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Theorems of Toeplitz–Silverman Type by Applying a Reduction Method

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Abstract In the paper (Vietnam J. Math. 42: 153–157, 2014) (cf. also the earlier appeared and more general paper (Kathmandu Univ. J. Sci. Eng. Technol. 8: 89–92, 2012)), Ganie and Sheikh characterized matrices $A = (a_{nk}) \in (bv(p), Y)$ in terms of the matrix coefficients a_{nk}, where $Y \in \{ac_\infty, ac, ac_0\}$ (the space of sequences being almost bounded, almost convergent, and almost convergent to 0, respectively). In this publication, we pursue two aims: The first one is to give an example showing that none of the results in (Vietnam J. Math. 42: 153–157, 2014) (and thus none of the Theorems 2.3 and 2.4 and Corallary 2.5 in (Kathmandu Univ. J. Sci. Eng. Technol. 8, 89–92, 2012)) is correct in general and to correct and extend these results. The second one is to apply the reduction method presented in (J. Anal. 9: 149–181, 2001). In this way, we get easily the (corrected) results of Ganie and Sheikh (cf. Remark 7) and many other theorems of Toeplitz–Silverman type (cf. Sections 5 and 6) by reduction to known theorems of Toeplitz–Silverman type. So it is not necessary to prove all the results completely anew, as Ganie and Sheikh tried.

Keywords Sequence spaces of non-absolute type · Theorems of toeplitz–silverman type for matrix maps or SM-maps · Double sequence spaces · Almost convergence

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1 Introduction, Notation, and Preliminaries

Throughout this note, we assume familiarity with summability and the standard sequence spaces (see, e.g., [3, 20]). So we denote by ω, ℓ_∞, c, c_0, cs, ℓ, and bv the set of all sequences

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in $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), of all bounded sequences, of all convergent sequences, of all sequences converging to 0, of all convergent series, of all absolutely summable sequences, and of all sequences with bounded variation, respectively. A sequence space X is defined to be a linear subspace of ω. If X is any sequence space, then its *β-dual* X^β is defined by

$$X^\beta := \left\{ (y_k) \in \omega \;\middle|\; \forall (x_k) \in X \,:\, \sum_k y_k x_k \text{ converges} \right\}.$$

If $A = (a_{nk})$ is an infinite matrix with scalar entries, then we consider the *application domain*

$$\omega_A := \left\{ (x_k) \in \omega \;\middle|\; \sum_k a_{nk} x_k \text{ converges for each } n \in \mathbb{N} \right\}$$

of A. We have the following obvious result:

Proposition 1 (cf. [3, 2.3.2(e)]) *If $A = (a_{nk})$ is any matrix, then $X \subset \omega_A$ if and only if $(a_{nk})_k \in X^\beta$ for each $n \in \mathbb{N}$.*

For fixed sequence spaces X and Y and any infinite matrix $A = (a_{nk})$, we say that A maps X into Y, if $X \subset \omega_A$ and $Ax := (\sum_k a_{nk} x_k)_n \in Y$ for all $x \in X$. The set of all matrices A mapping X into Y is denoted by (X, Y).

Let $p = (p_k)$ be a sequence of positive reals throughout this paper. In the following considerations, we deal with the following sets of sequences:

$$\ell_\infty(p) := \left\{ x \in \omega \;\middle|\; \sup_k |x_k|^{p_k} < \infty \right\},$$

$$c_0(p) := \left\{ x \in \omega \;\middle|\; \lim_k |x_k|^{p_k} = 0 \right\},$$

$$c(p) := \left\{ x \in \omega \;\middle|\; \exists \alpha \in \mathbb{K} : x - \alpha e \in c_0(p) \right\}, \quad \text{where } e := (1, 1, \ldots),$$

$$\ell(p) := \left\{ x \in \omega \;\middle|\; \sum_{k=1}^\infty |x_k|^{p_k} < \infty \right\},$$

$$bv(p) := \left\{ x \in \omega \;\middle|\; \sum_{k=1}^\infty |x_k - x_{k-1}|^{p_k} < \infty \right\}, \quad \text{where } x_0 := 0,$$

$$M(p) := \left\{ x \in \omega \;\middle|\; \exists N \in \mathbb{N} \,:\, \sum_k |x_k|^{q_k} N^{-\frac{q_k}{p_k}} < \infty \right\}, \quad \text{where } \frac{1}{p_k} + \frac{1}{q_k} = 1,$$

$$M_\infty(p) := \bigcap_{N=2}^\infty \left\{ x \in \omega \;\middle|\; \sum_k |x_k| N^{\frac{1}{p_k}} < \infty \right\},$$

$$M_0(p) := \bigcup_{N=2}^\infty \left\{ x \in \omega \;\middle|\; \sum_k |x_k| N^{\frac{-1}{p_k}} < \infty \right\}.$$

Remark 1 Let $p = (p_k)$ with $p_k > 0$ be given.

(a) Each of the sets $\ell_\infty(p)$, $c_0(p)$, $c(p)$, $\ell(p)$, and $bv(p)$ is a linear space if and only if $p \in \ell_\infty$ (cf. [8, p. 487]).

(b) $\ell_\infty(p) = \ell_\infty$ if and only if (cf. [8, p. 487])[1]

$$0 < \liminf_k p_k \le \limsup_k p_k < \infty. \tag{1}$$

(c) $c_0(p) = c_0$ if and only if (1) holds (cf. [12, Lemma 1]).
(d) $c(p) = c$ if and only if (1) holds, which is an immediate consequence of statement (c) since $c = c_0 \oplus \langle e \rangle$.
(e) Let $0 < p_k \le 1$ and q_k the conjugate index of p_k. Then $\ell(p) = \ell$ if and only if there exists an $N \in \mathbb{N}$ with $\sum_k N^{q_k} < \infty$ (cf. [18, Theorem 3]).
(f) $\ell(p) = \ell \iff bv(p) = bv$. △

Aiming to characterize $X \subset \omega_A$ where $A = (a_{nk})$ is any infinite matrix and $X \in \{\ell_\infty(p), c_0(p), c(p), \ell(p), bv(p)\}$, by Proposition 1, it is sufficient to know the β-duals of the spaces $\ell_\infty(p)$, $c_0(p)$, $c(p)$, $\ell(p)$, and $bv(p)$.

If $y = (y_k) \in cs$, then we put $R_k := \sum_{j=k}^{\infty} y_j$ $(k \in \mathbb{N})$.

Proposition 2 *Let* $p = (p_k)$ *with* $p_k > 0$ $(k \in \mathbb{N})$ *be fixed.*

(a) $\ell_\infty(p)^\beta = \begin{cases} M_\infty(p) & \text{(cf. [9, Theorem 2])}, \\ \ell_\infty^\beta = \ell & \textit{if } 0 < \liminf_k p_k \le \limsup_k p_k < \infty \text{ (cf. Remark 1(b))}. \end{cases}$

(b) $c_0(p)^\beta = \begin{cases} M_0(p) & \text{(cf. [13, Theorem 6])}, \\ c_0^\beta = \ell & \textit{if } 0 < \liminf_k p_k \le \limsup_k p_k < \infty \text{ (cf. Remark 1(c))}. \end{cases}$

(c) $c(p)^\beta = \begin{cases} cs \cap M_0(p) & \text{(cf. [8, Theorem 1])}, \\ c^\beta = \ell & \textit{if } 0 < \liminf_k p_k \le \limsup_k p_k < \infty \text{ (cf. Remark 1(d))}. \end{cases}$

(d) $\ell(p)^\beta = \begin{cases} \ell_\infty(p) & \textit{if } p_k \le 1 \textit{ for all } k \in \mathbb{N} \text{ (cf. [18, Theorem 7])}, \\ M(p) & \textit{if } 1 < p_k \textit{ for all } k \in \mathbb{N} \text{ (cf. [13, Theorem 1])}. \end{cases}$

(e) $bv(p)^\beta = \begin{cases} \{y \in \omega \mid \sup_n |R_n|^{p_n} < \infty\} & \textit{if } p_k \le 1 \textit{ for all } k \in \mathbb{N}, \\ \{y \in \omega \mid \exists N \in \mathbb{N} \setminus \{1\} : \sum_k |R_k|^{q_k} N^{-q_k} < \infty \textit{ and} \\ \qquad \sup_n \sum_{k=1}^{n} |R_n|^{q_k} N^{-q_k} < \infty\} & \textit{if } 1 < p_k \ (k \in \mathbb{N}) \textit{ and } p \in \ell_\infty \end{cases}$
(cf. [7, Theorem 2.2 and Theorem 2.1, respectively]).

General assumption: In the following, *let* $p = (p_k)$ *be a bounded sequence with* $p_k > 0$ $(k \in \mathbb{N})$ *and let* q_k *be such that* $1 = \frac{1}{p_k} + \frac{1}{q_k}$ $(k \in \mathbb{N})$. *Moreover, a strictly increasing sequence* (n_k) *in* $\mathbb{N}$ *will be called an index sequence.*

2 Some Remarks on the Claims by Ganie and Sheikh in [5] (and [17])

In [10], Lorentz defined

$$ac := \left\{ x = (x_k) \in \omega \;\middle|\; \exists \alpha_x \in \mathbb{K} : \frac{1}{n} \sum_{j=1}^{n} x_{\nu+j-1} \longrightarrow \alpha_x \text{ uniformly in } \nu \in \mathbb{N} \right\},$$

[1] Note: $0 < \inf_k p_k \le \sup_k p_k < \infty \iff 0 < \liminf_k p_k \le \limsup_k p_k < \infty$ since all $p_k > 0$.

the set of all *almost convergent sequences*. Analogously, we consider the sets

$$ac_0 := \left\{ x = (x_k) \in \omega \;\middle|\; \frac{1}{n}\sum_{j=1}^{n} x_{\nu+j-1} \longrightarrow 0 \text{ uniformly in } \nu \in \mathbb{N} \right\},$$

$$ac_\infty := \left\{ x = (x_k) \in \omega \;\middle|\; \sup_{n,\nu} \left| \frac{1}{n}\sum_{j=1}^{n} x_{\nu+j-1} \right| < \infty \right\},$$

$$wac_\infty := \left\{ x = (x_k) \in \omega \;\middle|\; \exists N \in \mathbb{N} : \sup_{n \geq N, \nu \in \mathbb{N}} \left| \frac{1}{n}\sum_{j=1}^{n} x_{\nu+j-1} \right| < \infty \right\},$$

the set of all sequences being almost convergent to 0, the set of all almost bounded sequences, and the set of all weakly almost bounded sequences. The space ac_∞ (and more generally $ac_\infty(p)$ where $p = (p_k) \in \ell_\infty$ with $p_k > 0$ $(k \in \mathbb{N})$) has been defined and studied by Nanda in [15] (cf. also [14]).

Proposition 3 $ac_\infty = \ell_\infty = wac_\infty$.[2]

Proof The inclusions $\ell_\infty \subset ac_\infty \subset wac_\infty$ are obvious (cf. [15, Proposition 2]). For a proof of $wac_\infty \subset \ell_\infty$, let $x \in wac_\infty$ and $N \in \mathbb{N}$ with

$$\sup_{n \geq N, \nu \in \mathbb{N}} \frac{1}{n} \left| \sum_{k=\nu}^{\nu+n-1} x_k \right| =: M < \infty$$

be given. Then we have for each $\nu \in \mathbb{N}$ the inequalities

$$|x_\nu| = \left| \sum_{k=\nu}^{\nu+N} x_k - \sum_{k=\nu+1}^{\nu+N} x_k \right| = \left| \sum_{k=\nu}^{\nu+N} x_k - \sum_{k=\nu+1}^{\nu+1+N-1} x_k \right| \leq (2N+1)M,$$

thus $x \in \ell_\infty$. □

In [5], Ganie and Sheikh aimed to characterize matrices A to be a member of $(bv(p), ac_\infty)$, $(bv(p), ac_0)$, and $(bv(p), ac)$, respectively. As an example, we cite [5, Theorem 1]:

Claim (cf. [5, Theorem 1]) Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (bv(p), ac_\infty)$ if and only if there exists an integer $C > 1$ such that[3]

$$\sup_{n,m \in \mathbb{N}} \sum_k |a(n,k,m)|^{q_k} C^{-q_k} < \infty, \tag{2}$$

where

$$a(n,k,m) := \frac{1}{m} \sum_{j=1}^{m} a_{n+j-1,k} \quad (n,k,m \in \mathbb{N}).$$

[2] In his review, MR3218851 (Mathematical Reviews) on the paper [5] Faruk Özger mentioned already $ac_\infty = \ell_\infty$ without proof or some citation.

[3] In contrast to the notation in [5], we use during this claim and the following example $\mathbb{N}$ instead of $\mathbb{N}^0$ as index set for sequences and matrices.

The following example shows that Theorems 1, 2, and 3 in [5] are not correct in general. In particular, the more general Theorems 2.3 and 2.4 in [17] are also not correct in general.

