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n-Multipliers and Their Relations with n-Homomorphisms

Javad Laali[1] · **Mohammad Fozouni**[2]

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Abstract Let A be a Banach algebra and X be a Banach A-bimodule. We introduce and study the notions of n-multipliers and approximately local n-multipliers by generalizing the classical concept of multipliers from A into X. As an algebraic result, we construct a Banach algebra consisting of n-multipliers on A and under some mild conditions, we give a nice relation of this algebra with n-homomorphisms from A into $\mathbb{C}$.

Keywords Banach algebra · Multiplier · Tensor product space · Banach module

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1 Introduction and Preliminaries

The concept of a multiplier first appears in harmonic analysis in connection with the theory of summability for Fourier series. Subsequently, the notion has been employed in other areas of harmonic analysis, such as the investigation of homomorphisms of group algebras, in the general theory of Banach algebras, and so on; see [5]. Many authors generalized the notion of a multiplier in different ways. See [1, 6], for one of this generalizations.

In this paper, our main concern will not be with these applications of the theory of multipliers and its generalizations. We only develop the theory of multipliers differently from the previous ways, by introducing a new class of operators from a Banach algebra A into a Banach A-bimodule X.

✉ Mohammad Fozouni
fozouni@hotmail.com

1 Department of Mathematics, Faculty of Mathematical Sciences and Computer, Kharazmi University, 43 Mofateh Ave, Tehran, Iran

2 Department of Mathematics, Faculty of Basic Sciences and Engineering, Gonbad Kavous University, P.O. Box 163, Gonbad-e Kavous, Golestan, Iran

Let A be a Banach algebra and $a, b \in A$. Define a bounded bilinear functional on $A^* \times A^*$ as

$$(a \otimes b)(f, g) = f(a)g(b) \qquad (f, g \in A^*).$$

The projective tensor product space $A \widehat{\otimes} A$ is a Banach algebra and a Banach A-bimodule that is characterized as follows

$$\left\{ \sum_{n=1}^{\infty} a_n \otimes b_n : n \in \mathbb{N}, a_n, b_n \in A, \sum_{n=1}^{\infty} \|a_n\| \|b_n\| < \infty \right\},$$

and its module actions are defined by

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \qquad (a, b, c \in A).$$

A Banach algebra A is called *nilpotent* if there exists an integer $n \geq 2$ such that

$$A^n = \{a_1 a_2 a_3 \dots a_n : a_1, a_2, a_3, \dots, a_n \in A\} = \{0\}.$$

The minimum of numbers n that $A^n = \{0\}$ is called the *index* of A which we denote by $I(A)$, i.e., if $I(A) = n$, then there exists $a_1, a_2, \dots, a_{n-1} \in A$ such that $a_1 a_2 \dots a_{n-1} \neq 0$.

To see an example of a nilpotent Banach algebra, suppose that B is a Banach algebra and let A be defined as follows

$$A = \begin{bmatrix} 0 & B & B & B & B \\ 0 & 0 & B & B & B \\ 0 & 0 & 0 & B & B \\ 0 & 0 & 0 & 0 & B \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, A is a Banach algebra equipped with the usual matrix-like operations and l_∞-norm such that A is nilpotent with $I(A) = 5$.

For undefined concepts and notations appearing in the sequel, one can consult [3].

2 *n*-Multipliers

We start this section with the main object of the paper.

Definition 1 Let A be a Banach algebra, X be a Banach A-bimodule and $T : A \to X$ be a bounded linear map. We say that T is an *n-multiplier* ($n \geq 2$) if

$$T(a_1 a_2 \dots a_n) = a_1 \cdot T(a_2 \dots a_n) \quad (a_1, a_2, a_3, \dots, a_n \in A).$$

We will denote by $\mathrm{Mul}_n(A, X)$ the set of all n-multipliers of Banach algebra A into X. Now, we study in more details the space $\mathrm{Mul}_n(A, X)$ when $n \geq 3$ (in the case $n = 2$ this is the space of all multipliers in the classical sense).

Let A be a Banach algebra and X be a Banach A-bimodule. The set $\mathrm{Mul}_n(A, X)$ is a vector subspace of $B(A, X)$; the space of all bounded linear maps from A into X.

As the first result, we show that $\mathrm{Mul}_n(A, X)$ is a closed vector subspace of $B(A, X)$.

Theorem 1 *Let A be a Banach algebra and X be a Banach A-bimodule. Then for all integers $n \geq 3$, the space* $\mathrm{Mul}_n(A, X)$ *is a closed vector subspace of* $B(A, X)$.

