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Boundary Value Problems of Hadamard-Type Fractional Differential Equations and Inclusions with Nonlocal Conditions

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Abstract In this paper, we obtain some new existence and uniqueness results for boundary value problems of Hadamard type fractional differential equations and inclusions with nonlocal boundary conditions. The contraction mapping principle and a fixed point theorem due to O'Regan are the main tools of our study for the single-valued case, while the multivalued case is based on the nonlinear alternative for contractive maps. The results are well illustrated with the aid of examples.

Keywords Fractional differential equations · Fractional differential inclusions · Hadamard fractional derivative · Nonlocal boundary conditions · Existence · Fixed point

Mathematics Subject Classification (2010) 34A08 · 34B10 · 34B15 · 34A60

1 Introduction

In this paper, we study boundary value problems of Hadamard type fractional differential equations and inclusions with nonlocal boundary conditions. Firstly, we discuss the existence and uniqueness of solutions for the following boundary value problem

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t)), \ 1 < t < e, 1 < \alpha \leq 2, \\ x(1) = 0, \qquad\qquad x(\eta) = g(x), 1 < \eta < e, \end{cases} \tag{1}$$

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where D^α is the Hadamard fractional derivative of order α, $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ and $g : C([0, 1], \mathbb{R}) \to \mathbb{R}$ are given continuous functions.

Several interesting and important results concerning existence and uniqueness of solutions, stability properties of solutions, analytic, and numerical methods of solutions for fractional differential equations can be found in the recent literature on the topic and the serge for investigating more and more results is in progress. Fractional-order operators are nonlocal in nature and take care of the hereditary properties of many phenomena and processes. Fractional calculus has also emerged as a powerful modeling tool for many real world problems. For examples and recent development of the topic, see ([1–6, 9, 10, 19, 23, 24, 28, 33]). However, it has been observed that most of the work on the topic involves either Riemann–Liouville or Caputo type fractional derivative. Besides these derivatives, Hadamard fractional derivative is another kind of fractional derivatives that was introduced by Hadamard in 1892 [17]. This fractional derivative differs from the other ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains logarithmic function of arbitrary exponent. For background material of Hadamard fractional derivative and integral, we refer the reader to the book [19], to the papers [11–13, 20, 21], while some recent work on Hadamard fractional differential equations can be found in [7, 8, 29–32].

In passing, we remark that the nonlocal conditions are more plausible than the standard initial conditions for the formulation of some physical phenomena. In (1), $g(x)$ may be regarded as $g(x) = \sum_{j=1}^{p} \alpha_j x(t_j)$ where α_j, $j = 1, \ldots, p$, are given constants and $0 < t_1 < \cdots < t_p \le 1$. Further details can be found in the work by Byszewski [14, 15].

Section 3 deals with the main results for the problem (1) which rely on Banach's contraction principle and a fixed point theorem due to O'Regan. Some examples illustrating the applicability of our results are presented.

In Section 4, we extend the results to cover the multi-valued case by considering the following boundary value problem of Hadamard type fractional differential inclusions

$$\begin{cases} D^\alpha x(t) \in F(t, x(t)), & 1 < t < e, 1 < \alpha \le 2, \\ x(1) = 0, & x(\eta) = g(x), 1 < \eta < e, \end{cases} \tag{2}$$

where $F : [1, e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$. We show the existence of solutions for the problem (2) by using the nonlinear alternative for contractive maps when the multivalued map $F(t, x)$ is convex valued.

Comparing our work with the available literature on the topic, we emphasize that we have addressed new problems on Hadamard type fractional differential equations and inclusions supplemented with a more general type of nonlocal condition (as explained above). The methods of proof employed in this paper are the standard ones, however, their imposition on the present problems is new. For instance, we notice that the existence of solutions for Hadamard-type fractional differential equations with nonlocal fractional integral boundary conditions in [30] was discussed by using Krasnoselskii fixed point theorem and nonlinear alternative of Leray–Schauder type. The authors in [7] obtained the existence results for Hadamard type fractional differential inclusions with integral boundary conditions via the nonlinear alternative of Leray–Schauder type, a selection theorem due to Bressan and Colombo for lower semi-continuous multivalued maps, and the fixed point theorem for contractive multivalued maps due to Covitz and Nadler. On the other hand, we have used a fixed point theorem due to O'Regan for problem (1) and nonlinear alternative for contractive maps due to Petryshyn and Fitzpatric [27] for inclusion problem (2) to obtain the existence results. Thus, our methods of proof for the given problems are different and the present work contributes to the enrichment of material on Hadamard fractional calculus.

