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Some Remarks on the Jacobson Radical Types of Semirings and Related Problems

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Abstract In this paper, we show that the J_s-radical and the nil-radical of zerosumfree commutative semirings coincide. Based on this result, a semiring version of Snapper's Theorem is given for polynomial semirings over zerosumfree commutative semirings, and zerosumfree commutative J_s-semisimple semirings are completely described. Finally, Problem 1 in Katsov and Nam (Commun. Algebra 42: 5065–5099, 2014) is solved for semisimple, additively π-regular, and anti-bounded artinian semirings.

Keywords Zerosumfree commutative semirings · Semisimple semirings · Additively π-regular semirings · Anti-bounded semirings · J_s-radical of semirings

Mathematics Subject Classification (2010) 16Y60 · 18G99 · 16N99 · 16D99

1 Introduction

Radical theory and radicals of algebraic structures were initiated by Wedderburn and Köthe for rings (see, e.g., [4]) and constitute important areas in research which, in turn, initiate research in new directions in other branches of mathematics. In 1945, Jacobson presented the concept of a radical for rings. Since then, the Jacobson radical becomes a useful tool in studying the structure of rings. A generalization of the structure of a ring is a hemiring.

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Hemirings were first introduced by Vandiver in 1934 [33], but did not immediately catch the attention of the mathematical community. Then, hemirings showed to be important in theoretical computer science [29, 30], as first recognized by Schützenberger [32] in his theory of weighted automata and rational power series [2, 31]. Recently, idempotent hemirings have entered into the mainstream mathematics because they are at the heart of the relatively new subject of tropical geometry [5, 16] and tropical algebra [17]. These things have led us to study the structure of hemirings. Similar to ring or group theory, we study the structure of hemirings using radical theory.

In [3], Bourne defined the Jacobson radical of a hemiring based on left (right) semiregular ideals and, subsequently, in [12], Iizuka proved that this radical can be determined via irreducible semimodules. Katsov and Nam [18] received some results on the structure of hemirings adopting this radical, such as Hopkins's Theorem for a left artinian semiring [18, Corollary 4.4] and a theorem on the structure for a primitive hemiring [18, Theorem 4.5]. However, a limitation of this radical operator is an additively idempotent hemiring (also known as an idempotent hemiring by some authors) belonged to its induced radical class; that is, if R is an additively idempotent hemiring then $J(R) = 0$ [18, Example 3.7]. This shows that the Jacobson radical of a hemiring is not a useful tool in studying the structure of additively idempotent hemirings. To remedy this problem, they defined the J_s-radical for hemirings using simple semimodules and obtained some results on the structure of additively idempotent hemirings through this radical, such as a theorem on the structure of finite additively idempotent hemirings [18, Theorem 3.11]. Simultaneously, they proved that these Jacobson radical types are discrimination (see, [18, Example 3.7]) and pointed out the relationship between them for additively regular hemirings and commutative hemirings.

This paper continues the above works. In Section 3, the J_s-radical for zerosumfree commutative semirings is calculated; it is exactly the nil-radical. Based on this result, a semiring "version" of Snapper's theorem is given for a polynomial semiring with coefficients from a zerosumfree commutative semiring, and the zerosumfree commutative J_s-semisimple semirings are described. In Section 4, the relationship between the two Jacobson radical types is considered. It is pointed out that the radicals are equal over "semisimple, additively π-regular, and anti-bounded antinian semirings" which then solves [18, Problem 1] for these classes of semirings. For the reader's convenience, Section 2 includes necessary notions and facts on semirings and semimodules, as well as on the radical theory of semirings; those used in this paper without any comment can be found in [3, 6, 12, 18, 24].

2 Preliminaries

Recall [6] that a *hemiring* R is an algebra $(R, +, \cdot, 0)$ such that the following conditions are satisfied:

(1) $(R, +, 0)$ is a commutative monoid with identity element 0;
(2) $(R, \cdot)$ is a semigroup;
(3) Multiplication distributes over addition on either side;
(4) $0r = 0 = r0$ for all $r \in R$.

A hemiring R is called a *semiring* if its multiplicative semigroup $(R, \cdot)$ is a monoid with identity element 1. A semiring R is said to be a *division semiring* if $(R \setminus \{0\}, \cdot, 1)$ is a group. When R is not a ring, it is often called a *proper semiring*. A hemiring R is said to be *additively cancellative* if $a + c = b + c$ implies $a = b$ for all $a, b, c \in R$. It is *zerosumfree* if and only if $a + b = 0$ implies $a = b = 0$ for all $a, b \in R$. It is *entire* if and only if $ab = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in R$. The notion of the *ideal* of a hemiring R is defined similarly as for rings. The *subtractive closure* $\overline{I} = \{r \in R \mid r + i \in I$ for some $i \in I\}$ of an ideal I is an ideal of R. An ideal I of a hemiring R is called *subtractive* if $\overline{I} = I$; that is, for all $x, a \in R$, if $x + a, a \in I$ then $x \in I$. It is *strong* if $x + y \in I$ then $x, y \in I$ for all $x, y \in R$. Denote by $\mathcal{I}(R)$ and $\mathcal{SI}(R)$ the sets of all ideals and all subtractive ideals of R, respectively.

As for rings, for any homomorphism $f : R \to S$ between hemirings R and S, there exists a subtractive ideal, which is the *kernel* of f and is defined as $\mathrm{Ker}(f) := \{r \in R \mid f(r) = 0\} \subseteq R$. A surjective hemiring homomorphism $f : R \to S$ is called a *semiisomorphism* if $\mathrm{Ker}(f) = 0$. As usual, the direct product $R = \prod_{i \in I} R_i$ of a family $(R_i)_{i \in I}$ of hemirings R_i consists of the elements $r = (r_i)_{i \in I}$ for $r_i \in R_i$ and is determined by the surjective homomorphisms $\pi_i : R \to R_i$ defined by $\pi_i(r) = r_i$; and a subhemiring S of R is called a *subdirect product* $S = \prod_{i \in I}^{\mathrm{sub}} R_i$ of $(R_i)_{i \in I}$ if, for each π_i, the restriction $\pi_i|_S : S \to R_i$ is also surjective.

Any ideal I of a hemiring R induces on R a congruence relation $\equiv_I$, which is referred to as *Bourne relation* [6, p. 78] and is given by: $r \equiv_I r'$ if and only if there exist elements $i_1, i_2 \in I$ such that $r + i_1 = r' + i_2$. Denote the factor hemiring $R/{\equiv_I}$ by R/I. It is easy to see that $\equiv_I$ and $\equiv_{\overline{I}}$ on R coincide for every ideal I of R, and hence $R/I = R/\overline{I}$ holds for every ideal I of R.

