for all $t \in [0, \infty)$, and $1_{[0,\infty)} = 0$, otherwise, we weaken the required conditions by considering weak contractions of Harjani and Sadarangani [14], and Nashine and Samet [27].

Since the base space does not necessarily have a vectorial structure, these fixed point theorems can be applied to prove the existence of solutions to ODEs, and partial differential equations (PDEs) in abstract spaces. We note that the space of fuzzy numbers is not a Banach space, but it is a quasilinear space having a partial ordering. Hence, there have been some recent results on the existence of solutions to fuzzy ODEs (see [25, 29, 36]) as applications of fixed point theory in partially ordered metric spaces.

In this paper, besides giving some new generalized results on the existence of coincidence points for a pair of mappings in partially ordered sets, we also show their applications in the field of fuzzy PDEs to illustrate the usability of our obtained results. The problem considered is

$$_kD_{xy}u(x, y) = f(x, y, u(x, y)), \quad (x, y) \in J := [0, a] \times [0, b], \quad k = 1, 2, \tag{1}$$

with condition

$$u(x, 0) = \eta_1(x), \quad x \in [0, a], \quad u(0, y) = \eta_2(y), \quad y \in [0, b], \tag{2}$$

where $u : J \to \mathbb{R}_{\mathcal{F}}$ is a fuzzy-valued mapping and $_kD_{xy}$ (for $k = 1, 2$) represents the gH-partial derivatives operators. This boundary value problem was considered in some previous research works [2, 20–24], in which the authors proved the validity of Picard's theorem. In these results, the Lipschitz contractivity of the function f is vital for the existence of the fuzzy solution. If f is just continuous or even not continuous, the situation is far different and some necessary conditions must be imposed in order to guarantee the existence of solutions (in the case of crisp ODEs we can see [3, 14, 15, 27, 28], and in the fuzzy case, we refer to [1, 29, 30, 36]).

In this paper, we show that, under the assumption of nondecreasing monotonicity and weak-contractivity of the mapping f only over comparable elements, the existence of just a lower or an upper solution is enough to guarantee the existence and uniqueness of two types of fuzzy solutions to the Problem (1)–(2). Some previous significant results for ODEs have been investigated in [29, 30, 36]. Our results presented here give some new approaches on the existence of two types of fuzzy solutions for some class of fuzzy PDEs under the gH-differentiability. One difficulty to be faced in the study of this problem is the existence of gH-differences, which also allows us to obtain a new solution to fuzzy PDEs with decreasing length of its support. In this case, the qualitative solutions may be better in comparison with those of crisp PDEs. Our results extend to a class of fuzzy PDEs some existing results for fuzzy ODEs by Alikhani and Bahrami [1], Nieto and Rodríguez-López [29], and Villamizar-Roa et al. [36].

The remainder of this paper is organized as follows. Section 2 presents our main results (Theorems 1 and 2), in which we prove the existence of coincidence points for a pair of mappings in a partially ordered metric space, and, in particular, we deduce a fixed point theorem. Our method is mainly based on the generalized contractive-like condition. Section 3 provides some results on the existence and uniqueness of solution for fuzzy partial differential equations as an effective application of our theorems presented in Section 2. Some necessary preliminaries about fuzzy analysis and gH-derivatives are shown in Sections 3.1 and 3.2. The boundary value problem of interest is stated in Section 3.3, and the study of the solvability of this problem is also included. Finally, some conclusions and future directions are discussed in Section 4.

2 Generalized Coincidence and Fixed Point Theorems

In this section, we provide some definitions and new results related to generalized coincidence and fixed point theorems in partially ordered metric spaces.

For $x \in \mathbb{R}$, $[x]$ is the greatest integer function or integer value, gives the largest integer less than or equal to x (the floor function).

By $\hat{C}([0, \infty))$, we denote the space of all nonnegative and continuous functions $\phi : [0, \infty) \to [0, \infty)$, for which the following property holds

$$\phi(t) = 0 \quad \text{if and only if} \quad t = 0.$$

Definition 1 [14] A nondecreasing function ψ in $\hat{C}([0, \infty))$ is called an altering distance function on $[0, \infty)$.

Some examples of altering distance functions on $[0, \infty)$ are t^2; $\ln(1+t)$; $t^2 - \ln(1+t^2)$.

Definition 2 [27] Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a metric space. We say that X is regular if, for an arbitrary nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x$ in X, then $x_n \leq x$ for all $n \in \mathbb{N}$.

Definition 3 [14] If $(X, \leq)$ is a partially ordered set and $f : X \to X$, we say that f is monotone nondecreasing (resp., nonincreasing) if $x, y \in X$, $x \leq y$ implies $f(x) \leq f(y)$ (resp., $f(y) \leq f(x)$).

Definition 4 [27] Let $(X, \leq)$ be a partially ordered set and let f, g be mappings from X to itself such that $f(X) \subset g(X)$. We say that f is weakly increasing with respect to g if, for all $x \in X$, we have $f(x) \leq f(y)$ for all $y \in g^{-1}(f(x))$, where

$$g^{-1}(f(x)) := \{u \in X \mid g(u) = f(x)\}.$$

Definition 5 [27] Let (X, d) be a metric space and $f, g : X \to X$. The pair $\{f, g\}$ is said to be compatible if $\lim_{n\to\infty} d(fg(x_n), gf(x_n)) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = x$ for some $x \in X$.

In this section, we extend the main results in [3, 14, 27] to get a generalized fixed point theorem in partially ordered metric spaces.

Theorem 1 *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f, g : X \to X$ be given mappings satisfying the following assumptions:*

i) $f(X) \subset g(X)$.
ii) *f is weakly increasing with respect to g.*
iii) *One of the two following conditions holds:*
   (a) *X is a regular metric space and $g(X)$ is a closed subspace of (X, d), or*
   (b) *f and g are continuous and the pair (f, g) is compatible.*
iv) *There exist a function $\beta \in \mathcal{S}$, $\phi \in \hat{C}([0, \infty))$, and ψ a strictly increasing altering distance function such that the following inequality holds*

$$\begin{aligned} \psi\left(d(f(x), f(y))\right) \leq\ & \beta\left(d(g(x), g(y))\right) \psi\left(d(g(x), g(y))\right) \\ & -\gamma\left(d(g(x), g(y))\right) \phi\left(d(g(x), g(y))\right) \end{aligned} \tag{3}$$

for all $(x, y) \in X \times X$ satisfying that $g(x)$ and $g(y)$ are comparable, where

$$\gamma(t) = [\beta(t)] \quad \text{for all } t \in [0, \infty).$$

Then, there exists a coincidence point x of f and g in X, i.e., $f(x) = g(x)$.

Proof We proceed in several steps.

Step 1. Firstly, we contribute a nondecreasing sequence $\{g(x_n)\}$ in X.

Let x_0 be an arbitrary point in X. Since $f(X) \subset g(X)$, we can construct a sequence $\{x_n\}$ in X defined by

$$g(x_{n+1}) = f(x_n) \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Since $x_1 \in g^{-1}(f(x_0))$, $x_2 \in g^{-1}(f(x_1))$ and f is weakly increasing with respect to g, we obtain

$$g(x_1) = f(x_0) \leq f(x_1) = g(x_2) \leq f(x_2) = g(x_3) \leq \cdots$$

Therefore, by recurrence, we obtain a nondecreasing sequence

$$g(x_1) \leq g(x_2) \leq g(x_3) \leq \cdots \leq g(x_n) \leq g(x_{n+1}) \leq \cdots$$

Since $g(x_n) \leq g(x_{n+1})$ for $n \geq 1$, it follows from (3) that

$$\begin{aligned}\psi(d(g(x_{n+1}), g(x_{n+2}))) &= \psi(d(f(x_n), f(x_{n+1}))) \\ &\leq \beta(d(g(x_n), g(x_{n+1})))\psi(d(g(x_n), g(x_{n+1}))) \\ &\quad -\gamma(d(g(x_n), g(x_{n+1})))\phi(d(g(x_n), g(x_{n+1}))) \\ &\leq \beta(d(g(x_n), g(x_{n+1})))\psi(d(g(x_n), g(x_{n+1}))) \\ &\leq \psi(d(g(x_n), g(x_{n+1})))\end{aligned}$$

for all $n \geq 1$. Hence, we have

$$\psi(d(g(x_{n+1}), g(x_{n+2}))) \leq \psi(d(g(x_n), g(x_{n+1}))) \quad \text{for all } n \geq 1.$$

Due to the strictly increasing character of the function ψ, $\{d(g(x_n), g(x_{n+1}))\}$ is a non-increasing and bounded from below sequence in $\mathbb{R}$. Therefore, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) = r. \tag{4}$$

We will prove that $r = 0$. In fact, from the continuity property of ψ and ϕ, we have

$$\lim_{n\to\infty} \psi(d(g(x_n), g(x_{n+1}))) = \psi\left(\lim_{n\to\infty} d(g(x_n), g(x_{n+1}))\right) = \psi(r)$$

and

$$\lim_{n\to\infty} \phi(d(g(x_n), g(x_{n+1}))) = \phi\left(\lim_{n\to\infty} d(g(x_n), g(x_{n+1}))\right) = \phi(r).$$

If $\beta = 1_{[0,\infty)}$, then $\gamma(t) = [\beta(t)] = 1$ for all $t \geq 0$. In this case, it follows from (3) that the following estimation holds

$$\psi(d(g(x_{n+1}), g(x_{n+2}))) \leq \psi(d(g(x_n), g(x_{n+1}))) - \phi(d(g(x_n), g(x_{n+1}))) \text{ for all } n \geq 1.$$

By taking limits on both sides when $n \to \infty$, we get

$$\psi(r) \leq \psi(r) - \phi(r),$$

which implies that $0 \leq -\phi(r)$, and using that $\phi \in \hat{C}([0, \infty))$, we obtain $\phi(r) = 0$ and $r = 0$.

On the other hand, if $\beta \in \mathcal{S}_0$, from (3) and the inequalities $g(x_n) \leq g(x_{n+1})$, $n \geq 1$, we have

$$\begin{aligned}\psi(d(g(x_{n+1}), g(x_{n+2}))) &= \psi(d(f(x_n), f(x_{n+1}))) \\ &\leq \beta(d(g(x_n), g(x_{n+1})))\psi(d(g(x_n), g(x_{n+1}))), \quad n \geq 1.\end{aligned}$$

By contradiction method, we assume that $r > 0$. It permits to affirm, from (4), the non-increasing character of the sequence $\{d(g(x_n), g(x_{n+1}))\}$ and the properties of ψ, that $\psi(d(g(x_n), g(x_{n+1}))) > 0$ for $n \geq 1$. Hence

$$\frac{\psi(d(g(x_{n+1}), g(x_{n+2})))}{\psi(d(g(x_n), g(x_{n+1})))} \leq \beta(d(g(x_n), g(x_{n+1}))) < 1$$

for $n \geq 1$. By taking limits on both sides of this equation, it leads to

$$\lim_{n\to\infty} \beta(d(g(x_n), g(x_{n+1}))) = 1.$$

Taking into account that $\beta \in \mathcal{S}_0$, the previous condition implies that $\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) = 0$, which is a contradiction.