Proof We claim that $\mathrm{Mul}_n(A, X)$ is closed in $B(A, X)$. Suppose that $\{T_m\}$ is a sequence in $\mathrm{Mul}_n(A, X)$ converging to $T \in B(A, X)$.

Let $a_1, a_2, \dots, a_n$ be arbitrary elements of A. So, we have

$$\begin{aligned}\|T(a_1 \dots a_n) - a_1 \cdot T(a_2 \dots a_n)\| &\le \|T(a_1 \dots a_n) - T_m(a_1 \dots a_n)\| \\ &\quad + \|T_m(a_1 \dots a_n) - a_1 \cdot T(a_2 \dots a_n)\| \\ &\le \|T - T_m\| \|a_1 \dots a_n\| \\ &\quad + \|a_1 \cdot T_m(a_2 \dots a_n) - a_1 \cdot T(a_2 \dots a_n)\| \\ &\le \|T - T_m\| \|a_1 \dots a_n\| + \|T - T_m\| \|a_1\| \|a_2 \dots a_n\|.\end{aligned}$$

If $m \to \infty$, we conclude that $T(a_1 \dots a_n) = a_1 \cdot T(a_2 \dots a_n)$. The rest of the proof is easy. □

Similarly, one can see that $\mathrm{Mul}_n(A, X)$ is complete in the *strong operator topology* (SOT), i.e., in the topology on $B(A, X)$ for which a net $\{T_\alpha\}$ converges to T if and only if for each $a \in A$, $\|Ta - T_\alpha a\| \to 0$.

In the next two theorems, we give some relations between the spaces of n-multipliers.

Theorem 2 *There exist a Banach algebra A and a Banach A-bimodule X such that for all positive integers $n \ge 3$*

$$\mathrm{Mul}_2(A, X) \subsetneq \mathrm{Mul}_3(A, X) \subsetneq \cdots \subsetneq \mathrm{Mul}_n(A, X).$$

Proof For every positive integer $n \ge 3$, take A being a nilpotent Banach algebra with $I(A) = n$ and $X = A\widehat{\otimes}A$. So, there exist non-zero elements $a_1, a_2, \dots, a_{n-1} \in A$ such that $a_1 a_2 \dots a_{n-1} \ne 0$.

The verification of the above chain of inclusions is easy. We only show each of the strict relations. For every integer number i such that $2 \le i < n$, define a linear map $T_i : A \to A\widehat{\otimes}A$ by

$$T_i(a) = a_1 a_2 \dots a_{n-1} \otimes a_1 a_2 \dots a_{n-(i+1)} a \qquad (a \in A).$$

Thus, T_i is an element of $\mathrm{Mul}_{i+1}(A, A\widehat{\otimes}A)$, but it does not belong to $\mathrm{Mul}_i(A, A\widehat{\otimes}A)$. To see this, let $f \in A^*$ be a functional such that $f(a_1 a_2 \dots a_{n-1}) \ne 0$. So

$$\begin{aligned}T_i(a_{n-i} \dots a_{n-1})(f, f) &= (a_1 a_2 \dots a_{n-1} \otimes a_1 a_2 \dots a_{n-1})(f, f) \\ &= f(a_1 a_2 \dots a_{n-1})^2 \ne 0.\end{aligned}$$

Therefore, $T_i(a_{n-i} \dots a_{n-1}) \ne 0$, but $a_{n-i} \cdot T_i(a_{(n-i)+1} \dots a_{n-1}) = 0$ and this completes the proof. □

For a Banach algebra A, let $A^2 = \mathrm{span}\{ab : a, b \in A\}$. The Banach algebra A is *essential* if $\overline{A^2} = A$.

Theorem 3 *Let A be an essential Banach algebra and X be a Banach A-bimodule. Then for all integers $n \ge 3$, we have*

$$\mathrm{Mul}_{n-1}(A, X) = \mathrm{Mul}_n(A, X).$$

Specially, $\mathrm{Mul}_2(A, X) = \mathrm{Mul}_n(A, X)$ *for all $n \ge 2$.*

Proof Let $T \in \mathrm{Mul}_n(A, X)$, A be essential and $a_1, \ldots, a_{n-1}$ be arbitrary elements of A. We show that $T \in \mathrm{Mul}_{n-1}(A, X)$. Since $a_2 \in A = \overline{A^2}$, there exists a net $\{a_{2,\alpha}\} \in A^2$ with $a_{2,\alpha} = \sum_{i_\alpha} \beta_{i_\alpha} b_{i_\alpha} c_{i_\alpha}$ for $b_{i_\alpha}, c_{i_\alpha} \in A$ and $\beta_{i_\alpha} \in \mathbb{C}$, such that $a_2 = \lim_\alpha a_{2,\alpha}$. So, we have

