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On Superlinear p-Laplace Equations

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Abstract We study the existence of non-trivial weak solutions in $W_0^{1,p}(\Omega)$ of the superlinear Dirichlet problem:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f satisfies the condition

$$|f(x,t)| \leq |\omega(x)t|^{r-1} + b(x) \qquad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $r \in (p, \frac{Np}{N-p})$, $b \in L^{\frac{r}{r-1}}(\Omega)$ and $|\omega|^{r-1}$ may be non-integrable on Ω.

Keywords Nemytskii operators · p-Laplacian · Multiplicity of solutions · Mountain-pass theorem

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1 Introduction

Let N be an integer ≥ 3, Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, p be in $[1, N)$ and $p^* = \frac{Np}{N-p}$. Let $W_0^{1,p}(\Omega)$ be the usual Sobolev space with the following norm

$$\|u\|_{1,p} = \left\{ \int_\Omega |\nabla u|^p dx \right\}^{\frac{1}{p}} \qquad \forall u \in W_0^{1,p}(\Omega).$$

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We consider the following Dirichlet problem:

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

In [3–5, 8], one has proved (1) has non-trivial solutions if f is continuous on $\overline{\Omega} \times \mathbb{R}$ and satisfies the following conditions

(C_1) There exist $r \in (p, p^* - 1)$ and a positive real number α such that

$$|f(x,t)| \le \alpha(1 + |t|^{r-1}) \qquad \forall (x,t) \in \Omega \times \mathbb{R}.$$

(C_2) $f(x,0) = 0$ for every x in Ω and $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0$ uniformly in Ω.

(C_3) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = \infty$ uniformly in Ω.

(C_4) There exist $C \in [0,\infty)$, $\theta > p$ such that

$$0 \le f(x,t)t - \theta F(x,t) \qquad \text{a.e. in } \Omega \times \{t \in \mathbb{R} : |t| > C\},$$

where $F(x,t) = \int_0^t f(x,\xi)d\xi$ for every (x,t) in $\Omega \times \mathbb{R}$.

In the present paper, we prove the following result.

Theorem 1 *Assume f is a Carathéodory function on $\Omega \times \mathbb{R}$ and satisfies the following conditions*

(f_1) *there exist $r \in (p, p^*)$, $\omega \in \mathcal{K}_{p,r}$ (see Definition 1) and $b \in L^{\frac{r}{r-1}}(\Omega)$ such that*

$$|f(x,t)| \le |\omega(x)t|^{r-1} + b(x) \qquad \forall (x,t) \in \Omega \times \mathbb{R},$$

(f_2) *there exists $d \in L^1(\Omega)$ such that $|f(x,t)| \le d(x)$ for every x in Ω and $|t| \le C$,*

(f_3) *there is a non-positive function d_1 in $L^{\frac{2N}{p}}(\Omega)$ such that $d_1(x) \le \frac{f(x,t)}{|t|^{p-2}t}$ for every $(x,t) \in \Omega \times \mathbb{R}$,*

(f_4) $f(x,0) = 0$ *for every x in Ω and* $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0$ *a.e. in Ω,*

(f_5) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = \infty$ *a.e. in Ω, and*

(f_6) *there exist $\theta > p$ and $d_2 \in L^1(\Omega)$ such that*

$$d_2(x) \le f(x,t)t - \theta F(x,t) \qquad \text{a.e. in } \Omega \times \{t \in \mathbb{R} : |t| > C\}.$$

Then there is a non-trivial weak solution in $W_0^{1,p}(\Omega)$ of the problem (1).

Remark 1 In many applications, $\frac{f(x,t)}{|t|^{p-2}t}$ is non-negative for $t \ne 0$ and $|f(x,t)|$ is well-controlled when $|t|$ is sufficiently small. This observation is the motivation of (f_2) and (f_3). Here, we consider the case, in which the positivity of $\frac{f(x,t)}{|t|^{p-2}t}$ can be disturbed by a non-positive function d_1 in $L^{\frac{2N}{p}}(\Omega)$.

Remark 2 If f is continuous on $\overline{\Omega} \times \mathbb{R}$ and satisfies the conditions (C_1), (C_2), (C_3), and (C_4), then f satisfies (f_1)–(f_6). Furthermore, $|w|^{r-1}$ may be not integrable on Ω and the convergences in (f_4) and (f_5) may be not uniform on Ω (see Example 4). Therefore our theorem improves the corresponding results in [3–5, 8].