2 Preliminaries

Definition 1 [19] The Hadamard derivative of fractional order q for a function $g : [1, \infty) \to \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \left(t \frac{d}{dt} \right)^n \int_1^t \left(\log \frac{t}{s} \right)^{n-q-1} \frac{g(s)}{s} ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2 [19] The Hadamard fractional integral of order q for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s} \right)^{q-1} \frac{g(s)}{s} ds, \quad q > 0,$$

provided the integral exists.

Lemma 1 *Given* $y \in AC([1, e], \mathbb{R})$*, the problem*

$$\begin{cases} D^\alpha x(t) = y(t), & 1 < t < e, 1 < \alpha \le 2, \\ x(1) = 0, & x(\eta) = y_0, \end{cases} \tag{3}$$

is equivalent to an integral equation

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{y(s)}{s} ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^\eta \left(\log \frac{\eta}{s} \right)^{\alpha-1} \frac{y(s)}{s} ds \\ & + \frac{(\log t)^{\alpha-1}}{(\log \eta)^{\alpha-1}} y_0, \quad x \in [1, e]. \end{aligned} \tag{4}$$

Proof As argued in [19], the solution of Hadamard differential equation in (3) can be written as

$$x(t) = I^\alpha y(t) + c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2}. \tag{5}$$

Using the given boundary conditions, we find that $c_2 = 0$ and

$$c_1 = \frac{1}{(\log \eta)^{\alpha-1}} \left[y_0 - \frac{1}{\Gamma(\alpha)} \int_1^\eta \left(\log \frac{\eta}{s} \right)^{\alpha-1} \frac{y(s)}{s} ds \right].$$

Substituting the values of c_1 and c_2 in (5), we obtain (4).

Conversely, applying the operator D^α on (4) it follows that $D^\alpha x(t) = y(t)$. From (4), it is easy to verify that the boundary conditions $x(1) = 0$, $x(\eta) = y_0$ are satisfied. This establishes the equivalence between (3) and (4). This completes the proof. □

3 Existence Results—The Singlevalued Case

We denote by $\mathcal{C} = C([1, e], \mathbb{R})$ the Banach space of all continuous functions from $[1, e] \to \mathbb{R}$ endowed with the topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [1, e]\}$.

In view of Lemma 1, we define an operator $\mathcal{Q} : C([1, e], \mathbb{R}) \to C([1, e], \mathbb{R})$ by

$$\begin{aligned} \mathcal{Q}x(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s, x(s))}{s} ds \\ &\quad - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{f(s, x(s))}{s} ds \\ &\quad + \frac{(\log t)^{\alpha-1}}{(\log \eta)^{\alpha-1}} g(x), \quad x \in [1, e]. \end{aligned} \tag{6}$$

Observe that the existence of a fixed point for the operator $\mathcal{Q}$ implies the existence of a solution for the problem (1).

Theorem 1 *Let $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ and $g : C([1, e], \mathbb{R}) \to \mathbb{R}$ be continuous functions. Assume that*

(A_1) $|f(t, x) - f(t, y)| \le L|x - y| \ \ \forall t \in [1, e], L > 0, x, y \in \mathbb{R}$;
(A_2) $|g(u) - g(v)| \le \ell \|u - v\|, 0 < \ell < (\log \eta)^{\alpha-1}$ *for all* $u, v \in C([1, e], \mathbb{R})$;
(A_3) $\gamma = \frac{L}{\Gamma(\alpha+1)}(1 + \log \eta) + \frac{\ell}{(\log \eta)^{\alpha-1}} < 1.$

Then the boundary value problem (1) has a unique solution.

Proof For $x, y \in \mathcal{C}$ and for each $t \in [1, e]$, from the definition of $\mathcal{Q}$ and assumptions (A_1) and (A_2), we obtain

$$\begin{aligned} |(\mathcal{Q}x)(t) - (\mathcal{Q}y)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{|f(s, x(s)) - f(s, y(s))|}{s} ds \\ &\quad + \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{|f(s, x(s)) - f(s, y(s))|}{s} ds \\ &\quad + \frac{(\log t)^{\alpha-1}}{(\log \eta)^{\alpha-1}} |g(x) - g(y)| \\ &\le L\|x - y\| \left[\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds \right. \\ &\quad \left. + \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{1}{s} ds \right] + \frac{\ell}{(\log \eta)^{\alpha-1}} \|x - y\| \\ &\le \|x - y\| \left\{ \frac{L}{\Gamma(\alpha + 1)}(1 + \log \eta) + \frac{\ell}{(\log \eta)^{\alpha-1}} \right\}. \end{aligned}$$