Hence, in both cases, we have that $r = 0$ and thus, $\{g(x_n)\}$ is a nondecreasing sequence satisfying that

$$\lim_{n\to\infty} d(g(x_n), g(x_{n+1})) = 0. \tag{5}$$

Step 2. Next, we prove that $\{g(x_n)\}$ is a Cauchy sequence.
Case 1: If there exists an $n \in \mathbb{N}$ such that $g(x_n) = g(x_{n+1})$, then, from (3), we have

$$\psi(d(f(x_n), f(x_{n+1}))) \leq \beta(d(g(x_n), g(x_{n+1})))\psi(d(g(x_n), g(x_{n+1}))) = 0.$$

This inequality implies, by the properties of ψ, that $f(x_n) = f(x_{n+1})$ or $g(x_{n+1}) = g(x_{n+2})$. So, for all $m \geq n$, we have that $g(x_m) = g(x_n)$. It obviously shows that $\{g(x_n)\}$ is a Cauchy sequence.
Case 2: Assume that all the successive terms of $\{g(x_n)\}$ are different, that is, $g(x_n) \neq g(x_{n+1})$ for every $n \in \mathbb{N}$. We prove that

$$\limsup_{m\to\infty} \sup_{n\geq m} d(g(x_n), g(x_m)) = 0.$$

Indeed, suppose that $\limsup_{m\to\infty} \sup_{n\geq m} d(g(x_n), g(x_m)) \neq 0$ and select $\varepsilon > 0$ such that

$$\limsup_{m\to\infty} \sup_{n\geq m} d(g(x_n), g(x_m)) > \varepsilon.$$

Then, we can choose two subsequences $\{g(x_{n_k})\}$, $\{g(x_{m_k})\}$ of $\{g(x_n)\}$ such that $n_k \geq m_k > k$ and

$$d(g(x_{n_k}), g(x_{m_k})) > \varepsilon. \tag{6}$$

For each fixed m_k, we choose n_k to be the smallest number such that $n_k \geq m_k$ satisfying (6). Note that (6) implies, in fact, that $n_k > m_k$. Hence, it follows that $n_k - 1 \geq m_k$ and

$$d(g(x_{n_k-1}), g(x_{m_k})) \leq \varepsilon.$$

Then, we get

$$\begin{aligned} \varepsilon < d(g(x_{n_k}), g(x_{m_k})) &\leq d(g(x_{n_k}), g(x_{n_k-1})) + d(g(x_{n_k-1}), g(x_{m_k})) \\ &\leq d(g(x_{n_k}), g(x_{n_k-1})) + \varepsilon. \end{aligned} \tag{7}$$

Taking into account (5) and letting $k \to \infty$ in (7), we have

$$\lim_{k\to\infty} d(g(x_{n_k}), g(x_{m_k})) = \varepsilon. \tag{8}$$

Since

$$\begin{aligned} d(g(x_{n_k}), g(x_{m_k})) &\leq d(g(x_{n_k}), g(x_{n_k-1})) + d(g(x_{n_k-1}), g(x_{m_k-1})) \\ &\quad + d(g(x_{m_k-1}), g(x_{m_k})), \quad k \geq 1, \end{aligned} \tag{9}$$

using (5), (6), and passing to the limit inferior when $k \to \infty$ in the inequality (9), we obtain

$$\liminf_{k\to\infty} d(g(x_{n_k-1}), g(x_{m_k-1})) \geq \varepsilon. \tag{10}$$

On the other hand, from the estimation

$$\begin{aligned} d(g(x_{n_k-1}), g(x_{m_k-1})) &\leq d(g(x_{n_k}), g(x_{n_k-1})) + d(g(x_{n_k}), g(x_{m_k})) \\ &\quad + d(g(x_{m_k-1}), g(x_{m_k})), \quad k \geq 1, \end{aligned}$$

we get, from (5) and (8), that

$$\limsup_{k\to\infty} d(g(x_{n_k-1}), g(x_{m_k-1})) \le \varepsilon. \tag{11}$$

Thus, by combining (10) and (11), we have

$$\lim_{k\to\infty} d(g(x_{n_k-1}), g(x_{m_k-1})) = \varepsilon. \tag{12}$$

Now $m_k \le n_k$ implies $m_k - 1 \le n_k - 1$ and, thus, $g(x_{m_k-1}) \le g(x_{n_k-1})$. Applying the inequality (3) once again, we have

$$\begin{aligned}\psi(d(g(x_{n_k}), g(x_{m_k}))) &= \psi(d(f(x_{n_k-1}), f(x_{m_k-1})))\\ &\le \beta(d(g(x_{n_k-1}), g(x_{m_k-1})))\psi(d(g(x_{n_k-1}), g(x_{m_k-1})))\\ &\quad -\gamma(d(g(x_{n_k-1}), g(x_{m_k-1})))\phi(d(g(x_{n_k-1}), g(x_{m_k-1}))).\end{aligned} \tag{13}$$

If $\beta = 1_{[0,\infty)}$, then $\gamma(t) = \beta(t) = 1$ for all $t \ge 0$. Since ψ and ϕ are continuous, by passing to the limit as $k \to \infty$ in (13), we have $\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon)$, that is, $0 \le -\phi(\varepsilon)$. Hence, by the properties of ϕ, it follows that $\phi(\varepsilon) = 0$ and $\varepsilon = 0$.

On the other hand, if $\beta \in \mathcal{S}_0$, then $0 \le \beta(t) < 1$ for all $t \ge 0$. Denote $t_k = d(g(x_{n_k-1}), g(x_{m_k-1}))$ for $k \ge 1$. Since $\{\beta(t_k)\} \subset [0, 1]$ and $[0, 1]$ is a compact set in $\mathbb{R}$, then there exists a subsequence $\{\beta(t_{k_j})\}$ converging to $\lambda \in [0, 1]$. Therefore, by choosing subsequences if necessary, we assume that

$$\lim_{k\to\infty} \beta(d(g(x_{n_k-1}), g(x_{m_k-1}))) = \lambda \in [0, 1].$$

If $\lambda = 1$, then $\lim_{k\to\infty} d(g(x_{n_k-1}), g(x_{m_k-1})) = 0$, which implies that $\varepsilon = 0$. If $0 \le \lambda < 1$, then, from

$$\begin{aligned}\psi(d(g(x_{n_k}), g(x_{m_k}))) &= \psi(d(f(x_{n_k-1}), f(x_{m_k-1})))\\ &\le \beta(d(g(x_{n_k-1}), g(x_{m_k-1})))\psi(d(g(x_{n_k-1}), g(x_{m_k-1}))), \quad k \ge 1,\end{aligned}$$

by passing to the limit as $k \to \infty$ and using the continuity property of ψ, we have that $\psi(\varepsilon) \le \lambda\psi(\varepsilon)$, or, equivalently, $(1 - \lambda)\psi(\varepsilon) \le 0$. Hence, $\psi(\varepsilon) = 0$ and $\varepsilon = 0$.

Therefore, it follows that

$$\limsup_{m\to\infty} \sup_{n\ge m} d(g(x_n), g(x_m)) = 0.$$

Thus, $\{g(x_n)\}$ is a Cauchy sequence in (X, d).

Step 3. We prove the existence of a coincidence point of f and g.

Case 1: Assume that X is a regular metric space and that $g(X)$ is a closed subspace of (X, d). Then $(g(X), d)$ is a complete metric subspace of (X, d). Since $\{g(x_n)\}$ is a Cauchy sequence in $(g(X), d)$, there exists $u = g(z) \in g(X)$ such that $g(x_n) \to u = g(z)$ as $n \to \infty$. Since $\{g(x_n)\}$ is a nondecreasing sequence and X is regular, then $g(x_n) \le g(z)$ for all $n \in \mathbb{N}$. Applying (3) once again, we have

$$\begin{aligned}0 &\le \psi(d(f(z), g(x_{n+1})))\\ &= \psi(d(f(z), f(x_n)))\\ &\le \beta(d(g(z), g(x_n)))\psi(d(g(z), g(x_n))) - \gamma(d(g(z), g(x_n)))\phi(d(g(z), g(x_n)))\\ &\le \beta(d(g(z), g(x_n)))\psi(d(g(z), g(x_n)))\\ &\le \psi(d(g(z), g(x_n))).\end{aligned}$$

By using the property of continuity of ψ and letting $n \to \infty$, we get

$$\psi(d(f(z), g(z))) = 0.$$

It clearly follows that $d(f(z), g(z)) = 0$ and $f(z) = g(z)$.
Case 2: Assume that f and g are continuous and that the pair (f, g) is compatible.

Since $\{g(x_n)\}$ is a Cauchy sequence in a complete metric space (X, d), there exists $z \in X$ such that $g(x_n) \to z$ and $f(x_n) = g(x_{n+1}) \to z$, as $n \to \infty$. Since f, g are continuous, we get

$$\lim_{n\to\infty} g(g(x_n)) = g(z); \quad \lim_{n\to\infty} f(g(x_n)) = f(z); \quad \lim_{n\to\infty} g(f(x_n)) = g(z).$$

Since $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ and the pair (f, g) is compatible, it follows that

$$\lim_{n\to\infty} d(g(f(x_n)), f(g(x_n))) = 0.$$

Thus, from

$$0 \leq d(g(z), f(z)) \leq d(g(z), g(g(x_{n+1}))) + d(g(f(x_n)), f(g(x_n))) + d(f(g(x_n), f(z)))$$

and letting $n \to \infty$, we have that $d(g(z), f(z)) = 0$, i.e., $f(z) = g(z)$.

In consequence, z is a coincidence point of f and g and the theorem is proved. □

Remark 1 Theorem 1 is actually an extension of some previous results in [3] and [27]. Indeed,

1. If we choose $\beta(\cdot) = 1_{[0,\infty)}(\cdot)$, then we get the context of Theorem 2.4 and Theorem 2.6 in [27];
2. If we choose $\beta \in \mathcal{S}_0$, then $\gamma(t) = [\beta(t)] = 0$ for all $t \in [0, \infty)$. Hence, we receive a generalized result connected to Theorem 2.1 in [3], with ψ an altering distance function and g a generalized function defined on X.

Theorem 2 *Assume that $(X, \leq)$ is a partially ordered set and that there exists a metric d on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping. Assume that:*

i) *There exists $\beta \in \mathcal{S}$ such that*

$$\psi(d(f(x), f(y))) \leq \beta(d(x, y))\psi(d(x, y)) - \gamma(d(x, y))\phi(d(x, y)) \tag{14}$$

for all $x \leq y$ in X, where ψ is a strictly increasing altering distance function, $\phi \in \hat{C}([0, \infty))$ and $\gamma(t) = [\beta(t)]$ for all $t \in [0, \infty)$.

ii) *There exists $x_0 \in X$ such that $x_0 \leq f(x_0)$ or $f(x_0) \leq x_0$.*

iii) *One of the two following conditions holds:*

 (a) *X is a regular metric space; or*
 (b) *f is continuous.*

Then f has a fixed point in X, that is, there exists a point $z \in X$ such that $f(z) = z$. Furthermore, if

$$\textit{for each } y, z \in X, \textit{ there exists } x \in X \textit{ which is comparable both to } y \textit{ and } z, \tag{15}$$

then the fixed point of f is unique.

Proof If $f(x_0) = x_0$, then x_0 is a fixed point of f. We consider the case when $x_0 < f(x_0)$, that is, $x_0 \leq f(x_0)$ but $x_0 \neq f(x_0)$. Since f is a nondecreasing mapping, by induction method, we construct a sequence

$$x_0 < f(x_0) \leq f^2(x_0) \leq \cdots \leq f^n(x_0) \leq f^{n+1}(x_0) \leq \cdots$$

Set $x_{n+1} = f(x_n)$ for all $n \geq 0$. We have that $\{x_n\}$ is a nondecreasing sequence in X. The existence of a fixed point for the mapping f is proved similarly to the proof of Theorem 1 when g is the identity mapping from X to itself, i.e., $g = Id_X$.