We study some method to construct weight functions in weighted Sobolev embeddings and the Nemytskii operator from Sobolev spaces into Lebesgue spaces (see Theorems 4 and 5) in Section 2. We apply these results to prove the existence of non-trivial solutions of a class of super-linear p-Laplace problems in the last section.

2 Nemytskii Operators

Definition 1 Let σ be a measurable function on Ω. We put

$$T_\sigma u = \sigma u \qquad \forall u \in W_0^{1,p}(\Omega).$$

We say

(i) σ is of class $\mathcal{C}_{p,s}$ if T_σ is a continuous mapping from $W_0^{1,p}(\Omega)$ into $L^s(\Omega)$;
(ii) σ is of class $\mathcal{K}_{p,s}$ if T_σ is a compact mapping from $W_0^{1,p}(\Omega)$ into $L^s(\Omega)$.

We have the following results.

Theorem 2 *Let α_1 and α_2 be in $[1, \infty)$ such that $\alpha_1 < \alpha_2$. Let $\omega_1 \in \mathcal{C}_{p,\alpha_1}$, $\omega_2 \in \mathcal{C}_{p,\alpha_2}$ be such that ω_1 and ω_2 are non-negative. Let $\beta \in (\alpha_1, \alpha_2)$ and $\omega = \omega_1^{\frac{\alpha_1(\alpha_2-\beta)}{\beta(\alpha_2-\alpha_1)}} \omega_2^{\frac{\alpha_2(\beta-\alpha_1)}{\beta(\alpha_2-\alpha_1)}}$. Then $w \in \mathcal{C}_{p,\beta}$.*

Proof There is a positive real number C_1 such that

$$\left\{ \int_\Omega \omega_i^{\alpha_i} |u|^{\alpha_i} dx \right\}^{1/\alpha_i} \le C_1 \|u\|_{1,p} \qquad \forall u \in W_0^{1,p}(\Omega), i = 1, 2. \tag{2}$$

Since $\beta = \frac{\alpha_2-\beta}{\alpha_2-\alpha_1}\alpha_1 + \frac{\beta-\alpha_1}{\alpha_2-\alpha_1}\alpha_2$, by Hölder's inequality and (2), we get

$$\begin{aligned}
\left\{ \int_\Omega \omega^\beta |u|^\beta dx \right\}^{1/\beta} &= \left\{ \int_\Omega \omega_1^{\frac{\alpha_2-\beta}{\alpha_2-\alpha_1}\alpha_1} |u|^{\frac{\alpha_2-\beta}{\alpha_2-\alpha_1}\alpha_1} \omega_2^{\frac{\beta-\alpha_1}{\alpha_2-\alpha_1}\alpha_2} |u|^{\frac{\beta-\alpha_1}{\alpha_2-\alpha_1}\alpha_2} dx \right\}^{1/\beta} \\
&\le \left\{ \left\{ \int_\Omega \omega_1^{\alpha_1} |u|^{\alpha_1} dx \right\}^{\frac{\alpha_2-\beta}{\alpha_2-\alpha_1}} \left\{ \int_\Omega \omega_2^{\alpha_2} |u|^{\alpha_2} dx \right\}^{\frac{\beta-\alpha_1}{\alpha_2-\alpha_1}} \right\}^{1/\beta} \\
&\le \left\{ \left\{ \int_\Omega \omega_1^{\alpha_1} |u|^{\alpha_1} dx \right\}^{\frac{1}{\alpha_1}\frac{\alpha_2-\beta}{\alpha_2-\alpha_1}\alpha_1} \left\{ \int_\Omega \omega_2^{\alpha_2} |u|^{\alpha_2} dx \right\}^{\frac{1}{\alpha_2}\frac{\beta-\alpha_1}{\alpha_2-\alpha_1}\alpha_2} \right\}^{1/\beta} \\
&\le C_1 \|u\|_{1,p} \quad \forall u \in W_0^{1,p}(\Omega).
\end{aligned}$$

□

Theorem 3 *Let s be in $[1, \frac{Np}{N-p})$, α be in $(0, 1)$, $\omega \in \mathcal{C}_{p,s}$ and θ be measurable functions on Ω such that $\omega \ge 0$ and $|\theta| \le \omega^\alpha$. Then θ is of class $\mathcal{K}_{p,s}$.*

Proof Since T_ω is in $\mathcal{C}_{p,s}$, T_ω is continuous from $W_0^{1,p}(\Omega)$ into $L^s(\Omega)$ and there is a positive real number C_2 such that

$$\left\{ \int_\Omega |u|^s \omega^s dx \right\}^{1/s} \le C_2 \|u\|_{1,p} \qquad \forall u \in W_0^{1,p}(\Omega). \tag{3}$$