$$\begin{aligned} T(a_1 a_2 \ldots a_{n-1}) &= \lim_\alpha T\left(a_1 \left(\sum_{i_\alpha} \beta_{i_\alpha} b_{i_\alpha} c_{i_\alpha}\right) a_3 \ldots a_{n-1}\right) \\ &= \lim_\alpha \sum_{i_\alpha} \beta_{i_\alpha} T(\overbrace{a_1 b_{i_\alpha} c_{i_\alpha} a_3 \ldots a_{n-1}}^{n}) \\ &= \lim_\alpha \sum_{i_\alpha} \beta_{i_\alpha} a_1 \cdot T(b_{i_\alpha} c_{i_\alpha} a_3 \ldots a_{n-1}) \\ &= a_1 \cdot T(a_2 \ldots a_{n-1}), \end{aligned}$$

which completes the proof. □

3 Relations with *n*-Homomorphisms

Let A be a Banach algebra and $n \geq 3$ be a positive integer. Here, we denote the space $\mathrm{Mul}_n(A, A)$ briefly by $\mathrm{Mul}_n(A)$.

We know that the space of all multipliers on A is a Banach subalgebra of $B(A) = B(A, A)$ with composition of operators as product and the operator norm. But in general, the space of n-multipliers on A is not an algebra with composition of operators. So, we should define another product on this space to make $\mathrm{Mul}_n(A)$ into a Banach algebra.

Now, let $a_0 \in A$ and consider $\bullet_{a_0} : \mathrm{Mul}_n(A) \times \mathrm{Mul}_n(A) \to \mathrm{Mul}_n(A)$ which is defined by

$$S \bullet_{a_0} T(a) := S(T(a) a_0^{n-2}) \qquad (a \in A). \tag{1}$$

Without losing the generality we assume that $\|a_0\| \leq 1$.

Theorem 4 *Let A be a Banach algebra. Then for all positive integers $n \geq 3$, $\mathrm{Mul}_n(A)$ is a Banach algebra, with the product "$\bullet_{a_0}$" and the operator norm.*

Proof Clearly, $\mathrm{Mul}_n(A)$ is a vector space with operations that inherit from $B(A)$. Let $S, T \in \mathrm{Mul}_n(A)$. First, we show that $S \bullet_{a_0} T$ is well-defined, i.e., $S \bullet_{a_0} T \in \mathrm{Mul}_n(A)$. Let $a_1, a_2, \ldots, a_n \in A$, we have

$$\begin{aligned} S \bullet_{a_0} T(a_1 \ldots a_n) = S(T(a_1 \ldots a_n) a_0^{n-2}) &= S(a_1 T(a_2 \ldots a_n) a_0^{n-2}) \\ &= a_1 S(T(a_2 \ldots a_n) a_0^{n-2}) \\ &= a_1 (S \bullet_{a_0} T)(a_2 \ldots a_n). \end{aligned}$$

Therefore, $S \bullet_{a_0} T$ is an n-multiplier on A.

Let T_1, T_2, and T_3 be elements of $\mathrm{Mul}_n(A)$. We have

$$\begin{aligned} (T_1 \bullet_{a_0} T_2) \bullet_{a_0} T_3(a) &= (T_1 \bullet_{a_0} T_2)(T_3(a) a_0^{n-2}) = T_1(T_2(T_3(a) a_0^{n-2}) a_0^{n-2}), \\ T_1 \bullet_{a_0} (T_2 \bullet_{a_0} T_3)(a) &= T_1((T_2 \bullet_{a_0} T_3)(a) a_0^{n-2}) = T_1(T_2(T_3(a) a_0^{n-2}) a_0^{n-2}). \end{aligned}$$

Hence, the product "$\bullet_{a_0}$" is associative.

On the other hand, Theorem 1 shows that $\mathrm{Mul}_n(A)$ is a closed vector subspace of $B(A)$ with the operator norm. The investigation of the other properties is easy. □

Recall that for Banach algebras A and B a linear map $\phi : A \to B$ is called an *n-homomorphism* if, $\phi(a_1a_2\dots a_n) = \phi(a_1)\phi(a_2)\dots\phi(a_n)$ for all $a_1, a_2, \dots, a_n \in A$ [4].