Now, we prove the uniqueness of the fixed point. Indeed, assume that y and z are two fixed points of f. From hypothesis (15), there exists a point $x \in X$ which is comparable both to y and z. From the monotonicity property of f, this implies that, for each $n \in \mathbb{N}$, $f^n(x)$ is comparable both to $f^n(y) = y$ and $f^n(z) = z$. Therefore, by applying the inequality (14), we have

$$\begin{aligned}
\psi(d(z, f^n(x))) &= \psi(d(f^n(z), f^n(x))) \\
&\leq \beta(d(f^{n-1}(z), f^{n-1}(x)))\psi(d(f^{n-1}(z), f^{n-1}(x))) \\
&\quad -\gamma(d(f^{n-1}(z), f^{n-1}(x)))\phi(d(f^{n-1}(z), f^{n-1}(x))) \\
&\leq \beta(d(f^{n-1}(z), f^{n-1}(x)))\psi(d(f^{n-1}(z), f^{n-1}(x))) \\
&\leq \psi(d(f^{n-1}(z), f^{n-1}(x))) \\
&= \psi(d(z, f^{n-1}(x))), \quad n \in \mathbb{N},\ n \geq 2.
\end{aligned}$$

Denote $\tau_n = d(z, f^n(x)) \in [0, \infty)$, $n \in \mathbb{N}$, $n \geq 1$. By the strict monotonicity of ψ, it follows that $0 \leq \tau_n \leq \tau_{n-1}$, $n \in \mathbb{N}$, $n \geq 2$. Consequently, the sequence τ_n is nonnegative and decreasing. So there exists $r \geq 0$ such that $\lim_{n\to\infty} \tau_n = r$. We prove that $r = 0$.

Case 1: If $\beta = 1_{[0,\infty)}$, then $\beta(t) = 1$ for all $t \geq 0$ and $\gamma(t) = 1$ for all $t \geq 0$. From (13), we have

$$\psi(\tau_n) \leq \psi(\tau_{n-1}) - \phi(\tau_{n-1}), \quad n \in \mathbb{N},\ n \geq 2.$$

Passing to the limit as $n \to \infty$, by the continuity of the mappings ψ and ϕ, we have $\psi(r) \leq \psi(r) - \phi(r)$ and $\phi(r) = 0$. That implies $r = 0$.

Case 2: If $\beta \in \mathcal{S}_0$, from (13), we get

$$\psi(\tau_n) \leq \beta(\tau_{n-1})\psi(\tau_{n-1}), \quad n \in \mathbb{N}, n \geq 2. \tag{16}$$

By choosing subsequences if necessary, we assume that

$$\lim_{n\to\infty} \beta(\tau_n) = \lambda \in [0, 1],$$

which allows to deduce, by letting $n \to \infty$ in (16), that $\psi(r) \leq \lambda\psi(r)$, that is, $\psi(r)(1 - \lambda) \leq 0$. If $\lambda < 1$, then $\psi(r) = 0$, i.e., $r = 0$. If $\lambda = 1$, then $\lim_{n\to\infty} \beta(\tau_n) = 1$. From the properties of the function $\beta \in \mathcal{S}_0$, one gets $\lim_{n\to\infty} \tau_n = 0$. By the uniqueness of the limit, we prove that $r = 0$.

By applying analogous arguments, we have $\lim_{n\to\infty} d(y, f^n(x)) = 0$. It follows that

$$0 \leq d(y, z) \leq d(y, f^n(x)) + d(f^n(x), z) \to 0 \quad \text{as } n \to \infty.$$

This means that $y = z$. It completes the proof. □

Remark 2 Theorem 2 is also connected with some previous results:

1. If we choose $\beta(\cdot) = 1_{[0,\infty)}(\cdot)$, we receive again Theorems 2.1, 2.2, and 2.3 in [14], with weaker conditions on the function ϕ (here, ϕ is not necessarily nondecreasing on $[0, \infty)$).
2. If we choose $\beta \in \mathcal{S}_0$, we obtain a generalized result connected to Theorem 2.1 in [3], with ψ a strictly increasing altering distance function.
3. If we choose $\beta \in \mathcal{S}_0$ and $\psi = Id_{[0,\infty)}$ the identity mapping, one has again Theorem 2.1 in [3].

Remark 3 It is well-known that the hypothesis (15) is equivalent to the following hypothesis in [28]:

for each $y, z \in X$, there exists in X a lower bound or an upper bound of y, z.

Remark 4 From the proof of Theorem 2, we deduce that, if z is a fixed point of f, then $\lim_{n\to\infty} d(f^n(x), z) = 0$ for any $x \in X$ comparable to z.

Remark 5 We can affirm from the proof of Theorem 2 that, in order to obtain the existence of a unique fixed point for some function f, it is not necessary for the function f to be continuous. Instead of the condition of continuity, we can consider the requirement that the space X is regular. This restriction is valid in the case where X is the space of fuzzy sets on $\mathbb{R}$ (see [29]).

In the next section, we investigate some applications of these fixed point theorems to prove the existence of solution for a class of fuzzy partial differential equations.

3 Application to Fuzzy Partial Differential Equations

3.1 Fuzzy Partially Ordered Metric Spaces

Let $\mathbb{R}_{\mathcal{F}}$ be the space of fuzzy sets on $\mathbb{R}$ that are nonempty subsets $\{(x, u(x)) : x \in \mathbb{R}\}$ in $\mathbb{R} \times [0, 1]$ of certain functions $u : \mathbb{R} \to [0, 1]$ being normal, fuzzy-convex, upper semi-continuous, and compact-supported.

Let $u \in \mathbb{R}_{\mathcal{F}}$. The α-cuts or level sets of u are defined by

$$[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\} \quad \text{for each } 0 < \alpha \leq 1,$$

which are nonempty, compact, and convex subsets of $\mathbb{R}$ for all $0 < \alpha \leq 1$. The same properties hold for $[u]^0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$, which is called the support of u. For $u \in \mathbb{R}_{\mathcal{F}}$, we denote the parametric form of u by $[u]^\alpha = [u_{l\alpha}, u_{r\alpha}]$ for all $0 \leq \alpha \leq 1$, and $\text{len}([u])^\alpha = u_{r\alpha} - u_{l\alpha}$.

In $\mathbb{R}_{\mathcal{F}}$, we define the supremum metric d_∞ as follows

$$d_\infty(u, v) = \sup_{0\leq\alpha\leq 1} d_H\left([u]^\alpha, [v]^\alpha\right) \quad \text{for all } u, v \in \mathbb{R}_{\mathcal{F}},$$

where d_H is the Hausdorff metric in the set consisting of all nonempty, compact, and convex subsets of $\mathbb{R}$. It is well-known that $(\mathbb{R}_{\mathcal{F}}, d_\infty)$ is a complete metric space (see, for instance, [19]).

The addition and the multiplication by a scalar in the space of fuzzy numbers $\mathbb{R}_{\mathcal{F}}$ is defined levelsetwise, that is, for all $u, v \in \mathbb{R}_{\mathcal{F}}$, $\alpha \in [0, 1]$, and $k \in \mathbb{R}$,

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha \quad \text{and} \quad [ku]^\alpha = k[u]^\alpha.$$

In the special case where $k = -1$, $(-1)[u]^\alpha = (-1)[u_{l\alpha}, u_{r\alpha}] = [-u_{r\alpha}, -u_{l\alpha}]$.

If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v + w$, we call $w = u \ominus v$ the Hukuhara difference (or H-difference) of u and v. If $u \ominus v$ exists, then $[u \ominus v]^\alpha = [u_{l\alpha} - v_{l\alpha}, u_{r\alpha} - v_{r\alpha}]$ for all $0 \leq \alpha \leq 1$.

Lemma 1 [17] *For all $u, v, w, e \in \mathbb{R}_{\mathcal{F}}$, if the H-differences $u \ominus v$, $w \ominus e$ exist, then*

$$d_\infty(u \ominus v, w \ominus e) \leq d_\infty(u, w) + d_\infty(v, e).$$

Definition 6 [29] In $\mathbb{R}_{\mathcal{F}}$, a partial ordering can be defined as follows:

$$x \leq y \quad \text{if} \quad x_{l\alpha} \leq y_{l\alpha} \text{ and } x_{r\alpha} \leq y_{r\alpha} \qquad \text{for all } \alpha \in [0, 1],$$

where $x, y \in \mathbb{R}_{\mathcal{F}}$, $[x]^\alpha = [x_{l\alpha}, x_{r\alpha}]$, $[y]^\alpha = [y_{l\alpha}, y_{r\alpha}]$, $\alpha \in [0, 1]$.

Lemma 2 [29] *Some properties of fuzzy sets with respect to the partial ordering $\leq$ are:*

1) *If $x \leq y$, then $x + z \leq y + z$ for $x, y, z \in \mathbb{R}_{\mathcal{F}}$.*
2) *For every nondecreasing sequence $\{x_n\} \subset \mathbb{R}_{\mathcal{F}}$, if $x_n \to x$ in $\mathbb{R}_{\mathcal{F}}$, then $x_n \leq x$ for all $n \in \mathbb{N}$.*
3) *Every pair of elements of $\mathbb{R}_{\mathcal{F}}$ has an upper bound and a lower bound in $\mathbb{R}_{\mathcal{F}}$.*

Lemma 3 *If $u, v, w \in \mathbb{R}_{\mathcal{F}}$ are such that $w \leq v$ and the H-differences $u \ominus v$, $u \ominus w$ exist, then $u \ominus v \leq u \ominus w$.*

Proof It is clear that $w_{l\alpha} \leq v_{l\alpha}$ and $w_{r\alpha} \leq v_{r\alpha}$, imply that $u_{l\alpha} - v_{l\alpha} \leq u_{l\alpha} - w_{l\alpha}$ and $u_{r\alpha} - v_{r\alpha} \leq u_{r\alpha} - w_{r\alpha}$ for all $\alpha \in [0, 1]$. □

For $J \subset \mathbb{R}^2$, we denote by $C(J, \mathbb{R}_{\mathcal{F}})$ the space of all continuous functions defined on J and fuzzy-valued in $\mathbb{R}_{\mathcal{F}}$. Set

$$H_\lambda(u, v) = \sup_{(x,y)\in J} \left\{ d_\infty(u(x, y), v(x, y))e^{-\lambda(x+y)} \right\}$$

for $u, v \in C(J, \mathbb{R}_{\mathcal{F}})$, where $\lambda > 0$. It is easy to see that $(C(J, \mathbb{R}_{\mathcal{F}}), H_\lambda)$ is a complete metric space [19].

Definition 7 Consider $f, g \in C(J, \mathbb{R}_{\mathcal{F}})$. We say that $f \leq g$ in $C(J, \mathbb{R}_{\mathcal{F}})$ if and only if $f(x, y) \leq g(x, y)$ for all $(x, y) \in J$. That means $f_{l\alpha}(x, y) \leq g_{l\alpha}(x, y)$ and $f_{r\alpha}(x, y) \leq g_{r\alpha}(x, y)$ for all $\alpha \in [0, 1]$ and $(x, y) \in J$.

Some of the following properties of fuzzy-valued continuous functions with respect to the partial ordering $\leq$ are inferred directly from the corresponding properties of fuzzy numbers in $(\mathbb{R}_{\mathcal{F}}, \leq)$ given in Lemma 2.