Hence,

$$\|\mathcal{Q}x - \mathcal{Q}y\| \le \gamma \|x - y\|.$$

As $\gamma < 1$, by (A_3), F is a contraction map from the Banach space $\mathcal{C}$ into itself. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 1 Consider the following fractional boundary value problem

$$\begin{cases} D^{3/2}x(t) = \dfrac{L}{2}\left(x + \dfrac{|\sin x|}{1+|\sin x|} + \cos t\right), & 1 < t < e, \\ x(1) = 0, \quad x\left(\dfrac{5}{4}\right) = \dfrac{1}{7}x\left(\dfrac{3}{2}\right) + \dfrac{1}{9}x\,(2) + \dfrac{1}{11}x\left(\dfrac{5}{2}\right), \end{cases} \tag{7}$$

where L will be fixed later. Clearly, $\eta = \frac{5}{4}$, $g(x) = \frac{1}{7}x\left(\frac{3}{2}\right) + \frac{1}{9}x(2) + \frac{1}{11}x\left(\frac{5}{2}\right)$. With the given values, it is found that $\ell \simeq 0.344877$ and the assumption (A_3) is satisfied for $L < 0.293351$. Thus, all the conditions of Theorem 1 are satisfied. Hence, the boundary value problem (7) has a unique solution on $[1, e]$.

Next, we introduce the fixed point theorem which was established by O'Regan in [25]. This theorem will be adopted to prove the next main result.

Lemma 2 *Let U be an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$. Also assume that $\mathcal{F}(\bar{U})$ is bounded and that $\mathcal{F} : \bar{U} \to C$ is given by $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, in which $\mathcal{F}_1 : \bar{U} \to E$ is continuous and completely continuous and $\mathcal{F}_2 : \bar{U} \to E$ is a nonlinear contraction (i.e., there exists a continuous nondecreasing function $\vartheta : [0, \infty) \to [0, \infty)$ satisfying $\vartheta(z) < z$ for $z > 0$, such that $\|\mathcal{F}_2(x) - \mathcal{F}_2(y)\| \leq \vartheta(\|x - y\|)$ for all $x, y \in \bar{U}$). Then, either*

(C1) *$\mathcal{F}$ has a fixed point $u \in \bar{U}$; or*
(C2) *there exist a point $u \in \partial U$ and $\kappa \in (0, 1)$ with $u = \kappa\mathcal{F}(u)$, where $\bar{U}$ and ∂U, respectively, represent the closure and boundary of U on C.*

In the sequel, we will use Lemma 2 by taking C to be E. For more details of such fixed point theorems, we refer the paper [26] by Petryshyn.

Let

$$\Omega_r = \{x \in C([1, e], \mathbb{R}) : \|x\| < r\}.$$

Theorem 2 *Let $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that (A_2) holds. In addition we assume that*

(A_4) $g(0) = 0$;
(A_5) *there exist a nonnegative function $p \in C([1, e], \mathbb{R})$ and a nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that*

$$|f(t, u)| \leq p(t)\psi(|u|) \quad \textit{for any} \quad (t, u) \in [1, e] \times \mathbb{R};$$

(A_6) $\sup_{r \in (0,\infty)} \frac{r}{p_0\psi(r)} > \frac{1}{1-\ell(\log\eta)^{1-\alpha}}$, *where*

$$p_0 = \frac{\|p\|}{\Gamma(\alpha+1)}(1 + \log\eta).$$

Then the boundary value problem (1) has at least one solution on $[1, e]$.

Proof Consider the operator $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ as that defined in (6). We decompose $\mathcal{Q}$ into the sum of two operators

$$(\mathcal{Q}x)(t) = (\mathcal{Q}_1 x)(t) + (\mathcal{Q}_2 x)(t), \quad t \in [1, e],$$

where

$$(\mathcal{Q}_1 x)(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s}ds, \quad t\in[1,e],$$

and

$$(\mathcal{Q}_2 x)(t) = \frac{(\log t)^{\alpha-1}}{(\log\eta)^{\alpha-1}}g(x), \quad t\in[1,e].$$

From (A_6), there exists a number $r_0 > 0$ such that

$$\frac{r_0}{p_0\psi(r_0)} > \frac{1}{1-\ell(\log\eta)^{\alpha-1}}. \tag{8}$$

We shall prove that the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy all the conditions in Lemma 2.