As usual, a *left R-semimodule* over a hemiring R is a commutative monoid $(M, +, 0_M)$ together with a scalar multiplication $(r, m) \mapsto rm$ from $R \times M$ to M that satisfies the following identities for all $r, r' \in R$ and $m, m' \in M$:

(1) $(rr')m = r(r'm)$;
(2) $r(m + m') = rm + rm'$;
(3) $(r + r')m = rm + r'm$;
(4) $r0_M = 0_M = 0m$.

Right semimodules over a hemiring R and homomorphisms between semimodules are defined in the standard manner. If a hemiring R is a semiring, then all semimodules over R are unitary ones. And, from now on, let $\mathcal{M}_R$ and ${}_R\mathcal{M}$ denote the categories of all right and left semimodules, respectively, over a semiring R. A left semimodule M over a hemiring R is *cancellative* if $x + z = y + z$ implies $x = y$ for all $x, y, z \in M$. A subsemimodule N of an R-semimodule M is *subtractive* if, for all $x, y \in M$, from $x + y, x \in N$ it follows that $y \in N$, too.

The usual concepts of the *Descending Chain Condition* and *artinian modules* of the theory of modules over rings, as well as results involving them, are easily extended in an obvious fashion to a context of semimodules over semirings (see, e.g., [19]). For a left R-semimodule ${}_RM$, the ideal $(0 : M)_R := \{r \in R \mid rM = 0\}$ of R is called the *annihilator* of M and ${}_RM$ is *faithful* if and only if $(0 : M)_R = 0$.

Congruences on a left R-semimodule M are defined in the standard manner, and we denote by $\mathrm{Cong}(M)$ the set of all congruences on ${}_RM$. This set is non-empty because it always contains at least two trivial congruences: the *diagonal congruence* $\Delta_M := \{(m, m) \mid m \in M\}$, and the *universal congruence* $M^2 := \{(m, n) \mid m, n \in M\}$. Any subsemimodule N of a left R-semimodule M induces a congruence $\equiv_N$ on M, known as the *Bourne congruence*, by setting $m \equiv_N m'$ if and only if $m + n = m' + n'$ for some $n, n' \in N$. Denote by M/N the factor left R-semimodule $M/\equiv_N$ that has the canonical R-surjection $\pi_N : M \to M/N$.

A nonzero cancellative left semimodule M over a hemiring R is *irreducible* [12, Definition 5] if, for an arbitrarily fixed pair of elements $m_1, m_2 \in M$ with $m_1 \neq m_2$ and any $m \in M$, there exist $r_1, r_2 \in R$ such that $m + r_1m_1 + r_2m_2 = r_1m_2 + r_2m_1$. The *Jacobson radical* was defined by Iizuka [12, Definition 6] as follows:

$$J(R) = \cap\{(0 : M)_R \mid M \in \mathcal{J}\},$$

where $\mathcal{J}$ is the class of all irreducible left semimodules over a hemiring R. When $\mathcal{J} = \emptyset$, $J(R) = R$ by convention. The hemiring R is said to be J-*semisimple* if $J(R) = 0$.

A left R-semimodule M is *simple* if $RM \neq 0$ and there are only trivial subsemimodules of, as well as congruences on, M. The J_s-radical was defined by Katsov and Nam [18, p. 5076] as follows:

$$J_s(R) = \cap\{(0 : M)_R \mid M \in \mathcal{J}'\},$$

where $\mathcal{J}'$ is the class of all simple left semimodules over a hemiring R. When $\mathcal{J}' = \emptyset$, $J_s(R) = R$ by convention. The hemiring R is J_s-*semisimple* if $J_s(R) = 0$.

Remark 1 If M is a simple semimodule over a semiring R then, according to [18, p. 5092], M is always unitary, that is, $1.m = m$ for all $m \in M$.

Katsov and Nam in [18] briefly reviewed the Kurosh–Amitsur radical theory of the category $\mathbb{H}$ of all hemirings, which was developed by various scientists in [9–11, 24–28]. In this theory, a nonempty subclass $\mathbb{U}$ of $\mathbb{H}$ is said to be *hereditary* if $R \in \mathbb{U}$ implies $\mathcal{I}(R) \subseteq \mathbb{U}$, and *homomorphically closed* if $R \in \mathbb{U}$ implies $\varphi(R) \in \mathbb{U}$ for each homomorphism φ of R. If $\mathbb{U}$ is both hereditary and homomorphically closed, then it is said to be *universal*. Similarly to the radical theory of rings, there are three equivalent approaches to the Kurosh–Amitsur radical theory of hemirings, by means of radical classes, of radical operators, and of semisimple classes. These approaches are independently defined in a fixed universal class $\mathbb{U} \subseteq \mathbb{H}$ of hemirings.

A nonempty subclass $\mathbb{R}$ of a fixed universal class $\mathbb{U} \subseteq \mathbb{H}$ is called a *radical class* of $\mathbb{U}$ if $\mathbb{R}$ satisfies the following two conditions [24, Definition 3.1]:

(1) $\mathbb{R}$ is homomorphically closed;
(2) For every hemiring $R \in \mathbb{U} \setminus \mathbb{R}$, there is a subtractive ideal $K \in \mathcal{SI}(R) \setminus \{R\}$ such that $\mathcal{I}(R/K) \cap \mathbb{R} = 0$.

A mapping $\rho : \mathbb{U} \to \mathbb{U}$ is called a *radical operator* in $\mathbb{U}$ if it assigns to each hemiring $R \in \mathbb{U}$ a subtractive ideal $\rho(R) \in \mathcal{SI}(R) \subseteq \mathbb{U}$ such that the following conditions are satisfied for all $S, T \in \mathbb{U}$ [24, Definition 4.1]:

(1) $\varphi(\rho(S)) \subseteq \rho(\varphi(S))$ for each homomorphism $\varphi : S \to T$;
(2) $\rho(S/\rho(S)) = 0$;
(3) For every nonzero ideal I of S, $\rho(I) = I$ implies that $I \subseteq \rho(S)$;
(4) $\rho(\rho(S)) = \rho(S)$.