Lemma 4 *Let $(\mathbb{R}_{\mathcal{F}}, \leq)$ be the space of fuzzy numbers equipped with the partial ordering defined, then we have*

1) *$(C(J, \mathbb{R}_{\mathcal{F}}), \leq)$ is a partial ordered space;*
2) *$(C(J, \mathbb{R}_{\mathcal{F}}), H_\lambda)$ is a regular metric space;*
3) *Every pair of elements of $C(J, \mathbb{R}_{\mathcal{F}})$ has an upper bound and a lower bound in $C(J, \mathbb{R}_{\mathcal{F}})$.*

Proof These properties have been established briefly in [29]. We include their proofs for the sake of completeness. The proofs of property 1) and property 3) are obvious, since they are true in $\mathbb{R}_{\mathcal{F}}$. So we can proceed for each $(x, y) \in J$, and these properties are satisfied in $C(J, \mathbb{R}_{\mathcal{F}})$ (note that we can select the upper and lower bounds to be continuous). Hence, we only give the proof of property 2).

2) Indeed, assume that $\{u_n\} \subset C(J, \mathbb{R}_{\mathcal{F}})$ is a nondecreasing sequence and convergent to u in $C(J, \mathbb{R}_{\mathcal{F}})$, then $\{u_n(x, y)\}$ is a nondecreasing sequence in $\mathbb{R}_{\mathcal{F}}$ for every $(x, y) \in J$. Moreover, for each $(x, y) \in J$,

$$e^{-\lambda(x+y)} d_\infty (u_n(x, y), u(x, y)) \leq \sup_J \left\{ d_\infty(u_n(x, y), u(x, y)) e^{-\lambda(x+y)} \right\} = H_\lambda(u_n, u).$$

Since $\lim_{n\to\infty} H_\lambda(u_n, u) = 0$, we have $\lim_{n\to\infty} d_\infty(u_n(x, y), u(x, y)) = 0$, or $u_n(x, y)$ converges to $u(x, y)$ in $\mathbb{R}_{\mathcal{F}}$ for every $(x, y) \in J$. From Lemma 2, we have $u_n(x, y) \leq u(x, y)$ for all $n \in \mathbb{N}$ and every $(x, y) \in J$. □

3.2 Some Preliminaries on Fuzzy Analysis

For $u, v \in \mathbb{R}_{\mathcal{F}}$, the generalized Hukuhara difference [4] (or gH-difference) of u and v, denoted by $u \ominus_{gH} v$ is defined as the element $w \in \mathbb{R}_{\mathcal{F}}$ such that

$$u \ominus_{gH} v = w \quad \Longleftrightarrow \quad \text{(i)}\ u = v + w \quad \text{or} \quad \text{(ii)}\ v = u + (-1)w.$$

Notice that, if $u \ominus v$ exists, then $u \ominus_{gH} v = u \ominus v$. If (i) and (ii) are satisfied simultaneously, then w is a crisp number. Also, $u \ominus_{gH} u = \hat{0}$ and if $u \ominus_{gH} v$ exists, it is unique.

The generalized Hukuhara partial derivatives (gH-p-derivatives, for short) of a fuzzy-valued mapping $f : I \subset \mathbb{R}^2 \to \mathbb{R}_{\mathcal{F}}$ are defined in Definitions 2.9 and 3.4 in [2]. Denote by $C^2(I, \mathbb{R}_{\mathcal{F}})$ the set of all functions $f \in C(I, \mathbb{R}_{\mathcal{F}})$ which have gH-p-derivatives up to order 2 with respect to x and y continuous on I.

Definition 8 [2] Let $f : I \to \mathbb{R}_{\mathcal{F}}$ be gH-p-differentiable with respect to x at $(x_0, y_0) \in I$. We say that f is (i)-gH differentiable with respect to x at $(x_0, y_0) \in I$ if

$$[f_x(x_0, y_0)]^\alpha = [\partial_x f_{l\alpha}(x_0, y_0), \partial_x f_{r\alpha}(x_0, y_0)] \quad \forall \alpha \in [0, 1]$$

and that f is (ii)-gH differentiable with respect to x at $(x_0, y_0) \in I$ if

$$[f_x(x_0, y_0)]^\alpha = [\partial_x f_{r\alpha}(x_0, y_0), \partial_x f_{l\alpha}(x_0, y_0)] \quad \forall \alpha \in [0, 1].$$

The (i) and (ii)-gH derivatives of f with respect to y are defined similarly.

Definition 9 Let $f \in C^2(I, \mathbb{R}_{\mathcal{F}})$ and f_y be gH-p-differentiable at $(x_0, y_0) \in I$ with respect to x and do not have any switching points on I. We say that

a) f_{xy} is in type 1 of gH-derivatives (denote ${}_1D_{xy} f$) if the type of gH-derivatives of both f and f_y are the same. Then, for $\alpha \in [0, 1]$,

$$\left[{}_1D_{xy} f(x_0, y_0)\right]^\alpha = \left[\partial_{xy} f_{l\alpha}(x_0, y_0), \partial_{xy} f_{r\alpha}(x_0, y_0)\right].$$

b) f_{xy} is in type 2 of gH-derivatives (denote ${}_2D_{xy} f$) if the type of gH-derivatives of both f and f_y are different. Then, for $\alpha \in [0, 1]$,

$$\left[{}_2D_{xy} f(x_0, y_0)\right]^\alpha = \left[\partial_{xy} f_{r\alpha}(x_0, y_0), \partial_{xy} f_{l\alpha}(x_0, y_0)\right].$$

It is a well-known result that, if f is continuous on U, then f is integrable on U. Moreover, we have the following properties.

Lemma 5 *Let U be a compact subset of $\mathbb{R}^2$, $u \leq v$ in $C(U, \mathbb{R}_{\mathcal{F}})$. Then*

$$\int_U u(x, y)dxdy \leq \int_U v(x, y)dxdy.$$

Proof From the definition of the fuzzy Aumann integral [19], we have

$$\left[\int_U u(x,y)dxdy\right]^\alpha = \left[\int_U u_{l\alpha}(x,y)dxdy, \int_U u_{r\alpha}(x,y)dxdy\right]$$

and

$$\left[\int_U v(x,y)dxdy\right]^\alpha = \left[\int_U v_{l\alpha}(x,y)dxdy, \int_U v_{r\alpha}(x,y)dxdy\right]$$

for every $\alpha \in [0,1]$.

Since $u \leq v$ in $C(U, \mathbb{R}_\mathcal{F})$, then $u(x,y) \leq v(x,y) \in \mathbb{R}_\mathcal{F}$ for all $(x,y) \in U$. That means, from Definition 6, that $(u(x,y))_{l\alpha} \leq (v(x,y))_{l\alpha}$, $(u(x,y))_{r\alpha} \leq (v(x,y))_{r\alpha}$ for all $\alpha \in [0,1]$. It implies that

$$\int_U u_{l\alpha}(x,y)dxdy \leq \int_U v_{l\alpha}(x,y)dxdy, \quad \int_U u_{r\alpha}(x,y)dxdy \leq \int_U v_{r\alpha}(x,y)dxdy$$

for all $\alpha \in [0,1]$. From Definition 6, we deduce that $\int_U u(x,y)dxdy \leq \int_U v(x,y)dxdy$. □

3.3 Statement of the Problems

In this part, we prove some new results on the existence of a unique solution for fuzzy partial differential equations with local boundary conditions by applying the theory presented in Section 2.

For arbitrary positive real numbers a, b, we denote $J_a = [0,a]$, $J_b = [0,b]$, $J = J_a \times J_b$. We recall Problem (1)–(2) with $\eta_1(\cdot) \in C(J_a, \mathbb{R}_\mathcal{F})$, $\eta_2(\cdot) \in C(J_b, \mathbb{R}_\mathcal{F})$ being given functions such that $\eta_1(0) = \eta_2(0)$ and the difference $\eta_2(y) \ominus \eta_1(0)$ exists for all $y \in J_b$ and the function $f : J \times \mathbb{R}_\mathcal{F} \to \mathbb{R}_\mathcal{F}$ has no switching points. This boundary value problem has been considered in some references such as [2, 20–22]. In these papers, the authors prove the Picard's theorem for Problem (1)–(2), i.e., when f is Lipschitz continuous, the problem has a unique fuzzy solution. By weakening the Lipschitz condition, now the function f only needs to satisfy a generalized contractive-like condition between comparable items, and we also prove the existence of fuzzy solutions.

For $(x,y) \in J$, let $I_{xy}f(x,y,u)$ denote the integral $\int_0^y \int_0^x f(s,t,u(s,t))dsdt$. We change the order of integration with respect to the notation in [22], since, in the derivatives ${}_kD_{xy}$, we first calculate a derivative with respect to y and then with respect to x, so that we integrate in the reverse order.

Lemma 6 [22] *Assume that f is a continuous function on $J \times \mathbb{R}_\mathcal{F}$ and that $u(\cdot,\cdot) \in C^2(J, \mathbb{R}_\mathcal{F})$ satisfies Problem (1)–(2) in J. Then $u(\cdot,\cdot)$ satisfies the following integral equations:*

1) *If $k = 1$ then $u(x,y) = p(x,y) + I_{xy}f(x,y,u)$ for $(x,y) \in J$; or*
2) *If $k = 2$ then $u(x,y) = p(x,y) \ominus (-1)I_{xy}f(x,y,u)$ for $(x,y) \in J$,*

where

$$p(x,y) = \eta_1(x) + \eta_2(y) \ominus \eta_1(0). \tag{17}$$

Definition 10 A function $u \in C(J, \mathbb{R}_\mathcal{F})$ is called an integral solution of type 1 of the Problem (1)–(2) if it satisfies the following integral equation

$$u(x,y) = p(x,y) + I_{xy}f(x,y,u) \quad \text{for all } (x,y) \in J$$

and $u \in C(J, \mathbb{R}_{\mathcal{F}})$ is called an integral solution of type 2 of the Problem (1)–(2) if it satisfies the following integral equation

$$u(x, y) = p(x, y) \ominus (-1) I_{xy} f(x, y, u) \quad \text{for all } (x, y) \in J,$$

where $p(\cdot, \cdot)$ is defined by (17).

Remark 6 Notice that Definition 10 makes sense via Lemma 6.

Definition 11 A fuzzy function $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}})$ is called a (k)-lower $(k = 1, 2)$ solution of the Problem (1)–(2) if

$$
\begin{aligned}
&{}_k D_{xy} \mu(x, y) \leq f(x, y, \mu(x, y)), \quad (x, y) \in J, \\
&\mu(x, 0) \leq \eta_1(x), \quad x \in J_a, \qquad \mu(0, y) \leq \eta_2(y), \quad y \in J_b, \qquad \mu(0, 0) = \eta_1(0).
\end{aligned}
$$

Analogously, a fuzzy function $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}})$ is called a (k)-upper $(k = 1, 2)$ solution of the Problem (1)–(2) if

$$
\begin{aligned}
&{}_k D_{xy} \mu(x, y) \geq f(x, y, \mu(x, y)), \quad (x, y) \in J, \\
&\mu(x, 0) \geq \eta_1(x), \quad x \in J_a, \qquad \mu(0, y) \geq \eta_2(y), \quad y \in J_b, \qquad \mu(0, 0) = \eta_1(0).
\end{aligned}
$$

Remark 7 The first steps in the theory of lower and upper solutions have been given by Picard for PDEs and ODEs [31, 32]. In both cases, the existence of a solution is guaranteed from a monotone iterative technique. Dragoni [10, 11] are the first ones that recognize explicitly the central role of lower and upper solutions for ordinary differential equations with Dirichlet boundary value conditions. In the monograph of Bernfeld and Lakshmikantham [5], Ladde et al. [18] the theory of the method of lower and upper solutions and the monotone iterative technique are presented in details.