Example 1 We consider the matrix $A = (a_{nk})$ defined by $a_{1k} = \frac{1}{k}$ and $a_{nk} = 0$ for $n > 1$ ($k \in \mathbb{N}$). Set also $p_k = 2$ ($k \in \mathbb{N}$), then $q_k = 2$ ($k \in \mathbb{N}$). First, we verify that the condition (2) (that is (2) in [5, Theorem 1]) holds. By the definition of A, we get

$$a(n,k,m) = \frac{1}{m}\sum_{j=1}^{m} a_{n+j-1,k} = 0 \quad \text{for } n > 1$$

and

$$a(1,k,m) = \frac{1}{m}\cdot\frac{1}{k} \quad (k, m \in \mathbb{N}).$$

Therefore, for any $C \in \mathbb{N}$, we obtain

$$\sup_{m,n\in\mathbb{N}} \sum_k |a(n,k,m)|^{q_k} C^{-q_k} = C^{-2} \sup_{m\in\mathbb{N}} \sum_k \left(\frac{1}{m}\cdot\frac{1}{k}\right)^2 = C^{-2}\sum_k \left(\frac{1}{k}\right)^2 < \infty.$$

So the condition (2) in [5, Theorem 2] holds. In view of

$$\lim_{m\to\infty} a(n,k,m) = \lim_{m\to\infty} 0 = 0 \quad (n, k \in \mathbb{N})$$

and

$$\lim_{m\to\infty} a(1,k,m) = \lim_{m\to\infty} \frac{1}{m}\cdot\frac{1}{k} = 0 \quad (k \in \mathbb{N})$$

the condition (3) in [5, Theorem 1] holds with $\beta_k = 0$ ($k \in \mathbb{N}$).

Now let us consider the sequence $x = (1, 1, \ldots)$. Since

$$\sum_k |x_k - x_{k-1}|^2 = 1 < \infty,$$

then $x \in bv(p)$. On the other hand $x \notin \omega_A$, since

$$[Ax]_1 = \sum_k a_{1k}x_k = \sum_k \frac{1}{k}\cdot 1 = \infty.$$

Therefore $A \notin (bv(p), ac_\infty)$, thus $A \notin (bv(p), ac)$ and $A \notin (bv(p), ac_0)$. △

Remark 2 The given example shows that $bv(p) \subset \omega_A$ does not hold in general under the conditions of [5, Theorems 1–3]. As we will state in Section 6, even under the additional assumption $bv(p) \subset \omega_A$, the condition (2) in Theorem 2 of [5] is not correct in general. Moreover, in Applications 1(e), we will state that the condition (2) in Theorem 1 of [5] characterizes the matrices $A \in (\ell(p), ac_\infty)$. Thus, because of $(\ell(p), ac_\infty) \subsetneq (bv(p), ac_\infty)$, the condition (2) in Theorem 1 of [5] is not sufficient for $A \in (bv(p), ac_\infty)$ with $bv(p) \subset \omega_A$. Note that both parts of the proof of [5, Theorem 1] contain wrong conclusions. △

In [5], Ganie and Sheikh tried to prove Theorems 1, 2, and 3 entirely anew, in particular, they do not use related results like those in [7]. In the following, we use a simple method, developed in [4], to reduce results corresponding to those in [5] to results in [7]; we call it the *reduction method*. Moreover, in this way, we are able to characterize matrices A to be a member of (X, ac_∞), (X, ac_0), and (X, ac) where X denotes one of the sequence spaces $\ell_\infty(p)$, $c_0(p)$, $c(p)$, $\ell(p)$, and $bv(p)$ for any given sequence $p = (p_k) \in \ell_\infty$ with $p_k > 0$ $(k \in \mathbb{N})$. *We will do it in three steps and—following the expositions in* [4]*—we do it in a very general as well as simple way.*

3 Characterization of $A \in (X, Y)$ for Special X and Y

For the application of the reduction method, we need a characterization of $A \in (X, Y)$ in terms of the coefficients of A where $X \in \{\ell_\infty(p), c_0(p), c(p), \ell(p), bv(p)\}$ and $Y \in \{\ell_\infty, c, c_0\}$. For that, we searched the bibliography for theorems of Toeplitz–Silverman type in the case of the spaces X and Y being in consideration.

Proposition 4 ($X = \ell_\infty(p)$ and $Y \in \{\ell_\infty, c_0, c\}$) *For any matrix $A = (a_{nk})$, the following statements hold:*

(a) $A \in (\ell_\infty(p), \ell_\infty) \iff \sup_n \sum_k |a_{nk}| N^{\frac{1}{p_k}} < \infty$ *for every* $N \in \mathbb{N}\backslash\{1\}$ (cf. [9, Theorem 3]).

(b) $$A \in (\ell_\infty(p), c) \iff \begin{cases} \text{(i) } \sum_k |a_{nk}| N^{\frac{1}{p_k}} < \infty & \textit{converges uniformly in } n \\ & \textit{for every } N \in \mathbb{N}\backslash\{1\}, \\ \text{(ii) } (a_{nk})_n \in c \quad (k \in \mathbb{N}) \end{cases}$$
(cf. [9, Corollary of Theorem 3]).

(c) $$A \in (\ell_\infty(p), c_0) \iff \begin{cases} \text{(i) } \sum_k |a_{nk}| N^{\frac{1}{p_k}} < \infty & \textit{converges uniformly in } n \\ & \textit{for every } N \in \mathbb{N}\backslash\{1\}, \\ \text{(ii) } (a_{nk})_n \in c_0 \ (k \in \mathbb{N}) \end{cases}$$
(immediate corollary of Part (b) and its proof).

Proposition 5 ($X = c_0(p)$ and $Y \in \{\ell_\infty, c_0, c\}$) *For any matrix $A = (a_{nk})$, the following statements hold:*

(a) $A \in (c_0(p), \ell_\infty) \iff \sup_n \sum_k |a_{nk}| M^{\frac{-1}{p_k}} < \infty$ *for some* $M \in \mathbb{N} \setminus \{1\}$ (cf. [11, Theorem 1 with $q := e$]).

(b) $$A \in (c_0(p), c) \iff \begin{cases} \text{(i) } \sup_n \sum_k |a_{nk}| M^{\frac{-1}{p_k}} < \infty \textit{ for some } M \in \mathbb{N} \setminus \{1\}, \\ \text{(ii) } (a_{nk})_n \in c \quad (k \in \mathbb{N}) \end{cases}$$
(cf. [8, Corollary 2 of Theorem 9]).

(c) $$A \in (c_0(p), c_0) \iff \begin{cases} \text{(i) } \sup_n \sum_k |a_{nk}| M^{\frac{-1}{p_k}} < \infty \textit{ for some } M \in \mathbb{N} \setminus \{1\}, \\ \text{(ii) } (a_{nk})_n \in c_0 \quad (k \in \mathbb{N}) \end{cases}$$
(immediate corollary of Part (b)).

Proposition 6 ($X = c(p)$ and $Y \in \{\ell_\infty, c_0, c\}$) *For any matrix $A = (a_{nk})$, the following statements hold:*

(a) $$A \in (c(p), \ell_\infty) \iff \begin{cases} \text{(i) } \sup_n \sum_k |a_{nk}| M^{\frac{-1}{p_k}} < \infty \quad \textit{for some } M > 1, \\ \text{(ii) } (a_{nk})_k \in cs \ \ (n \in \mathbb{N}) \quad \textit{and} \quad \sup_n \left|\sum_k a_{nk}\right| < \infty \end{cases}$$
(note $e \in c(p)$, cf. [11, Theorem 1 with $q := e$]*).*

(b) $A \in (c(p), c) \iff \begin{cases} \text{(i) } \sup_n \sum_k |a_{nk}| M^{\frac{-1}{p_k}} < \infty & \textit{for some } M \in \mathbb{N} \setminus \{1\}, \\ \text{(ii) } (a_{nk})_n \in c \quad (k \in \mathbb{N}), \\ \text{(iii) } (a_{nk})_k \in cs \ \ (n \in \mathbb{N}) \quad \textit{and} \quad \lim_n \sum_k a_{nk} \ \textit{exists} \end{cases}$

(cf. [8, Theorem 9]).

(c) $A \in (c(p), c_0) \iff \begin{cases} \text{(i) } \sup_n \sum_k |a_{nk}| M^{\frac{-1}{p_k}} < \infty & \textit{for some } M \in \mathbb{N} \setminus \{1\}, \\ \text{(ii) } (a_{nk})_n \in c_0 \quad (k \in \mathbb{N}), \\ \text{(iii) } (a_{nk})_k \in cs \ \ (n \in \mathbb{N}) \quad \textit{and} \quad \lim_n \sum_k a_{nk} = 0 \end{cases}$

(cf. Part (b)).

Remark 3 If $0 < \liminf_k p_k \leq \limsup_k p_k < \infty$, then $\ell_\infty(p) = \ell_\infty$, $c(p) = c$, and $c_0(p) = c_0$ by Remark 1. Thus, on account of the corresponding well-known Toeplitz–Silverman theorems, we may consider $p_k = q_k = 1$ and $N = M = 1$ in Propositions 4, 5, and 6.

Proposition 7 ($X = \ell(p)$ and $Y \in \{\ell_\infty, c_0, c\}$) *Let* $1 < p_k$ $(k \in \mathbb{N})$. *Then for any matrix* $A = (a_{nk})$, *the following statements hold:*

(a) $A \in (\ell(p), \ell_\infty) \iff \sup_n \sum_k |a_{nk}|^{q_k} M^{-q_k} < \infty$ *for some* $M \in \mathbb{N} \setminus \{1\}$
(cf. [9, Theorem 1]).

(b) $A \in (\ell(p), c) \iff \begin{cases} \text{(i) } \sup_n \sum_k |a_{nk}|^{q_k} M^{-q_k} < \infty & \textit{for some } M \in \mathbb{N} \setminus \{1\}, \\ \text{(ii) } (a_{nk})_n \in c \ (k \in \mathbb{N}) \end{cases}$

(cf. [9, Corollary 1]).

(c) $A \in (\ell(p), c_0) \iff \begin{cases} \text{(i) } \sup_n \sum_k |a_{nk}|^{q_k} M^{-q_k} < \infty & \textit{for some } M \in \mathbb{N} \setminus \{1\}, \\ \text{(ii) } (a_{nk})_n \in c_0 \ (k \in \mathbb{N}) \end{cases}$

(cf. [9, Corollary 1 with $\alpha_k = 0$]).

Proposition 8 ($X = \ell(p)$ and $Y \in \{\ell_\infty, c_0, c\}$) *Let* $0 < p_k \leq 1$ $(k \in \mathbb{N})$. *Then for any matrix* $A = (a_{nk})$, *the following statements hold:*

(a) $A \in (\ell(p), \ell_\infty) \iff \sup_{n,k} |a_{nk}|^{p_k} < \infty$ (cf. [9, Theorem 1]).

(b) $A \in (\ell(p), c) \iff \begin{cases} \text{(i) } \sup_{n,k} |a_{nk}|^{p_k} < \infty, \\ \text{(ii) } (a_{nk})_n \in c \ \ (k \in \mathbb{N}) \end{cases}$ (cf. [9, Corollary 1]).

(c) $A \in (\ell(p), c_0) \iff \begin{cases} \text{(i) } \sup_{n,k} |a_{nk}|^{p_k} < \infty, \\ \text{(ii) } (a_{nk})_n \in c_0 \ \ (k \in \mathbb{N}) \end{cases}$

(cf. [9, Corollary 1 with $\alpha_k = 0$]).

Proposition 9 ($X = bv(p)$ and $Y \in \{\ell_\infty, c_0, c\}$) *Let* $1 < p_k$ $(k \in \mathbb{N})$ *and* $A = (a_{nk})$ *be an infinite matrix with* $bv(p) \subset \omega_A$, *that is,*

$$\forall n \in \mathbb{N},\ \exists N_n \in \mathbb{N}\setminus\{1\} : \ \sum_r \left|\sum_{k=r}^{\infty} a_{nk}\right|^{q_r} N_n^{-q_r} < \infty \ \ \textit{and} \ \ \sup_r \sum_{j=1}^{r} \left|\sum_{k=r}^{\infty} a_{nk}\right|^{q_j} N_n^{-q_j} < \infty.$$

Then the following statements hold:

(a) $A \in (bv(p), \ell_\infty) \iff \exists M \in \mathbb{N} \setminus \{1\} : \ \sup_n \sum_k \left|\sum_{j=k}^{\infty} a_{nj}\right|^{q_k} M^{-q_k} < \infty$
(cf. [7, Theorem 3.2(2.)]).

(b) $A \in (bv(p), c_0) \iff \begin{cases} \textit{(i)}\ \exists M \in \mathbb{N} \setminus \{1\} :\ \sup_n \sum_k \left|\sum_{j=k}^{\infty} a_{nj}\right|^{q_k} M^{-q_k} < \infty, \\ \textit{(ii)}\ \forall k \in \mathbb{N} :\ \left(\sum_{j=k}^{\infty} a_{nj}\right)_n \in c_0 \end{cases}$

(cf. [7, Theorem 3.2(3.)]).