Step 1. The operator $\mathcal{Q}_2 : \bar{\Omega}_{r_0} \to C([1,e],\mathbb{R})$ is contractive. Indeed, we have

$$\begin{aligned} |(\mathcal{Q}_2 x)(t) - (\mathcal{Q}_2 y)(t)| &= \frac{(\log t)^{\alpha-1}}{(\log\eta)^{\alpha-1}}|g(x)-g(y)| \\ &\le \frac{\ell}{(\log\eta)^{\alpha-1}}\|x-y\| \end{aligned}$$

and hence by (A_2), $\mathcal{Q}_2$ is contractive.

Step 2. The operator $\mathcal{Q}_1$ is continuous and completely continuous. We first show that $\mathcal{Q}_1(\bar{\Omega}_{r_0})$ is bounded. For any $x\in\bar{\Omega}_{r_0}$, we have

$$\begin{aligned} \|\mathcal{Q}_1 x\| &\le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{|f(s,x(s))|}{s}ds \\ &\quad + \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1} \frac{|f(s,x(s))|}{s}ds \\ &\le \|p\|\psi(r_0)\left[\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{1}{s}ds\right] \\ &\le \|p\|\psi(r_0)\frac{1}{\Gamma(\alpha+1)}(1+\log\eta). \end{aligned}$$

This proves that $\mathcal{Q}_1(\bar{\Omega}_{r_0})$ is uniformly bounded.

In addition for any $\tau_1, \tau_2 \in [1, e]$, $\tau_1 < \tau_2$, we have

$$\begin{aligned}
&|(\mathcal{Q}_1 x)(\tau_2) - (\mathcal{Q}_1 x)(\tau_1)| \\
&\le \frac{\psi(r_0)\|p\|}{\Gamma(\alpha)} \left| \int_1^{\tau_1} \left(\log \frac{\tau_1}{s}\right)^{\alpha-1} \frac{1}{s} ds - \int_1^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{1}{s} ds \right| \\
&\quad + \frac{\psi(r_0)\|p\| |(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}|}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{1}{s} ds \\
&\le \frac{\psi(r_0)\|p\|}{\Gamma(\alpha)} \left| \int_1^{\tau_1} \left[\left(\log \frac{\tau_1}{s}\right)^{\alpha-1} - \left(\log \frac{\tau_2}{s}\right)^{\alpha-1}\right] \frac{1}{s} ds \right| \\
&\quad + \frac{\psi(r_0)\|p\|}{\Gamma(\alpha)} \left| \int_{\tau_1}^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{1}{s} ds \right| \\
&\quad + \frac{\psi(r_0)\|p\| |(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}|}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{1}{s} ds,
\end{aligned}$$

which is independent of x and tends to zero as $\tau_2 - \tau_1 \to 0$. Thus, $\mathcal{Q}_1$ is equicontinuous. Hence, by the Arzelá–Ascoli theorem, $\mathcal{Q}_1(\bar{\Omega}_{r_0})$ is a relatively compact set. Now, let $x_n \subset \bar{\Omega}_{r_0}$ with $\|x_n - x\| \to 0$. Then the limit $\|x_n(t) - x(t)\| \to 0$ uniformly valid on $[1, e]$. From the uniform continuity of $f(t, x)$ on the compact set $[1, e] \times [-r_0, r_0]$, it follows that $\|f(t, x_n(t)) - f(t, x(t))\| \to 0$ is uniformly valid on $[1, e]$. Hence, $\|\mathcal{Q}_1 x_n - \mathcal{Q}_1 x\| \to 0$ as $n \to \infty$ which proves the continuity of $\mathcal{Q}_1$. Hence, Step 1 is completely proved.

Step 3. The set $F(\bar{\Omega}_{r_0})$ is bounded. The conditions (A_2) and (A_4) imply that

$$\|\mathcal{Q}_2(x)\| \le \frac{1}{(\log \eta)^{\alpha-1}} \ell r_0$$

for any $x \in \bar{\Omega}_{r_0}$. This, with the boundedness of the set $\mathcal{Q}_1(\bar{\Omega}_{r_0})$, implies that the set $\mathcal{Q}(\bar{\Omega}_{r_0})$ is bounded.