Since $\omega^\alpha(x) \le 1 + \omega(x)$ for every x in Ω and 1 and ω are in $\mathcal{C}_{p,s}$, ω^α belongs to $\mathcal{C}_{p,s}$. Thus, T_θ is in $\mathcal{C}_{p,s}$. Let M be a positive real number and $\{u_n\}$ be a sequence in $W_0^{1,p}(\Omega)$, such that $\|u_n\|_{1,p} \le M$ for any n. By Rellich–Kondrachov's theorem (Theorem 9.16 in [2]), $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ converging to u in $L^s(\Omega)$ and $\{u_{n_k}\}$ converging weakly to u in $W_0^{1,p}(\Omega)$, therefore $\|u\|_{1,p} \le \liminf_{k\to\infty} \|u_{n_k}\|_{1,p} \le M$. We shall prove $\{T_\theta(u_{n_k})\}$ converges to $T_\theta(u)$ in $L^s(\Omega)$.

Let ε be a positive real number. Choose a positive real number δ such that

$$(2C_2M)^s\delta^{(\alpha-1)s} < \frac{\varepsilon^s}{2}. \tag{4}$$

Put $\Omega' = \{x \in \Omega : \omega(x) > \delta\}$. By (3) and (4), we have

$$\begin{aligned}
\int_\Omega |\theta(u_{n_k} - u)|^s dx &= \int_\Omega |u_{n_k} - u|^s |\theta|^s dx \\
&\le \int_{\Omega'} |u_{n_k} - u|^s \omega^{\alpha s} dx + \int_{\Omega\setminus\Omega'} |u_{n_k} - u|^s \omega^{\alpha s} dx \\
&\le \delta^{(\alpha-1)s} \int_{\Omega'} |u_{n_k} - u|^s \omega^s dx + \delta^{\alpha s} \int_{\Omega\setminus\Omega'} |u_{n_k} - u|^s dx \\
&\le \delta^{(\alpha-1)s} \int_{\Omega} |u_{n_k} - u|^s \omega^s dx + \delta^{\alpha s} \int_{\Omega} |u_{n_k} - u|^s dx \\
&\le \delta^{(\alpha-1)s} \left(C_2\|u_{n_k} - u\|_{1,p}\right)^s + \delta^{\alpha s} \int_{\Omega} |u_{n_k} - u|^s dx \\
&\le \delta^{(\alpha-1)s} (2C_2M)^s + \delta^{\alpha s} \int_{\Omega} |u_{n_k} - u|^s dx \\
&\le \frac{\varepsilon^s}{2} + \delta^{\alpha s} \int_{\Omega} |u_{n_k} - u|^s dx.
\end{aligned} \tag{5}$$

Since $\{u_{n_k}\}$ converges in $L^s(\Omega)$, there is an integer k_0 such that

$$\int_\Omega |u_{n_k} - u|^s dx \le \delta^{-\alpha s}\frac{\varepsilon^s}{2} \qquad \forall k \ge k_0. \tag{6}$$

Combining (5) and (6), we get the theorem. □

Corollary 1 *Let $p \in [1, N)$, $s \in \left(1, \frac{Np}{N-p}\right)$, $\eta \in \left(\frac{sNp}{Np-s(N-p)}, \infty\right)$ and $\theta \in L^\eta(\Omega)$. Then θ is in $\mathcal{K}_{p,s}$.*

Proof Let $\beta \in (0, 1)$ be such that $\beta\eta = \frac{sNp}{Np-s(N-p)}$ and $\omega = |\theta|^{1/\beta}$. Then ω is in $L^{\frac{sNp}{Np-s(N-p)}}(\Omega)$. Since $\frac{Np-s(N-p)}{Np} + \frac{s(N-p)}{Np} = 1$, by Hölder's inequality, we have

$$\int_\Omega |\omega u|^s dx \le \int_\Omega \left(|\omega|^{\frac{sNp}{Np-s(N-p)}}\right)^{\frac{Np-s(N-p)}{Np}} \left(\int_\Omega |u|^{\frac{Np}{N-p}}\right)^{\frac{s(N-p)}{Np}} \qquad \forall u \in W_0^{1,p}(\Omega),$$

which implies that T_ω is continuous at 0 in $W_0^{1,p}(\Omega)$. Thus, T_ω is a linear continuous map from $W_0^{1,p}(\Omega)$ into $L^s(\Omega)$. By Theorem 3, θ is of class $\mathcal{K}_{p,r}$. □

Example 1 Let $N = 5$, $p = 3$, $s = 4$ and $\Omega = \{x \in \mathbb{R}^5 : |x| < 1\}$. Then $\frac{sNp}{Np-s(N-p)} = \frac{4\cdot5\cdot3}{5\cdot3-4(5-3)} = \frac{60}{7} < 10$. Put $\omega_0 = |x|^{-\frac{1}{30}}\cos(16|x|)$. Then ω_0 is in $L^{10}(\Omega)$. Thus by Corollary 1, ω_0 is of class $\mathcal{K}_{p,s}$.