The following observations will prove to be useful in the sequel:

Proposition 1 *Let $\mathbb{R}$ be a radical class in a universal class $\mathbb{U}$ and ρ be a corresponding radical operator. Then $\rho(A \oplus B) = \rho(A) \oplus \rho(B)$ for every $A, B \in \mathbb{U}$.*

Proof Let $x \in \rho(A) \oplus \rho(B)$, then $x = a + b$ for $a \in \rho(A), b \in \rho(B)$. Suppose that $x = a + b \notin \rho(A \oplus B)$, by [24, Theorem 4.9], there exists an ideal I of $A \oplus B$ such that

$x = a + b \notin I$ and $\rho((A \oplus B)/I) = 0$. Since I is an ideal of $A \oplus B$, there exist ideals I_1 of A, I_2 of B such that $I = I_1 \oplus I_2$. For $a + b \notin I$, it induces $a \notin I_1$ or $b \notin I_2$. In addition, we have $(A \oplus B)/I \cong (A/I_1) \oplus (B/I_2)$, and so A/I_1, B/I_2 are ideals of $(A \oplus B)/I$. Since $\rho((A \oplus B)/I) = 0$, it implies that $\rho(A/I_1) = \rho(B/I_2) = 0$; hence, $a \in I_1$ and $b \in I_2$ (contradiction). Thus, $x = a + b \in \rho(A \oplus B)$, that is, $\rho(A) \oplus \rho(B) \subseteq \rho(A \oplus B)$.

Conversely, let $x \in \rho(A \oplus B)$. Suppose that $x = a + b \notin \rho(A) \oplus \rho(B)$, it induces that $a \notin \rho(A)$ or $b \notin \rho(B)$. Assume $a \notin \rho(A)$, there exists an ideal I of A such that $a \notin I$ and $\rho(A/I) = 0$. On the other hand, $(A \oplus B)/(I \oplus B) \cong A/I$. Since $\rho(A/I) = 0$, we have $\rho((A \oplus B)/(I \oplus B)) = 0$; and hence, $x = a + b \in I \oplus B$, it implies that $a \in I$ (contradiction). Therefore, $x \in \rho(A) \oplus \rho(B)$, that is, $\rho(A \oplus B) \subseteq \rho(A) \oplus \rho(B)$. Thus, $\rho(A \oplus B) = \rho(A) \oplus \rho(B)$. □

By [3, Theorems 5 and 6], the mapping $\varrho : \mathbb{H} \to \mathbb{H}$ given by $R \mapsto J(R)$ is, in fact, a radical operator in $\mathbb{H}$. So is the mapping $\varrho : \mathbb{H} \to \mathbb{H}$ given by $R \mapsto J_s(R)$ (see [18]). From these observations and using Proposition 1, we have the following.

Corollary 1 *Let R be a hemiring and R_1, R_2 are its subhemirings. If $R = R_1 \oplus R_2$ then $J(R) = J(R_1) \oplus J(R_2)$ and $J_s(R) = J_s(R_1) \oplus J_s(R_2)$.*

3 Zerosumfree Commutative Semirings

The aim of this section is to calculate the J_s-radical of a zerosumfree commutative semiring. First, recall [6, Corollary 7.5] that an ideal I of a commutative semiring R is prime if and only if, for all elements $a, b \in R$, $ab \in I$ implies that $a \in I$ or $b \in I$. Moreover, every prime ideal I of a semiring R contains a minimal prime ideal [6, Proposition 7.14]. Denote by $Pr(R)$ and $Pr_m(R)$, the sets of all prime ideals and all minimal prime ideals of a commutative semiring R, respectively. The prime radical (or simply the radical) of a proper ideal I in R is known to be the intersection of all prime ideals of R containing I and denoted by $\sqrt{I}$. The radical $\sqrt{(0)}$ is the *nil-radical* of R, denoted by $Nil(R)$ [6, p. 91]. On the other hand, if I is an ideal of a commutative semiring R then $\sqrt{I} = \{r \in R \mid \exists n \in \mathbb{N} : r^n \in I\}$ [6, Proposition 7.28]. From these observations, we have the following result: If R is a commutative semiring then

$$Nil(R) = \cap_{P \in Pr(R)} P = \cap_{P \in Pr_m(R)} P = \{r \in R \mid \exists n \in \mathbb{N} : r^n = 0\}.$$

To show that the J_s-radical and the nil-radical are equivalent for zerosumfree commutative semirings (Theorem 1), first we prove that the nil-radical is contained in the J_s-radical for commutative semirings.

Lemma 1 *Let R be a commutative semiring. Then $Nil(R) \subseteq J_s(R)$.*

Proof Suppose that M is a simple left R-semimodule. We then have $M = Rm$ for $0 \neq m \in M$, and the R-homomorphism $\varphi : {}_R R \to {}_R M$ defined by $\varphi(r) = rm$ for all $r \in R$, is surjective, and hence, it induces an R-isomorphism $\theta : R/\ker\varphi \to M$, defined by $\theta(\overline{r}) = rm$, where $\ker\varphi := \{(x, y) \in R^2 \mid \varphi(x) = \varphi(y)\}$ is a congruence on ${}_R R$. Since R is commutative, the congruence $\ker\varphi$ is also a congruence on the semiring R, that is, $\overline{R} := R/\ker\varphi$ is a commutative semiring. Next, we will show that $\overline{R}$ is a semifield. Indeed, let I be an ideal of $\overline{R}$. Then we get that I is a subsemimodule of the R-semimodule $\overline{R}$. So $\theta(I)$ is a subsemimodule of ${}_R M$. Since ${}_R M$ is simple, $\theta(I) = 0$ or $\theta(I) = M$, and hence,

$I = 0$ or $I = \overline{R}$. This implies that $\overline{R}$ is a semifield. Now, for any $r \in Nil(R)$, there exists $n \in \mathbb{N}$ such that $r^n = 0$, and so $\overline{r}^n = \overline{0}$ in $\overline{R}$. It implies that $\overline{r} = \overline{0}$, since $\overline{R}$ is a semifield; that means, $r \in \mathrm{Ker}(\varphi) = \varphi^{-1}(0_M)$, and so $r \in J_s(R)$. Therefore $Nil(R) \subseteq J_s(R)$. □