(c) $A \in (bv(p), c) \iff \begin{cases} \textit{(i)}\ \exists M \in \mathbb{N} \setminus \{1\} :\ \sup_n \sum_k \left|\sum_{j=k}^{\infty} a_{nj}\right|^{q_k} M^{-q_k} < \infty, \\ \textit{(ii)}\ \forall k \in \mathbb{N} : \left(\sum_{j=k}^{\infty} a_{nj}\right)_n \in c \end{cases}$

(cf. [7, Theorem 3.2(4.)]).

Proposition 10 ($X = bv(p)$ and $Y \in \{\ell_\infty, c_0, c\}$) *Let* $0 < p_k \leq 1$ $(k \in \mathbb{N})$. *Then for any matrix* $A = (a_{nk})$, *the following statements hold:*

(a) $A \in (bv(p), \ell_\infty) \iff \sup_{n,k} \left|\sum_{j=k}^{\infty} a_{nj}\right|^{p_k} < \infty$ (cf. [7, Theorem 3.2(2.)]).

(b) $A \in (bv(p), c_0) \iff \begin{cases} \textit{(i)}\ \sup_{n,k} \left|\sum_{j=k}^{\infty} a_{nj}\right|^{p_k} < \infty, \\ \textit{(ii)}\ \forall k \in \mathbb{N} :\ \left(\sum_{j=k}^{\infty} a_{nj}\right)_n \in c_0 \end{cases}$

(cf. [7, Theorem 3.2(3.)]).

(c) $A \in (bv(p), c) \iff \begin{cases} \textit{(i)}\ \sup_{n,k} \left|\sum_{j=k}^{\infty} a_{nj}\right|^{p_k} < \infty, \\ \textit{(ii)}\ \forall k \in \mathbb{N} :\ \left(\sum_{j=k}^{\infty} a_{nj}\right)_n \in c \end{cases}$

(cf. [7, Theorem 3.2(4.)]).

4 Theorems of Toeplitz–Silverman Type for SM-Maps

In the following, we deal with double sequence spaces with a, in some sense, 'uniform structure', in particular we consider the following double sequence spaces:

$$\Omega := \left\{x = (x_{n\nu}) \,\middle|\, \forall\, n, \nu \in \mathbb{N} :\ x_{n\nu} \in \mathbb{K}\right\} \quad \text{(set of all double sequences)},$$

$$\mathcal{M}_p := \left\{x = (x_{n\nu}) \in \Omega \,\middle|\, \exists\, N_x \in \mathbb{N} :\ \sup_{n,\nu \geq N_x} |x_{n\nu}| < \infty\right\}$$

(double sequences being bounded in the sense of Pringsheim),

$$\mathcal{M}_{au} := \left\{x = (x_{n\nu}) \in \Omega \,\middle|\, \exists\, N_x \in \mathbb{N} :\ \sup_{n \geq N_x;\ \nu \in \mathbb{N}} |x_{n\nu}| < \infty\right\}$$

(almost uniformly bounded double sequences),

$$\mathcal{M}_u := \left\{x = (x_{n\nu}) \in \Omega \,\middle|\, \sup_{n,\nu \in \mathbb{N}} |x_{n\nu}| < \infty\right\}$$

(uniformly bounded double sequences),

$$\mathcal{C} := \prod_{n \in \mathbb{N}} c = \{x = (x_{n\nu}) \in \Omega \mid \forall \nu \in \mathbb{N} :\ (x_{n\nu})_n \in c\},$$

$$\mathcal{C}^t := \{x = (x_{n\nu}) \in \Omega \mid (x_{\nu n}) \in \mathcal{C}\},$$

$$\mathcal{C}_p := \left\{x = (x_{n\nu}) \in \Omega \,\middle|\, \exists\, p_x \in \mathbb{K} : \lim_r \sup_{n,\nu \geq r} |x_{n\nu} - p_x| = 0\right\}$$

(double sequences being convergent in the sense of Pringsheim),

$$\mathcal{C}_{p0} := \left\{x = (x_{n\nu}) \in \Omega \,\middle|\, \lim_r \sup_{n,\nu \geq r} |x_{n\nu}| = 0\right\},$$

$$\mathcal{C}_r := \left\{x = (x_{n\nu}) \in \mathcal{C}_p \mid (x_{n\nu})_\nu \in c \ (n \in \mathbb{N}) \text{ and } (x_{n\nu})_n \in c \ (\nu \in \mathbb{N})\right\}$$
$$= \mathcal{C}_p \cap \mathcal{C} \cap \mathcal{C}^t$$

(double sequences being regularly convergent (in the sense of Hardy)),

$$\mathcal{C}_{r0} := \mathcal{C}_{p0} \cap \mathcal{C}_r,$$

$$\mathcal{C}_h := \left\{x = (x_{n\nu}) \in \Omega \,\middle|\, \exists\, h_x \in \mathbb{K} : \lim_r \sup_{\max\{n,\nu\} \geq r} |x_{n\nu} - h_x| = 0\right\},$$

(note, by mistake, in [4] this convergence was identified with Hardy convergence),

$$\mathcal{C}_{h0} := \left\{x = (x_{n\nu}) \in \Omega \,\middle|\, \lim_r \sup_{\max\{n,\nu\} \geq r} |x_{n\nu}| = 0\right\},$$

$$\mathcal{C}_{uc} := \left\{x = (x_{n\nu}) \in \Omega \,\middle|\, \exists\, u_x \in \mathbb{K} : \lim_r \sup_{n \geq r;\, \nu \in \mathbb{N}} |x_{n\nu} - u_x| = 0\right\}$$

(double sequences being uniformly convergent to a constant value),

$$\mathcal{C}_{u0} := \left\{x = (x_{n\nu}) \in \Omega \,\middle|\, \lim_r \sup_{n \geq r;\, \nu \in \mathbb{N}} |x_{n\nu}| = 0\right\}$$

(double sequences being uniformly convergent to 0),

$$\mathcal{F} := \mathcal{M}_u \cap \mathcal{C}_{uc},$$
$$\mathcal{F}_0 := \mathcal{M}_u \cap \mathcal{C}_{u0}.$$

As it was stated in [4, Proposition 3], the members of the double sequence spaces defined above may be easily characterized in terms of sequences:

Proposition 11 (cf. [4, Proposition 3]) *Let* $x = (x_{n\nu}) \in \Omega$ *be given. Then:*

(a) $x \in \mathcal{M}_p \iff \forall$ *index sequences* (ν_r) *and* (n_r) *in* $\mathbb{N}$: $(x_{n_r \nu_r}) \in \ell_\infty$.
(b) $x \in \mathcal{M}_{au} \iff \forall\, (\nu_r)$ *and index sequences* (n_r) *in* $\mathbb{N}$: $(x_{n_r \nu_r}) \in \ell_\infty$.
(c) $x \in \mathcal{M}_u \iff \forall\, (\nu_r)$ *and* (n_r) *in* $\mathbb{N}$: $(x_{n_r \nu_r}) \in \ell_\infty$.
(d) $x \in \mathcal{C}_p \iff \exists\, p_x \in \mathbb{K}$, $\forall$ *index sequences* (ν_r) *and* (n_r) *in* $\mathbb{N}$: $\lim_r x_{n_r \nu_r} = p_x$.
(e) $x \in \mathcal{C}_{p0} \iff \forall$ *index sequences* (ν_r) *and* (n_r) *in* $\mathbb{N}$: $(x_{n_r \nu_r}) \in c_0$.
(f) $x \in \mathcal{C}_h \iff \exists\, h_x \in \mathbb{K}$, $\forall\, (\nu_r)$ *and* (n_r) *in* $\mathbb{N}$ *with* $\max\{n_r, \nu_r\} \nearrow \infty$: $\lim_r x_{n_r \nu_r} = h_x$.
(g) $x \in \mathcal{C}_{h0} \iff \forall\, (\nu_r)$ *and* (n_r) *in* $\mathbb{N}$ *with* $\max\{n_r, \nu_r\} \nearrow \infty$: $(x_{n_r \nu_r}) \in c_0$.
(h) $x \in \mathcal{C}_{uc} \iff \exists\, u_x \in \mathbb{K}$, $\forall\, (\nu_r)$ *and index sequences* (n_r) *in* $\mathbb{N}$: $\lim_r x_{n_r \nu_r} = u_x$.
(i) $x \in \mathcal{C}_{u0} \iff \forall\, (\nu_r)$ *and index sequences* (n_r) *in* $\mathbb{N}$: $(x_{n_r \nu_r}) \in c_0$.

Now, we define maps defined by sequences of matrices (SM-maps).

Definition 1 (cf. [4]) Let $\mathcal{A} = (A^{(\nu)})$ be a sequence of infinite matrices $A^{(\nu)} = (a_{nk}^{(\nu)})$, X be a sequence space, and $\mathcal{Y}$ be a double sequence space.

$$\Omega_{\mathcal{A}} := \left\{ x = (x_k) \in \omega \,\middle|\, \forall\, n, \nu \in \mathbb{N}: \ \sum_k a_{nk}^{(\nu)} x_k \ \text{ converges} \right\} = \bigcap_\nu \omega_{A^{(\nu)}},$$

$$\mathcal{Y}_{\mathcal{A}} := \left\{ x \in \Omega_{\mathcal{A}} \,\middle|\, \mathcal{A}x := \left(A^{(\nu)}x\right)_\nu \in \mathcal{Y} \ \text{ with } \ A^{(\nu)}x := \left(\sum_k a_{nk}^{(\nu)} x_k\right) \right\}$$

(domain of $\mathcal{A}$ with respect to $\mathcal{Y}$).

Then the map

$$\mathcal{A} : \Omega_{\mathcal{A}} \longrightarrow \Omega, \quad x \longmapsto \mathcal{A}x$$

is well-defined and called an *SM-map*. We use the notation

$$\mathcal{A} \in (X, \mathcal{Y}) : \iff X \subset \mathcal{Y}_{\mathcal{A}} \text{ (that is, } \mathcal{A} \text{ maps } X \text{ into } \mathcal{Y}).$$

Example 2 Let $\mathcal{A}_\sigma := (A^{(\nu)})$ where $A^{(\nu)} = (a_{nk}^{(\nu)})$ is defined by

$$a_{nk}^{(\nu)} := \begin{cases} \frac{1}{n} & \text{if } k = \nu, \dots, \nu + n - 1, \\ 0 & \text{otherwise} \end{cases} \qquad (n, \nu, k \in \mathbb{N}). \tag{3}$$

Then $\ell_\infty = ac_\infty = \mathcal{M}_{u\mathcal{A}_\sigma} = \mathcal{M}_{au\mathcal{A}_\sigma} = wac_\infty$ (cf. Proposition 3), $ac = \mathcal{C}_{uc\mathcal{A}_\sigma}$, $ac_0 = \mathcal{C}_{u0\mathcal{A}_\sigma}$. △

The observations in Proposition 11 give us the tool to reduce theorems of Toeplitz–Silverman type for SM-maps to corresponding Toeplitz–Silverman theorems for matrix maps.

Corollary 1 (cf. Proposition 11, [4, Corollary 4]) *Let $\mathcal{A} = (A^{(\nu)})$ be a sequence of infinite matrices $A^{(\nu)} = (a_{nk}^{(\nu)})$ and let X be a sequence space with $X \subset \Omega_{\mathcal{A}}$. Then the following equivalences are valid:*

(a) $\mathcal{A} \in (X, \mathcal{M}_p) \iff \forall$ *index sequences* (ν_r) *and* (n_r) *in* $\mathbb{N}$: $(a_{n_r k}^{(\nu_r)}) \in (X, \ell_\infty)$.

(b) $\mathcal{A} \in (X, \mathcal{M}_{au}) \iff \forall\, (\nu_r)$ *and index sequences* (n_r) *in* $\mathbb{N}$: $(a_{n_r k}^{(\nu_r)}) \in (X, \ell_\infty)$.

(c) $\mathcal{A} \in (X, \mathcal{M}_u) \iff \forall\, (\nu_r)$ *and* (n_r) *in* $\mathbb{N}$: $(a_{n_r k}^{(\nu_r)}) \in (X, \ell_\infty)$.

(d) $\mathcal{A} \in (X, \mathcal{C}_p) \iff \forall$ *index sequences* (ν_r) *and* (n_r) *in* $\mathbb{N}$: $(a_{n_r k}^{(\nu_r)}) \in (X, c)$
and all these matrices are pairwise consistent[4] *on* X.

(e) $\mathcal{A} \in (X, \mathcal{C}_{p0}) \iff \forall$ *index sequences* (ν_r) *and* (n_r) *in* $\mathbb{N}$: $(a_{n_r k}^{(\nu_r)}) \in (X, c_0)$.

(f) $\mathcal{A} \in (X, \mathcal{C}_h) \iff \forall\, (\nu_r), (n_r)$ *in* $\mathbb{N}$ *with* $\max\{n_r, \nu_r\} \nearrow \infty$: $(a_{n_r k}^{(\nu_r)}) \in (X, c)$
and all these matrices are pairwise consistent on X.