Step 4. Finally, it is to show that the case (C2) in Lemma 2 does not occur. To this end, we suppose that (C2) holds. Then, we have that there exist $\lambda \in (0, 1)$ and $x \in \partial\Omega_{r_0}$ such that $x = \lambda \mathcal{Q} x$. So, we have $\|x\| = r_0$ and

$$\begin{aligned}
x(t) = \lambda \Bigg\{ &\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s, x(s))}{s} ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \\
&\times \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{f(s, x(s))}{s} ds + \frac{(\log t)^{\alpha-1}}{(\log \eta)^{\alpha-1}} g(x) \Bigg\}, \quad t \in [1, e].
\end{aligned}$$

With hypotheses (A_4)–(A_6), we have

$$\begin{aligned}
|x(t)| \le \|p\|\psi(\|x\|) &\left[\frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \right. \\
&\left. + \frac{1}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{1}{s} ds \right] + \frac{1}{(\log \eta)^{\alpha-1}} \ell \|x\|.
\end{aligned}$$

Taking supremum over $t \in [1, e]$, we get $r_0 \le p_0 \psi(r_0) + (\log \eta)^{1-\alpha} \ell r_0$. Thus,

$$\frac{r_0}{p_0 \psi(r_0)} \le \frac{1}{1 - \ell(\log \eta)^{1-\alpha}},$$

which contradicts (8). Thus, it follows that the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy all the conditions of Lemma 2. Hence, the operator $\mathcal{Q}$ has at least one fixed point $x \in \bar{\Omega}_{r_0}$,which is the solution of the boundary value problem (1). □

Example 2 Consider the following fractional boundary value problem

$$\begin{cases} D^{3/2}x(t) = \dfrac{1}{\sqrt{63+t^2}} 2^{1+|\sin x|}, \ 1 < t < e, \\ x(1) = 0, \quad x\left(\dfrac{5}{4}\right) = \dfrac{1}{7}x\left(\dfrac{3}{2}\right) + \dfrac{1}{9}x\,(2) + \dfrac{1}{11}x\left(\dfrac{5}{2}\right), \end{cases} \tag{9}$$

Since $\left|\frac{1}{\sqrt{63+t^2}} 2^{1+|\sin x|}\right| \le \frac{1}{2}$, we take $\|p\| = \frac{1}{2}$ and $\psi(|u|) = 1$. By the condition (A_6), we find that $r_0 > 1.70445$. Obviously, all the conditions of Theorem 2 are satisfied. Therefore, the conclusion of Theorem 2 applies to the problem (9).

4 Existence Results—The Multivalued Case

Here, we outline some basic definitions and results for multivalued maps [16, 18].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A multi-valued map $G : X \to \mathcal{P}(X)$

(i) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$;
(ii) is *bounded* on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e., $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$);
(iii) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$;
(iv) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$;
(v) is said to be *measurable* if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable;
(vi) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by Fix G.

For each $x \in C([1, e], \mathbb{R})$, define the set of selections of F by

$$S_{F,x} := \{v \in L^1([1, e], \mathbb{R}) : v(t) \in F(t, x(t)) \text{ for a.e. } t \in [1, e]\}.$$

Definition 3 A multivalued map $F : [1, e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t, x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \longmapsto F(t, x)$ is upper semicontinuous for almost all $t \in [1, e]$.

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([1, e], \mathbb{R}^+)$ such that

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \le \varphi_\alpha(t)$$

for all $\|x\| \le \alpha$ and for a.e. $t \in [1, e]$.

We define the graph of G to be the set $\mathrm{Gr}(G) = \{(x, y) \in X \times Y, y \in G(x)\}$ and recall two results for closed graphs and upper-semicontinuity.

Lemma 3 [16, Proposition 1.2] *If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $\mathrm{Gr}(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 4 [22] *Let X be a separable Banach space. Let $F : [1, e] \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory function. Then the operator*

$$\Theta \circ S_F : C([1, e], X) \to \mathcal{P}_{cp,c}(C([1, e], X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x,y})$$

is a closed graph operator in $C([1, e], X) \times C([1, e], X)$.

Definition 4 A function $x \in AC^2([1, e], \mathbb{R})$ is called a solution of problem (2) if there exists a function $f \in L^1([1, e], \mathbb{R})$ with $f(t) \in F(t, x(t))$ a.e. on $[1, e]$ such that $D^\alpha x(t) = f(t)$ a.e. on $[1, e]$ and $x(1) = 0$, $x(\eta) = g(x)$.

To prove our main result in this section, we will use the following form of the nonlinear alternative for contractive maps [27, Corollary 3.8].