Corollary 2 *Let* $p \in [1, N)$, $s \in \left(1, \frac{Np}{N-p}\right)$, α *be in* $(0, 1)$ *and* $\eta \in \mathcal{C}_{p,p}$. *Then* $\theta = \eta^{\alpha \frac{p(p^*-s)}{s(p^*-p)}}$ *is of class* $\mathcal{K}_{p,s}$.

Proof Put $\omega_1 = \eta$, $\omega_2 = 1$, $\alpha_1 = p$, $\alpha_2 = p^*$, $\beta = s$. By the Embedding theorem of Sobolev, $\omega_2 \in \mathcal{C}_{p,p^*}$. By Theorem 2, we see that $\eta^{\frac{p(p^*-s)}{s(p^*-p)}} \in \mathcal{C}_{p,s}$. Thus by Theorem 4, $\eta^{\alpha \frac{p(p^*-s)}{s(p^*-p)}}$ is of class $\mathcal{K}_{p,s}$. □

Example 2 Let $\Omega = \{x \in \mathbb{R}^5 : \|x\| < 1\}$, $p = 3$, $s = 4$, $\alpha = \frac{3}{4}$ and $\eta(x) = (1 - \|x\|^2)^{-1}$ for every x in Ω. By Theorem 8.4 in [7], $\eta \in \mathcal{C}_{p,p}$. Note that $p^* = \frac{Np}{N-p} = \frac{15}{2}$ and

$$\alpha \frac{p(p^*-s)}{s(p^*-p)} = \frac{3}{4}\frac{3}{4}\frac{7}{9} = \frac{7}{16}.$$

Put $\theta(x) = (1 - \|x\|^2)^{-\frac{7}{16}}$ for every x in Ω. Then $\theta \in \mathcal{K}_{3,4}$.

Theorem 4 *Let* s *be in* $(1, p^*)$, ω *be in* $\mathcal{K}_{p,s}$, b *be in* $L^{\frac{s}{s-1}}(\Omega)$ *and* g *be a Carathéodory function from* $\Omega \times \mathbb{R}$ *into* $\mathbb{R}$. *Assume*

$$|g(x, z)| \le |\omega(x)|^{s-1}|z|^{s-1} + b(x) \qquad \forall (x, z) \in \Omega \times \mathbb{R}. \tag{7}$$

Put

$$N_g(v)(x) = g(x, v(x)) \qquad \forall v \in W_0^{1,p}(\Omega), x \in \Omega.$$

We have

(i) N_g *is a continuous mapping from* $W_0^{1,p}(\Omega)$ *into* $L^{\frac{s}{s-1}}(\Omega)$.
(ii) *If* A *is a bounded subset in* $W_0^{1,p}(\Omega)$, *then* $\overline{N_g(A)}$ *is compact in* $L^{\frac{s}{s-1}}(\Omega)$.

Proof (i) Put $\mu = s$, $q = \frac{s}{s-1}$ and

$$g_1(x, \zeta) = g(x, \omega(x)^{-1}\zeta) \qquad \forall (x, \zeta) \in \Omega \times \mathbb{R},$$

By (7), we have

$$|g_1(x, \zeta)| \le |\zeta|^{s-1} + b(x) \qquad \forall (x, \zeta) \in \Omega \times \mathbb{R}.$$

On the other hand

$$N_g(v) = N_{g_1} \circ T_{|\omega|}(v) \qquad \forall v \in W_0^{1,p}(\Omega).$$

Since $w \in \mathcal{K}_{p,s}$, applying Theorem 2.3 in [5], we get the theorem. □

Theorem 5 *Let* $s \in (1, p^*)$, ω *be in* $\mathcal{K}_{p,s}$, *a function* $b \in L^{\frac{s}{s-1}}(\Omega)$ *and* g *be a Carathéodory function from* $\Omega \times \mathbb{R}$ *into* $\mathbb{R}$. *Assume*

$$|g(x, z)| \le |\omega(x)|^{s-1}|z|^{s-1} + b(x) \qquad \forall (x, z) \in \Omega \times \mathbb{R}.$$

Put

$$G(x, t) = \int_0^t g(x, \xi)d\xi \qquad \forall (x, t) \in \Omega,$$

$$\Psi_g(u) = \int_\Omega G(x, t)dx \qquad \forall u \in W_0^{1,p}(\Omega).$$

We have

(i) $\{N_G(w_n)\}$ *converges to* $N_G(w)$ *in* $L^1(\Omega)$ *when* $\{w_n\}$ *weakly converges to* w *in* $W_0^{1,p}(\Omega)$.