(g) $\mathcal{A} \in (X, \mathcal{C}_{h0}) \iff \forall\, (\nu_r), (n_r)$ *in* $\mathbb{N}$ *with* $\max\{n_r, \nu_r\} \nearrow \infty$: $(a_{n_r k}^{(\nu_r)}) \in (X, c_0)$.

(h) $\mathcal{A} \in (X, \mathcal{C}_{uc}) \iff \forall\, (\nu_r)$ *and index sequences* (n_r) *in* $\mathbb{N}$: $(a_{n_r k}^{(\nu_r)}) \in (X, c)$
and all these matrices are pairwise consistent on X.

(i) $\mathcal{A} \in (X, \mathcal{C}_{u0}) \iff \forall\, (\nu_r)$ *and index sequences* (n_r) *in* $\mathbb{N}$: $(a_{n_r k}^{(\nu_r)}) \in (X, c_0)$.

[4] If $X \subset \omega$ and A, B are matrices with $X \subset c_A \cap c_B$, then A and B are called *consistent on* X, if $\lim_A x = \lim_B x$ for each $x \in X$.

Table 1 Summary of the results of Corollary 2

$X \setminus \mathcal{Y}$	$\mathcal{M}_p$	$\mathcal{M}_{au}$	$\mathcal{M}_u$	$\mathcal{C}_p$	$\mathcal{C}_{p0}$	$\mathcal{C}_{uc}$	$\mathcal{C}_{u0}$	$\mathcal{F}$	$\mathcal{F}_0$
$\ell_\infty(p)$	1)	8)	15)	22)	29)	36)	43)	50)	57)
$c_0(p)$	2)	9)	16)	23)	30)	37)	44)	51)	58)
$c(p)$	3)	10)	17)	24)	31)	38)	45)	52)	59)
$\ell(p), (0<p_k\leq 1)$	4)	11)	18)	25)	32)	39)	46)	53)	60)
$\ell(p), (1<p_k<\infty)$	5)	12)	19)	26)	33)	40)	47)	54)	61)
$bv(p), (0<p_k\leq 1)$	6)	13)	20)	27)	34)	41)	48)	55)	62)
$bv(p), (1<p_k<\infty)$	7)	14)	21)	28)	35)	42)	49)	56)	63)

Using the reduction method, the first author and Seydel characterized SM-maps $\mathcal{A} \in (X, \mathcal{Y})$ in terms of the matrix coefficients, where $X \in \{\varphi, \ell_\infty, \dots, bs\}$ and $\mathcal{Y} \in \{\mathcal{M}_p, \dots, \mathcal{F}_0\}$ (cf. the double sequence spaces listed in the headline of the following table). In the following, we complete these results for $X \in \{\ell_\infty(p), c_0(p), c(p), \ell(p), bv(p)\}$ and $\mathcal{Y} \in \{\mathcal{M}_p, \dots, \mathcal{F}_0\}$. Further, at the end of this section, we will consider in Remark 4 the cases of the double sequence spaces $\mathcal{C}, \mathcal{C}^t, \mathcal{C}_r, \mathcal{C}_{r0}, \mathcal{C}_h$, and $\mathcal{C}_{h0}$.

For example, $\mathcal{A} \in (c(p), \mathcal{C}_{uc})$ is characterized in part 38) of Corollary 2.

In all cases we assume $X \subset \Omega_{\mathcal{A}}$, or equivalently, $(a_{nk}^{(\nu)})_k \in X^\beta$ $(n, \nu \in \mathbb{N})$ which is a necessary condition for $X \subset \mathcal{Y}_{\mathcal{A}}$. Note that in almost all cases the characterizing conditions are already contained in those listed below.

To present clearly the characterizations 1)–63), we proceed as follows: We list all these characterizations in the following corollary, prove the second one and give for the remaining—completely analogous—proofs only some hints (in square brackets). Unusually, for the sake of clarity, we place directly the proof and the hints behind the respective characterization (Table 1).

Corollary 2 *Let* $p = (p_k) \in \ell_\infty$ *with* $p_k > 0$ $(k \in \mathbb{N})$.

1) $\boldsymbol{\mathcal{A} \in (\ell_\infty(p), \mathcal{M}_p)}$ $\iff$ (4), *where*

$$\forall N \in \mathbb{N}\backslash\{1\},\ \exists K_N \in \mathbb{N}: \quad \sup_{n,\nu \geq K_N} \sum_k \left| a_{nk}^{(\nu)} \right| N^{\frac{1}{p_k}} < \infty. \tag{4}$$

[Apply Corollary 1(a) and Proposition 4(a).]

2) $\boldsymbol{\mathcal{A} \in (c_0(p), \mathcal{M}_p)}$ $\iff$ (5), *where*

$$\exists M \in \mathbb{N} \setminus \{1\},\ \exists N \in \mathbb{N}: \quad \sup_{n,\nu \geq N} \sum_k \left| a_{nk}^{(\nu)} \right| M^{\frac{-1}{p_k}} < \infty. \tag{5}$$

Proof By Corollary 1(a) (cf. [4, Corollary 4]) and Proposition 5(a), the statement $\mathcal{A} \in (c_0(p), \mathcal{M}_p)$ is equivalent to

(∇) $\forall$ index sequences (ν_r), (n_r) in $\mathbb{N}$, $\exists M \in \mathbb{N} \setminus \{1\}$: $\sup_r \sum_k |a_{n_r k}^{(\nu_r)}| M^{\frac{-1}{p_k}} < \infty$.
Obviously, (5) implies (∇). For a proof of the converse implication, we assume that (∇) holds and that (5) fails, that is,

$$\forall M \in \mathbb{N} \setminus \{1\},\ \forall N \in \mathbb{N}: \quad \sup_{n,\nu \geq N} \sum_k \left| a_{nk}^{(\nu)} \right| M^{\frac{-1}{p_k}} = \infty.$$

By that, we can find index sequences (ν_r) and (n_r) such that

$$\sum_k \left|a_{n_r k}^{(\nu_r)}\right| r^{\frac{-1}{p_k}} > r \quad (r \in \mathbb{N}\backslash\{1\}).$$

By (∇), we may choose an $M \in \mathbb{N} \setminus \{1\}$ such that $\sup_r \sum_k |a_{n_r k}^{(\nu_r)}| M^{\frac{-1}{p_k}} < \infty$. Then

$$\infty = \sup_{r \geq M} \sum_k \left|a_{n_r k}^{(\nu_r)}\right| r^{\frac{-1}{p_k}} \leq \sup_{r \geq M} \sum_k \left|a_{n_r k}^{(\nu_r)}\right| M^{\frac{-1}{p_k}} < \infty,$$

which contradicts our assumption. □

3) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{c(p)}, \boldsymbol{\mathcal{M}_p}) \iff$ (5) *and* (6), *where*

$$\exists N_e \in \mathbb{N}: \quad \sup_{n,\nu \geq N_e} \left|\sum_k a_{nk}^{(\nu)}\right| < \infty. \tag{6}$$

[Apply Corollary 1(a) and Proposition 6(a).]

4) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell(p)}, \boldsymbol{\mathcal{M}_p}) \iff$ (7), *where*

$$\exists N \in \mathbb{N}: \quad \sup_{n,\nu \geq N,\, k \in \mathbb{N}} \left|a_{nk}^{(\nu)}\right|^{p_k} < \infty. \tag{7}$$

[Apply Corollary 1(a) and Proposition 8(a).]

5) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell(p)}, \boldsymbol{\mathcal{M}_p}) \iff$ (8), *where*

$$\exists M \in \mathbb{N} \setminus \{1\},\ \exists N \in \mathbb{N}: \quad \sup_{n,\nu \geq N} \sum_k \left|a_{nk}^{(\nu)}\right|^{q_k} M^{-q_k} < \infty. \tag{8}$$

[Apply Corollary 1(a) and Proposition 7(a).]

6) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{bv(p)}, \boldsymbol{\mathcal{M}_p}) \iff$ (9), *where*

$$\exists N \in \mathbb{N}: \quad \sup_{n,\nu \geq N,\, k \in \mathbb{N}} \left|\sum_{j=k}^{\infty} a_{nj}^{(\nu)}\right|^{p_k} < \infty. \tag{9}$$

[Apply Corollary 1(a) and Proposition 10(a).]

7) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{bv(p)}, \boldsymbol{\mathcal{M}_p}) \iff$ (10) *and* (11), *where*

$$\forall n, \nu \in \mathbb{N},\ \exists M_{n\nu} > 1: \quad \sup_m \sum_{k=1}^{m} \left|\sum_{j=m}^{\infty} a_{nj}^{(\nu)}\right|^{q_k} M_{n\nu}^{-q_k} < \infty, \tag{10}$$

$$\exists M \in \mathbb{N} \setminus \{1\},\ \exists N \in \mathbb{N}: \quad \sup_{n,\nu \geq N} \sum_k \left|\sum_{j=k}^{\infty} a_{nj}^{(\nu)}\right|^{q_k} M^{-q_k} < \infty. \tag{11}$$

[Apply Corollary 1(a) and Proposition 9(a).]

8) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell_\infty(p)}, \boldsymbol{\mathcal{M}_{au}}) \iff$ (12), *where*

$$\forall N \in \mathbb{N}\backslash\{1\},\ \exists K_N \in \mathbb{N}: \quad \sup_{n \geq K_N,\, \nu \in \mathbb{N}} \sum_k \left|a_{nk}^{(\nu)}\right| N^{\frac{1}{p_k}} < \infty. \tag{12}$$

[Apply Corollary 1(b) and Proposition 4(a).]

9) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{c_0(p)}, \boldsymbol{\mathcal{M}_{au}}) \iff$ (13), *where*

$$\exists M \in \mathbb{N} \setminus \{1\},\ \exists N \in \mathbb{N}: \quad \sup_{n \geq N,\, \nu \in \mathbb{N}} \sum_k \left|a_{nk}^{(\nu)}\right| M^{\frac{-1}{p_k}} < \infty. \tag{13}$$

[Apply Corollary 1(b) and Proposition 5(a).]

10) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{c}(\boldsymbol{p}), \boldsymbol{\mathcal{M}_{au}}) \iff$ (13) *and* (14), *where*

$$\exists N_e \in \mathbb{N} : \quad \sup_{n \geq N_e,\, \nu \in \mathbb{N}} \left| \sum_k a_{nk}^{(\nu)} \right| < \infty. \tag{14}$$

[Apply Corollary 1(b) and Proposition 6(a).]

11) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell}(\boldsymbol{p}), \boldsymbol{\mathcal{M}_{au}}) \iff$ (15), *where*

$$\exists N \in \mathbb{N} : \quad \sup_{n \geq N;\, k, \nu \in \mathbb{N}} \left| a_{nk}^{(\nu)} \right|^{p_k} < \infty. \tag{15}$$

[Apply Corollary 1(b) and Proposition 8(a).]

12) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell}(\boldsymbol{p}), \boldsymbol{\mathcal{M}_{au}}) \iff$ (16), *where*

$$\exists M \in \mathbb{N} \setminus \{1\},\ \exists N \in \mathbb{N} : \quad \sup_{n \geq N,\, \nu \in \mathbb{N}} \sum_k \left| a_{nk}^{(\nu)} \right|^{q_k} M^{-q_k} < \infty. \tag{16}$$

[Apply Corollary 1(b) and Proposition 7(a).]

13) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{bv}(\boldsymbol{p}), \boldsymbol{\mathcal{M}_{au}}) \iff$ (17), *where*

$$\exists N \in \mathbb{N} : \quad \sup_{n \geq N;\, k, \nu \in \mathbb{N}} \left| \sum_{j=k}^{\infty} a_{nj}^{(\nu)} \right|^{p_k} < \infty. \tag{17}$$

[Apply Corollary 1(b) and Proposition 10(a).]

14) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{bv}(\boldsymbol{p}), \boldsymbol{\mathcal{M}_{au}}) \iff$ (10) *and* (18), *where*

$$\exists M \in \mathbb{N} \setminus \{1\},\ \exists N \in \mathbb{N} : \quad \sup_{n \geq N,\, \nu \in \mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} a_{nj}^{(\nu)} \right|^{q_k} M^{-q_k} < \infty. \tag{18}$$

[Apply Corollary 1(b) and Proposition 9(a).]

15) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell_\infty}(\boldsymbol{p}), \boldsymbol{\mathcal{M}_{u}}) \iff$ (19), *where*

$$\forall N \in \mathbb{N} \backslash \{1\} : \quad \sup_{n, \nu \in \mathbb{N}} \sum_k \left| a_{nk}^{(\nu)} \right| N^{\frac{1}{p_k}} < \infty. \tag{19}$$

[Apply Corollary 1(c) and Proposition 4(a).]

16) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{c_0}(\boldsymbol{p}), \boldsymbol{\mathcal{M}_{u}}) \iff$ (20), *where*

$$\exists M \in \mathbb{N} \setminus \{1\} : \quad \sup_{n, \nu \in \mathbb{N}} \sum_k \left| a_{nk}^{(\nu)} \right| M^{\frac{-1}{p_k}} < \infty. \tag{20}$$

[Apply Corollary 1(c) and Proposition 5(a).]

17) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{c}(\boldsymbol{p}), \boldsymbol{\mathcal{M}_{u}}) \iff$ (20) *and* (21), *where*

$$\sup_{n, \nu \in \mathbb{N}} \left| \sum_k a_{nk}^{(\nu)} \right| < \infty. \tag{21}$$

[Apply Corollary 1(c) and Proposition 6(a).]

18) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell}(\boldsymbol{p}), \boldsymbol{\mathcal{M}_{u}}) \iff$ (22), *where*

$$\sup_{k, n, \nu \in \mathbb{N}} \left| a_{nk}^{(\nu)} \right|^{p_k} < \infty. \tag{22}$$

[Apply Corollary 1(c) and Proposition 8(a).]

19) *Let* $1 < p_k$. *Then* $\mathcal{A} \in (\ell(p), \mathcal{M}_u) \iff$ (23), *where*

$$\exists M \in \mathbb{N} \setminus \{1\}: \quad \sup_{n,\nu \in \mathbb{N}} \sum_k \left|a_{nk}^{(\nu)}\right|^{q_k} M^{-q_k} < \infty. \tag{23}$$

[Apply Corollary 1(c) and Proposition 7(a).]

20) *Let* $0 < p_k \leq 1$. *Then* $\mathcal{A} \in (bv(p), \mathcal{M}_u) \iff$ (24), *where*

$$\sup_{n,k,\nu \in \mathbb{N}} \left|\sum_{j=k}^{\infty} a_{nj}^{(\nu)}\right|^{p_k} < \infty. \tag{24}$$

[Apply Corollary 1(c) and Proposition 10(a).]

21) *Let* $1 < p_k$. *Then* $\mathcal{A} \in (bv(p), \mathcal{M}_u) \iff$ (10) *and* (25), *where*

$$\exists M \in \mathbb{N} \setminus \{1\}: \quad \sup_{n,\nu \in \mathbb{N}} \sum_k \left|\sum_{j=k}^{\infty} a_{nj}^{(\nu)}\right|^{q_k} M^{-q_k} < \infty. \tag{25}$$

[Apply Corollary 1(c) and Proposition 9(a).]

22) $\mathcal{A} \in (\ell_\infty(p), \mathcal{C}_p) \iff$ (26) *and* (27), *where*

$$\forall N \in \mathbb{N}\setminus\{1\}, \exists K_N \in \mathbb{N}: \quad \sum_k \left|a_{nk}^{(\nu)}\right| N^{\frac{1}{p_k}} \text{ converges uniformly in } n, \nu \geq K_N, \tag{26}$$

$$\forall k \in \mathbb{N}, \exists \alpha_k \in \mathbb{K}: \quad \lim_r \sup_{n,\nu \geq r} \left|a_{nk}^{(\nu)} - \alpha_k\right| = 0. \tag{27}$$

[Apply Corollary 1(d) and Proposition 4(b).]

23) $\mathcal{A} \in (c_0(p), \mathcal{C}_p) \iff$ (5) *and* (27).
[Apply Corollary 1(d) and Proposition 5(b).]

24) $\mathcal{A} \in (c(p), \mathcal{C}_p) \iff$ (5), (27) *and* (28), *where*

$$\exists \alpha \in \mathbb{K}: \quad \lim_r \sup_{n,\nu \geq r} \left|\sum_k a_{nk}^{(\nu)} - \alpha\right| = 0. \tag{28}$$

[Apply Corollary 1(d) and Proposition 6(b).]

25) *Let* $0 < p_k \leq 1$. *Then* $\mathcal{A} \in (\ell(p), \mathcal{C}_p) \iff$ (7) *and* (27).
[Apply Corollary 1(d) and Proposition 8(b).]

26) *Let* $1 < p_k$. *Then* $\mathcal{A} \in (\ell(p), \mathcal{C}_p) \iff$ (8) *and* (27).
[Apply Corollary 1(d) and Proposition 7(b).]

27) *Let* $0 < p_k \leq 1$. *Then* $\mathcal{A} \in (bv(p), \mathcal{C}_p) \iff$ (9) *and* (29), *where*

$$\forall k \in \mathbb{N}, \ \exists A_k \in \mathbb{K}: \quad \lim_r \sup_{n,\nu \geq r} \left|\sum_{j=k}^{\infty} a_{nj}^{(\nu)} - A_k\right| = 0. \tag{29}$$

[Apply Corollary 1(d) and Proposition 10(b).]

28) *Let* $1 < p_k$. *Then* $\mathcal{A} \in (bv(p), \mathcal{C}_p) \iff$ (10), (11), *and* (29).
[Apply Corollary 1(d) and Proposition 9(b).]

29) $\mathcal{A} \in (\ell_\infty(p), \mathcal{C}_{p0}) \iff$ (26) *and* (30), *where*

$$\forall k \in \mathbb{N}: \quad \lim_r \sup_{n,\nu \geq r} \left|a_{nk}^{(\nu)}\right| = 0. \tag{30}$$

[Apply Corollary 1(e) and Proposition 4(c).]

30) $\boldsymbol{\mathcal{A} \in (c_0(p), \mathcal{C}_{p0})} \iff$ (5) *and* (30).
[Apply Corollary 1(e) and Proposition 5(c).]

31) $\boldsymbol{\mathcal{A} \in (c(p), \mathcal{C}_{p0})} \iff$ (5), (30), *and* (31), *where*

$$\lim_r \sup_{n,\nu \geq r} \left| \sum_k a_{nk}^{(\nu)} \right| = 0. \tag{31}$$

[Apply Corollary 1(e) and Proposition 6(c).]

32) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A} \in (\ell(p), \mathcal{C}_{p0})} \iff$ (7) *and* (30).
[Apply Corollary 1(e) and Proposition 8(c).]

33) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A} \in (\ell(p), \mathcal{C}_{p0})} \iff$ (8) *and* (30).
[Apply Corollary 1(e) and Proposition 7(c).]

34) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A} \in (bv(p), \mathcal{C}_{p0})} \iff$ (9) *and* (32), *where*

$$\forall k \in \mathbb{N}: \quad \lim_r \sup_{n,\nu \geq r} \left| \sum_{j=k}^{\infty} a_{nj}^{(\nu)} \right| = 0. \tag{32}$$

[Apply Corollary 1(e) and Proposition 10(c).]

35) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A} \in (bv(p), \mathcal{C}_{p0})} \iff$ (10), (11), *and* (32).
[Apply Corollary 1(e) and Proposition 9(c).]

36) $\boldsymbol{\mathcal{A} \in (\ell_\infty(p), \mathcal{C}_{uc})} \iff$ (33) *and* (34), *where*

$$\forall N \in \mathbb{N}\backslash\{1\}, \exists K_N \in \mathbb{N}: \quad \sum_k \left| a_{nk}^{(\nu)} \right| N^{\frac{1}{p_k}} \tag{33}$$

converges uniformly in $n \geq K_N$, $\nu \in \mathbb{N}$,

$$\forall k \in \mathbb{N}, \exists \alpha_k \in \mathbb{K}: \quad \lim_r \sup_{n \geq r, \nu \in \mathbb{N}} \left| a_{nk}^{(\nu)} - \alpha_k \right| = 0. \tag{34}$$

[Apply Corollary 1(h) and Proposition 4(b).]

37) $\boldsymbol{\mathcal{A} \in (c_0(p), \mathcal{C}_{uc})} \iff$ (13) *and* (34).
[Apply Corollary 1(h) and Proposition 5(b).]

38) $\boldsymbol{\mathcal{A} \in (c(p), \mathcal{C}_{uc})} \iff$ (13), (34), *and* (35), *where*

$$\exists \alpha \in \mathbb{K}: \quad \lim_r \sup_{n \geq r, \nu \in \mathbb{N}} \left| \sum_k a_{nk}^{(\nu)} - \alpha \right| = 0. \tag{35}$$

[Apply Corollary 1(h) and Proposition 6(b).]

39) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A} \in (\ell(p), \mathcal{C}_{uc})} \iff$ (15) *and* (34).
[Apply Corollary 1(h) and Proposition 8(b).]

40) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A} \in (\ell(p), \mathcal{C}_{uc})} \iff$ (16) *and* (34).
[Apply Corollary 1(h) and Proposition 7(b).]

41) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A} \in (bv(p), \mathcal{C}_{uc})} \iff$ (17) *and* (36), *where*

$$\forall k \in \mathbb{N}, \ \exists A_k \in \mathbb{K}: \quad \lim_r \sup_{n \geq r, \nu \in \mathbb{N}} \left| \sum_{j=k}^{\infty} a_{nj}^{(\nu)} - A_k \right| = 0. \tag{36}$$

[Apply Corollary 1(h) and Proposition 10(b).]

42) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A} \in (bv(p), \mathcal{C}_{uc})} \iff$ (10), (18), *and* (36).
[Apply Corollary 1(h) and Proposition 9(b).]

43) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell_\infty(p)}, \boldsymbol{\mathcal{C}_{u0}}) \iff$ (33) *and* (37), *where*

$$\forall k \in \mathbb{N}: \quad \lim_r \sup_{n \geq r, \nu \in \mathbb{N}} \left| a_{nk}^{(\nu)} \right| = 0. \tag{37}$$

[Apply Corollary 1(i) and Proposition 4(c).]

44) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{c_0(p)}, \boldsymbol{\mathcal{C}_{u0}}) \iff$ (13) *and* (37).
[Apply Corollary 1(i) and Proposition 5(c).]

45) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{c(p)}, \boldsymbol{\mathcal{C}_{u0}}) \iff$ (13), (37), *and* (38), *where*

$$\lim_r \sup_{n \geq r, \nu \in \mathbb{N}} \left| \sum_k a_{nk}^{(\nu)} \right| = 0. \tag{38}$$

[Apply Corollary 1(i) and Proposition 6(c).]

46) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell(p)}, \boldsymbol{\mathcal{C}_{u0}}) \iff$ (15) *and* (37).
[Apply Corollary 1(i) and Proposition 8(c).]

47) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell(p)}, \boldsymbol{\mathcal{C}_{u0}}) \iff$ (16) *and* (37).
[Apply Corollary 1(i) and Proposition 7(c).]

48) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{bv(p)}, \boldsymbol{\mathcal{C}_{u0}}) \iff$ (17) *and* (39), *where*

$$\forall k \in \mathbb{N}: \quad \lim_r \sup_{n \geq r, \nu \in \mathbb{N}} \left| \sum_{j=k}^{\infty} a_{nj}^{(\nu)} \right| = 0. \tag{39}$$

[Apply Corollary 1(i) and Proposition 10(c).]

49) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{bv(p)}, \boldsymbol{\mathcal{C}_{u0}}) \iff$ (10), (18), *and* (39).
[Apply Corollary 1(i) and Proposition 9(c).]

50) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell_\infty(p)}, \boldsymbol{\mathcal{F}}) \iff$ (19), (33), *and* (34). [Cf. 15) and 36).]

51) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{c_0(p)}, \boldsymbol{\mathcal{F}}) \iff$ (20) *and* (34). [Cf. 16) and 37).]

52) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{c(p)}, \boldsymbol{\mathcal{F}}) \iff$ (20), (34), *and* (35). [Cf. 17) and 38).]

53) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell(p)}, \boldsymbol{\mathcal{F}}) \iff$ (22) *and* (34). [Cf. 18) and 39).]

54) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell(p)}, \boldsymbol{\mathcal{F}}) \iff$ (23) *and* (34). [Cf. 19) and 40).]

55) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{bv(p)}, \boldsymbol{\mathcal{F}}) \iff$ (24) *and* (36). [Cf. 20) and 41).]

56) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{bv(p)}, \boldsymbol{\mathcal{F}}) \iff$ (10), (25), *and* (36). [Cf. 21) and 42).]

57) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell_\infty(p)}, \boldsymbol{\mathcal{F}_0}) \iff$ (19), (33), *and* (37). [Cf. 15) and 43).]

58) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{c_0(p)}, \boldsymbol{\mathcal{F}_0}) \iff$ (20) *and* (37). [Cf. 16) and 44).]

59) $\boldsymbol{\mathcal{A}} \in (\boldsymbol{c(p)}, \boldsymbol{\mathcal{F}_0}) \iff$ (20), (37), *and* (38). [Cf. 17) and 45).]

60) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell(p)}, \boldsymbol{\mathcal{F}_0}) \iff$ (22) *and* (37). [Cf. 18) and 46).]

61) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{\ell(p)}, \boldsymbol{\mathcal{F}_0}) \iff$ (23) *and* (37). [Cf. 19) and 47).]

62) *Let* $0 < p_k \leq 1$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{bv(p)}, \boldsymbol{\mathcal{F}_0}) \iff$ (24) *and* (39). [Cf. 20) and 48).]

63) *Let* $1 < p_k$. *Then* $\boldsymbol{\mathcal{A}} \in (\boldsymbol{bv(p)}, \boldsymbol{\mathcal{F}_0}) \iff$ (10), (25), *and* (39). [Cf. 21) and 49).]