In this paper, the existence of lower solutions or upper solutions of considered problem is used as a sufficient condition in generalized contractive-like theorems in Section 2 to ensure the existence and uniqueness of two types of fuzzy solutions to the Problem (1)–(2). For more about the method of lower and upper solutions, we refer the reader to the classical work of Mawhin [26] and the surveys in this field of De Coster and Habets [6–8] in which we can find historical and bibliographical references together with recent results and open problems.

3.4 Existence and Uniqueness of Fuzzy Solutions

Lemma 7 *For an arbitrary strictly increasing altering distance function γ and for all positive real numbers a, b, there exists $\lambda > 0$ such that the function*

$$\Phi(t) = \gamma(t) - \gamma\left(\frac{1}{\lambda^2}\left(1 - e^{-\lambda a}\right)\left(1 - e^{-\lambda b}\right) t\right), \quad t \in [0, \infty),$$

belongs to $\hat{C}([0, \infty))$.

Proof From the continuity of γ, Φ is a continuous function on $[0, \infty)$. Choose $\lambda > 0$ such that

$$\frac{1}{\lambda^2}\left(1 - e^{-\lambda a}\right)\left(1 - e^{-\lambda b}\right) < 1.$$

Then, for all $t \geq 0$, we have $\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b})t \leq t$. Since γ is increasing, it follows that $\gamma\left(\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b})t\right) \leq \gamma(t)$ for all $t \geq 0$. Hence $\Phi(t) \geq 0$ for all $t \geq 0$.

Now, we consider $t > 0$. From $\frac{1}{\lambda^2}(1-e^{-\lambda a})(1-e^{-\lambda b})t < t$ and the strict increase property of γ, it implies that $\Phi(t) > 0$. It follows that, if $\Phi(t) = 0$, then $t = 0$ (and conversely). It completes the proof. □

Theorem 3 *Let f be a continuous function that satisfies the following two hypotheses:*

(h_1) *$f : J \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is nondecreasing in the third variable, i.e., if $\nu \le \xi \in \mathbb{R}_{\mathcal{F}}$, then $f(x, y, \nu) \le f(x, y, \xi)$ for all $(x, y) \in J$.*

(h_2) *f is weakly contractive over comparable elements, that is, for some altering distance function ψ and $\phi \in \hat{C}([0,\infty))$, the following estimation*

$$\psi(d_\infty(f(x, y, \nu), f(x, y, \xi))) \le \psi(d_\infty(\nu, \xi)) - \phi(d_\infty(\nu, \xi))$$

holds for all $(x, y) \in J$, $\nu \le \xi$ in $\mathbb{R}_{\mathcal{F}}$.

Suppose that there exists a (1)-lower solution $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}})$ for the Problem (1)–(2). Then the Problem (1)–(2) has a unique integral solution of type 1 on J.

Proof Define the operator $T_1 : C(J, \mathbb{R}_{\mathcal{F}}) \to C(J, \mathbb{R}_{\mathcal{F}})$ by

$$(T_1 u)(x, y) = p(x, y) + I_{xy} f(x, y, u), \quad (x, y) \in J, \tag{18}$$

for $u \in C(J, \mathbb{R}_{\mathcal{F}})$, where $p(\cdot,\cdot)$ is defined by (17).

Step 1: We prove that T_1 is a nondecreasing operator in $C(J, \mathbb{R}_{\mathcal{F}})$.

Assume that $u \le v$ in $C(J, \mathbb{R}_{\mathcal{F}})$, which means $u(s,t) \le v(s,t)$ for all $(s,t) \in J$. From hypothesis (h_1), that is, the nondecreasing character of f with respect to the third variable, we have that $f(s, t, u(s,t)) \le f(s, t, v(s,t))$ for all $(s,t) \in J$. Then, from Lemma 5, we have

$$I_{xy} f(x, y, u) \le I_{xy} f(x, y, v) \quad \text{for } (x, y) \in J.$$

It means that $(T_1 u)(x, y) \le (T_1 v)(x, y)$ for all $(x, y) \in J$. Hence, $T_1 u \le T_1 v$.

Step 2: Now, we prove that

$$d_\infty(f(x, y, \nu), f(x, y, \eta)) \le d_\infty(\nu, \eta) \quad \text{for all } \nu \le \eta \text{ in } \mathbb{R}_{\mathcal{F}} \text{ and } (x, y) \in J.$$

Indeed, assume that $\nu \le \eta$ in $\mathbb{R}_{\mathcal{F}}$ but $d_\infty(\nu, \eta) < d_\infty(f(x, y, \nu), f(x, y, \eta))$ for some $(x, y) \in J$. Due to the nondecrease property of ψ, we have

$$\psi(d_\infty(\nu, \eta)) \le \psi(d_\infty(f(x, y, \nu), f(x, y, \eta))). \tag{19}$$

On the other hand, from the hypothesis (h_2), we have

$$\begin{aligned} \psi(d_\infty(f(x, y, \nu), f(x, y, \eta))) &\le \psi(d_\infty(\nu, \eta)) - \phi(d_\infty(\nu, \eta)) \\ &\le \psi(d_\infty(\nu, \eta)) \end{aligned} \tag{20}$$

for all $\nu \le \eta$ in $\mathbb{R}_{\mathcal{F}}$. From (19) and (20), one has

$$\psi(d_\infty(\nu, \eta)) = \psi(d_\infty(f(x, y, \nu), f(x, y, \eta))).$$

It follows from (20) that $0 \le -\phi(d_\infty(\nu, \eta))$ or $\phi(d_\infty(\nu, \eta)) = 0$. Thanks to $\phi \in \hat{C}([0,\infty))$, that implies $d_\infty(\nu, \eta) = 0$. Hence

$$\psi(d_\infty(f(x, y, \nu), f(x, y, \eta))) = \psi(d_\infty(\nu, \eta)) = 0.$$

It implies $d_\infty(f(x, y, \nu), f(x, y, \eta)) = 0$, leading to a contradiction.

Step 3: We check the generalized contractive-like property of the operator T_1.

For all $u \leq v$ in $C(J, \mathbb{R}_{\mathcal{F}})$, we have $u(x, y) \leq v(x, y)$ for all $(x, y) \in J$. It is known from Step 2 that

$$d_\infty(f(x, y, u(x, y)), f(x, y, v(x, y))) \leq d_\infty(u(x, y), v(x, y)) \quad \text{for all } (x, y) \in J.$$

Thus

$$\begin{aligned} d_\infty((T_1u)(x, y), (T_1v)(x, y)) &= d_\infty\left(p(x, y) + I_{xy}f(x, y, u), p(x, y) + I_{xy}f(x, y, v)\right) \\ &= d_\infty\left(I_{xy}f(x, y, u), I_{xy}f(x, y, v)\right) \\ &\leq \int_0^y\int_0^x d_\infty(f(s, t, u(s, t)), f(s, t, v(s, t)))dsdt \\ &\leq \int_0^y\int_0^x d_\infty(u(s, t), v(s, t))dsdt \\ &\leq \int_0^y\int_0^x H_\lambda(u, v)e^{\lambda(s+t)}dsdt \\ &= \frac{1}{\lambda^2}H_\lambda(u, v)(e^{\lambda x} - 1)(e^{\lambda y} - 1). \end{aligned}$$

Then, for all $(x, y) \in J$, we have

$$d_\infty((T_1u)(x, y), (T_1v)(x, y))e^{-\lambda(x+y)} \leq \frac{1}{\lambda^2}H_\lambda(u, v)(1 - e^{-\lambda x})(1 - e^{-\lambda y}).$$

Therefore

$$H_\lambda(T_1u, T_1v) \leq \frac{1}{\lambda^2}H_\lambda(u, v)(1 - e^{-\lambda a})(1 - e^{-\lambda b}). \tag{21}$$

For an arbitrary strictly increasing altering distance function γ, from (21), we have

$$\begin{aligned} &\gamma(H_\lambda(T_1u, T_1v)) \leq \gamma\left(\frac{1}{\lambda^2}H_\lambda(u, v)(1 - e^{-\lambda a})(1 - e^{-\lambda b})\right) \\ &\quad = \gamma(H_\lambda(u, v)) - \left[\gamma(H_\lambda(u, v)) - \gamma\left(\frac{1}{\lambda^2}H_\lambda(u, v)(1 - e^{-\lambda a})(1 - e^{-\lambda b})\right)\right]. \end{aligned}$$

Denote $\Phi(t) = \gamma(t) - \gamma\left(\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b})t\right)$, $t \in [0, \infty)$. From Lemma 7, there exists $\lambda > 0$ such that Φ belongs to $\hat{C}([0, \infty))$ and

$$\gamma(H_\lambda(T_1u, T_1v)) \leq \gamma(H_\lambda(u, v)) - \Phi(H_\lambda(u, v)) \quad \text{for all } u \leq v \text{ in } C(J, \mathbb{R}_{\mathcal{F}}).$$

This means that the operator T_1 satisfies the contractive-like property.

Step 4: Since there exists a (1)-lower solution $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}})$ for the Problem (1)–(2), then

$$\begin{aligned} \mu_{l\alpha}(x, y) &\leq \mu_{l\alpha}(x, 0) + \mu_{l\alpha}(0, y) - \mu_{l\alpha}(0, 0) + \int_0^y\int_0^x f_{l\alpha}(s, t, \mu(s, t))dsdt \\ &\leq (\eta_1)_{l\alpha}(x) + (\eta_2)_{l\alpha}(y) - (\eta_1)_{l\alpha}(0) + \int_0^y\int_0^x f_{l\alpha}(s, t, \mu(s, t))dsdt, \\ \mu_{r\alpha}(x, y) &\leq \mu_{r\alpha}(x, 0) + \mu_{r\alpha}(0, y) - \mu_{r\alpha}(0, 0) + \int_0^y\int_0^x f_{r\alpha}(s, t, \mu(s, t))dsdt \\ &\leq (\eta_1)_{r\alpha}(x) + (\eta_2)_{r\alpha}(y) - (\eta_1)_{r\alpha}(0) + \int_0^y\int_0^x f_{r\alpha}(s, t, \mu(s, t))dsdt, \end{aligned}$$

for $\alpha \in [0, 1]$ and $(x, y) \in J$, so that

$$\mu(x, y) \leq \eta_1(x) + \eta_2(y) \ominus \eta_1(0) + I_{xy}f(x, y, \mu) = (T_1\mu)(x, y)$$

for all $(x, y) \in J$. It follows that $\mu \leq T_1\mu$ in $C(J, \mathbb{R}_{\mathcal{F}})$.

It is easy to see from Steps 1–4 that the operator T_1 satisfies all the hypotheses of Theorem 2 in case $\beta = 1_{[0,\infty)}$. In consequence, T_1 has a fixed point in $C(J, \mathbb{R}_{\mathcal{F}})$. Note that $C(J, \mathbb{R}_{\mathcal{F}})$ satisfies that every pair of elements of $C(J, \mathbb{R}_{\mathcal{F}})$ have an upper bound and a lower bound in $C(J, \mathbb{R}_{\mathcal{F}})$ (Lemma 4). It follows that the operator T_1 has a unique fixed point, which is the unique integral solution of type 1 to Problem (1)–(2). □

Remark 8 The existence of an integral solution of type 1 is guaranteed by the weakly non-decreasing character and the generalized weak contractivity property of function f. The existence of an integral solution of type 2 is more difficult to obtain due to the requirement of the existence of Hukuhara differences.