Theorem 1 *Let R be a zerosumfree commutative semiring. Then*

$$J_s(R) = Nil(R).$$

Proof Applying Lemma 1, we only need to show that $J_s(R) \subseteq Nil(R)$. Assume that P is a minimal prime ideal of R, from [34, Corollary 3.2], it implies that P is strong. Then, the map $\varphi : R \to \mathbb{B}$ such that $\varphi(x) = 0$ if $x \in P$ and $\varphi(x) = 1$ if $x \in R \setminus P$ is a surjective homomorphism, where $\mathbb{B} = \{0, 1\}$ is the *Boolean semiring*. It implies that $\mathbb{B}$ is a simple left R-semimodule and $(0 : \mathbb{B})_R = P$. Thus, $J_s(R) = \cap_{M \in \mathcal{J}'}(0 : M)_R \subseteq \cap_{P \in Pr(R)} P = Nil(R)$. Hence, $J_s(R) = Nil(R)$. □

Remark 2

(i) The "zerosumfree" condition in Theorem 1 cannot be omitted. For example, if D is a local domain, then D has a unique maximal ideal I. Consider the semiring $R := D \oplus \mathbb{B}$ which is commutative but non-zerosumfree. Obviously,

$$Nil(R) = 0 \subsetneq I \oplus 0 = J_s(R).$$

(ii) By a (commutative, unitary) $\mathbb{B}$-semialgebra A ($\mathbb{B}$-semialgebra is also called B_1-algebra in [21–23]), we mean the data of a $\mathbb{B}$-semimodule A and of an associative and commutative multiplication on A that has a neutral element 1_A and is bilinear with respect to the operations of $\mathbb{B}$-semimodule (see [21, Definition 4.1] or [22, Definition 1.2]). Since $\mathbb{B}$-semialgebra A is a zerosumfree commutative semiring, so [23, Theorem 5.1] in fact is a corollary of the above Theorem 1.

By [20, Theorem 5.1], Snapper has proved that for a commutative ring R, $J(R[x]) = Nil(R[x]) = Nil(R)[x]$. As a corollary of Theorem 1, in the following result, we are now ready to present a semiring version of the famous Snapper's theorem, which is fundamental in the theory of rings and modules, for zerosumfree commutative semirings.

Corollary 2 (cf. [20, Theorem 5.1]) *Let R be a zerosumfree commutative semiring and let R[x] be a polynomial semiring over R. Then*

$$J_s(R[x]) = Nil(R[x]) = Nil(R)[x].$$

Proof Since $R[x]$ is a zerosumfree commutative semiring, $J_s(R[x]) = Nil(R[x])$, by Theorem 1. Since R is commutative, $Nil(R)[x] \subseteq Nil(R[x])$. Conversely, for any $f(x) = a_0 + a_1x + \cdots + a_nx^n \in Nil(R[x])$, there exists $k \in \mathbb{N}$ such that $f(x)^k = 0$, and so $a_0^k + \cdots + a_1^kx^k + \cdots + a_n^kx^{kn} = 0$, it induces $a_0^k = a_1^k = \cdots = a_n^k = 0$ because R is zerosumfree, that is, $f(x) \in Nil(R)[x]$; and then, $Nil(R[x]) \subseteq Nil(R)[x]$. Thus, $J_s(R[x]) = Nil(R)[x]$. □

In [18, Theorem 3.11], the authors have used the J_s-radical and proved that a finite additively idempotent hemiring R is J_s-semisimple if and only if it is semiisomorphic to a subdirect product of some hemirings S_i $(i \in I)$ such that each of the hemirings S_i $(i \in I)$, in turn, is isomorphic to a dense subhemiring of the endomorphism hemiring $End(M_i)$ $(i \in I)$ of a finite semilattice M_i $(i \in I)$ with zero. We conclude this section by describing zerosumfree commutative J_s-semisimple semirings. To do this, first of all, recall [34, pp. 396 and 398] that a zerosumfree semiring R is *quasi-positive* if $r \neq 0$ then $r^n \neq 0$ for all natural numbers $n \in \mathbb{N}$. A congruence ρ on a semiring R is *entire* if and only if R/ρ is an entire semiring; in this case, R/ρ is called an *entire quotient* of R. An entire congruence ρ is *minimal* if and only if $\rho \geq \mu$ implies $\rho = \mu$ for any entire congruence μ on R. Note that ρ is entire if and only if the set $\overline{0} := \{r \in R \mid r\rho\ 0\}$ is a prime ideal of R, and it is minimal entire if and only if $\overline{0}$ is a minimal prime ideal of R. Also, if ρ is a minimal entire congruence then the semiring R/ρ is a *maximal entire quotient* of R. From these results, together with Theorem 1 and [34, Theorem 3.3 and Theorem 3.4], we have the following characterization of a zerosumfree commutative J_s-semisimple semiring.

Corollary 3 *The following conditions on a zerosumfree commutative semiring R are equivalent:*

(1) *R is J_s-semisimple;*
(2) $\cap_{P \in Pr_m(R)} P = 0$*;*
(3) $\cap_{P \in Pr(R)} P = 0$*;*
(4) *R is quasi-positive;*
(5) *R is semiisomorphic to a subdirect product of its maximal entire quotients.*

Proof (1) $\Rightarrow$ (2). This follows from Theorem 1.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4). This follows from [34, Theorem 3.3].

(4) $\Rightarrow$ (5). This follows from [34, Theorem 3.4].

(5) $\Rightarrow$ (1). Let $\{R_i \mid i \in I\}$ be the family of all maximal entire quotients of R and $\varphi : R \to \prod_{i \in I}^{\text{sub}} R_i$ be a semiisomorphic. Since R_i $(i \in I)$ is a maximal entire quotient of R, there exists a minimal entire congruence ρ_i on R such that $R_i = R/\rho_i$. Then, the set $\overline{0} := \{r \in R \mid r\rho_i\ 0\}$ is a minimal prime ideal of R. By [34, Corollary 3.2], $\overline{0}$ is a strong prime ideal. It implies that R_i is zerosumfree. Indeed, let $\overline{x}, \overline{y} \in R_i$ be such that $\overline{x} + \overline{y} = \overline{0}$, that is, $\overline{x + y} = \overline{0}$, and hence, $x + y \in \overline{0}$. Since $\overline{0}$ is strong, $x, y \in \overline{0}$, that is, $\overline{x} = \overline{y} = \overline{0}$. Since R_i is entire and zerosumfree, there exists the surjective semiring homomorphism $f : R_i \to \mathbb{B}$, defined by $f(\overline{x}) = 0$ if $\overline{x} = \overline{0}$ and $f(\overline{x}) = 1$ if $\overline{0} \neq \overline{x} \in R_i$. We have that $\mathbb{B}$ is a simple left R_i-semimodule with $(0 : \mathbb{B})_{R_i} = 0$. Therefore, $J_s(R_i) = 0$ for any $i \in I$. From this observation and [11, Theorem 4.3(a)], we obtain that $J_s(\prod_{i \in I}^{sub} R_i) = 0$. Then, applying [11, Theorem 3.7(b)], one gets immediately that $J_s(R) = 0$, which ends our proof. □

4 Semisimple, Additively π-Regular and Anti-bounded Artinian Semirings

Obviously, on the subclass of all rings of the class $\mathbb{H}$, both radicals $J(R)$ and $J_s(R)$ coincide. However, in general, they are different (see [18, Example 3.7]). And a problem raised [18, Problem 1] as follows: Describe the subclass of all hemirings R of the class $\mathbb{H}$ with $J_s(R) \subseteq J(R)$, particularly, with $J_s(R) = J(R)$. In this section, we solve [18, Problem 1] for the above-mentioned three important classes of semirings.