Remark 4 (a) Let $\mathcal{A} = (A^{(\nu)})$ be a sequence of infinite matrices $A^{(\nu)}$ and let X and $Y^{(\nu)}$, $\nu \in \mathbb{N}$, be arbitrary sequence spaces. If $\mathcal{Y}$ is the cartesian product of $Y^{(\nu)}$ ($\nu \in \mathbb{N}$), that is $\mathcal{Y} := \prod_\nu Y^{(\nu)}$, then obviously

$$\mathcal{A} \in (X, \mathcal{Y}) \quad \iff \quad \forall \nu \in \mathbb{N}: \ A^{(\nu)} \in (X, Y^{(\nu)}).$$

For instance, we may apply this equivalence to sequence spaces X and $Y^{(\nu)}$ under consideration in [6, 11, 16, 19]. In particular, we may consider the space $\mathcal{C}$ of all double sequences with all columns being convergent. Obviously, $\mathcal{C} = \prod_\nu Y^{(\nu)}$ with $Y^{(\nu)} := c$ ($\nu \in \mathbb{N}$) is a simple application example. Considering $\mathcal{Y} = \mathcal{C}^t$ for $\mathcal{A} = (A^{(\nu)})$ with $A^{(\nu)} = (a^{\nu}_{nk})$, we get obviously

$$\mathcal{A} \in (X, \mathcal{C}^t) \quad \iff \quad \forall n \in \mathbb{N} : \ B^{(n)} \in (X, c) \text{ with } B^{(n)} := (a^{(\nu)}_{nk})_{\nu,k}, \ n \in \mathbb{N}.$$

(b) Let $\mathcal{A} = (A^{(\nu)})$ be a sequence of infinite matrices $A^{(\nu)} = (a^{(\nu)}_{nk})$ and let X be an arbitrary sequence space. If $\mathcal{Y} = \bigcap_{i\in I} \mathcal{Y}_i$, where $\{\mathcal{Y}_i \mid i \in I\}$ is a family of double sequence spaces, then we obviously have

$$(X, \mathcal{Y}) = \bigcap_{i\in I}(X, \mathcal{Y}_i), \quad \text{thus } \mathcal{A} \in (X, \mathcal{Y}) \iff \forall i \in I : \ \mathcal{A} \in (X, \mathcal{Y}_i).$$

For instance, we may apply this observation to the double sequence space $\mathcal{C}_r$ of all regularly convergent double sequences where $\mathcal{C}_r = \mathcal{C}_p \cap \mathcal{C} \cap \mathcal{C}^t$.

Now, if $\mathcal{A} = (A^{(\nu)})$ is a sequence of infinite matrices $A^{(\nu)} = (a^{(\nu)}_{nk})$, then

$$\mathcal{A} \in (X, \mathcal{C}_r) \iff \mathcal{A} \in (X, \mathcal{C}_p) \wedge \mathcal{A} \in (X, \mathcal{C}) \wedge \mathcal{A} \in (X, \mathcal{C}^t).$$

Here, we should note that $\mathcal{A} \in (X, \mathcal{C}^t)$ if and only if $\mathcal{B} \in (X, \mathcal{C})$ where $\mathcal{B} = (B^{(n)})$ is the sequence of the matrices $B^{(n)} := (a^{(\nu)}_{nk})_{\nu k}$, $n \in \mathbb{N}$. Thus, in both cases, we may apply Part (a) of these remarks.

(c) Quite similarly, we may handle the cases $\mathcal{A} \in (X, \mathcal{C}_h)$ and $\mathcal{A} \in (X, \mathcal{C}_{h0})$ since $\mathcal{C}_{h0} = \mathcal{C}_{u0} \cap \mathcal{C}^t_{u0}$ and $\mathcal{C}_h = \langle \widehat{\mathbf{e}} \rangle \oplus \mathcal{C}_{h0}$ where $\mathcal{C}^t_{u0} := \{x = (x_{n\nu}) \in \Omega \mid (x_{\nu n}) \in \mathcal{C}_{u0}\}$ and $\widehat{\mathbf{e}}$ denotes the double sequence with '1' in each position.

5 Characterization of $A \in (X, \mathcal{Y}_{\mathcal{A}})$ where $\mathcal{Y} \in \{\mathcal{M}_{au}, \mathcal{M}_u, \mathcal{C}_{uc}\}$

We start—similarly as in the previous sections—with some more general considerations: We aim to characterize $A \in (X, \mathcal{Y}_{\mathcal{A}})$ where A is any matrix, $\mathcal{A}$ is a sequence of row finite matrices $A^{(\nu)} = (a^{(\nu)}_{nk})$, $X \in \{\ell_\infty(p), c_0(p), c(p), \ell(p), bv(p)\}$, and $\mathcal{Y}$ is at first any double sequence space and then, specially, $\mathcal{M}_{au}$, $\mathcal{M}_u$, and $\mathcal{C}_{uc}$, respectively. Under these assumptions, we have

$$A^{(\nu)}(Ax) = \left(A^{(\nu)}A\right)x \qquad (x \in \omega_A, \nu \in \mathbb{N}). \tag{40}$$

Now, as an easy consequence of (40), we get

Proposition 12 (cf. [4, Proposition 7]) *Let X be a sequence space, $\mathcal{Y}$ be a double sequence space, and $A^{(\nu)}$ ($\nu \in \mathbb{N}$) be row-finite. Then for each matrix A with $X \subset \omega_A$, we have $X \subset \Omega_{\mathcal{A}}$ and*

$$A \in (X, \mathcal{Y}_{\mathcal{A}}) \quad \iff \quad \mathcal{A}A \in (X, \mathcal{Y}),$$

where $\mathcal{A}A := (A^{(\nu)}A)$.

For further considerations, we apply Proposition 12 to the special case $X \in \{\ell_\infty(p), c_0(p), c(p), \ell(p), bv(p)\}$ and, as three examples, $\mathcal{Y} \in \{\mathcal{M}_{au}, \mathcal{M}_u, \mathcal{C}_{uc}\}$. Applications to other special spaces X and $\mathcal{Y}$ are also straightforward. Note, $X \subset \omega_A$ implies $X \subset \omega_{\mathcal{A}A}$, if $A^{(\nu)}$ is row-finite. So the results in Section 4 are applicable to $\mathcal{A}A$ here.

We aim to obtain results like Corollary 8(a) in [4] where the space c_0 is replaced with any one of the spaces $bv(p)$, $\ell(p)$, $\ell_\infty(p)$, $c_0(p)$, and $c(p)$.

The same simple procedure is naturally also possible for all double sequence spaces $\mathcal{Y}$ considered in Section 4 and for all sequence spaces X with known characterizations of $A \in (X, Y)$ where $Y \in \{\ell_\infty, c, c_0\}$.

Let $\mathcal{Y} = \mathcal{M}_{au}$. The following Corollaries are immediate applications of Proposition 12 and Corollary 2, parts 8)–14).

Corollary 3 *Let $A^{(\nu)}$ $(\nu \in \mathbb{N})$ be row-finite and let A be any matrix with $\ell_\infty(p) \subset \omega_A$, that is $(a_{nk})_k \in M_\infty(p)$ for each $n \in \mathbb{N}$. Then the following statements are equivalent:*

(a) $A \in (\ell_\infty(p), \mathcal{M}_{au\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (\ell_\infty(p), \mathcal{M}_{au})$.
(c) *The condition (41) holds, where*

$$\forall N \in \mathbb{N} \setminus \{1\},\ \exists K_N \in \mathbb{N} : \quad \sup_{n \geq K_N, \nu \in \mathbb{N}} \sum_k \left| \sum_\mu a_{n\mu}^{(\nu)} a_{\mu k} \right| N^{\frac{1}{p_k}} < \infty. \tag{41}$$

Proof The statements (a) and (b) are equivalent by Proposition 12 and (b) and (c) are equivalent by part 8) of Corollary 2, applied to $\mathcal{A}A$. □

Corollary 4 *Let $A^{(\nu)}$ $(\nu \in \mathbb{N})$ be row-finite and let A be any matrix with $c_0(p) \subset \omega_A$, that is $(a_{nk})_k \in M_0(p)$ for each $n \in \mathbb{N}$. Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 9))*:*

(a) $A \in (c_0(p), \mathcal{M}_{au\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (c_0(p), \mathcal{M}_{au})$.
(c) *The condition (42) holds, where*

$$\exists M \in \mathbb{N} \setminus \{1\},\ \exists N \in \mathbb{N} : \quad \sup_{n \geq N,\ \nu \in \mathbb{N}} \sum_k \left| \sum_\mu a_{n\mu}^{(\nu)} a_{\mu k} \right| M^{\frac{-1}{p_k}} < \infty. \tag{42}$$

Corollary 5 *Let $A^{(\nu)}$ $(\nu \in \mathbb{N})$ be row-finite and let A be any matrix with $c(p) \subset \omega_A$, that is $(a_{nk})_k \in cs \cap M_0(p)$ for each $n \in \mathbb{N}$. Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 10))*:*

(a) $A \in (c(p), \mathcal{M}_{au\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (c(p), \mathcal{M}_{au})$.
(c) *The conditions (42) and (43) hold, where*

$$\exists N_e \in \mathbb{N} : \quad \sup_{n \geq N_e, \nu \in \mathbb{N}} \left| \sum_k \sum_\mu a_{n\mu}^{(\nu)} a_{\mu k} \right| < \infty. \tag{43}$$

Corollary 6 *Let $0 < p_k \leq 1$, $A^{(\nu)}$ $(\nu \in \mathbb{N})$ be row-finite matrices, and A be any matrix with $\ell(p) \subset \omega_A$, that is $(a_{nk})_k \in M(p)$ for each $n \in \mathbb{N}$. Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 11))*:*

(a) $A \in (\ell(p), \mathcal{M}_{au\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (\ell(p), \mathcal{M}_{au})$.

(c) *The condition* (44) *holds, where*

$$\exists N \in \mathbb{N}: \qquad \sup_{n \geq N;\, k, \nu \in \mathbb{N}} \left| \sum_{\mu} a_{n\mu}^{(\nu)} a_{\mu k} \right|^{p_k} < \infty. \tag{44}$$

Corollary 7 *Let* $1 < p_k$, $A^{(\nu)}$ $(\nu \in \mathbb{N})$ *be row-finite matrices, and* A *be any matrix with* $\ell(p) \subset \omega_A$, *that is* $(a_{nk})_k \in \ell_\infty(p)$ *for each* $n \in \mathbb{N}$. *Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 12))*:*

(a) $A \in (\ell(p), \mathcal{M}_{au\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (\ell(p), \mathcal{M}_{au})$.
(c) *The condition* (45) *holds, where*

$$\exists M \in \mathbb{N} \setminus \{1\},\ \exists N \in \mathbb{N}: \qquad \sup_{n \geq N,\, \nu \in \mathbb{N}} \sum_k \left| \sum_{\mu} a_{n\mu}^{(\nu)} a_{\mu k} \right|^{q_k} M^{-q_k} < \infty. \tag{45}$$

Corollary 8 *Let* $0 < p_k \leq 1$, $A^{(\nu)}$ $(\nu \in \mathbb{N})$ *be row-finite matrices, and let* A *be any matrix with* $bv(p) \subset \omega_A$, *that is,* $\sup_r |\sum_{k=r}^{\infty} a_{nk}|^{p_r} < \infty$ *for each* $n \in \mathbb{N}$. *Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 13))*:*

(a) $A \in (bv(p), \mathcal{M}_{au\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (bv(p), \mathcal{M}_{au})$.
(c) *The condition* (46) *holds, where*

$$\exists N \in \mathbb{N}: \qquad \sup_{n \geq N;\, k, \nu \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \sum_{\mu} a_{n\mu}^{(\nu)} a_{\mu j} \right|^{p_k} < \infty. \tag{46}$$

Corollary 9 *Let* $1 < p_k$, $A^{(\nu)}$ $(\nu \in \mathbb{N})$ *be row-finite matrices, and* A *be any matrix with* $bv(p) \subset \omega_A$, *that is,*

$$\forall n \in \mathbb{N},\ \exists N \in \mathbb{N} \setminus \{1\}: \ \sum_r \left| \sum_{k=r}^{\infty} a_{nk} \right|^{q_r} N^{-q_r} < \infty \quad \text{and} \quad \sup_r \sum_{j=1}^{r} \left| \sum_{k=r}^{\infty} a_{nk} \right|^{q_j} N^{-q_j} < \infty.$$

Then the following statements are equivalent (by application of Proposition 12 and Corollary 2, part 14))*:*

(a) $A \in (bv(p), \mathcal{M}_{au\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (bv(p), \mathcal{M}_{au})$.
(c) *The condition* (47) *holds, where*

$$\exists M \in \mathbb{N} \setminus \{1\},\ \exists N \in \mathbb{N}: \qquad \sup_{n \geq N,\, \nu \in \mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} \sum_{\mu} a_{n\mu}^{(\nu)} a_{\mu j} \right|^{q_k} M^{-q_k} < \infty. \tag{47}$$

Let $\mathcal{Y} = \mathcal{M}_u$. The following corollaries are immediate applications of Proposition 12 and Corollary 2, parts 15)–21); for an example of a proof see the proof of Corollary 3.