We denote

$$\hat{C}(J, \mathbb{R}_{\mathcal{F}}) = \{u \in C(J, \mathbb{R}_{\mathcal{F}}) : p(x, y) \ominus (-1)I_{xy} f(x, y, u) \text{ exists for all } (x, y) \in J\},$$

where $p(x, y)$ is defined by (17).

Lemma 8 *Consider $(C(J, \mathbb{R}_{\mathcal{F}}), d)$ a complete metric space. If f is a continuous function and $\hat{C}(J, \mathbb{R}_{\mathcal{F}}) \neq \emptyset$, then $(\hat{C}(J, \mathbb{R}_{\mathcal{F}}), d)$ is a complete metric space.*

Proof Let $\{u_m\}_{m=1}^{\infty}$ be a sequence in $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$ converging towards u (in $C(J, \mathbb{R}_{\mathcal{F}})$). Then, for all $(x, y) \in J$, the following differences exist

$$p(x, y) \ominus (-1)I_{xy} f(x, y, u_m).$$

For simplicity of exposition, let

$$F(u_m)(x, y) = (-1)I_{xy} f(x, y, u_m).$$

From Proposition 21 in [35], we know that, for each fixed $(x, y) \in J$,

$$\begin{cases} \text{len}[p(x, y)]^\alpha \geq \text{len}[F(u_m)(x, y)]^\alpha, \quad 0 \leq \alpha \leq 1, \\ (p(x, y))_{l\alpha} - (F(u_m)(x, y))_{l\alpha} \text{ is monotonically increasing in } \alpha \in [0, 1], \\ (p(x, y))_{r\alpha} - (F(u_m)(x, y))_{r\alpha} \text{ is monotonically decreasing in } \alpha \in [0, 1]. \end{cases}$$

Since f is continuous and $\{u_m\}_{m=1}^{\infty}$ converges uniformly to u, then

$$\text{len}\left[\int_0^y \int_0^x f(s, t, u_m(s, t))dsdt\right]^\alpha$$

is convergent towards

$$\text{len}\left[\int_0^y \int_0^x f(s, t, u(s, t))dsdt\right]^\alpha$$

for each $\alpha \in [0, 1]$. Therefore, $\text{len}[F(u_m)(x, y)]^\alpha$ converges to $\text{len}[F(u)(x, y)]^\alpha$, where

$$F(u)(x, y) = (-1)I_{xy} f(x, y, u) = (-1)\int_0^y \int_0^x f\,(s, t, u(s, t))\,dsdt.$$

Hence, from the inequality

$$\text{len}[p(x, y)]^\alpha \geq \text{len}[F(u_m)(x, y)]^\alpha, \quad 0 \leq \alpha \leq 1,$$

we derive that, for each fixed $(x, y) \in J$,

$$\text{len}[p(x, y)]^\alpha \geq \text{len}[F(u)(x, y)]^\alpha, \quad 0 \leq \alpha \leq 1.$$

Moreover, for arbitrary $0 \leq \alpha \leq \gamma \leq 1$, we have

$$(p(x, y))_{l\alpha} - (F(u_m)(x, y))_{l\alpha} \leq (p(x, y))_{l\gamma} - (F(u_m)(x, y))_{l\gamma}.$$

Taking the limits when $m \to \infty$ and using similar arguments as above, we receive

$$(p(x,y))_{l\alpha} - (F(u)(x,y))_{l\alpha} \le (p(x,y))_{l\gamma} - (F(u)(x,y))_{l\gamma}.$$

By analogous arguments, one has

$$(p(x,y))_{r\alpha} - (F(u)(x,y))_{r\alpha} \ge (p(x,y))_{r\gamma} - (F(u)(x,y))_{r\gamma}$$

for all $0 \le \alpha \le \gamma \le 1$.

Therefore, the difference

$$p(x,y) \ominus (-1) I_{xy} f(x,y,u)$$

exists for all $(x,y) \in J$. It shows that $u \in \hat{C}(J, \mathbb{R}_{\mathcal{F}})$ and $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$ is a closed subset of the space $C(J, \mathbb{R}_{\mathcal{F}})$. Since $(C(J, \mathbb{R}_{\mathcal{F}}), d)$ is a complete metric space, $(\hat{C}(J, \mathbb{R}_{\mathcal{F}}), d)$ is also a complete metric space. □

By changing the solution space to $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$, we can prove the existence of solution of type 2 to the Problem (1)–(2).

Theorem 4 *Let f be a continuous function satisfying the hypotheses* (h$_1$)–(h$_2$) *in Theorem 3. Moreover, suppose that the following hypotheses are fulfilled:*

(h$_3$) $\hat{C}(J, \mathbb{R}_{\mathcal{F}}) \neq \emptyset$.

(h$_4$) *If $u \in C(J, \mathbb{R}_{\mathcal{F}})$ satisfies that $u \in \hat{C}(J, \mathbb{R}_{\mathcal{F}})$, then the Hukuhara difference*

$$p(x,y) \ominus (-1) I_{xy} f(x,y,\nu)$$

also exists for every $(x,y) \in J$, where

$$\nu(x,y) = p(x,y) \ominus (-1) I_{xy} f(x,y,u), \quad (x,y) \in J.$$

Suppose that there exists a (2)*-lower solution $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}}) \cap \hat{C}(J, \mathbb{R}_{\mathcal{F}})$ for the Problem* (1)–(2)*. Then the Problem* (1)–(2) *has an integral solution of type* 2 *on J.*

Furthermore, if the following condition holds:

(h$_5$) *For each pair $u, v \in \hat{C}(J, \mathbb{R}_{\mathcal{F}})$ fixed, there exists $\xi \in C(J, \mathbb{R}_{\mathcal{F}})$ an upper or a lower bound of u, v such that the Hukuhara difference $p(x,y) \ominus (-1) I_{xy} f(x,y,\xi)$ exists for all $(x,y) \in J$,*

then the Problem (1)–(2) *has a unique integral solution of type* 2 *on J.*

Proof By the hypothesis (h$_3$), $\hat{C}(J, \mathbb{R}_{\mathcal{F}}) \neq \emptyset$ and it is clear that, for every $u \in \hat{C}(J, \mathbb{R}_{\mathcal{F}})$, the Hukuhara difference $p(x,y) \ominus (-1) I_{xy} f(x,y,u)$ exists for all $(x,y) \in J$. By the assumption (h$_4$), it is reasonable to build the operator $T_2 : \hat{C}(J, \mathbb{R}_{\mathcal{F}}) \to \hat{C}(J, \mathbb{R}_{\mathcal{F}})$ defined by

$$(T_2 u)(x,y) = p(x,y) \ominus (-1) I_{xy} f(x,y,u), \quad (x,y) \in J.$$

Similarly to Step 2 in the proof of Theorem 3, we receive from hypotheses (h$_1$)–(h$_2$) that

$$d_\infty(f(x,y,\nu), f(x,y,\eta)) \le d_\infty(\nu, \eta)$$

for all $\nu \le \eta$ in $\mathbb{R}_{\mathcal{F}}$ and $(x,y) \in J$.

Using analogous arguments as in the proof of (21) and combining with Lemma 1, for all $u \leq v$ in $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$, we have

$$\begin{aligned}
&d_\infty((T_2u)(x, y), (T_2v)(x, y)) \\
&= d_\infty \big(p(x, y) \ominus (-1)I_{xy} f(x, y, u), p(x, y) \ominus (-1)I_{xy} f(x, y, v)\big) \\
&\leq d_\infty(I_{xy} f(x, y, u), I_{xy} f(x, y, v)) \\
&\leq \frac{1}{\lambda^2} H_\lambda(u, v)(e^{\lambda x} - 1)(e^{\lambda y} - 1),
\end{aligned}$$

and it follows that

$$H_\lambda(T_2u, T_2v) \leq \frac{1}{\lambda^2} H_\lambda(u, v)(1 - e^{-\lambda a})(1 - e^{-\lambda b}). \tag{22}$$

Now, assume that $u \leq v$ in $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$. We need to indicate the nondecreasing character of the operator T_2, proving that $T_2u \leq T_2v$. Since $u(s, t) \leq v(s, t)$ for all $(s, t) \in J$, and using the hypothesis of the nondecreasing character of f in the third variable, we have $f(s, t, u(s, t)) \leq f(s, t, v(s, t))$ for all $(s, t) \in J$. It follows from Lemma 5 that

$$\int_0^y \int_0^x f(s, t, u(s, t))dsdt \leq \int_0^y \int_0^x f(s, t, v(s, t))dsdt,$$

or

$$(-1)\int_0^y \int_0^x f(s, t, v(s, t))dsdt \leq (-1)\int_0^y \int_0^x f(s, t, u(s, t))dsdt$$

for all $(x, y) \in J$. Hence, by Lemma 3, since the differences involved exist, we have

$$\begin{aligned}
(T_2v)(x, y) &= p(x, y) \ominus (-1)\int_0^y \int_0^x f(s, t, v(s, t))dsdt \\
&\geq p(x, y) \ominus (-1)\int_0^y \int_0^x f(s, t, u(s, t))dsdt = (T_2u)(x, y)
\end{aligned}$$

for all $(s, t) \in J$, and the consequence is that T_2 is a nondecreasing operator on $\hat{C}(J, \mathbb{R}_{\mathcal{F}})$.

From (22), for an arbitrary strictly increasing altering distance function γ, we have

$$\begin{aligned}
\gamma(H_\lambda(T_2u, T_2v)) &\leq \gamma\left(\frac{1}{\lambda^2} H_\lambda(u, v)(1 - e^{-\lambda a})(1 - e^{-\lambda b})\right) \\
&= \gamma(H_\lambda(u, v)) - \left[\gamma(H_\lambda(u, v)) - \gamma\left(\frac{1}{\lambda^2} H_\lambda(u, v)(1 - e^{-\lambda a})(1 - e^{-\lambda b})\right)\right].
\end{aligned}$$

Denote $\Phi(t) = \gamma(t) - \gamma\left(\frac{1}{\lambda^2}(1 - e^{-\lambda a})(1 - e^{-\lambda b})t\right)$, $t \in [0, \infty)$. Then, from Lemma 7, there exists $\lambda > 0$ such that Φ is in $\hat{C}([0, \infty))$ and T_2 satisfies the generalized contractive-like condition

$$\gamma(H_\lambda(T_2u, T_2v)) \leq \gamma(H_\lambda(u, v)) - \Phi(H_\lambda(u, v)) \quad \text{for all } u, v \in \hat{C}(J, \mathbb{R}_{\mathcal{F}}) \text{ with } u \leq v.$$

Next, since there exists a (2)-lower solution $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}}) \cap \hat{C}(J, \mathbb{R}_{\mathcal{F}})$ for the Problem (1)–(2), we prove that $\mu \leq T_2\mu$. Note that the difference

$$(T_2\mu)(x, y) = p(x, y) \ominus (-1)\int_0^y \int_0^x f(s, t, \mu(s, t))dsdt$$

exists for all $(x, y) \in J$, since $\mu \in \hat{C}(J, \mathbb{R}_{\mathcal{F}})$.