First, we calculate the J_s-radical and the Jacobson radical of a division semiring (Proposition 2) and provide a necessary and sufficient condition under which two radicals are equal for the semisimple semirings class (Theorem 2).

Next, we give a complete description of additively π-regular J-semisimple semirings (Proposition 3). We prove that J_s-radical is contained in Jacobson radical (Corollary 5), and we provide a necessary and sufficient condition under which two radicals are equal (Theorem 3) for additively π-regular semirings.

Finally, for anti-bounded semirings, we prove that if M is an additively idempotent and simple left R-semimodule, then $(0 : M)_R = V(R)$ and if the two radicals are equal then the zeroid $Z(R) = 0$ (Lemma 3). The J_s-radical is contained in Jacobson radical (Theorem 4). In addition, we provide a necessary and sufficient condition under which two radicals are equal for the anti-bounded artinian semirings class (Theorem 5).

4.1 On Semisimple Semirings

As usual, a semiring R is said to be *left (right) semisimple* if the regular semimodule ${}_RR$ (R_R) is a direct sum of minimal left (right) ideals. Recall (see, for example, [8, Theorem 7.8] or [19, Theorem 4.5]) that a semiring R is (left, right) semisimple if and only if

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

where $M_{n_1}(D_1), \ldots, M_{n_r}(D_r)$ are the semirings of $n_1 \times n_1, \ldots, n_r \times n_r$ matrices for suitable division semirings $D_1, \ldots, D_r$ and positive integers $n_1, \ldots, n_r$, respectively. Moreover, $Z(R) = \{r \in R \mid r + x = x \text{ for some } x \in R\}$ denotes the *zeroid* of a hemiring R (see [6, p. 50]).

Proposition 2 *Let R be a division semiring. Then $J_s(R) = 0$ and $J(R) = Z(R)$.*

Proof Because R is a division semiring, we have R is a division ring or a zerosumfree division semiring, according to [6, Proposition 4.34]. If R is a division ring then $J_s(R) = 0$ is obvious. If R is a zerosumfree division semiring then there exists a semiisomorphism $f : R \to \mathbb{B}$ such that $f(0) = 0$, $f(r) = 1$ for any $0 \neq r \in R$. Since $\mathbb{B}$ is a simple left $\mathbb{B}$-semimodule, $\mathbb{B}$ is also a simple left R-semimodule. Thus, $J_s(R) = 0$.

From [12, p. 420, Section 4e], $J(R) = \{x \in R \mid x^* \in J(R^*)\}$ and $J(R^*) = J(D(R)) \cap R^*$ for $R^* = R/_{\equiv} = \{x^* \mid x \in R\}$, where the congruence $\equiv$ is defined by $x \equiv y$ if and only if $x+a = y+a$ for $x, y, a \in R$, and the ring of differences $D(R) = (R \times R)/W = \{\overline{(x, y)} \mid (x, y) \in R \times R\}$, where $W = \{(r, r) \mid r \in R\}$ is an ideal of $R \times R$ (see, e.g., [6, Chapter 8] or [18, p. 5083]). Since R is a division semiring, and so $D(R)$ is a division ring or $D(R) = 0$, it induces $J(D(R)) = 0$, and so $J(R^*) = 0$. This shows that $J(R) = Z(R)$. □

Theorem 2 *The following conditions for a semisimple semiring R are equivalent:*

(1) $J_s(R) = J(R)$;
(2) $Z(R) = 0$.

Proof Since R is a semisimple semiring,

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

where $M_{n_1}(D_1), \dots, M_{n_r}(D_r)$ are the semirings of $n_1 \times n_1, \dots, n_r \times n_r$ matrices for suitable division semirings $D_1, \dots, D_r$ and positive integers $n_1, \dots, n_r$, respectively. From Corollary 1, Theorem 5.8 in [18] and Proposition 2, we have $J_s(R) \cong J_s(\oplus M_{n_i}(D_i)) = \oplus J_s(M_{n_i}(D_i)) = \oplus M_{n_i}(J_s(D_i)) = 0$ and $J(R) \cong J(\oplus M_{n_i}(D_i)) = \oplus J(M_{n_i}(D_i)) = \oplus M_{n_i}(J(D_i)) = \oplus M_{n_i}(Z(D_i))$. Thus, $J_s(R) = J(R)$ if and only if $\oplus M_{n_i}(Z(D_i)) = 0$ if and only if $M_{n_i}(Z(D_i)) = 0$ for $i = 1, \dots, r$ if and only if $Z(D_i) = 0$ for $i = 1, \dots, r$. Next, we prove $Z(R) = 0$ if and only if $Z(D_i) = 0$ for $i = 1, \dots, r$. It is easy to show that

$$Z(R) \cong Z(\oplus M_{n_i}(D_i)) = \oplus Z(M_{n_i}(D_i)) = \oplus M_{n_i}(Z(D_i)).$$

Therefore, $Z(R) = 0$ if and only if $Z(D_i) = 0$ for $i = 1, \dots, r$. □

4.2 On Additively π-Regular Semirings

Now, recall some notions for subsequent needs. A commutative monoid $(R, +, 0)$ is called *π-regular* (or a *epigroup*) if every element of it has a power in some subgroup of R (see, for example, [7]). Using Clifford representations of commutative inverse monoids [7, Theorem 4.2.1], it is easy to show that the last condition is equivalent to the condition that for any $a \in R$, there exist a natural number n and an element $x \in R$ such that $na + x + na = na$. A semiring R is called *additively π-regular* if and only if its additive reduct $(R, +, 0)$ is a π-regular monoid. Then, it is clear that a semiring R is additively π-regular if and only if for any $a \in R$, there exist a natural number n and an element $x \in R$ such that $na+x+na = na$. Note that the element x in the last equation can be chosen to be *mutually inverse* with the element na, that is, $na + x + na = na$ and $x + na + x = x$. Indeed, if $na + x + na = na$ for $x \in R$, then one can immediately verify that na and $x + na + x$ are mutually inverse. Moreover, as every our semirings contain a multiplicative identity 1, we can just define an *additively π-regular semiring* as a semiring R for which there exist a natural number n and an element $x \in R$ such that $n1$ and x are mutually inverse, that is, $n1 + x + n1 = n1$ and $x+n1+x = x$. Note that the class of additively π-regular semirings is sufficiently abundant. It includes the classes of associative rings, additively regular (particularly, additively idempotent) semirings, finite and *finite locally* semirings whose finitely generated subsemirings are finite.