Corollary 10 *Let* $A^{(\nu)}$ $(\nu \in \mathbb{N})$ *be row-finite and let* A *be any matrix with* $\ell_\infty(p) \subset \omega_A$, *that is* $(a_{nk})_k \in M_\infty(p)$ *for each* $n \in \mathbb{N}$. *Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 15))*:*

(a) $A \in (\ell_\infty(p), \mathcal{M}_{u\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (\ell_\infty(p), \mathcal{M}_u)$.
(c) *The condition* (48) *holds, where*

$$\forall N \in \mathbb{N} \setminus \{1\}: \quad \sup_{n,\nu \in \mathbb{N}} \sum_k \left| \sum_\mu a_{n\mu}^{(\nu)} a_{\mu k} \right| N^{\frac{1}{p_k}} < \infty. \tag{48}$$

Corollary 11 *Let $A^{(\nu)}$ $(\nu \in \mathbb{N})$ be row-finite and let A be any matrix with $c_0(p) \subset \omega_A$, that is $(a_{nk})_k \in M_0(p)$ for each $n \in \mathbb{N}$. Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 16))*:*

(a) $A \in (c_0(p), \mathcal{M}_{u\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (c_0(p), \mathcal{M}_u)$.
(c) *The condition* (49) *holds, where*

$$\exists M \in \mathbb{N} \setminus \{1\}: \quad \sup_{n,\nu \in \mathbb{N}} \sum_k \left| \sum_\mu a_{n\mu}^{(\nu)} a_{\mu k} \right| M^{\frac{-1}{p_k}} < \infty. \tag{49}$$

Corollary 12 *Let $A^{(\nu)}$ $(\nu \in \mathbb{N})$ be row-finite and let A be any matrix with $c(p) \subset \omega_A$, that is $(a_{nk})_k \in cs \cap M_0(p)$ for each $n \in \mathbb{N}$. Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 17))*:*

(a) $A \in (c(p), \mathcal{M}_{u\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (c(p), \mathcal{M}_u)$.
(c) *The conditions* (49) *and* (50) *hold, where*

$$\sup_{n,\nu \in \mathbb{N}} \left| \sum_k \sum_\mu a_{n\mu}^{(\nu)} a_{\mu k} \right| < \infty. \tag{50}$$

Corollary 13 *Let $0 < p_k \leq 1$, $A^{(\nu)}$ $(\nu \in \mathbb{N})$ be row-finite matrices, and A be any matrix with $\ell(p) \subset \omega_A$, that is $(a_{nk})_k \in M(p)$ for each $n \in \mathbb{N}$. Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 18))*:*

(a) $A \in (\ell(p), \mathcal{M}_{u\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (\ell(p), \mathcal{M}_u)$.
(c) *The condition* (51) *holds, where*

$$\sup_{n,k,\nu \in \mathbb{N}} \left| \sum_\mu a_{n\mu}^{(\nu)} a_{\mu k} \right|^{p_k} < \infty. \tag{51}$$

Corollary 14 *Let $1 < p_k$, $A^{(\nu)}$ $(\nu \in \mathbb{N})$ be row-finite matrices, and A be any matrix with $\ell(p) \subset \omega_A$, that is $(a_{nk})_k \in \ell_\infty(p)$ for each $n \in \mathbb{N}$. Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 19))*:*

(a) $A \in (\ell(p), \mathcal{M}_{u\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (\ell(p), \mathcal{M}_u)$.
(c) *The condition* (52) *holds, where*

$$\exists M \in \mathbb{N} \setminus \{1\}: \quad \sup_{n,\nu \in \mathbb{N}} \sum_k \left| \sum_\mu a_{n\mu}^{(\nu)} a_{\mu k} \right|^{q_k} M^{-q_k} < \infty. \tag{52}$$

Corollary 15 *Let* $0 < p_k \leq 1$, $A^{(\nu)}$ $(\nu \in \mathbb{N})$ *be row-finite matrices, and let A be any matrix with* $bv(p) \subset \omega_A$, *that is,* $\sup_r |\sum_{k=r}^{\infty} a_{nk}|^{p_r} < \infty$ *for each* $n \in \mathbb{N}$. *Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 20))*:*

(a) $A \in (bv(p), \mathcal{M}_{u\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (bv(p), \mathcal{M}_u)$.
(c) *The condition* (53) *holds, where*

$$\sup_{n,k,\nu\in\mathbb{N}} \left| \sum_{j=k}^{\infty} \sum_{\mu} a_{n\mu}^{(\nu)} a_{\mu j} \right|^{p_k} < \infty. \tag{53}$$

Corollary 16 *Let* $1 < p_k$, $A^{(\nu)}$ $(\nu \in \mathbb{N})$ *be row-finite matrices, and A be any matrix with* $bv(p) \subset \omega_A$, *that is*

$$\forall n \in \mathbb{N},\ \exists N \in \mathbb{N}\backslash\{1\} : \sum_r \left| \sum_{k=r}^{\infty} a_{nk} \right|^{q_r} N^{-q_r} < \infty \quad \textit{and} \quad \sup_r \sum_{j=1}^{r} \left| \sum_{k=r}^{\infty} a_{nk} \right|^{q_j} N^{-q_j} < \infty.$$

Then the following statements are equivalent (by application of Proposition 12 and Corollary 2, part 21))*:*

(a) $A \in (bv(p), \mathcal{M}_{u\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (bv(p), \mathcal{M}_u)$.
(c) *The condition* (54) *holds, where*

$$\exists M \in \mathbb{N} \setminus \{1\} : \quad \sup_{n,\nu\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} \sum_{\mu} a_{n\mu}^{(\nu)} a_{\mu j} \right|^{q_k} M^{-q_k} < \infty. \tag{54}$$

Let $\mathcal{Y} = \mathcal{C}_{uc}$**.** The following corollaries are immediate applications of Proposition 12 and Corollary 2, parts 36)–42); for an example of a proof see the proof of Corollary 3.

Corollary 17 *Let* $A^{(\nu)}$ $(\nu \in \mathbb{N})$ *be row-finite and let A be any matrix with* $\ell_\infty(p) \subset \omega_A$, *that is* $(a_{nk})_k \in M_\infty(p)$ *for each* $n \in \mathbb{N}$. *Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 36))*:*

(a) $A \in (\ell_\infty(p), \mathcal{C}_{uc\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (\ell_\infty(p), \mathcal{C}_{uc})$.
(c) *The conditions* (55) *and* (56) *hold, where*

$$\forall N \in \mathbb{N}\backslash\{1\}, \exists K_N \in \mathbb{N} : \quad \sum_k \left| \sum_{\mu} a_{n\mu}^{(\nu)} a_{\mu k} \right| N^{\frac{1}{p_k}} \tag{55}$$

converges uniformly in $n \geq K_N$ *and* $\nu \in \mathbb{N}$,

$$\forall k \in \mathbb{N}, \exists \beta_k \in \mathbb{K} : \quad \lim_r \sup_{n\geq r,\nu\in\mathbb{N}} \left| \sum_{\mu} a_{n\mu}^{(\nu)} a_{\mu k} - \beta_k \right| = 0. \tag{56}$$

Corollary 18 *Let* $A^{(\nu)}$ $(\nu \in \mathbb{N})$ *be row-finite and let A be any matrix with* $c_0(p) \subset \omega_A$, *that is* $(a_{nk})_k \in M_0(p)$ *for each* $n \in \mathbb{N}$. *Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 37))*:*

(a) $A \in (c_0(p), \mathcal{C}_{uc\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (c_0(p), \mathcal{C}_{uc})$.
(c) *The conditions* (56) *and* (42) *hold.*

Corollary 19 *Let $A^{(\nu)}$ $(\nu \in \mathbb{N})$ be row-finite and let A be any matrix with $c(p) \subset \omega_A$, that is $(a_{nk})_k \in cs \cap M_0(p)$ for each $n \in \mathbb{N}$. Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 38))*:*

(a) $A \in (c(p), \mathcal{C}_{uc\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (c(p), \mathcal{C}_{uc})$.
(c) *The conditions* (56), (42), *and* (57) *hold, where*

$$\exists \beta \in \mathbb{K}: \quad \lim_r \sup_{n \geq r, \nu \in \mathbb{N}} \left| \sum_k \sum_\mu a_{n\mu}^{(\nu)} a_{\mu k} - \beta \right| = 0. \tag{57}$$

Corollary 20 *Let $0 < p_k \leq 1$, $A^{(\nu)}$ $(\nu \in \mathbb{N})$ be row-finite matrices, and A be any matrix with $\ell(p) \subset \omega_A$, that is $(a_{nk})_k \in M(p)$ for each $n \in \mathbb{N}$. Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 39))*:*

(a) $A \in (\ell(p), \mathcal{C}_{uc\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (\ell(p), \mathcal{C}_{uc})$.
(c) *The conditions* (56) *and* (44) *hold.*

Corollary 21 *Let $1 < p_k$, $A^{(\nu)}$ $(\nu \in \mathbb{N})$ be row-finite matrices, and A be any matrix with $\ell(p) \subset \omega_A$, that is $(a_{nk})_k \in \ell_\infty(p)$ for each $n \in \mathbb{N}$. Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 40))*:*

(a) $A \in (\ell(p), \mathcal{C}_{uc\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (\ell(p), \mathcal{C}_{uc})$.
(c) *The conditions* (56) *and* (45) *hold.*

Corollary 22 *Let $0 < p_k \leq 1$, $A^{(\nu)}$ $(\nu \in \mathbb{N})$ be row-finite matrices, and let A be any matrix with $bv(p) \subset \omega_A$, that is, $\sup_r |\sum_{k=r}^{\infty} a_{nk}|^{p_r} < \infty$ for each $n \in \mathbb{N}$. Then the following statements are equivalent* (by application of Proposition 12 and Corollary 2, part 41))*:*

(a) $A \in (bv(p), \mathcal{C}_{uc\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (bv(p), \mathcal{C}_{uc})$.
(c) *The conditions* (46) *and* (58) *hold, where*

$$\forall k \in \mathbb{N},\ \exists B_k \in \mathbb{K}: \quad \lim_r \sup_{n \geq r, \nu \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \sum_\mu a_{n\mu}^{(\nu)} a_{\mu j} - B_k \right| = 0. \tag{58}$$

Corollary 23 *Let $1 < p_k$, $A^{(\nu)}$ $(\nu \in \mathbb{N})$ be row-finite matrices, and A be any matrix with $bv(p) \subset \omega_A$, that is,*

$$\forall n \in \mathbb{N},\ \exists N \in \mathbb{N}\backslash\{1\}: \sum_r \left| \sum_{k=r}^{\infty} a_{nk} \right|^{q_r} N^{-q_r} < \infty \quad and \quad \sup_r \sum_{j=1}^{r} \left| \sum_{k=r}^{\infty} a_{nk} \right|^{q_j} N^{-q_j} < \infty.$$

Then the following statements are equivalent (by application of Proposition 12 and Corollary 2, part 42))*:*

(a) $A \in (bv(p), \mathcal{C}_{uc\mathcal{A}})$.
(b) $\mathcal{B} = \mathcal{A}A \in (bv(p), \mathcal{C}_{uc})$.
(c) *The conditions* (58) *and* (47) *hold.*

Remark 5 For the case $\mathcal{Y} := \mathcal{C}_{u0}$ consider Corollary 2, part 43)–49) and replace in Corollaries 17–23 the space $\mathcal{C}_{uc}$ by $\mathcal{C}_{u0}$ as well as β_k, $B_k (k \in \mathbb{N})$ and β by 0.

6 Characterization of $A \in (X, Y)$ where $Y \in \{ac_\infty, ac, ac_0\}$

Now we are ready to deduce the characterizations of $A \in (X, Y)$ where $X \in \{\ell_\infty(p), c_0(p), c(p), \ell(p), bv(p)\}$ and $Y \in \{ac_\infty, ac, ac_0\}$ from the results in Section 4; in particular, we get the *correct versions* of the claims by Ganie and Sheikh in [5].

The following results are simple applications of the corresponding results in Section 5 in the case of $\mathcal{A}_\sigma$ (cf. Example 2) and also in Section 3. So we state the results without proofs.

First of all, let $Y = ac_\infty$, that is, $Y = ac_\infty = wac_\infty = \ell_\infty$ by Proposition 3. Thus we may characterize $A \in (X, Y)$ where X is a fixed member of $\{\ell_\infty(p), c_0(p), c(p), \ell(p), bv(p)\}$. Through this, we get three equivalent statements characterizing $A \in (X, ac_\infty)$; in the cases, ac_∞ and wac_∞, we deduce the characterization from our results in Section 5 and in the case ℓ_∞ from the results of Jarrah and Malkowsky in [7] (which we presented in Section 3). Without doubt, in any case of $X \in \{\ell_\infty(p), c_0(p), c(p), \ell(p), bv(p)\}$, the third characterization is much easier to handle than the others.