Besides, from ${}_2D_{xy}\mu(x,y) \leq f(x,y,\mu(x,y))$, we deduce that

$$\int_0^y\int_0^x {}_2D_{xy}\mu(s,t)dsdt \leq \int_0^y\int_0^x f(s,t,\mu(s,t))dsdt$$

for all $(x,y) \in J$. The previous inequality together with $\mu(x,0) \leq \eta_1(x)$, $\mu(0,y) \leq \eta_2(y)$, and $\mu(0,0) = \eta_1(0)$, implies that

$$\begin{aligned}\mu_{r\alpha}(x,y) &\leq \mu_{r\alpha}(x,0) + \mu_{r\alpha}(0,y) - \mu_{r\alpha}(0,0) + \int_0^y\int_0^x f_{l\alpha}(s,t,\mu(s,t))dsdt\\ &\leq (\eta_1)_{r\alpha}(x) + (\eta_2)_{r\alpha}(y) - (\eta_1)_{r\alpha}(0) + \int_0^y\int_0^x f_{l\alpha}(s,t,\mu(s,t))dsdt,\\ \mu_{l\alpha}(x,y) &\leq \mu_{l\alpha}(x,0) + \mu_{l\alpha}(0,y) - \mu_{l\alpha}(0,0) + \int_0^y\int_0^x f_{r\alpha}(s,t,\mu(s,t))dsdt\\ &\leq (\eta_1)_{l\alpha}(x) + (\eta_2)_{l\alpha}(y) - (\eta_1)_{l\alpha}(0) + \int_0^y\int_0^x f_{r\alpha}(s,t,\mu(s,t))dsdt\end{aligned}$$

for $\alpha \in [0,1]$ and $(x,y) \in J$, which proves that

$$\begin{aligned}\mu(x,y) &\leq \eta_1(x) + \eta_2(y) \ominus \eta_1(0) \ominus (-1)\int_0^y\int_0^x f(s,t,\mu(s,t))dsdt\\ &= p(x,y) \ominus (-1)\int_0^y\int_0^x f(s,t,\mu(s,t))dsdt = (T_2\mu)(x,y)\end{aligned}$$

for all $(x,y) \in J$. Therefore, $\mu \leq T_2\mu$ in $\hat{C}(J,\mathbb{R}_{\mathcal{F}})$.

Because of Lemma 8, since $\hat{C}(J,\mathbb{R}_{\mathcal{F}})$ is a closed subspace of $C(J,\mathbb{R}_{\mathcal{F}})$, then $(\hat{C}(J,\mathbb{R}_{\mathcal{F}}), H_\lambda)$ is a complete metric space. Besides, the properties 1) and 2) in Lemma 4 are valid in $\hat{C}(J,\mathbb{R}_{\mathcal{F}})$. Then the operator T_2 satisfies all the hypotheses of Theorem 2 in $\hat{C}(J,\mathbb{R}_{\mathcal{F}})$. Hence, T_2 has a fixed point in $\hat{C}(J,\mathbb{R}_{\mathcal{F}})$. The uniqueness of fixed point comes from the existence of an upper or a lower bound in $\hat{C}(J,\mathbb{R}_{\mathcal{F}})$ for each pair of fixed elements in $\hat{C}(J,\mathbb{R}_{\mathcal{F}})$, which comes from ($h_5$). This completes the proof. □

Theorem 5 *The conclusions of Theorems 3 and 4 are still valid if instead of a* (k)*-lower solution, a* (k)*-upper solution* ($k = 1,2$) *of Problem* (1)–(2) *is supposed to be exist.*

Proof If μ is a (1)-upper solution to the Problem (1)–(2), then

$$\mu(x,y) \geq \eta_1(x) + \eta_2(y) \ominus \eta_1(0) + I_{xy}f(x,y,\mu(x,y)) = (T_1\mu)(x,y)$$

for all $(x,y) \in J$, from which it follows that $\mu \geq T_1\mu$. Hence, the existence of a unique integral solution of type 1 for Problem (1)–(2) is derived from Theorem 2. The proof of the solvability of Problem (1)–(2) with a unique integral solution of type 2 is obtained similarly by taking a (2)-upper solution μ in $\hat{C}(J,\mathbb{R}_{\mathcal{F}})$. □

Finally, we prove the existence of solutions to Problem (1)–(2) by applying the generalized results obtained in Section 2 for the case $\beta \in \mathcal{S}_0$.

In the space $C(J,\mathbb{R}_{\mathcal{F}})$, we consider the metric

$$d(u,v) = \sup_{(x,y)\in J}\{d_\infty(u(x,y),v(x,y))\}.$$

Due to the compactness of J in $\mathbb{R}^2$, it is easy to see that $(C(J,\mathbb{R}_{\mathcal{F}}),d)$ is a complete metric space.

For an arbitrary altering distance function η, we denote by $\mathcal{B}_\eta$ the class of functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

i) φ is monotonic increasing.
ii) $\varphi(t) < t$ for $t > 0$.
iii) The function $\beta : [0, \infty) \to [0, 1)$ defined as $\beta(t) = \begin{cases} \frac{\varphi\circ\eta(t)}{\eta(t)}, & t > 0, \\ 0, & t = 0 \end{cases}$ is in $\mathcal{S}_0$.

Theorem 6 *Consider Problem* (1)–(2)*, with a continuous function f satisfying the hypothesis* (h_1)*, and suppose that there exist a strictly increasing altering distance function ψ satisfying $\psi(t) \le t$ if $t > 0$, and $\varphi \in \mathcal{B}_\psi$ such that the following inequality holds*

$$d_\infty(f(x, y, u(x, y)), f(x, y, v(x, y))) \le \frac{1}{ab}\varphi\left(\psi(d_\infty(u(x, y), v(x, y)))\right), \quad (x, y) \in J, \tag{23}$$

for $u \le v$ in $C(J, \mathbb{R}_\mathcal{F})$. Then the existence of a (1)*-lower solution (or a* (1)*-upper solution) $\mu \in C^2(J, \mathbb{R}_\mathcal{F})$ for the Problem* (1)–(2) *provides the existence of a unique integral solution of type* 1 *to the Problem* (1)–(2).

Proof Consider the operator $T_1 : (C(J, \mathbb{R}_\mathcal{F}), d) \to (C(J, \mathbb{R}_\mathcal{F}), d)$ defined by (18).

Using (h_1) and following the same reasoning as in Step 1 of Theorem 3, we obtain the nondecreasing character of the operator T_1 in $C(J, \mathbb{R}_\mathcal{F})$.

For all $u \le v$ in $C(J, \mathbb{R}_\mathcal{F})$, we have, from (23),

$$\begin{aligned} d_\infty((T_1u)(x, y), (T_1v)(x, y)) &= d_\infty(I_{xy}f(x, y, u(x, y)), I_{xy}f(x, y, v(x, y))) \\ &\le \int_0^y \int_0^x d_\infty(f(s, t, u(s, t)), f(s, t, v(s, t)))dsdt \\ &\le \frac{1}{ab}\int_0^y \int_0^x \varphi\left(\psi(d_\infty(u(x, y), v(x, y)))\right) dsdt. \end{aligned}$$

Since $d_\infty(u(x, y), v(x, y)) \le d(u, v)$ for all $(x, y) \in J$, by using the nondecrease property of ψ and φ, we get $\psi(d_\infty(u(x, y), v(x, y))) \le \psi(d(u, v))$ and

$$\varphi(\psi(d_\infty(u(x, y), v(x, y)))) \le \varphi(\psi(d(u, v)))$$

for all $(x, y) \in J$. It follows, for all $(x, y) \in J$, that

$$\begin{aligned} d_\infty((T_1u)(x, y), (T_1v)(x, y)) &\le \frac{1}{ab}\varphi\left(\psi(d(u, v))\right)\int_0^y \int_0^x dsdt \\ &= \frac{1}{ab}xy\varphi\left(\psi(d(u, v))\right) \le \varphi\left(\psi(d(u, v))\right). \end{aligned}$$

Thus, for $u \le v$ in $C(J, \mathbb{R}_\mathcal{F})$,

$$d(T_1u, T_1v) \le \varphi\left(\psi(d(u, v))\right).$$

From the nondecreasing character of ψ, we get, for $u \le v$ in $C(J, \mathbb{R}_\mathcal{F})$,

$$\begin{aligned} \psi\left(d(T_1u, T_1v)\right) &\le \psi\left(\varphi\left(\psi(d(u, v))\right)\right) \le \varphi\left(\psi(d(u, v))\right) \\ &= \frac{\varphi\left(\psi(d(u, v))\right)}{\psi(d(u, v))}\psi(d(u, v)) = \beta(d(u, v))\psi(d(u, v)), \end{aligned}$$

if $d(u,v) > 0$, and the inequality is trivially valid if $d(u,v) = 0$. Here, we have

$$\beta(t) = \begin{cases} \frac{\varphi\circ\psi(t)}{\psi(t)} & \text{if } t > 0, \\ 0 & \text{if } t = 0, \end{cases}$$

which belongs to $\mathcal{S}_0$, by hypothesis.

Finally, let $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}})$ be a (1)-lower solution for the Problem (1)–(2). It is clear again that $\mu \leq T_1\mu$, since $\mu(x,y) \leq \eta_1(x) + \eta_2(y) \ominus \eta_1(0) + I_{xy} f(x,y,\mu) = (T_1\mu)(x,y)$, $(x,y) \in J$. Similarly, if there exists a (1)-upper solution μ for the Problem (1)–(2), then we have $\mu \geq T_1\mu$. Note that $(C(J, \mathbb{R}_{\mathcal{F}}), d)$ is also regular.

Overall, the operator T_1 satisfies all the hypotheses of Theorem 2 in case $\beta \in \mathcal{S}_0$. In consequence, T_1 has a fixed point in $C(J, \mathbb{R}_{\mathcal{F}})$. Noticing that every pair of elements of $C(J, \mathbb{R}_{\mathcal{F}})$ has an upper and a lower bound, it follows that the operator T_1 has a unique fixed point. □

Theorem 7 *Consider Problem* (1)–(2) *with f continuous satisfying the hypotheses* (h_1)*,* (h_3)*,* (h_4) *and suppose that there exist a strictly increasing altering distance function ψ satisfying $\psi(t) \leq t$ if $t > 0$, and $\varphi \in \mathcal{B}_\psi$ such that the inequality* (23) *holds for $u \leq v$ in $C(J, \mathbb{R}_{\mathcal{F}})$.*

Then the existence of a (2)*-lower solution (or a* (2)*-upper solution) $\mu \in C^2(J, \mathbb{R}_{\mathcal{F}}) \cap \hat{C}(J, \mathbb{R}_{\mathcal{F}})$ for the Problem* (1)–(2) *provides the existence of a fuzzy integral solution of type* 2 *to the Problem* (1)–(2)*.*

Furthermore, if the condition (h_5) *holds, then the Problem* (1)–(2) *has a unique integral solution of type* 2 *on J.*

Proof Using analogous arguments for the operator T_2 in Theorem 4, we deduce the existence of a (unique) integral solution of type 2 to the Problem (1)–(2). □

Example 1 Denote $\mathbb{R}_{\mathcal{F}}{}^+ = \{z \in \mathbb{R}_{\mathcal{F}} : \hat{0} \leq z\}$, where $\hat{0}$ is defined by $\hat{0}(t) = 1$ if $t = 0$ and $\hat{0}(t) = 0$ in other cases. In this example, we consider the following fuzzy partial hyperbolic equation under generalized Hukuhara derivatives

$$\begin{cases} {}_kD_{xy}u = f(x,y,u(x,y)), & (x,y) \in J = [0,a] \times [0,b], \\ u(x,0) = 0, & x \in J_a, \\ u(0,y) = 0, & y \in J_b, \end{cases} \tag{24}$$

where $f : J \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}{}^+$. Note that $u(0,0) = 0$ is deduced for a solution.