For each integer $n \geq 2$, suppose that $\Delta_n(A)$ denotes the *n-character space* of A, i.e., the space consisting of all non-zero n-homomorphisms from A into $\mathbb{C}$. It is clear that for every integer $n \geq 3$, $\Delta_2(A) \subseteq \Delta_n(A)$. The last inclusion may be strict. As an example for $n = 3$, if $\phi \in \Delta_2(A)$, then $\varphi := -\phi$ is in $\Delta_3(A)$, but φ is not a 2-character from A into $\mathbb{C}$.

Let $\phi_n \in \Delta_n(A)$. Define $\widetilde{\phi_n} : (\mathrm{Mul}_n(A), \bullet_{a_0}) \to \mathbb{C}$ by

$$\widetilde{\phi_n}(T) = \phi_n(T(a_0^{n-1})) \quad (T \in \mathrm{Mul}_n(A)).$$

Clearly, $\widetilde{\phi_n}$ is a linear operator. We say that $\widetilde{\phi_n}$ extends ϕ_n if, $\widetilde{\phi_n}(L_a) = \phi_n(a)$ for all $a \in A$.

In the next theorem, under some mild conditions, we show that $\Delta_n(\mathrm{Mul}_n(A)) \neq \emptyset$. Recall that $Z(A)$ denotes the center of A, i.e.,

$$Z(A) = \{a \in A : ab = ba \quad (b \in A)\}.$$

Theorem 5 *Let A be a Banach algebra and let $a_0 \in Z(A) \setminus \{0\}$. Then $\widetilde{\phi_n} \in \Delta_n(\mathrm{Mul}_n(A))$ is an extension of $\phi_n \in \Delta_n(A)$ if $\phi_n(a_0) = 1$.*

Proof Suppose that there exists $\phi_n \in \Delta_n(A)$ with $\phi_n(a_0) = 1$. We must show that $\widetilde{\phi_n}$ is a non-zero n-homomorphism. For each $a \in A$, we have

$$\widetilde{\phi_n}(L_a) = \phi_n(L_a(a_0^{n-1})) = \phi_n(aa_0^{n-1}) = \phi_n(a)\phi_n(a_0)^{n-1} = \phi_n(a).$$

Therefore, in the special case when $a = a_0$, we have $\widetilde{\phi_n}(L_{a_0}) = 1$. So, $\widetilde{\phi_n}$ is a non-zero extension of ϕ_n.

On the other hand, for $T_1, T_2, \dots, T_n \in \mathrm{Mul}_n(A)$, we have

$$\begin{aligned}
\widetilde{\phi_n}(T_1 \bullet_{a_0} \dots \bullet_{a_0} T_n) &= \phi_n(T_1 \bullet_{a_0} \dots \bullet_{a_0} T_n(a_0^{n-1})) \\
&= \phi_n(T_1(T_2 \overbrace{(\dots(}^{n-2} T_n(a_0^{n-1})a_0^{n-2} \overbrace{)\dots)}^{n-2} a_0^{n-2})) \\
&= \phi_n(a_0)^{n-1}\phi_n(T_1(T_2 \overbrace{(\dots(}^{n-2} T_n(a_0^{n-1})a_0^{n-2} \overbrace{)\dots)}^{n-2} a_0^{n-2})) \\
&= \phi_n(a_0^{n-1}T_1(T_2 \overbrace{(\dots(}^{n-2} T_n(a_0^{n-1})a_0^{n-2} \overbrace{)\dots)}^{n-2} a_0^{n-2})) \\
&= \phi_n(T_1(T_2 \overbrace{(\dots(}^{n-2} T_n(a_0^{n-1})a_0^{n-1} \overbrace{)\dots)}^{n-2} a_0^{n-1})) \\
&= \cdots \\
&= \phi_n(T_1(a_0^{n-1})T_2(a_0^{n-1})\dots T_n(a_0^{n-1})) \\
&= \phi_n(T_1(a_0^{n-1}))\phi_n(T_2(a_0^{n-1}))\dots\phi_n(T_n(a_0^{n-1})) \\
&= \widetilde{\phi_n}(T_1)\widetilde{\phi_n}(T_2)\dots\widetilde{\phi_n}(T_n).
\end{aligned}$$

Therefore, $\widetilde{\phi_n}$ is an n-homomorphism and this completes the proof. □

4 Approximate Local *n*-Multipliers

In [7], Samei investigated the approximately local left 2-multipliers and study some of its relations with left 2-multipliers on a Banach algebra A. In this section, we give two theorems similar to Theorem 2.2 and Proposition 2.3 of [7] for n-multipliers. Indeed, we are interested in determining when an approximately local n-multiplier (Definition 2) is an n-multiplier. First, we give the following definition.