Theorem 3 *Let X be a Banach space, and D a bounded neighborhood of $0 \in X$. Let $Z_1 : X \to \mathcal{P}_{cp,c}(X)$ and $Z_2 : \bar{D} \to \mathcal{P}_{cp,c}(X)$ be two multi-valued operators satisfying*

(a) *Z_1 is a contraction, and*
(b) *Z_2 is u.s.c and compact.*

Then, if $G = Z_1 + Z_2$, either

(i) *G has a fixed point in $\bar{D}$, or*
(ii) *there are a point $u \in \partial D$ and $\lambda \in (0, 1)$ with $u \in \lambda G(u)$.*

Theorem 4 *Assume that (A_2) holds. In addition, we suppose that*

(H_1) *$F : [1, e] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is a L^1-Carathéodory multivalued map;*
(H_2) *there exist a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $p \in C([1, e], \mathbb{R}^+)$ such that*

$$\|F(t, x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t, x)\} \le p(t)\psi(\|x\|) \quad \textit{for each } (t, x) \in [1, e]\times\mathbb{R};$$

(H_3) *there exists a number $M > 0$ such that*

$$\frac{(1 - \ell(\log \eta)^{1-\alpha})M}{\|p\|\psi(M)\frac{(1+\log \eta)}{\Gamma(\alpha+1)}} > 1. \tag{10}$$

Then the boundary value problem (2) has at least one solution on $[1, e]$.

Proof To transform the problem (2) into a fixed point problem, we define an operator $\mathcal{F}: C([1,e],\mathbb{R}) \longrightarrow \mathcal{P}(C([1,e],\mathbb{R}))$ as

$$\mathcal{F}(x) = \Bigg\{ h \in C([1,e],\mathbb{R}): \quad h(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds + \frac{(\log t)^{\alpha-1}}{(\log\eta)^{\alpha-1}}g(x), \quad t \in [1,e] \Bigg\}$$

for $f \in S_{F,x}$.

Next, we introduce two operators $\mathcal{A}: C([1,e],\mathbb{R}) \longrightarrow C([1,e],\mathbb{R})$ and $\mathcal{B}: C([1,e],\mathbb{R}) \longrightarrow \mathcal{P}(C([1,e],\mathbb{R}))$ as follows:

$$\mathcal{A}x(t) = \frac{(\log t)^{\alpha-1}}{(\log\eta)^{\alpha-1}}g(x), \quad t \in [1,e]$$

and

$$\mathcal{B}(x) = \left\{ \begin{array}{l} h \in C([1,e],\mathbb{R}): \\ h(t) = \left\{ \begin{array}{l} \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds \\ -\frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds, \quad t \in [1,e] \end{array}\right. \end{array}\right\}.$$

Observe that $\mathcal{F} = \mathcal{A} + \mathcal{B}$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 3 on $[1,e]$. First, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ define the multivalued operators $\mathcal{A},\mathcal{B}: B_r \to \mathcal{P}_{cp,c}(X)$, where $B_r = \{x \in X : \|x\|_X \le r\}$ is a bounded set in $C([1,e],\mathbb{R})$. First, we prove that $\mathcal{B}$ is compact-valued on B_r. Note that the operator $\mathcal{B}$ is equivalent to the composition $\mathcal{L}\circ S_F$, where $\mathcal{L}$ is the continuous linear operator on $L^1([1,e],\mathbb{R})$ into X, defined by

$$\mathcal{L}(v)(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds.$$

Suppose that $x \in B_r$ is arbitrary and let $\{v_n\}$ be a sequence in $S_{F,x}$. Then, by the definition of $S_{F,x}$, we have $v_n(t) \in F(t,x(t))$ for almost all $t \in [1,e]$. Since $F(t,x(t))$ is compact for all $t \in J$, there is a convergent subsequence of $\{v_n(t)\}$ (we denote it by $\{v_n(t)\}$ again) that converges in measure to some $v(t) \in S_{F,x}$ for almost all $t \in J$. On the other hand, $\mathcal{L}$ is continuous, so $\mathcal{L}(v_n)(t) \to \mathcal{L}(v)(t)$ point-wise on $[1,e]$.

In order to show that the convergence is uniform, we have to show that $\{\mathcal{L}(v_n)\}$ is an equi-continuous sequence. Let $\tau_1, \tau_2 \in [1, e]$ with $\tau_1 < \tau_2$. Then, we have

$$\begin{aligned}
&|\mathcal{L}(v_n)(\tau_2) - \mathcal{L}(v_n)(\tau_1)| \\
&\quad \le \frac{\psi(r)\|p\|}{\Gamma(\alpha)} \left| \int_1^{\tau_1} \left(\log \frac{\tau_1}{s}\right)^{\alpha-1} \frac{1}{s} ds - \int_1^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{1}{s} ds \right| \\
&\qquad + \frac{\psi(\rho)\|p\| |(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}|}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{1}{s} ds \\
&\quad \le \frac{\psi(\rho)\|p\|}{\Gamma(\alpha)} \left| \int_1^{\tau_1} \left[\left(\log \frac{\tau_1}{s}\right)^{\alpha-1} - \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \right] \frac{1}{s} ds \right| \\
&\qquad + \frac{\psi(\rho)\|p\|}{\Gamma(\alpha)} \left| \int_{\tau_1}^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{1}{s} ds \right| \\
&\qquad + \frac{\psi(r)\|p\| |(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}|}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{1}{s} ds.
\end{aligned}$$