In [18, Corollary 4.6], given such a general result on characterization of J-semisimple hemirings that a hemiring R is J-semisimple if and only if it is semiisomorphic to a subdirect product of some additively cancellative hemirings S whose rings of differences $D(S)$ are isomorphic to dense subrings of linear transformations of vector spaces over division rings. Applying this result, we receive a characterization of additively π-regular J-semisimple semirings.

Proposition 3 *For an additively π-regular semiring R, the following conditions are equivalent:*

(1) *R is J-semisimple;*
(2) *R is a ring with $J(R) = 0$;*
(3) *R is a ring isomorphic to a subdirect product of primitive rings.*

Proof (1) ⇒ (2) Assume that R is a J-semisimple semiring. Applying [18, Corollary 4.6], R is semiisomorphic to a subdirect product $\prod_{i\in I}^{\text{sub}} R_i$ of some additively cancellative semirings R_i for $i \in I$. Moreover, since R is an additively π-regular semiring and R_i for $i \in I$ are additively cancellative semirings, then $\prod_{i\in I}^{\text{sub}} R_i$ is an additively π-regular, additively cancellative semiring. Therefore, $\prod_{i\in I}^{\text{sub}} R_i$ is a ring. This implies that R is also a ring and is isomorphic to $\prod_{i\in I}^{\text{sub}} R_i$. Finally, $J(R) = 0$ is obvious.

(2) $\Rightarrow$ (1) It is obvious.

(2) $\Leftrightarrow$ (3) It follows immediately from [18, Corollary 3.8]. □

It is easily verified that if R is a finite semiring then R is an additively π-regular semiring. From Proposition 3 and applying the Wedderburn–Artin theorem in ring theory (see, for example, [20]), we obtain the following result.

Corollary 4 *A finite semiring R is J-semisimple if and only if*

$$R \cong M_{n_1}(F_1) \times M_{n_2}(F_2) \times \cdots \times M_{n_k}(F_k),$$

where $F_1, \ldots, F_k$ are finite fields and $n_1, \ldots, n_k$ are positive integers.

The following result is extended from [18, Proposition 4.8] for additively π-regular semirings.

Corollary 5 *Let R be an additively π-regular semiring. Then*

$$J_s(R) \subseteq J(R).$$

Proof Since R is an additively π-regular semiring, so is R/I. By [24, Theorem 4.9], we have $J(R) = \cap\{I \in \mathcal{SI}(R) \mid J(R/I) = 0\}$. Therefore, R/I is a ring, according to Proposition 3, and hence $J_s(R/I) = J(R/I) = 0$. Then, applying [24, Theorem 4.9] again, we obtain $J(R) = \cap\{I \in \mathcal{SI}(R) \mid J(R/I) = 0\} \supseteq \cap\{I \in \mathcal{SI}(R) \mid J_s(R/I) = 0\} = J_s(R)$, which means $J(R) \supseteq J_s(R)$. □

Lemma 2 *If R is an additively idempotent semiring then there exists a simple left R-semimodule.*

Proof By [6, Proposition 23.5], an additively idempotent semiring R can be embedded in a finitary complete semiring S. Moreover, according to [6, Proposition 22.27], any complete semiring S has an infinite element ∞, that is, $s + \infty = \infty + s = \infty$ for all $s \in S$. Let $M := R\infty$, then M is an additively idempotent cyclic subsemimodule of the left R-semimodule S, and therefore, there exists a maximal congruence ρ on M, and so the left R-semimodule $\overline{M} := M/\rho$ has only trivial congruences. Next, we show that $\overline{M}$ has no nonzero proper subsemimodules. Assume that there is a nonzero proper subsemimodule N of $\overline{M}$. Then, Bourne congruence $\equiv_N$ on $\overline{M}$ is not trivial (contradiction). Thus, $\overline{M}$ is a simple left R-semimodule. □

Theorem 3 *The following conditions for an additively π-regular semiring R are equivalent:*

(1) $J_s(R) = J(R)$;

(2) *R is a ring.*

Proof Given (2), it is obvious that (1) is satisfied. Now we show that (1) implies (2). Assume that R is a proper semiring. First, we show that there exists a nonzero additively idempotent ideal I of R. Let $n1$ and x be mutually inverse elements of a proper additively π-regular semiring R. Denote $I := (n1 + x)R = \{(n1 + x)r \mid r \in R\}$. By the proof of [14, Theorem 3.3], I is a nonzero additively idempotent subsemiring of R and the element $n1 + x$ is a central multiplicative idempotent in R. Therefore, I is also a nonzero additively idempotent ideal of R.

By [12, Theorem 2], $J(I) = I \cap J(R)$. In addition, by [15, Theorem 4.5], $J_s(I) = I \cap J_s(R)$. Since $J_s(R) = J(R)$, we have $J_s(I) = J(I)$. Furthermore, since I is additively idempotent, $J(I) = I$ [18, Example 3.7] and $J_s(I) \subsetneq I$, according to Lemma 2. It follows that $J_s(I) \subsetneq J(I)$ (contradiction). Therefore, R is exactly a ring. □

We note that the "additively π-regular" condition in Theorem 3 cannot be omitted. For example, the semiring $\mathbb{N}$ of nonnegative integers has $J(\mathbb{N}) = J_s(\mathbb{N}) = 0$, but $\mathbb{N}$ is not a ring, because it is not additively π-regular.

4.3 On Anti-Bounded Semirings

In [1, p. 4637], the authors introduced a quite interesting class of semirings that naturally extend the class of all rings as follows: For any semiring R, let $P(R) = V(R) \cup \{1 + r \mid r \in R\}$, where $V(R) = \{r \in R \mid r + r' = 0 \text{ for some } r' \in R\}$. It is easy to see that $P(R)$ is always a subsemiring of R. When $P(R) = R$, we say that the semiring R is *anti-bounded*. To solve the above mentioned problem for the class of anti-bounded semirings, we first prove the following lemma.