(ii) Ψ_g *is a continuously Fréchet differentiable mapping from* $W_0^{1,p}(\Omega)$ *into* $\mathbb{R}$ *and*

$$D\Psi_g(u)(\phi) = \int_\Omega g(x,\xi)\phi dx \qquad \forall u, \phi \in W_0^{1,p}(\Omega).$$

(iii) *If* A *is a bounded subset in* $W_0^{1,p}(\Omega)$, *then there is a positive real number* M *such that*

$$|\Psi_g(v)| + \|D\Psi_g(v)\| \le M \qquad \forall v \in A.$$

Proof Let $\mu = s$, $q = \frac{s}{s-1}$ and g_1 be as in the proof of Theorem 4. Put

$$G_1(x,t) = \int_0^t g(x,\xi)d\xi \qquad \forall (x,t) \in \Omega,$$

$$\Psi_{g_1}(u) = \int_\Omega \int_0^{u(x)} g_1(x,\xi)d\xi dx \qquad \forall u \in L^p(\Omega).$$

By [5, Theorem 2.8], N_{G_1} is continuous from $L^{\frac{s}{s-1}}(\Omega)$ into $L^1(\Omega)$ and Ψ_{g_1} is a continuously Fréchet differentiable mapping from $L^{\frac{s}{s-1}}(\Omega)$ into $\mathbb{R}$. We see that $N_G = N_{G_1} \circ T_\omega$ and $\Psi_g = \Psi_{g_1} \circ T_\omega$. By Theorem 3, we get the theorem. □

Remark 3 If $\omega = 1$, Theorems 4 and 5 have been proved in [1, 5, 6].

Example 3 Let $\Omega = \{x \in \mathbb{R}^5 : \|x\| < 1\}$, $p = 3$, $s = 4$, $\alpha = \frac{3}{4}$ and $\rho(x) = (\frac{1}{2} - \|x\|^2)^2(1-\|x\|^2)^{-\frac{7}{16}}$ for every x in Ω. By Example 2, $\rho \in \mathcal{K}_{3,4}$. Put $a(x) = \rho(x)^{s-1} = (\frac{1}{2} - \|x\|^2)^6(1 - \|x\|^2)^{-\frac{21}{16}}$ for every x in Ω. Thus, a is not integrable on Ω and Theorem 5 improves corresponding results in [1, 5, 6].

3 Proof of Theorem 1

Put

$$J(u) = \frac{1}{p}\|u\|_{1,p}^p - \int_\Omega F(x,u)dx \qquad \forall u \in W_0^{1,p}(\Omega). \tag{8}$$

By [3, Theorem 9], Theorem 5 and (f_1), J is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$ and

$$DJ(u)(v) = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx - \int_\Omega f(x,u) \cdot v dx \qquad \forall u, v \in W_0^{1,p}(\Omega). \tag{9}$$

In order to prove the theorem, we need the following lemmas.

Lemma 1 *Under condition* (f_3) *and* (f_4), *there exist positive numbers* ρ *and* η *such that* $J(u) \ge \eta$ *for all* u *in* $W_0^{1,p}(\Omega)$ *with* $\|u\| = \rho$.

Proof Suppose on the contrary that

$$\inf\left\{J(u) : u \in W_0^{1,p}(\Omega), \|u\|_{1,p} = \frac{1}{n}\right\} \le 0 \qquad \forall n \in \mathbb{N}.$$