Applications 1 *As above we assume in any case that $X \subset \omega_A$ where X is a fixed member of $\{\ell_\infty(p), c_0(p), c(p), \ell(p), bv(p)\}$.*

(a) $A \in (\ell_\infty(p), ac_\infty) \iff$ (59) *holds* (*by* (48)), *where*

$$\forall N \in \mathbb{N} \setminus \{1\} : \quad \sup_{n,\nu \in \mathbb{N}} \sum_k \left| \frac{1}{n} \sum_{\mu=\nu}^{\nu+n-1} a_{\mu k} \right| N^{\frac{1}{p_k}} < \infty \tag{59}$$

$\iff$ (60) *holds* (*by* (41)), *where*

$$\forall N \in \mathbb{N} \setminus \{1\},\ \exists K_N \in \mathbb{N} : \quad \sup_{n \geq K_N, \nu \in \mathbb{N}} \sum_k \left| \frac{1}{n} \sum_{\mu=\nu}^{\nu+n-1} a_{\mu k} \right| N^{\frac{1}{p_k}} < \infty \tag{60}$$

$\iff$ (61) *holds* (*by Proposition 4(a)*), *where*

$$\forall N \in \mathbb{N} \setminus \{1\} : \quad \sup_n \sum_k |a_{nk}| N^{\frac{1}{p_k}} < \infty. \tag{61}$$

(b) $A \in (c_0(p), ac_\infty) \iff$ (62) *holds* (*by* (49)), *where*

$$\exists M \in \mathbb{N} \setminus \{1\} : \quad \sup_{n,\nu \in \mathbb{N}} \sum_k \left| \frac{1}{n} \sum_{\mu=\nu}^{\nu+n-1} a_{\mu k} \right| M^{\frac{-1}{p_k}} < \infty \tag{62}$$

$\iff$ (63) *holds* (*by* (42)), *where*

$$\exists M \in \mathbb{N} \setminus \{1\},\ \exists N \in \mathbb{N} : \quad \sup_{n \geq N, \nu \in \mathbb{N}} \sum_k \left| \frac{1}{n} \sum_{\mu=\nu}^{\nu+n-1} a_{\mu k} \right| M^{\frac{-1}{p_k}} < \infty \tag{63}$$

$\iff$ (64) *holds* (*by Proposition 5(a)*), *where*

$$\exists M \in \mathbb{N} \setminus \{1\}: \quad \sup_n \sum_k |a_{nk}| M^{\frac{-1}{p_k}} < \infty. \tag{64}$$

(c) $A \in (c(p), ac_\infty) \iff$ (62) *and* (65) *holds* (*by Corollary 12*(*c*)), *where*

$$\sup_{n,\nu \in \mathbb{N}} \left| \frac{1}{n} \sum_k \sum_{\mu=\nu}^{\nu+n-1} a_{\mu k} \right| < \infty \tag{65}$$

$\iff$ (63) *and* (66) *hold* (*by Corollary 5*(*c*)), *where*

$$\exists N_e \in \mathbb{N}: \quad \sup_{n \geq N_e, \nu \in \mathbb{N}} \left| \frac{1}{n} \sum_k \sum_{\mu=\nu}^{\nu+n-1} a_{\mu k} \right| < \infty \tag{66}$$

$\iff$ (64) *and* (67) *hold* (*by Proposition 6(a)*), *where*

$$(a_{nk})_k \in cs \;\; (n \in \mathbb{N}) \quad \text{and} \quad \sup_n \left| \sum_k a_{nk} \right| < \infty. \tag{67}$$

(d) *Let* $0 < p_k \leq 1$. *Then* $A \in (\ell(p), ac_\infty) \iff$ (68) *holds* (*by* (51)), *where*

$$\sup_{n,k,\nu \in \mathbb{N}} \left| \frac{1}{n} \sum_{\mu=\nu}^{\nu+n-1} a_{\mu k} \right|^{p_k} < \infty \tag{68}$$

$\iff$ (69) *holds* (*by* (44)), *where*

$$\exists N \in \mathbb{N}: \quad \sup_{n \geq N, k, \nu \in \mathbb{N}} \left| \frac{1}{n} \sum_{\mu=\nu}^{\nu+n-1} a_{\mu k} \right|^{p_k} < \infty \tag{69}$$

$\iff$ (70) *holds* (*by Proposition 8(a)*), *where*

$$\sup_{n,k \in \mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{70}$$

(e) *Let* $1 < p_k$. *Then* $A \in (\ell(p), ac_\infty) \iff$ (71) *holds* (*by* (52)), *where*

$$\exists M \in \mathbb{N} \setminus \{1\}: \quad \sup_{n,\nu \in \mathbb{N}} \frac{1}{n} \sum_k \left| \sum_{\mu=\nu}^{\nu+n-1} a_{\mu k} \right|^{q_k} M^{-q_k} < \infty \tag{71}$$

$\iff$ (72) *holds* (*by* (45)), *where*

$$\exists M \in \mathbb{N} \setminus \{1\}, \; \exists N \in \mathbb{N}: \quad \sup_{n \geq N, \nu \in \mathbb{N}} \frac{1}{n} \sum_k \left| \sum_{\mu=\nu}^{\nu+n-1} a_{\mu k} \right|^{q_k} M^{-q_k} < \infty \tag{72}$$

$\iff$ (73) *holds* (*by Proposition 7(a)*), *where*

$$\exists M \in \mathbb{N} \setminus \{1\}: \quad \sup_n \sum_k |a_{nk}|^{q_k} M^{-q_k} < \infty. \tag{73}$$

(f) *Let* $0 < p_k \leq 1$. *Then* $A \in (bv(p), ac_\infty) \iff$ (74) *holds* (*by* (53)), *where*

$$\sup_{n,k,\nu \in \mathbb{N}} \frac{1}{n} \left| \sum_{j=k}^{\infty} \sum_{\mu=\nu}^{\nu+n-1} a_{\mu j} \right|^{p_k} < \infty \tag{74}$$

$\Longleftrightarrow$ (75) *holds* (*by* (46)), *where*

$$\exists N \in \mathbb{N}: \quad \sup_{n\geq N,k,\nu\in\mathbb{N}} \frac{1}{n}\left|\sum_{j=k}^{\infty}\sum_{\mu=\nu}^{\nu+n-1} a_{\mu j}\right|^{p_k} < \infty \tag{75}$$

$\Longleftrightarrow$ (76) *holds* (*by Proposition 10(a)*), *where*

$$\sup_{n,k}\left|\sum_{j=k}^{\infty} a_{nj}\right|^{p_k} < \infty. \tag{76}$$

(g) *Let* $1 < p_k$. *Then* $A \in (bv(p), ac_\infty) \Longleftrightarrow$ (77) *holds* (*by Corollary 16(c)*), *where*

$$\exists M \in \mathbb{N}\setminus\{1\}: \quad \sup_{n,\nu\in\mathbb{N}} \frac{1}{n}\sum_k\left|\sum_{j=k}^{\infty}\sum_{\mu=\nu}^{\nu+n-1} a_{\mu j}\right|^{q_k} M^{-q_k} < \infty \tag{77}$$

$\Longleftrightarrow$ (78) *holds* (*by Corollary* 9(*c*)), *where*

$$\exists M \in \mathbb{N}\setminus\{1\},\ \exists N \in \mathbb{N}: \quad \sup_{n\geq N,\nu\in\mathbb{N}} \frac{1}{n}\sum_k\left|\sum_{j=k}^{\infty}\sum_{\mu=\nu}^{\nu+n-1} a_{\mu j}\right|^{q_k} M^{-q_k} < \infty \tag{78}$$

$\Longleftrightarrow$ (79) *holds* (*by Proposition 9(a)*), *where*

$$\exists M \in \mathbb{N}\setminus\{1\}: \quad \sup_n \sum_k\left|\sum_{j=k}^{\infty} a_{nj}\right|^{q_k} M^{-q_k} < \infty. \tag{79}$$

Now, **let** $Y = ac$ or $Y = ac_0$.

Applications 2 *As above we assume in any case that* $X \subset \omega_A$, *where* X *is a fixed member of* $\{\ell_\infty(p), c_0(p), c(p), \ell(p), bv(p)\}$.

(a) $A \in (\ell_\infty(p), ac) \Longleftrightarrow$ (80) *and* (81) *hold* (*by Corollary 17(c)*), *where*

$$\forall N \in \mathbb{N}\setminus\{1\},\ \exists K_N \in \mathbb{N}: \quad \sum_k\left|\frac{1}{n}\sum_{\mu=\nu}^{\nu+n-1} a_{\mu k}\right| N^{\frac{1}{p_k}}$$

$$\text{converges uniformly in } n \geq K_N \text{ and } \nu \in \mathbb{N}, \tag{80}$$

$$\forall k \in \mathbb{N},\ \exists \beta_k \in \mathbb{K}: \quad \lim_r \sup_{n\geq r,\nu\in\mathbb{N}}\left|\frac{1}{n}\sum_{\mu=\nu}^{\nu+n-1} a_{\mu k} - \beta_k\right| = 0. \tag{81}$$

(b) $A \in (c_0(p), ac) \Longleftrightarrow$ (81) *and* (63) *hold* (*by Corollary 18(c)*).

(c) $A \in (c(p), ac) \Longleftrightarrow$ (81), (63) *and* (82) *hold* (*by Corollary 19(c)*), *where*

$$\exists \beta \in \mathbb{K}: \quad \lim_r \sup_{n\geq r,\nu\in\mathbb{N}}\left|\frac{1}{n}\sum_k\sum_{\mu=\nu}^{\nu+n-1} a_{\mu k} - \beta\right| = 0. \tag{82}$$

(d) *Let* $0 < p_k \leq 1$. *Then* $A \in (\ell(p), ac) \Longleftrightarrow$ (81) *and* (69) *hold* (*by Corollary 20(c)*).

(e) *Let* $1 < p_k$. *Then* $A \in (\ell(p), ac) \Longleftrightarrow$ (81) *and* (83) *hold* (*by Corollary 21(c)*), *where*

$$\exists M \in \mathbb{N}\setminus\{1\},\ \exists N \in \mathbb{N}: \quad \sup_{n\geq N,\ \nu\in\mathbb{N}} \frac{1}{n}\sum_k\left|\sum_{\mu=\nu}^{\nu+n-1} a_{\mu k}\right|^{q_k} M^{-q_k} < \infty. \tag{83}$$

(f) *Let* $0 < p_k \leq 1$. *Then* $A \in (bv(p), ac) \iff$ (84) *and* (85) *hold* (*by Corollary 22(c)*), *where*

$$\exists N \in \mathbb{N}: \qquad \sup_{n \geq N;\, k,\nu \in \mathbb{N}} \frac{1}{n} \left| \sum_{j=k}^{\infty} \sum_{\mu=\nu}^{\nu+n-1} a_{\mu j} \right|^{p_k} < \infty, \tag{84}$$

$$\forall k \in \mathbb{N},\ \exists B_k \in \mathbb{K}: \qquad \lim_r \sup_{n \geq r, \nu \in \mathbb{N}} \frac{1}{n} \left| \sum_{j=k}^{\infty} \sum_{\mu=\nu}^{\nu+n-1} a_{\mu j} - B_k \right| = 0. \tag{85}$$

(g) *Let* $1 < p_k$. *Then* $A \in (bv(p), ac) \iff$ (85) *and* (86) *hold* (*by Corollary 23(c)*), *where*

$$\exists M \in \mathbb{N} \setminus \{1\},\ \exists N \in \mathbb{N}: \qquad \sup_{n \geq N,\, \nu \in \mathbb{N}} \frac{1}{n} \sum_k \left| \sum_{j=k}^{\infty} \sum_{\mu=\nu}^{\nu+n-1} a_{\mu j} \right|^{q_k} M^{-q_k} < \infty. \tag{86}$$

Remark 6 For the case $Y = ac_0$ replace in Applications 2 (a)–(g), the space ac by ac_0 as well as β_k, B_k ($k \in \mathbb{N}$) and β by 0.

Remark 7 (Claims by Ganie and Sheikh)

(a) The first equivalence in Applications 1(g) corresponds to Theorem 1 in [5], where the (wrong) characterizing condition in [5, Theorem 1] has been replaced by (77). The further equivalences in this application may be considered as an extension of [5, Theorem 1].
(b) Application 2(g) corresponds to Theorem 2 in [5], where the (wrong) characterizing conditions in [5, Theorem 2] have been replaced by (85) and (86).
(c) Following Remark 6, we get a corresponding theorem to [5, Theorem 3] if we replace in Applications 2(g) the space ac by ac_0 as well as β_k, B_k ($k \in \mathbb{N}$) and β by 0.
(d) Ganie and Sheikh assumed in [5, Theorems 1–3] that $p \in \ell_\infty$ and $1 < p_k$ ($k \in \mathbb{N}$). Analogously to (a), (b), and (c), we get the corresponding results in the case $0 < p_k \leq 1$ ($k \in \mathbb{N}$) by Applications 1(f) and 2(f) as well as Remark 6, respectively.

Remark 8 Differently than in Section 3, we did not consider the special case $0 < \liminf_k p_k \leq \limsup_k p_k < \infty$ in Sections 4–6. The interested reader should note Remarks 1 and 3 and may consult [4] for the corresponding results.

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