Definition 2 Let X be a Banach A-module and $T : A \to X$ be a bounded linear operator. We say that T is an *approximately local n-multiplier* if, for each $a \in A$, there exists a sequence $\{T_{a,m}\}$ of n-multipliers such that $T(a) = \lim_m T_{a,m}(a)$.

We recall the algebraic reflexivity from [2]. Let X and Y be Banach spaces and S be a subset of $B(X, Y)$. Put

$$\mathrm{ref}(S) = \{T \in B(X,Y) : T(x) \in \overline{\{s(x) : s \in S\}} (x \in X)\}.$$

Then S is algebraically reflexive if, $S = \mathrm{ref}(S)$ or just $\mathrm{ref}(S) \subseteq S$.

Theorem 6 *Let A be a Banach algebra and X be a Banach A-module. Then the following statements are equivalent.*

1. *Every approximately local n-multiplier from A into X is an n-multiplier* ($n \geq 3$).
2. Mul_n*(A,X) is algebraically reflexive.*

Proof (1) $\Rightarrow$ (2) Let $T \in \mathrm{ref}(\mathrm{Mul}_n(A, X))$. So, for all $a \in A$, there exists a sequence $\{T_m\}$ in $\mathrm{Mul}_n(A, X)$ such that, $T(a) = \lim_m T_{a,m}(a)$. Hence, T is an approximately local n-multiplier. Therefore, T is an n-multiplier by assumption and this shows that Mul_n(A,X) is algebraically reflexive.

(2) $\Rightarrow$ (1) Let $T : A \to X$ be an approximately local n-multiplier. So, for all $a \in A$, there exists a sequence $\{T_{a,m}\}$ such that, $T(a) = \lim_m T_{a,m}(a)$. Hence, $T \in \mathrm{ref}(\mathrm{Mul}_n(A, X))$ and reflexivity of $\mathrm{Mul}_n(A, X)$ implies that T is an n-multiplier. □

Let A be a Banach algebra and X be a Banach A-module. Then for each $x \in X$, the *left annihilator* of x in A is defined by $x^{\perp} = \{a \in A : a \cdot x = 0\}$.

Theorem 7 *Suppose that A is a Banach algebra such that* $\mathrm{Mul}_n(A, A^*)$ *is algebraically reflexive and X is a left Banach A-module with* $\{x \in X : x^{\perp} = A\} = 0$. *Then every approximately local n-multiplier from A into X is an n-multiplier.*

Proof Let $T : A \to X$ be an approximately local n-multiplier and $f \in X^*$. Define a map $\mathfrak{M}_f : X \to A^*$ as follows

$$\mathfrak{M}_f(x) = x \bullet f \quad (x \in X),$$

where $x \bullet f \in A^*$ is defined by $x \bullet f(a) = f(a \cdot x)$ for all $a \in A$. Therefore, $\mathfrak{M}_f$ is a bounded right A-module morphism. Because, for $a \in A$ and $x \in X$, we have

$$\mathfrak{M}_f(a \cdot x) = (a \cdot x) \bullet f = a \cdot (x \bullet f) = a \cdot \mathfrak{M}_f(x),$$

so

$$\mathfrak{M}_f \circ T \in \mathrm{ref}(\mathrm{Mul}_n(A, A^*)) = \mathrm{Mul}_n(A, A^*).$$

Thus, $\mathfrak{M}_f \circ T \in \mathrm{Mul}_n(A, A^*)$. Now, for $a_1, a_2, a_3, \ldots, a_n \in A$, we have

$$\begin{aligned}\mathfrak{M}_f(T(a_1a_2a_3\ldots a_n)) = \mathfrak{M}_f \circ T(a_1a_2a_3\ldots a_n) &= a_1 \cdot \mathfrak{M}_f \circ T(a_2a_3\ldots a_n)\\ &= a_1 \cdot \mathfrak{M}_f(T(a_2a_3\ldots a_n))\\ &= \mathfrak{M}_f(a_1 \cdot T(a_2a_3\ldots a_n)).\end{aligned}$$

Therefore, $\mathfrak{M}_f(T(a_1a_2a_3\dots a_n) - a_1 \cdot T(a_2a_3\dots a_n)) = 0$. If we put

$$u = T(a_1a_2a_3\dots a_n) - a_1 \cdot T(a_2a_3\dots a_n),$$

then $f(a \cdot u) = 0$ for all $a \in A$. So, by Hahn-Banach's theorem, we have $a \cdot u = 0$ for all $a \in A$. So, $u^{\perp} = A$ and this implies that $u = 0$. Hence, T is an n-multiplier. □

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