Theorem 8 *Consider $f : J \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}{}^+$ continuous and nondecreasing with respect to the third variable and suppose that, if $u \leq v$ in $C(J, \mathbb{R}_{\mathcal{F}})$, then*

$$\begin{aligned} &d_\infty(f(x,y,u(x,y)), f(x,y,v(x,y))) \\ &\leq \frac{1}{ab} \ln\Big(1 + \min\{d_\infty^2(u(x,y), v(x,y)), d_\infty(u(x,y), v(x,y))\}\Big) \end{aligned} \tag{25}$$

for all $(x,y) \in J$. Then Problem (24) *has a unique nonnegative fuzzy integral solution of type* 1*. In addition to the hypotheses, if* (h_3) *and* (h_4) *are satisfied, then Problem* (24) *has a nonnegative integral solution of type* 2 *(unique if* (h_5) *holds).*

Proof Consider the cone $P = \{u \in C(J, \mathbb{R}_{\mathcal{F}}) : u \geq \hat{0}\}$, where we also denote by $\hat{0}$ the constant function equal to $\hat{0}$ at any point. Obviously, (P, d) is a complete metric space (and regular). The operator T_1 defined as $(T_1 u)(x, y) = I_{xy} f(x, y, u)$ is nondecreasing and maps P into itself since $f(x, y, u(x, y))$ is a nonnegative continuous function for each $u \in P$. Besides, $T_1(\hat{0}) \geq \hat{0}$ ($\hat{0}$ is a lower solution). From Theorem 6 with $\varphi(t) = \ln(1 + t)$, $\psi(t) = \min\{t^2, t\}$, we derive the conclusion.

Note that the condition $f : J \times \mathbb{R}_{\mathcal{F}} \to {\mathbb{R}_{\mathcal{F}}}^+$ can be relaxed to $f : J \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ if we impose that $f(x, y, \hat{0}) \geq \hat{0}$ for every $(x, y) \in J$, due to the nondecreasing character of T_1, which yields $T_1 u \geq T_1(\hat{0}) \geq \hat{0}$ for $u \in P$.

Note also that, in this example, the weak solution of type 2 is sought in the space of functions $u \in C(J, \mathbb{R}_{\mathcal{F}})$ such that $u \geq \hat{0}$ and $f(x, y, u(x, y))$ is crisp for every $(x, y) \in J$, so condition (h_4) (and, hence, (h_3)) is satisfied if $f(x, y, z)$ is crisp for each $(x, y) \in J$ and $z \in \mathbb{R}_{\mathcal{F}}$ crisp. Under this restriction, (h_5) also holds since, given $u, v \geq \hat{0}$, we can take as a crisp lower bound of u, v the constant function $\hat{0}$. □

4 Conclusions

In this study, we have firstly presented some new generalized theorems on fixed points for nondecreasing mappings from a partially ordered metric space to itself. These results develop some previous results of [3, 14, 27] and admit them as special cases. Secondly, we have investigated the existence and uniqueness of fuzzy solutions to a boundary value problem for a class of fuzzy partial hyperbolic equation under generalized Hukuhara derivatives. Via these results, the function placed in the right-hand side of the equation does not need to be Lipschitz continuous. In spite of this condition, f is only demanded to satisfy a generalized contractive-like condition. However, a hypothesis of existing a lower or upper solution of considered problem is required. In real world applications, the use of lower and upper solutions method is hampered by the difficulty to exhibit such functions. This method does not require to find a solution of a boundary value problem but find lower and upper solutions. This replacement reminds us to the Liapunov's second method. Furthermore, in many theorems, the assumptions at hand provide lower and upper solutions and their use simplifies the argument. The questions arise whether it is easy to recognize that a set of assumptions provides such lower and upper solutions? Is it easy to find them? In general, there is no clue to finding these solutions. This drawback motivated more works to study the way to construct the lower as well as upper solutions in differential equations theory. Some efforts to offer a construction of lower and upper solutions can be seen Lemma 1.5.2 in [15] for initial value problems of first order ordinary differential equations, Chapters VI to X in [8] for showing how to build in specific cases appropriate lower and upper solutions of some classes of two points boundary value problems. For partial differential equations, we can cite here some works [12, 13]. This observation is our primary motivation in future work for stating conditions that ensure a given function is a lower or an upper solution of our considered problems.

Acknowledgements This work is supported by the NAFOSTED, Vietnam, under contract No. 101.02-2015.08, partially supported by Ministerio de Economía y Competitividad, project MTM2013-43014-P (co-financed by FEDER), and partially supported by Research project of the Ministry level B2017-SPH-33.

The authors are greatly indebted to Editor-in-Chiefs (Prof. Hoang Xuan Phu), Associate Editor and anonymous referees for their comments and valuable suggestions that greatly improve the quality and clarity of the paper.

This work is dedicated to Prof. Nguyen Manh Hung on the occasion of his 60th birthday.

References

1. Alikhani, R., Bahrami, F.: Global solutions of fuzzy integro-differential equations under generalized differentiability by the method of upper and lower solutions. Inform. Sci. **295**, 600–608 (2015)
2. Allahviranloo, T., Gouyandeh, Z., Armand, A., Hasanoglu, A.: On fuzzy solutions for heat equation based on generalized Hukuhara differentiability. Fuzzy Sets Syst. **265**, 1–23 (2015)
3. Amini-Harandi, A., Emami, H.: A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations. Nonlinear Anal. **72**, 2238–2242 (2010)
4. Bede, B., Stefanini, L.: Generalized differentiability of fuzzy-valued functions. Fuzzy Sets Syst. **230**, 119–141 (2013)
5. Bernfeld, S.R., Lakshmikantham, V.: An Introduction to Nonlinear Boundary Value Problems. Mathematics in Science and Engineering, vol. 10. Academic Press, New York (1974)
6. De Coster, C., Habets, P.: An overview of the method of lower and upper solutions for ODEs. In: Grossinho, M.R. et al. (eds.) Nonlinear Analysis and Its Applications to Differential Equations, pp. 3–22. Birkhäuser, Boston (2001)
7. De Coster, C., Habets, P.: The lower and upper solutions method for boundary value problems. In: Cañada, A. et al. (eds.) Handbook of Differential Equations: Ordinary Differential Equations, vol. 1, pp. 69–160. Elsevier/North-Holland, Amsterdam (2004)
8. De Coster, C., Habets, P.: Two-Point Boundary Value Problems: Lower and Upper Solutions. Mathematics in Science and Engineering, vol. 205. Elsevier, Amsterdam (2006)
9. Dutta, P.N., Choudhury, B.S.: A generalisation of contraction principle in metric spaces. Fixed Point Theory Appl. **2008**, 406368 (2008)
10. Dragoni, G.S.: II Problema dei valori ai limiti studiato in grande per gli integrali di una equazione differenziale del secondo ordine. G. Mat. (Battaglini) **69**, 77–112 (1931)
11. Dragoni, G.S.: II Problema dei valori ai limiti studiato in grande per le equazioni differenziali del secondo ordine. Math. Ann. **105**, 133–143 (1931)
12. Gossez, J.P., Omari, P.: Non-ordered lower and upper solutions in semilinear elliptic problems. Commun. Partial Differ. Equ. **19**, 1163–1184 (1994)
13. Habets, P., Omari, P.: Existence and localization of solutions of second order elliptic problems using lower and upper solutions in the reversed order. Topol. Methods Nonlinear Anal. **8**, 25–56 (1996)
14. Harjani, J., Sadarangani, K.: Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations. Nonlinear Anal. **72**, 1188–1197 (2010)
15. Heikkilä, S., Lakshmikantham, V.: Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations. Marcel Dekker Inc., New York (1994)
16. Khan, M.S., Swaleh, M., Sessa, S.: Fixed point theorems by altering distances between the points. Bull. Aust. Math. Soc. **30**, 1–9 (1984)
17. Khastan, A., Nieto, J.J., Rodríguez-lópez, R.: Fuzzy delay differential equations under generalized differentiability. Inf. Sci. **275**, 145–167 (2014)
18. Ladde, G.S., Lakshmikantham, V., Vatsala, A.S.: Monotone Iterative Techniques for Nonlinear Differential Equations. Monographs and Studies in Mathematics, vol. 27. Pitman Publishing, Boston (1985)
19. Lakshmikantham, V., Mohapatra, R.: Theory of Fuzzy Differential Equations and Inclusions. Taylor & Francis, London (2003)
20. Long, H.V., Son, N.T.K., Ha, N.T.M., Son, L.H.: The existence and uniqueness of fuzzy solutions for hyperbolic partial differential equations. Fuzzy Optim. Decis. Mak. **13**, 435–462 (2014)
21. Long, H.V., Son, N.T.K., Tam, H.T.T., Cuong, B.C.: On the existence of fuzzy solutions for partial hyperbolic functional differential equations. Int. J. Comput. Intell. Syst. **7**, 1159–1173 (2014)
22. Long, H.V., Son, N.T.K., Tam, H.T.T.: Global existence of solutions to fuzzy partial hyperbolic functional differential equations with generalized Hukuhara derivatives. J. Intell. Fuzzy Syst. **29**, 939–954 (2015)
23. Long, H.V., Nieto, J.J., Son, N.T.K.: New approach for studying nonlocal problems related to differential systems and partial differential equations in generalized fuzzy metric spaces. Fuzzy Sets Syst. doi:10.1016/j.fss.2016.11.008 (2016)
24. Long, H.V., Son, N.T.K., Tam, H.T.T.: The solvability of fuzzy fractional partial differential equations under Caputo gH-differentiability. Fuzzy Sets Syst. **309**, 35–63 (2017)
25. Long, H.V., Son, N.T.K., Hoa, N.V.: Fuzzy fractional partial differential equations in partially ordered metric spaces. Iran. J. Fuzzy Syst. **14**, 107–126 (2017)
26. Mawhin, J.: Bounded solutions of nonlinear ordinary differential equations. In: Zanolin, F. (ed.) Non Linear Analysis and Boundary Value Problems for Ordinary Differential Equations, pp. 121–147. Springer, Vienna (1996)

27. Nashine, H.K., Samet, B.: Fixed point results for mappings satisfying (ψ, φ)-weakly contractive condition in partially ordered metric spaces. Nonlinear Anal. **74**, 2201–2209 (2011)
28. Nieto, J.J., Rodríguez-López, R.: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223–239 (2005)
29. Nieto, J.J., Rodríguez-López, R.: Applications of contractive-like mapping principles to fuzzy equations. Rev. Mat. Complut. **19**, 361–383 (2006)
30. Nieto, J.J., Rodríguez-López, R.: Bounded solutions for fuzzy differential and integral equations. Chaos Solitons Fractals **27**, 1376–1386 (2006)
31. Picard, E.: Mémoire sur la théorie des équations aux dérivées partielles et la méthode des approximations succesives. J. Math. Pures Appl. **6**, 145–210 (1890)
32. Picard, E.: Sur l'application des méthodes d'approximations successives à l'étude de certaines équations différentielles ordinaires. J. Math. Pures Appl. **9**, 217–271 (1893)
33. Ran, A.C.M., Reurings, M.C.B.: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**, 1435–1443 (2004)
34. Rhoades, B.E.: Some theorems on weakly contractive maps. Nonlinear Anal. **47**, 2683–2693 (2001)
35. Stefanini, L.: A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. Fuzzy Sets Syst. **161**, 1564–1584 (2010)
36. Villamizar-Roa, E.J., Angulo-Castillo, V., Chalco-Cano, Y.: Existence of solutions to fuzzy differential equations with generalized Hukuhara derivative via contractive-like mapping principles. Fuzzy Sets Syst. **265**, 24–38 (2015)