Then, there is a sequence $\{u_n\}$ in $W_0^{1,p}(\Omega)$ such that $\|u_n\|_{1,p} = \frac{1}{n}$ and $J(u_n) < \frac{1}{n^{p+1}}$. Note that $p < \frac{2Np}{2N-p} < \frac{Np}{N-p}$. By replacing $\{u_n\}$ by its subsequence, by [2, Theorem 4.9], we can suppose that $\lim_{n\to\infty} u_n(x) = 0$ for every x in Ω, $\left\{\frac{u_n}{\|u_n\|_{1,p}}\right\}$ strongly (resp. pointwise) converges to w in $L^{\frac{2Np}{2N-p}}(\Omega)$ (resp. on Ω) and $\frac{|u_n|}{\|u_n\|_{1,p}} \le v$ with a function v in $L^{\frac{2Np}{2N-p}}(\Omega)$. We have

$$\frac{1}{n} > \frac{J(u_n)}{\|u_n\|_{1,p}^p} = \frac{1}{p} - \int_\Omega \frac{F(x,u_n(x))}{\|u\|_{1,p}^p} dx = \frac{1}{p} - \int_\Omega \int_0^1 f(x, su_n(x)) \frac{u_n(x)}{\|u\|_{1,p}^p} ds dx$$
$$= \frac{1}{p} - \int_\Omega \int_0^1 \frac{f(x, su_n(x))}{(su_n(x))^{p-2} su_n(x)} s^p \frac{|u_n(x)|^p}{\|u\|_{1,p}^p} ds dx.$$

Since $d_1 \in L^{\frac{2N}{p}}(\Omega)$, $d_1 v^p$ is integrable on Ω and, by (f_3)

$$\frac{f(x, su_n(x))}{(su_n(x))^{p-2} su_n(x)} s^p \frac{|u_n(x)|^p}{\|u\|_{1,p}^p} \ge s^p d_1(x) \frac{|u_n(x)|^p}{\|u\|_{1,p}^p} \ge s^p d_1(x) v^p(x)$$

for all $x \in \Omega$, $s \in (0,1)$, $n \in \mathbb{N}$.

Hence, by the generalized Fatou lemma ([9, p.85]), and (f_4)

$$0 = \liminf_{n\to\infty} \frac{1}{n} = \frac{1}{p} - \limsup_{n\to\infty} \int_\Omega \int_0^1 \frac{f(x, su_n(x))}{(su_n(x))^{p-2} su_n(x)} s^p \frac{|u_n(x)|^p}{\|u\|_{1,p}^p} ds dx$$
$$\ge \frac{1}{p} - \int_\Omega \int_0^1 \limsup_{n\to\infty} \left[\frac{f(x, su_n(x))}{(su_n(x))^{p-2} su_n(x)} s^p \frac{|u_n(x)|^p}{\|u\|_{1,p}^p} \right] ds dx = \frac{1}{p}.$$

This contradiction implies the lemma. □

Lemma 2 *Let ρ be as in Lemma 1. Under conditions (f_3) and (f_5), there is e in $W_0^{1,p}(\Omega) \setminus B(0,\rho)$ such that $J(e) < 0$.*

Proof Let $u \in W_0^{1,p}(\Omega)$ be such that $\|u\|_{1,p} = 1$ and $u > 0$ on Ω. By (8), we have

$$J(nu) = \frac{n^p}{p} - \int_\Omega \int_0^{nu(x)} f(x,s) ds dx = \frac{n^p}{p} - \int_\Omega \int_0^1 f(x, \xi nu(x)) nu(x) d\xi dx$$
$$= \frac{n^p}{p} \left[1 - p \int_\Omega \int_0^1 \frac{f(x, \xi nu(x))}{(\xi nu(x))^{p-1}} \xi^{p-1} u(x)^p d\xi dx \right].$$

By Sobolev's embedding theorem, u belongs to $L^{\frac{2Np}{2N-p}}(\Omega)$. By (f_3), $d_1|u|^p$ is integrable and $\frac{f(x,\xi nu(x))}{|\xi nu(x)|^{p-2}\xi nu(x)} \xi^{p-1}|u(x)|^p \ge \xi^{p-1} d_1(x)|u(x)|^p$ for every integer n, $x \in \Omega$ and $\xi \in (0,1)$. Hence, by the generalized Fatou lemma and (f_5), one has

$$\limsup_{n\to\infty} \left[1 - p \int_\Omega \int_0^1 \frac{f(x, \xi nu(x))}{|\xi nu(x)|^{p-2} \xi nu(x)} \xi^{p-1} |u(x)|^p d\xi dx \right]$$
$$= 1 - \liminf_{n\to\infty} \left[p \int_\Omega \int_0^1 \frac{f(x, \xi nu(x))}{|\xi nu(x)|^{p-2} \xi nu(x)} \xi^{p-1} |u(x)|^p d\xi dx \right]$$
$$\le 1 - p \int_\Omega \int_0^1 \liminf_{n\to\infty} \left[\frac{f(x, \xi nu(x))}{|\xi nu(x)|^{p-2} \xi nu(x)} \xi^{p-1} |u(x)|^p \right] d\xi dx = -\infty,$$

which implies $\lim_{n\to\infty} J(nu) = -\infty$. Hence, we get the lemma. □

Lemma 3 *Assume* (f_1), (f_2), (f_3), (f_5) *and* (f_6) *hold. Let* $\{u_n\}$ *be a sequence in* $W_0^{1,p}(\Omega)$ *such that* $\{J(u_n)\}$ *is bounded and* $\lim_{n\to\infty}(1+||u_n||_{1,p})\|DJ(u_n)\| = 0$. *Then* $\{u_n\}$ *has a subsequence converging in* $W_0^{1,p}(\Omega)$.