We see that the right-hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$. Thus, the sequence $\{\mathcal{L}(v_n)\}$ is equi-continuous and by using the Arzelá–Ascoli theorem, we get that there is a uniformly convergent subsequence. So, there is a subsequence of $\{v_n\}$ (we denote it again by $\{v_n\}$) such that $\mathcal{L}(v_n) \to \mathcal{L}(v)$. Note that, $\mathcal{L}(v) \in \mathcal{L}(S_{F,x})$. Hence, $\mathcal{B}(x) = \mathcal{L}(S_{F,x})$ is compact for all $x \in B_r$. So $\mathcal{B}(x)$ is compact.

Now, we show that $\mathcal{B}(x)$ is convex for all $x \in X$. Let $z_1, z_2 \in \mathcal{B}(x)$. We select $f_1, f_2 \in S_{F,x}$ such that

$$\begin{aligned}
z_i(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f_i(s)}{s} ds \\
&\quad - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{f_i(s)}{s} ds, \quad i = 1, 2,
\end{aligned}$$

for almost all $t \in [1, e]$. Let $0 \le \lambda \le 1$. Then, we have

$$\begin{aligned}
[\lambda z_1 + (1-\lambda) z_2](t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{[\lambda f_1(s) + (1-\lambda) f_2(s)]}{s} ds \\
&\quad - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{[\lambda f_1(s) + (1-\lambda) f_2(s)]}{s} ds.
\end{aligned}$$

Since F has convex values, so $S_{F,u}$ is convex and $\lambda f_1(s) + (1-\lambda) f_2(s) \in S_{F,x}$. Thus,

$$\lambda z_1 + (1-\lambda) z_2 \in \mathcal{B}(x).$$

Consequently, $\mathcal{B}$ is convex-valued. Obviously, $\mathcal{A}$ is compact and convex-valued.

For the sake of clarity, we split the rest of the proof into a number of steps and claims.

Step 1. *$\mathcal{A}$ is a contraction on $C([1, e], \mathbb{R})$.* This is a consequence of (A_2).

Step 2. *$\mathcal{B}$ is compact and upper semicontinuous.* This will be established in several claims.

CLAIM 1 $\mathcal{B}$ maps bounded sets into bounded sets in $C([1,e],\mathbb{R})$. For that, let $B_\rho = \{x \in C([1,e],\mathbb{R}) : \|x\| \leq \rho\}$ be a bounded set in $C([1,e],\mathbb{R})$. Then, for each $h \in \mathcal{B}(x)$, $x \in B_\rho$, we have

$$\begin{aligned}|h(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|f(s)|}{s}ds + \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{|f(s)|}{s}ds \\ &\leq \|p\|\psi(\rho)\left[\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{1}{s}ds\right] \\ &\leq \|p\|\psi(\rho)\frac{1}{\Gamma(\alpha+1)}(1+\log\eta)\end{aligned}$$

and consequently, for each $h \in \mathcal{B}(B_\rho)$, we have

$$\|h\| \leq \|p\|\psi(r)\frac{1}{\Gamma(\alpha+1)}(1+\log\eta).$$

CLAIM 2 $\mathcal{B}$ maps bounded sets into equicontinuous sets. As before, let B_ρ be a bounded set and let $h \in \mathcal{B}(x)$ for $x \in B_\rho$. Let $\tau_1, \tau_2 \in [1,e]$ with $\tau_1 < \tau_2$ and $x \in B_\rho$. For each $h \in \mathcal{B}(x)$, we obtain

$$\begin{aligned}|h(\tau_2) - h(\tau_1)| &\leq \frac{\psi(r)\|p\|}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{1}{s}ds - \int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}ds\right| \\ &\quad + \frac{\psi(\rho)\|p\||(\log\tau_2)^{\alpha-1} - (\log\tau_1)^{\alpha-1}|}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{1}{s}ds \\ &\leq \frac{\psi(\rho)\|p\|}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left[\left(\log\frac{\tau_1}{s}\right)^{\alpha-1} - \left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\right]\frac{1}{s}ds\right| \\ &\quad + \frac{\psi(\rho)\|p\|}{\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}ds\right| \\ &\quad + \frac{\psi(r)\|p\||(\log\tau_2)^{\alpha-1} - (\log\tau_1)^{\alpha-1}|}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{1}{s}ds,\end{aligned}$$

which is independent of x and tends to zero as $\tau_2 - \tau_1 \to 0$. Therefore, it follows by the Arzelá–Ascoli theorem that $\mathcal{B} : C([1,e],\mathbb{R}) \to \mathcal{P}(C[1,e],\mathbb{R}))$ is completely continuous.