Lemma 3 *For an anti-bounded semiring R, the following statements hold:*

(1) *If M is an additively idempotent and simple left R-semimodule, then $(0 : M)_R = V(R)$;*
(2) *If $J_s(R) = J(R)$ then $Z(R) = 0$.*

Proof (1) Since R is an anti-bounded semiring, we have $R = V(R) \cup \{1 + r \mid r \in R\}$. Assume that there exists element $1 + r \in (0 : M)_R$ for $r \in R$, i.e., $(1 + r)m = 0$ for all $m \in M$, hence $m + rm = 0$. It implies that $(M, +)$ is a group (contradiction). Thus, $(0 : M)_R \subseteq V(R)$. In addition, since $(M, +)$ is idempotent, i.e., $m + m = m$ for all $m \in M$, we have $rm + rm = rm$ implies $rm + rm + (-r)m = rm + (-r)m$ for any $r \in V(R)$. Then, $rm = 0$; that is, $r \in (0 : M)_R$ or $V(R) \subseteq (0 : M)_R$. Therefore, $(0 : M)_R = V(R)$.

(2) If R is a ring then $Z(R) = 0$ is true. If R is a proper semiring then $V(R) \neq R$, and thus $\overline{R} := R/V(R)$ is a nonzero zerosumfree, entire quotient semiring. Therefore, $\overline{R}$ is semiisomorphic to the Boolean semiring $\mathbb{B}$. Since $\mathbb{B}$ is a simple left $\mathbb{B}$-semimodule, it is also a simple left R-semimodule and $(0 : \mathbb{B})_R = V(R)$. Thus, $J_s(R) \subseteq V(R)$. Moreover, for any $z \in Z(R)$, we have $z + x = x$ for some $x \in R$. Suppose that M is any irreducible left R-semimodule, then $zm = 0$ for any $m \in M$. Therefore, $z \in (0 : M)_R$; that is, $z \in J(R)$ induces $Z(R) \subseteq J(R)$. Thus, if $Z(R) \neq 0$ then $J_S(R) \neq J(R)$ (contradiction). This shows that $Z(R) = 0$. □

Theorem 4 *Let R be an anti-bounded semiring. Then $J_s(R) \subseteq J(R)$.*

Proof By [13, Proposition 1.2] and Lemma 3(1), $J_s(R) = \cap\{(0 : M) \mid M$ are simple left R-semimodules with $(M, +)$ being groups$\} \cap V(R)$. Let $0 \neq x \in J_s(R)$; that is, $x \in \cap\{(0 :$

$M) \mid M$ are simple left R-semimodules with $(M, +)$ being groups} and $x \in V(R)$. Assume that $x \notin J(R)$, there exists an irreducible left R-semimodule M such that $xM \neq 0$; that is, there exists $0 \neq m \in M$ such that $xm \neq 0$.

Let $\widetilde{M} = \{rm \mid r \in V(R)\}$. Then, it is easy to show that $\widetilde{M} \neq 0$ and $\widetilde{M}$ is an R-subsemimodule of M with $(\widetilde{M}, +)$ being a group. In addition, for any $m' \in M$, the differences left R-module $D(M)$ of M contains m' and is a simple left $D(R)$-module (see [18, p. 5083]). Therefore, for any $0 \neq sm \in \widetilde{M}$, we have $D(M) = D(R)sm$, and hence $m' = (r - r')sm = (rs - r's)m \in \widetilde{M}$ since $rs - r's \in V(R)$. This shows that $\widetilde{M} = M$.

Let $\widetilde{N} = \{rn \mid 0 \neq n \in \widetilde{M}, r \in V(R)\}$, then $\widetilde{N} \neq 0$. Indeed, since $M = D(R)n$, therefore, $m = (r - r')n$ for $r, r' \in R$, and then $xm = (xr)n - (xr')n$ for $xr, xr' \in V(R)$. Since $xm \neq 0$, we have $(xr)n \neq 0$ or $(xr')n \neq 0$, it implies that $\widetilde{N} \neq 0$. Similarly, it is easy to show that $\widetilde{N} = M$. Now, we will prove M is minimal, indeed, $Rn \subseteq \widetilde{M}$ for some $0 \neq n \in \widetilde{M}$. Conversely, for any $sm \in \widetilde{M} = \widetilde{N}$, there exists $s' \in V(R)$ such that $sm = s'n \in Rn$, therefore, $M = Rn$. This shows that M has no nonzero proper subsemimodules. Since M is cyclic, there exists a maximal congruence ρ over M such that $\overline{M} = M/\rho$ is a simple left R-semimodule and $(\overline{M}, +)$ is a group. By $xM \neq 0$, it implies that $x\overline{M} \neq 0$ (contradiction). Therefore, $x \in J(R)$ deduces $J_s(R) \subseteq J(R)$. □

Theorem 5 *The following conditions for an anti-bounded artinian semiring R are equivalent:*

(1) $J_s(R) = J(R)$;
(2) *R is an artinian ring.*

Proof Given (2), it is obvious that (1) is satisfied. Now we show that (1) implies (2). Suppose that R is a proper semiring. Then, we have the maximal subring $V(R) \neq R$. In addition,

$$S := R/V(R) = \{[0]\} \cup \{[1 + r] \mid r \in R\}$$

is a nonzero zerosumfree, entire quotient semiring. Let $S^* := S/\equiv$, where the congruence $\equiv$ on S is defined by $[x] \equiv [y]$ if and only if $[x]+[z] = [y]+[z]$ for $[x], [y], [z] \in S$. Then, S^* is an additively cancellative semiring. Let $a := [1]^* + [1]^* \in S^*$. If $[1]^* + [1]^* = [0]^*$, then $[1]+[1]+[x] = [x]$ for $[x] \in S$. Therefore, $1+1+x+v_1 = x+v_2$ for $v_1, v_2 \in V(R)$; that is, $1 + 1 + v_1 - v_2 \in Z(R)$. Since $J_s(R) = J(R)$, we have $Z(R) = 0$, according to Lemma 3(2). This leads to $V(R) = R$ (contradiction). Thus, $[1]^* + [1]^* \neq [0]^*$. We have

$$S^*a \supseteq S^*a^2 \supseteq \cdots \supseteq S^*a^n \supseteq \cdots,$$

since R is an artinian semiring, S^* is also an artinian semiring. It implies that $S^*a^n = S^*a^{n+1}$; that is, $a^n = ba^{n+1}$ for $b \in S^*$. Therefore,

$$([1]^* + [1]^*)^n = ([1]^* + [s]^*)([1]^* + [1]^*)^{n+1},$$

which implies that $([1]^* + [1]^*)^n = ([1]^* + [1]^*)^{n+1} + [s]^*([1]^* + [1]^*)^{n+1}$. We then have

$$\sum_{i=0}^{n} C_n^i[1]^* = \sum_{i=0}^{n+1} C_{n+1}^i[1]^* + [s]^* \sum_{i=0}^{n+1} C_{n+1}^i[1]^*.$$