Proof Put $\Omega_n = \{x \in \Omega : |u_n(x)| \le C\}$ for every $n \in \mathbb{N}$. By (f_2) and (f_6), we get

$$\begin{aligned}\int_\Omega [f(x,u_n)u_n - \theta F(x,u_n)]dx &= \left(\int_{\Omega\setminus\Omega_n} + \int_{\Omega_n}\right)[f(x,u_n)u_n - \theta F(x,u_n)]dx\\ &\ge \int_{\Omega\setminus\Omega_n} d_2 dx + \int_{\Omega_n}\left[f(x,u_n)u_n - \theta\int_0^{u_n(x)} f(x,t)dt\right]dx\\ &\ge -\int_\Omega |d_2|dx - C(1+\theta)\int_{\Omega_n}|d(x)|dx\\ &\ge -\|d_2\|_{L^1(\Omega)} - C(1+\theta)\|d\|_{L^1(\Omega)},\end{aligned}$$

which implies

$$\begin{aligned}&\int_\Omega\left[\left(\frac{\theta}{p}-1\right)|\nabla u_n|^p - \theta F(x,u_n) + f(x,u_n)u_n)\right]dx\\ &\ge \int_\Omega\left(\frac{\theta}{p}-1\right)|\nabla u_n|^p dx - \|d_2\|_{L^1(\Omega)} - C(1+\theta)\|d\|_{L^1(\Omega)} \qquad \forall n\in\mathbb{N}. \qquad (10)\end{aligned}$$

By (8) and (9), there are a positive real number M and a sequence $\{u_n\}$ in $W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}-M &\le \int_\Omega\left(\frac{1}{p}|\nabla u_n|^p - F(x,u_n)\right)dx \le M \qquad \forall n\in\mathbb{N},\\ -M &\le \int_\Omega(|\nabla u_n|^p - f(x,u_n)u_n)dx \le M \qquad \forall n\in\mathbb{N}.\end{aligned}$$

It follows that

$$\int_\Omega\left[\left(\frac{\theta}{p}-1\right)|\nabla u_n|^p - \theta F(x,u_n) + f(x,u_n)u_n)\right]dx \le (1+\theta)M \quad \forall n\in\mathbb{N}. \qquad (11)$$

Combining (10) and (11), we get

$$\int_\Omega\left(\frac{\theta}{p}-1\right)|\nabla u_n|^p dx \le (1+\theta)M + \|d_2\|_{L^1(\Omega)} + C(1+\theta)\|d\|_{L^1(\Omega)} \qquad \forall n\in\mathbb{N},$$

which implies $\{u_m\}$ is bounded in $W_0^{1,p}(\Omega)$. By Theorem 4, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ weakly (resp. strongly) converges to u in $W_0^{1,p}(\Omega)$ (resp. in $L^{\frac{p}{p-1}}(\Omega)$) and $\{N_f(u_{n_k})\}$ is bounded in $L^p(\Omega)$. Since $\lim_{n\to\infty}\|DJ(u_{n_k})\| = 0$ and $\{u_{n_k}-u\}_k$ is bounded in $W^{1,p}(\Omega)$, we have

$$\lim_{k\to\infty}\int_\Omega f(x,u_{n_k})(u_{n_k}-u)dx = \lim_{k\to\infty}\int_\Omega N_f(u_{n_k})(u_{n_k}-u)dx = 0$$

and

$$\begin{aligned}&\lim_{k\to\infty}\left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_{n_k}\nabla(u_{n_k}-u)dx - \int_\Omega f(x,u_{n_k})(u_{n_k}-u)dx\right|\\ &\le \lim_{k\to\infty}\|Du_{n_k}\|\|u_{n_k}-u\|_{1,p} = 0.\end{aligned}$$

Hence

$$\lim_{k\to\infty}\int_\Omega |\nabla u_n|^{p-2}\nabla u_{n_k}\nabla(u_{n_k}-u)dx = 0.$$