By Claims 1 and 2, $\mathcal{B}$ is completely continuous. By Lemma 3, $\mathcal{B}$ will be upper semicontinuous (since it is completely continuous) if we prove that it has a closed graph.

CLAIM 3 $\mathcal{B}$ has a closed graph. Let $x_n \to x_*$, $h_n \in \mathcal{B}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{B}(x_*)$. Associated with $h_n \in \mathcal{B}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [1,e]$,

$$h_n(t) = \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f_n(s)}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f_n(s)}{s}ds.$$

Then we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [1,e]$,

$$h_*(t) = \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f_*(s)}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f_*(s)}{s}ds.$$

Let us consider the continuous linear operator $\Theta : L^1([1,e],\mathbb{R}) \to C([1,e],\mathbb{R})$ given by

$$f \mapsto \Theta(f)(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds.$$

Observe that

$$\|h_n(t)-h_*(t)\| = \left\|\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{(f_n(s)-f_*(s))}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{(f_n(s)-f_*(s))}{s}ds\right\| \to 0$$

as $n\to\infty$. Thus, it follows by Lemma 4 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, therefore, we have

$$h_*(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f_*(s)}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f_*(s)}{s}ds$$

for some $f_*\in S_{F,x_*}$. Hence, $\mathcal{B}$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathcal{B}$ is compact and upper semicontinuous.

Thus, the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy hypotheses of Theorem 3 and therefore, by its application, it follows that either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x\in\lambda\mathcal{A}(x)+\lambda\mathcal{B}(x)$ for $\lambda\in(0,1)$, then there exists $f\in S_{F,x}$ such that

$$x(t) = \lambda\left\{\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds + \frac{(\log t)^{\alpha-1}}{(\log\eta)^{\alpha-1}}g(x)\right\}.$$

Consequently, we have

$$\begin{aligned}|x(t)| &\le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|f(s)|}{s}ds \\ &\quad + \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{|f(s)|}{s}ds + \frac{1}{(\log\eta)^{\alpha-1}}|g(x)| \\ &\le \|p\|\psi(\|x\|)\left[\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{1}{s}ds\right] \\ &\le \|p\|\psi(\|x\|)\frac{1}{\Gamma(\alpha+1)}(1+\log\eta) + \frac{1}{(\log\eta)^{\alpha-1}}\ell\|x\|.\end{aligned}$$

If condition (ii) of Theorem 3 holds, then there exist $\lambda\in(0,1)$ and $x\in\partial B_M$ with $x=\lambda\mathcal{F}(x)$. Then, x is a solution of (6) with $\|x\|=M$. Now, by the last inequality, we have

$$\frac{(1-\ell(\log\eta)^{1-\alpha})M}{\|p\|\psi(M)\frac{(1+\log\eta)}{\Gamma(\alpha+1)}} \le 1,$$

which contradicts (10). Hence, $\mathcal{F}$ has a fixed point in $[1, e]$ by Theorem 3, and consequently the problem (2) has a solution. This completes the proof. □

Example 3 Consider the following fractional boundary value problem

$$\begin{cases} D^{3/2}x(t) \in F(t,x), \quad 1 < t < e, \\ x(1) = 0, \quad x\left(\frac{5}{4}\right) = \frac{1}{7}x\left(\frac{3}{2}\right) + \frac{1}{9}x(2) + \frac{1}{11}x\left(\frac{5}{2}\right), \end{cases} \tag{11}$$

where

$$F(t,x) = \left[\frac{1}{(t+2)}\frac{|\sin x|^3}{8(|\sin x|^3+3)} + \frac{1}{10}, \frac{t}{3\sqrt{(t+1)}}\frac{|x|}{|x|+1}\right].$$

Clearly, $\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq p(t)\psi(\|x\|)$ for each $(t,x) \in [1,e]\times\mathbb{R}$ with $p(t) = \frac{t}{3\sqrt{(t+1)}}$, $\psi(\|x\|) = 1$. By the condition (H_3), it is found that $M > 1.601815$. Thus, all the conditions of Theorem 4 are satisfied and in consequence, there exists a solution for the problem (11) on $[1, e]$.

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