Since S^* is additively cancelllative, we have $[1]^* + [y]^* = [0]^*$ for $[y]^* \in S^*$. It implies that $1+y+z+v_1 = z+v_2$ for $z \in R$ and $v_1, v_2 \in V(R)$. Then, $1+y+v_1-v_2 \in Z(R) = 0$, and hence $V(R) = R$ (contradiction). Thus, R is exactly a ring. $\square$

We note that the "artinian" condition in Theorem 5 cannot be omitted. For example, the semiring $\mathbb{N}$ of nonnegative integers is an anti-bounded one that has $J(\mathbb{N}) = J_s(\mathbb{N}) = 0$, but $\mathbb{N}$ is not a ring, because it is not artinian.

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References

1. Abuhlail, J.Y., Il'in, S.N., Katsov, Y., Nam, T.G.: On V-semirings and semirings all of whose cyclic semimodules are injective. Commun. Algebra **43**, 4632–4654 (2015)
2. Berstel, J., Reutenauer, C.: Rational Series and Their Languages. Monogr. Theoret. Comput. Sci. EATCS Ser., vol. 12. Springer-Verlag, Berlin (1988)
3. Bourne, S.: The Jacobson radical of a semiring. Proc. Nat. Acad. Sci. USA **37**, 163–170 (1951)
4. Gardner, J., Wiegandt, R.: Radical Theory of Rings. Marcel Dekker, New York–Basel (2004)
5. Gathmann, A.: Tropical algebraic geometry. Jahresber. Dtsch. Math.-Ver. **108**, 3–32 (2006)
6. Golan, J.: Semirings and their Applications. Kluwer Academic Publishers, Dordrecht–Boston–London (1999)
7. Grillet, P.A.: On free commutative semigroups. J. Nat. Sci. Math. **9**, 71–78 (1969)
8. Hebisch, U., Weinert, H.J.: Semirings and semifields. In: Hazewinkel, M. (ed.) Handbook of Algebra, vol. 1, pp. 435–462. North-Holland (1996)
9. Hebisch, U., Weinert, H.J.: Radical theory for semirings. Quaest. Math. **20**, 647–661 (1997)
10. Hebisch, U., Weinert, H.J.: On the interrelation between radical theories for semirings and rings. Commun. Algebra **29**, 109–129 (2001)
11. Hebisch, U., Weinert, H.J.: Semisimple classes of semirings. Algebra Colloq. **9**, 177–196 (2002)
12. Iizuka, K.: On the Jacobson radical of a semiring. Tohoku Math. J. **11**, 409–421 (1959)
13. Il'in, S.N.: V-semirings. Sib. Math. J. **53**, 222–231 (2012)
14. Il'in, S.N., Katsov, Y.: On p-Schreier varieties of semimodules. Commun. Algebra **39**, 1491–1501 (2011)
15. Inassaridze, H., Mai, L.H., Tuyen, N.X.: On radical classes of hemirings. Tbilisi Math. J. **7**, 69–74 (2014)
16. Itenberg, I., Mikhalkin, G., Shustin, E.: Tropical Algebraic Geometry, 2nd edn. Oberwolfach Seminars, vol. 35. In: Basel, Birkhäuser Verlag (2009)
17. Izhakian, Z., Rowen, L.: Supertropical algebra. Adv. Math. **225**, 2222–2286 (2010)
18. Katsov, Y., Nam, T.G.: On radicals of semirings and related problems. Commun. Algebra **42**, 5065–5099 (2014)
19. Katsov, Y., Nam, T.G., Tuyen, N.X.: On subtractive semisimple semirings. Algebra Colloq. **16**, 415–426 (2009)
20. Lam, T.Y. A First Course in Noncommutative Rings, 2nd edn. Springer-Verlag, New York (2001)
21. Lescot, P.: Algèbre absolue. Ann. Sci. Math. Québec **33**, 63–82 (2009)
22. Lescot, P.: Absolute algebra II—ideals and spectra. J. Pure Appl. Algebra **215**, 1782–1790 (2011)
23. Lescot, P.: Absolute algebra III—the saturated spectrum. J. Pure Appl. Algebra **216**, 1004–1015 (2012)
24. Morak, B.: On the radical theory for semirings. Contrib. Algebra Geom. **40**, 533–549 (1999)
25. Olson, D.M., Heyman, G.A.P., LeRoux, L.H.: Weakly special classes of hemirings. Quaest. Math. **15**, 119–126 (1992)
26. Olson, D.M., Jenkins, T.L.: Radical theory for hemirings. J. Nat. Sci. Math. **23**, 23–32 (1983)
27. Olson, D.M., LeRoux, L.H., Heyman, G.A.P.: Three special classes for hemirings. Quaest. Math. **17**, 205–215 (1994)
28. Olson, D.M., Nance, A.C.: A note on radicals for hemirings. Quaest. Math. **12**, 307–314 (1989)
29. Polák, L.: Syntactic semiring and language equations. In: Champarnaud, J.-M., Maurel, D. (eds.) Implementation and Application of Automata. Lecture Notes in Computer Science, vol. 2608, pp. 182–193. Springer, Berlin–Heidelberg (2003)
30. Polák, L.: A classification of rational languages by semilattice-ordered monoids. Arch. Math. (Brno) **40**, 395–406 (2004)

31. Salomaa, A., Soittola, M.: Automata-Theoretic Aspects of Formal Power Series. Monographs in Computer Science. Springer-Verlag, New York (1978)
32. Schützenberger, M.P.: On the definition of a family of automata. Inf. Control **4**, 245–270 (1961)
33. Vandiver, H.S.: Note on a simple type of algebra in which the cancellation law of addition does not hold. Bull. Am. Math. Soc. **40**, 914–920 (1934)
34. Wang, H.: On characters of semirings. Houst. J. Math. **23**, 391–405 (1997)