Thus, by [3, Theorem 10], $\{u_{n_k}\}$ strongly converges to u in $W^{1,p}(\Omega)$. □

Proof of Theorem 1 Using the Mountain-pass theorem with the Palais–Smale condition, by Lemmas 1, 2, and 3, we obtain a non-trivial weak solution for the problem (1). □

Example 4 Let $N = 5$, $p = 3$, $r = 4$, $\alpha > 0$, $\Omega = \{x \in \mathbb{R}^5 : \|x\| < 1\}$,

$$\begin{aligned}
\omega_0(x) &= |x|^{-\frac{1}{30}}\cos(16|x|) \qquad \forall x \in \Omega,\\
\omega_1(x) &= \left(\frac{1}{2} - \|x\|^2\right)^2 (1 - \|x\|^2)^{-\frac{7}{6}} \qquad \forall x \in \Omega,\\
\varphi_0(t) &= \begin{cases} |t|^{r-2}t(1-|t|) & \text{if } |t| \le 1,\\ 0 & \text{if } |t| \in \mathbb{R}\setminus[-1,1],\end{cases}\\
\varphi_1(t) &= \begin{cases} 0 & \text{if } |t| \le 1,\\ |t|-1 & \text{if } |t| \in [1,2],\\ 1 & \text{if } |t| \ge 2.\end{cases}\\
f(x,t) &= \omega_0(x)^{r-1}\varphi_0(t) + \omega_1(x)^{r-1}|t|^{p-2}t\varphi_1(t) \qquad \forall (x,t) \in \Omega\times\mathbb{R}.
\end{aligned}$$

Let $\omega = |\omega_0| + \omega_1$, $C = 1$, $d(x) = |x|^{-\frac{1}{30}}$, $d_1(x) = -d(x)$ and $d_2(x) = |x|^{-\frac{1}{30}}$ for every x in Ω. We see that $d_1 \in L^{\frac{2N}{p}}(\Omega)$, $d_2 \in L^1(\Omega)$ and $d \in L^1(\Omega)$. By Examples 1 and 2, ω is in $\mathcal{K}_{p,r}$. Thus, f satisfies conditions (f_1)–(f_5). Since $\lim_{|x|\to 0}\omega_0(x) = \infty$ and $\lim_{|x|\to\frac{1}{2}}\omega_1(x) = 0$, the convergences in (f_4) and (f_5) are not uniform on Ω.

Let $\theta = 4$. For every x in Ω, we have

$$\begin{aligned}
\theta F(x,t) &\le \theta\left(\int_0^1 + \int_1^t\right) f_1(x,\xi)d\xi \le 4|\omega_0(x)|^{r-1} + 4\omega_1^{r-1}(x)\int_0^t (|\xi|^3\xi - \xi^3)d\xi\\
&= 4|\omega_0(x)|^3 + 4\omega_1(x)^3\int_1^{|t|}(\xi^4 - \xi^3)d\xi\\
&= 4|\omega_0(x)|^3 + \omega_1(x)^3\left[\frac{4}{5}|t|^5 - \frac{4}{5} - t^4 + 1\right]\\
&= 4|\omega_0(x)|^3 + \omega_1(x)^3\left[\frac{4}{5}|t^5| + \frac{1}{5} - t^4\right]\\
&\le 4|\omega_0(x)|^3 + \omega_1(x)^3[|t|^5 - t^4]\\
&= 4|\omega_0(x)|^3 + \omega_1(x)^3 t^4[|t| - 1]\\
&\le 4|\omega_0(x)|^{r-1} + \omega_1(x)^3 t^4\\
&= 4|\omega_0(x)|^3 + f_1(x,t)t \qquad \forall |t| \in [1,2],\\
\theta F(x,t) &\le 4|\omega_0(x)|^{r-1} + \theta\int_0^t \varphi_1(t)\omega_1(x)^{r-1}|\xi|^2\xi d\xi\\
&\le 4|\omega_0(x)|^3 + \theta\int_0^t \omega_1(x)^3|\xi|^2\xi d\xi\\
&= 4|\omega_0(x)|^3 + \omega_1(x)^3 t^4\\
&= 4|\omega_0(x)|^3 + f_1(x,t)t \qquad \forall |t| \ge 2.
\end{aligned}$$

Thus, we get (f_6).

Therefore, we can apply Theorem 1 to f with $C = 1$. Since $\omega^{r-1}(x) \geq (1 - \|x\|^2)^{-\frac{21}{16}}$ for every x in Ω, ω^{r-1} is not integrable on Ω. Therefore, the results in [3–5, 8] can not be applied to solve (1) in this case.

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